Circuit Quantum Electrodynamics (cQED) and Superconducting Qubits

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July 03, 2025

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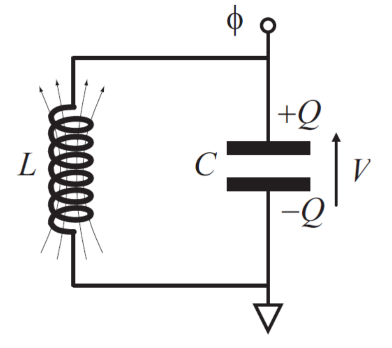
Quantum Circuit Elements

Classical LC Oscillator



- ϕ is the node flux defined as $\phi(t) = \int_{-\infty}^{t} d\tau V(\tau)$

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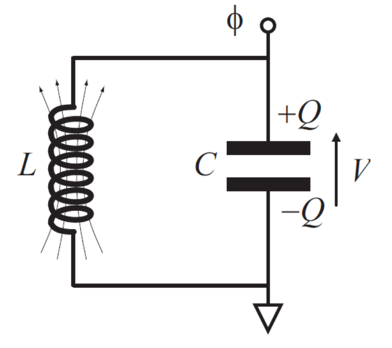

- ϕ is the node flux defined as $\phi(t) = \int_{-\infty}^{t} d\tau V(\tau)$
- **Hamiltonian**: $H = \frac{Q^2}{2C} + \frac{1}{2} C\omega_r^2 \phi^2$
  This is in the form of a mechanical oscillator of coordinate ϕ, conjugate momentum Q, and mass C

Quantum LC Oscillator

- Promote ϕ and Q to non-commuting operators, obeying the canonical commutation relation $[\hat{\phi}, \hat{Q}] = i\hbar$.

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$$H = \hbar\Omega \left\{ \hat{a}^\dagger \hat{a} + \frac{1}{2} \right\}$$

in terms of standard annihilation and creation operators

$$\hat{a} = +i\frac{1}{\sqrt{2C\hbar\Omega}}\hat{Q} + \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\phi}$$

$$\hat{a}^\dagger = -i\frac{1}{\sqrt{2C\hbar\Omega}}\hat{Q} + \frac{1}{\sqrt{2L\hbar\Omega}}\hat{\phi}$$

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that obey the relation
$$[\hat{a}, \hat{a}^\dagger] = 1$$
- The $\hat{a}^\dagger$ operator creates a quantized excitation of the flux and charge, which is interpreted as a photon of frequency Ω stored in the circuit.

Quantum LC Oscillator (Cont.)

Conditions on LC oscillator to work in the quantum regime

- The energy separation $\hbar\Omega$ must be significantly greater than the thermal energy $k_B T$.
- The oscillator must be sufficiently decoupled from uncontrolled degrees of freedom.

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Conditions on LC oscillator to work in the quantum regime

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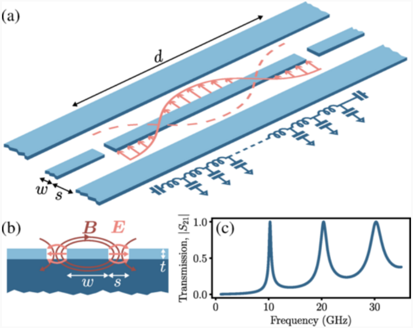
- The oscillator must be sufficiently decoupled from uncontrolled degrees of freedom.

Using superconducting material allows us to achieve these conditions (in addition to other advantages).
The energy gap between any two levels is $\hbar\Omega$. This does not give us the freedom to isolate 2 energy levels.

Transmission-Line Resonator



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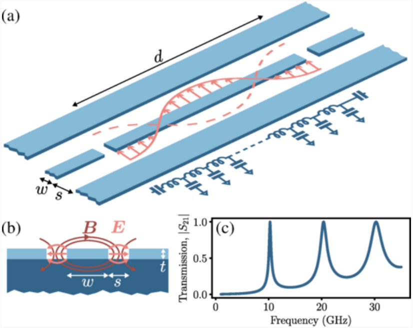

The classical Hamiltonian corresponding to the lumped-element circuit can be written as

$$H = \int_0^d dx \left\{ \frac{1}{2c} Q^2 + \frac{1}{2l} (\partial_x \phi)^2 \right\}$$

where c and l are capacitance and inductance per unit length, $\phi \equiv \int_{-\infty}^{t} d\tau V(x, \tau)$ is the generalized flux and Q being its conjugate momentum.

Transmission-Line Resonator (Cont.)

The solutions to the Hamilton's equation can be expressed in terms of the normal modes

$$\phi(x,t) = \sum_{m=0}^{\infty} u_m(t)\phi_m(x)$$

with

$$\ddot{u}_m(t) = -\omega_m^2 u_m(t)$$
$$\phi_m(x) = \sqrt{2}cos(k_m x)$$

where $m \in \{0, 1, 2, ...\}$ and $k_m = m\pi/d$. These solutions are for an open-ended $\lambda/2$ resonator.

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we can express the Hamiltonian as a sum over independent harmonic oscillators

$$H = \sum_{m=0}^{\infty} \left\{ \frac{1}{2dc} Q_m^2 + \frac{dc}{2}\omega_m^2 u_m^2 \right\}$$

Transmission-Line Resonator (Cont.)

Following the quantization procedure as before, we get

$$H = \sum_{m=0}^{\infty} \hbar\omega_m \hat{a}_m^\dagger \hat{a}_m$$

in terms of annihilation and creation operators

$$\hat{a}_m = +i\sqrt{\frac{2Z_m}{\hbar}}\hat{Q}_m + \sqrt{\frac{2}{\hbar Z_m}}\hat{u}_m, \quad \hat{a}_m^\dagger = -i\sqrt{\frac{2Z_m}{\hbar}}\hat{Q}_m + \sqrt{\frac{2}{\hbar Z_m}}\hat{u}_m$$

that obey the relation

$$[\hat{a}_m, \hat{a}_m^\dagger] = 1$$

with $Z_m = \sqrt{L_m/dc}$ the characteristic impedance of mode m, $\omega_m = (m+1)\omega_0$ the mode frequency and $\omega_0/2\pi = v_0/2d$ the fundamental frequency of the $\lambda/2$ transmission-line resonator.

Superconductivity and Josephson Junction

Superconductivity

- Superconductivity is a physical phenomenon where, in certain metals and compounds, an electric current flows with zero resistance below a critical temperature T_c.
- For an integrated circuit, using metallic superconductors fulfills two important requirements:
    1. The absence of dissipation
    2. The typical energy of thermal fluctuations is much smaller than the energy quantum associated with the transitions between states.

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- For an integrated circuit, using metallic superconductors fulfills two important requirements:
  1. The absence of dissipation
  2. The typical energy of thermal fluctuations is much smaller than the energy quantum associated with the transitions between states.
- This phenomenon is explained by considering Cooper pair of electrons which are pairs of opposite spin electrons that bind together via an effective attraction mediated by phonons.

Josephson Junction

- A Josephson tunnel junction consists of two metallic electrodes separated by a thin oxide barrier.
- For such a system, Josephson showed that the supercurrent is given by

$$I = I_c \sin \phi$$

where I_c is the maximum possible dissipationless current whose magnitude is determined by the junction size and material parameters, and ϕ is the gauge-invariant phase difference across the junction. Josephson also showed that, in presence of a potential difference V across the junction, the phase difference obeys

$$\partial_t \phi = \frac{2\pi}{\Phi_0} V$$

where Φ_0 is the flux quantum.

Josephson Junction

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- We define the differential Josephson inductance as

$$L_J(\Phi) = \left\{\frac{\partial I}{\partial \Phi}\right\}^{-1} = \frac{\Phi_0}{2\pi I_c}\frac{1}{cos(2\pi\Phi/\Phi_0)}$$

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  This shows that, under the critical temperature, the Josephson junction acts as a non-linear inductor.
- The non-linearity will allow us to isolate two energy levels of our artificial atom.

Superconducting Qubits

Transmon Artificial Atom

Replace the geometric inductance L of the LC oscillator with a Josephson junction.

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The energy of the non-linear inductance takes the form

$$E = -E_J \cos\left(\frac{2\pi\Phi}{\Phi_0}\right)$$

with $E_J = \Phi_0 I_c / 2\pi$ the Josephson energy.

The energy of the capacitor, including a possible offset charge term, takes the form

$$E = \frac{(Q - Q_g)}{2C_\Sigma}$$

with $C_\Sigma = C_J + C_S$ the total capacitance, including the junction's capacitance and the shunt capacitance and Q_g is the possible offset charge term representing the effect of an external electric field bias or some microscopic junction asymmetry which breaks the degeneracy between positive and negative charge transfers

Transmon Artificial Atom (Cont.)

The quantized Hamiltonian of the capacitively shunted Josephson junction becomes

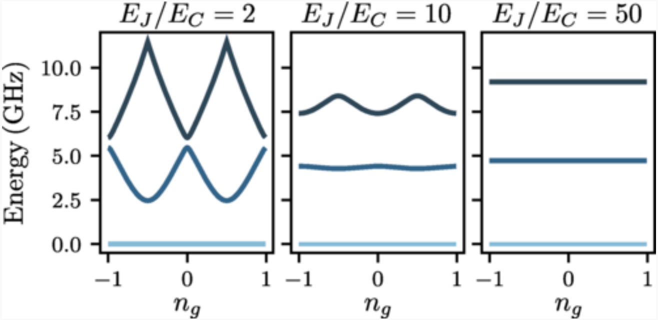
$$\hat{H} = \frac{(\hat{Q} - Q_g)}{2C_\Sigma} - E_J \cos\left(\frac{2\pi\hat{\Phi}}{\Phi_0}\right)$$
$$= 4E_C(\hat{n} - n_g)^2 - E_J \cos\hat{\phi}$$

where we define the charging energy $E_C = e^2/2C_\Sigma$, the charge number density $\hat{n} = \hat{Q}/2e$, the offset charge number density $n_g = Q_g/2e$ and the phase operator $\hat{\phi} = (2\pi/\Phi_0)\hat{\Phi}$.

Transmon Artificial Atom (Cont.)

The spectrum of $\hat{H}$ is controlled by the ratio E_J/E_C

- $E_J/E_C \ll 1$ corresponding to charge qubits
- $E_J/E_C \sim 1$ corresponding to the quantronium
- $E_J/E_C \gg 1$ corresponding to the transmon.



Transmon Artificial Atom (Cont.)

In the transmon regime the charge degree of freedom is highly delocalized due to large E_J and the first energy levels essentially become independent of the gate charge. The approximate transmon Hamiltonian takes the form

$$\hat{H}_q = 4E_C\hat{n}^2 + \frac{1}{2}E_J\hat{\phi}^2 - \frac{1}{4!}E_J\hat{\phi}^4$$

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Introducing the creation and annihilation operators, to diagonalize the first two terms, as

$$\hat{b} = +2i\left(\frac{2E_C}{E_J}\right)^{1/4}\hat{n} + \left(\frac{E_J}{2E_C}\right)^{1/4}\hat{\phi}, \quad \hat{b}^\dagger = -2i\left(\frac{2E_C}{E_J}\right)^{1/4}\hat{n} + \left(\frac{E_J}{2E_C}\right)^{1/4}\hat{\phi}$$

and writing the approximate transmon Hamiltonian as

$$\hat{H}_q = \hbar\omega_q\hat{b}^\dagger\hat{b} - \frac{E_C}{2}\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}$$

where $\hbar\omega_q = \sqrt{8E_C E_J} - E_C$

Other Superconducting Qubit Designs

- A useful variant is the **flux-tunable transmon**, where the single Josephson junction is replaced by two parallel junctions forming a **SQUID**.
- This allows the transmon frequency to be tuned by an **external magnetic flux** threading the SQUID loop. This tunability enables fast changes in qubit frequency, useful for quantum logical gates.
- However, flux-tunable transmons are susceptible to **dephasing due to flux noise**.

Other Superconducting Qubit Designs (Cont.)

Other types of superconducting qubits include:

- **Charge qubits:** Low E_J/E_C, sensitive to charge noise.
- **Flux qubits:** Based on superconducting loop with Josephson junctions.
- **Phase qubits:** Josephson junction shunted by a lumped element inductor.
- **Quantronium:** An intermediate E_J/E_C regime.
- **Fluxonium qubit:** Small Josephson junction shunted by a high inductance from a series array of large-capacitance tunnel junctions. The fluxonium can maintain large anharmonicity while suppressing offset charge noise effects.

Circuit QED

Jaynes-Cummings Model

- The **Jaynes-Cummings Hamiltonian** is the foundational model describing the interaction between a two-level atom and a single mode of the electromagnetic field.

$$\hat{H} = \hbar\omega_r \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_q}{2}\hat{\sigma}_z + \hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-)$$

where ω_r is the frequency of the mode that transmon interacts with primarily, g is the coupling constant, $\hat{\sigma}_- = |g\rangle\langle e|$, $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$.

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- In this model, the uncoupled states are qubit-photon states (e.g., $|g, n\rangle$, $|e, n\rangle$), and the coupled system forms **dressed states** or **polaritons**, which are the true eigenstates of the system.
- In the resonant regime ($\Delta = \omega_q - \omega_r = 0$), the degeneracy of states with $n+1$ quanta is lifted by $2g\sqrt{n+1}$ due to atom-photon interaction, leading to **vacuum Rabi splitting**.

Jaynes-Cummings Model (Cont.)

In the dispersive regime ($\Delta \gg g$), coherent exchange of quanta is suppressed, and interaction occur via virtual photon processes. The effective dispersive Hamiltonian takes the form

$$\hat{H}_{disp} \approx \hbar\omega_r^0 \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_q^0}{2}\hat{\sigma}_z + \hbar\chi\hat{a}^\dagger \hat{a}\hat{\sigma}_z$$

Here, ω_r^0 and ω_q^0 are the dressed (renormalized) resonator and qubit frequencies, respectively, and χ is the qubit-state-dependent dispersive cavity shift.

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This dispersive coupling is Quantum Non-Demolition (QND) with respect to photon number and qubit polarization, meaning it commutes with both, making it ideal for qubit readout.

Jaynes-Cummings Model (Cont.)

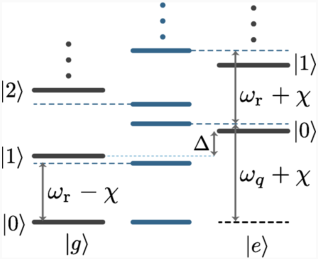

Figure 1: Energy spectrum of uncoupled and dressed states in the dispersive regime

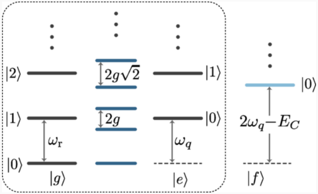

Figure 2: Energy spectrum of uncoupled states in the resonant regime

Wiring up Quantum Systems with Transmission Lines

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- A complete description of quantum electrical circuits must account for their coupling to the environment, including measurement apparatus and control circuitry.
- The environment plays a **dual role**: unavoidable unwanted coupling leading to decoherence, but also necessary coupling for control and observation.
- A common model is the semi-infinite coplanar waveguide transmission line. This configuration leads to a densely packed, continuous frequency spectrum of modes.

$$\hat{H}_{\text{tml}} = \int_0^\infty d\omega \hbar\omega \hat{b}_\omega^\dagger \hat{b}_\omega$$

where the mode operators satisfy $[\hat{b}_\omega, \hat{b}_{\omega'}^\dagger] = \delta(\omega - \omega')$

Wiring up Quantum Systems with Transmission Lines (Cont.)

Considering capacitive coupling of the line to the oscillator at $x = 0$, the total Hamiltonian takes the form

$$\hat{H} = \hat{H}_{\text{tml}} + \hbar\omega_r \hat{a}^\dagger \hat{a} + \hbar \int_0^\infty d\omega \lambda(\omega)(\hat{b}^\dagger_\omega - \hat{b}_\omega)(\hat{a}^\dagger - \hat{a})$$

where $\lambda(\omega) = (C_\kappa/\sqrt{cC_r})\sqrt{\omega_r\omega/2\pi\nu}$ is the frequency-dependent coupling strength, with C_κ the coupling capacitance and C_r the resonator capacitance.

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where $\lambda(\omega) = (C_\kappa/\sqrt{cC_r})\sqrt{\omega_r\omega/2\pi\nu}$ is the frequency-dependent coupling strength, with C_κ the coupling capacitance and C_r the resonator capacitance.
Assuming $\lambda(\omega)$ to be sufficiently small relative to ω_r

$$\hat{H} \approx \hat{H}_{\text{tml}} + \hbar\omega_r \hat{a}^\dagger \hat{a} + \hbar \int_0^\infty d\omega \lambda(\omega_r)(\hat{a}\hat{b}_\omega^\dagger + \hat{a}^\dagger \hat{b}_\omega)$$

Input-Output Theory

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- We divide the transmission line signal into left moving and right moving fields and define the input and output fields as

$$\hat{b}_{\text{in}}(t) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \hat{b}_{L\omega} e^{-i(\omega-\omega_r)t}$$

$$\hat{b}_{\text{out}}(t) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \hat{b}_{R\omega} e^{-i(\omega-\omega_r)t}$$

satisfying $\left[\hat{b}_{\text{in}}(t), \hat{b}_{\text{in}}^{\dagger}(t')\right] = \left[\hat{b}_{\text{out}}(t), \hat{b}_{\text{out}}^{\dagger}(t')\right] = \delta(t-t')$

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- A key insight from this theory is that a semi-infinite transmission line acts as a simple resistor by carrying energy away from the system as propagating waves.
- Conversely, the input field can be used to drive and control the system.

Thank you