Non-perturbative Insights into QCD Phase Structure via Functional Renormalization Group

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Motivation

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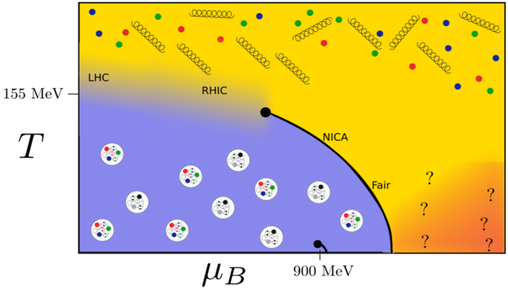
- Field theories, and in particular QCD, change their behaviour changing energy scale $\Longrightarrow$ Phase transitions:

- turn from weak to strong coupling;
- change in the relevant degrees of freedom
- Different realization of the fundamental symmetries.



Guenther, J.N. Overview of the QCD phase diagram. Eur. Phys. J. A 57, 136 (2021).

- A non-perturbative approach is needed.
- Possible solution $\Longrightarrow$ Functional Renormalization Group (FRG)

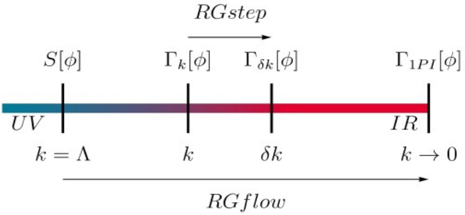
Motivation

- **Why FRG?**
  - Non-perturbative;
  - Fluctuations are taken into account not all at once but from scale to scale;
  - No a priori limitations. However...
- Difficult application to full QCD $\Rightarrow$ Effective field theories and models.
- **Advantages**:
  - Capture the (expected) essential features of the system in a given regime;
  - Insight on the relevant degrees of freedom;
  - Simpler calculations;
- **Disadvantage**:
  - Not the full theory $\Rightarrow$ loss of information.
- In this work we focus on chiral symmetry of QCD:
  - Quark-Meson model.

The Functional Renormalization Group

FRG Flow Equation

- FRG implements Wilson's RG approach $\Rightarrow$ Fluctuations integrated by momentum shells.
- We consider the *(scale dependent) effective average action* Γ_k



$$\Gamma_{k\to\Lambda} = S_{bare} \qquad \Gamma_{k\to 0} = \Gamma$$

- Γ_k can be constructed defining an IR regulated generating functional

$$e^{W_k[J]} \equiv Z_k[J] := \int_\Lambda \mathcal{D}\varphi \, e^{-S[\varphi]-\Delta S_k[\varphi]-\int J\varphi}$$

- where ΔS_k is a regulator term of the form

$$\Delta S_k[\varphi] = \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\, \varphi(-p)\, R_k(p)\, \varphi(p)$$

- The effective average action is given by:

$$\Gamma_k[\phi] = \sup_J \left(\int J\phi - W_k[J] \right) - \Delta S_k[\phi]$$

FRG Flow Equation

The **Wetterich flow equation** describes the k-(or t-)evoultion of Γ_k:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\partial_t R_k (\Gamma_k^{(2)}[\phi] + R_k)^{-1}\right] \qquad t = -\ln\frac{k}{\Lambda}$$

Key features:

- Exact one-loop structure;
- The purpose of the regulator is twofold:
  - IR Regularization;
  - Implements the idea of integrating over momentum shells $p^2 \sim k^2$;
- The flow equation is a functional integro-differential equation for Γ_k;
- Difficult to solve exactly $\Rightarrow$ we need some ansatz.
- We will use a *derivative expansion*:

$$\Gamma_k[\phi] = \int d^D x \left[V_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial_\mu \phi)^2 + \mathcal{O}(\partial^2)\right].$$

[1] C. Wetterich, Phys. Lett. B 301 (1993) 90-94.
[2] K. G. Wilson, Phys. Rev. B 4, (1971) 3174, Phys. Rev. B 4, (1971) 3184.
[3] J. Berges, N. Tetradis, C. Wetterich, Phys.Rept. 363 (2002) 223-386.

The Quark-Meson model

Quark-Meson model

- The $N_f = 2$ QM model uses as fundamental degrees of freedom mesons coupled to quarks

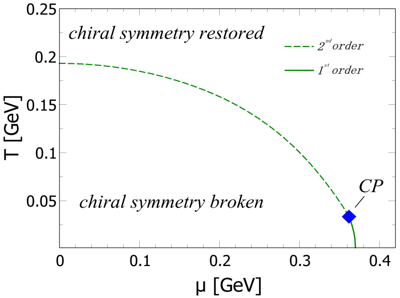
$$\mathcal{L}_{QM}^{E} = \bar{\psi}(\gamma_\mu \partial^\mu + h(\sigma + i\gamma^5 \vec{\tau}\vec{\pi}))\psi + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \vec{\pi})^2 + U(\sigma^2 + \vec{\pi}^2)$$

- Chiral phase transition: SSB $O(4) \to O(3)$

$$\langle \bar{\psi}\psi \rangle \simeq \langle \sigma \rangle \begin{cases} > 0 \Leftrightarrow & \text{symmetry breaking} \quad T < T_c \\ = 0 \Leftrightarrow & \text{symmetry restoration} \quad T > T_c \end{cases}$$

- 3 massless Goldstone bosons (pions).

- Expected features of the QM model phase diagram:
  - 2nd order phase transition at $\mu = 0$;
  - 1st order phase transition at $T = 0$;
  - critical endpoint.



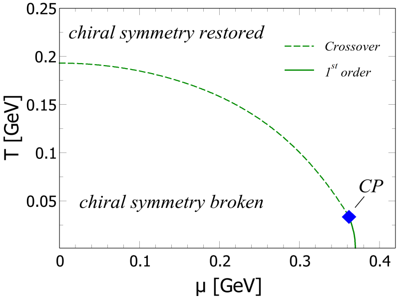
Finite quark mass

- In order to mimic the presence of a finite current quark mass we use a term

$$\mathcal{L}_m = -c\sigma$$

- The $\mathrm{O}(4)$ symmetry is (also) explicitly broken by the term $-c\sigma \Rightarrow$
  - spontaneous symmetry-breaking pattern is not exact;
  - $\langle\sigma\rangle \to 0$ never exactly;
  - the $\mathrm{O}(4)$ symmetry is never exactly restored;
  - Pions turn into massive pseudo-Goldstone mesons.

- Phase diagram:
  - Crossover at $\mu = 0$;
  - 1st-order phase transition at $T = 0$;
  - critical endpoint.



Quark-Meson model: FRG setup

- Ansatz for effective action: LPA

$$\Gamma_k[\bar{\Psi},\Psi,\phi] = \int_0^\beta dx_0 \int d^3\mathbf{x} \left\{ \bar{\psi}\left(\gamma_\mu \partial^\mu + h(\sigma + i\gamma^5 \vec{\tau}\vec{\pi}) - \mu\gamma_0\right)\psi + \frac{1}{2}(\partial_\mu \phi)^2 + U_k(\phi^2) \right\}$$

- We can express the flow equation in terms of $u_k(\sigma) = \partial_\sigma U_t(\sigma)$

$$\partial_t u_k(\sigma) + \partial_\sigma f_k(\sigma, u_k(\sigma)) = \partial_\sigma g_k(u_k'(\sigma)) + N_c \partial_\sigma S_k(\sigma)$$

- Advection and diffusion fluxes

$$f_k(\sigma, u_k) = f_k(E_{k,\pi}) \qquad g_k(u_k') = g_k(E_{k,\sigma})$$

where

$$E_{k,\pi} = \sqrt{k^2 + u_k(\sigma)/\sigma} \qquad E_{k,\sigma} = \sqrt{k^2 + u_k'(\sigma)}\,.$$

- Source term:

$$S_k(\sigma) = S_k(E_{k,\Psi}) \qquad E_{k,\,\Psi} = \sqrt{k^2 + (h\sigma)^2}$$

[4] E. Grossi and N. Wink (2019), arXiv:1903.09503.
[5] A. Koenigstein, M. J. Steil, N. Wink, E. Grossi, J. Braun, M. Buballa, and Dirk H. Rischke, Phys. Rev. D 106, 065012 (2022)

Thermodynamic geometry

Thermodynamic geometry: The concept

- An equilibrium state for a thermodynamic system can be characterized by the pair $(\beta = 1/T\,,\, \gamma = -\mu/T)$.
- **Key idea**: we consider the (β, γ)-space as a two-dimensional manifold.
- We introduce a distance in this space

$$dl^2 = g_{\beta\beta} d\beta d\beta + 2 g_{\beta\gamma} d\beta d\gamma + g_{\gamma\gamma} d\gamma d\gamma \ ,$$

  where the metric tensor is

$$g_{ij} = \frac{\partial^2 \log \mathcal{Z}}{\partial \beta^i \partial \beta^j} = \frac{\partial^2 \phi}{\partial \beta^i \partial \beta^j} \equiv \phi_{,ij} \ ,$$

  with $\phi = \beta P$, $P = -\Omega$, $\beta^1 = \beta$ and $\beta^2 = \gamma$.
- One can define the Riemann tensor as

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl},$$

  with the Christoffel symbols

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right).$$

- Ricci tensor $R_{ij} = R^k_{ikj}$, and scalar curvature $R = R^i_i$. Within thermodynamic geometry, R is called the **thermodynamic curvature**.

Thermodynamic geometry: The concept

- For our two-dimensional manifold we have

$$R = -\frac{1}{2\,g^2}\begin{vmatrix} \phi_{,\beta\beta} & \phi_{,\beta\gamma} & \phi_{,\gamma\gamma} \\ \phi_{,\beta\beta\beta} & \phi_{,\beta\beta\gamma} & \phi_{,\beta\gamma\gamma} \\ \phi_{,\beta\beta\gamma} & \phi_{,\beta\gamma\gamma} & \phi_{,\gamma\gamma\gamma} \end{vmatrix},$$

- R depends on the second- and third-order moments of the thermodynamic variables $\Rightarrow$information about the fluctuation of the physical quantities.
- Close to a second-order phase transition $|R| \propto \xi^3 \to \infty \Rightarrow$ information on the correlation volume.
- R can convey details about the nature of the interaction:
  - $R > 0$ indicates an attractive interaction;
  - $R < 0$ corresponds to a repulsive one.
- These interactions include also the statistical attraction and repulsion in phase space :
  1. $R < 0$ for an ideal Fermi gas;
  2. $R > 0$ for an ideal Bose Gas;
  3. $R = 0$ for an ideal classical gas.

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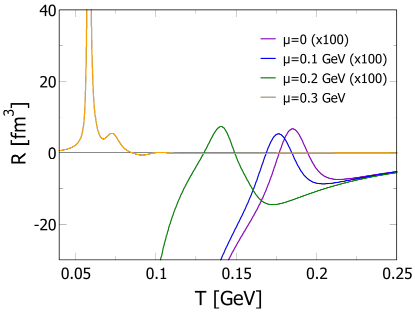
- **Crossover region** ($\mu \ll \mu_c$):
  - R peaked around the pseudo-critical temperature $\Rightarrow$ R sensitive to the chiral crossover;
  - MF positive peaks, FRG negative ones $\Rightarrow$ the sign is sensitive to the approximation;
- **Critical region** ($\mu \sim \mu_c$):
  - R enhanced close to the critical point $\Rightarrow$ R sensitive to the chiral phase transition;
  - For both MF and FRG, R shows a positive peak $\Rightarrow$ Qualitative behavior of R independent of the approximation.



[6]Murgana et al Phys. Rev. D 109, no.9, 096017 (2024)

Testing reconstruction from imaginary chemical potential

Testing reconstruction: Motivation

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- **Motivation**: Study the QCD phase diagram using both FRG (applied to effective models) and lQCD.
- **Challenge**:
  - lQCD faces the "sign problem" at finite μ, making direct simulations difficult;
  - Reconstruction techniques from imaginary μ can be used.
- **Key Questions**:
  - How reliable is the extrapolation from imaginary μ?
  - How to test it?
- **Idea**: Use the FRG (applied to the QM model):
  - Non-perturbative.
  - No a priori limitations (sign problem)
- In this framework one has access to both:
  - Direct results at both real and imaginary μ.
  - Extrapolated results from imaginary μ.
- Comparison and test is possible and well controlled.

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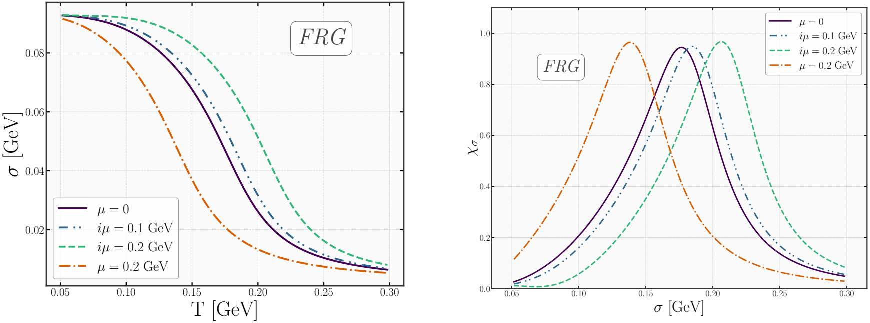
- **Phase Boundary**:
  - Direct calculation: T_c defined by the peak of the chiral susceptibility:

$$\chi_\sigma = -\frac{\partial\langle\sigma\rangle}{\partial T}$$

  - Used for both real and imaginary μ.



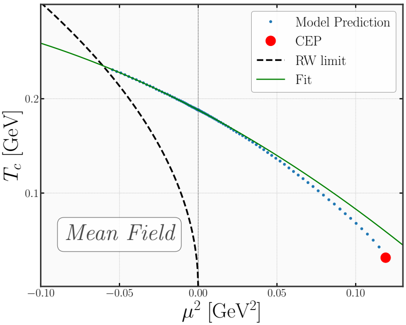
[7] F. Murgana and M. Ruggieri arXiv 2505.04569 (2025)

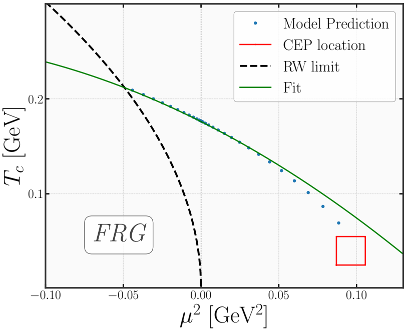
Testing reconstruction: Results

- **Phase Boundary**:
  - Reconstructed from imaginary μ:

$$\frac{T_c(\mu)}{T_c} = 1 - \kappa_2 \left(\frac{\mu}{T_c}\right)^2 - \kappa_4 \left(\frac{\mu}{T_c}\right)^4$$

- **Results**:
  - Excellent agreement at low μ (crossover region).
  - Growing discrepancy near CEP.





[7]F. Murgana and M. Ruggieri arXiv 2505.04569 (2025)

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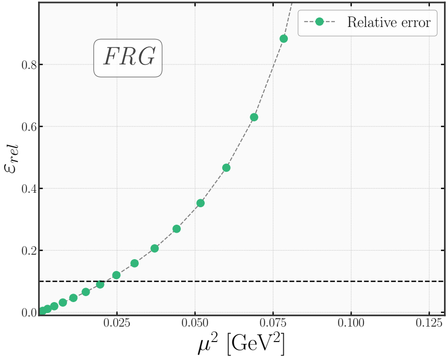
- **Convergence Radius**:

$$\varepsilon_{rel} = \frac{|T_c - T_c^{(fit)}|}{T_c};$$

  - μ_{conv} such that $\varepsilon_{rel} \leq 0.1$;
  - Effective radius $\mu_{\text{conv}} \approx 146$ MeV (both MF and FRG);
  - $\varepsilon_{\text{rel}} \approx 1.5$ near CEP (model dependent), limiting extrapolation reliability.



[7]F. Murgana and M. Ruggieri arXiv 2505.04569 (2025)

Conclusions and Outlook

Conclusions and Outlook

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- We described the **FRG** approach to QFT and in particular FRG flow equation;
- We applied an **hydrodynamic approach** to the FRG method in order to study the phase diagram of the **Quark-Meson model**;
- We explored the QM model phase diagram using the **Thermodynamic Geometry technique**:
  - R is peaked around the pseudo-critical temperature$\Rightarrow$ R sensitive to chiral crossover;
  - R exhibits a positive sharper peak around the critical temperature$\Rightarrow$ R sensitive to chiral phase transition;
  - Qualitative behavior of R independent of the approximation close to criticality.
- We tested **reconstruction techniques** from imaginary μ with FRG:
  - within FRG results from both real and imaginary μ are available for comparison;
  - Extrapolation from imaginary μ works well for $\mu < \mu_{\text{conv}} \simeq 146$ MeV;
  - Near CEP, non-analytic effects cause large errors.

What's next?

- Possible generalizations to **higher order truncations** (LPA', $O(\partial^2)$, ...):
  - Extension of the hydrodynamic formulation beyond LPA (inhomogeneous phases);
  - Study the Thermodynamic Geometry for the QM model beyond LPA;
  - Study the influence of truncation on the reconstruction procedure of the phase diagram.
- Inclusion of **different chemical potential axes** in the thermodynamic geometry framework (μ_I and exact comparison with lattice at $\mu_B \sim 0$);
- Improve reconstruction techniques for **critical regions**;
- Extend to **Polyakov-loop** models for confinement-deconfinement transition;
- More realistic QCD description with FRG to cross-validate with other methods (DSE, lattice).

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Thanks for your attention!

Appendix

Functional approach to Quantum field theory

Quantum field theory

In quantum field theory (QFT), all physical information is stored in the *generating functional* $Z[J]$

$$Z[J] \equiv \mathcal{N} \int \mathcal{D}\varphi \, e^{-S[\varphi]+\int J\varphi} \tag{1}$$

since all $n-$point functions can be obtained via a functional differentiation

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle := \mathcal{N} \int \mathcal{D}\varphi \, \varphi(x_1) \cdots \varphi(x_n) e^{-S[\varphi]} = \frac{\partial^n}{\partial J^n} Z[J] \bigg|_{J=0} \tag{2}$$

Functional approach to QFT

One introduces the *generating functional of connected correlators* $W[J]$:

$$Z[J] \equiv e^{W[J]} = \int \mathcal{D}\varphi \, e^{-S[\varphi] + \int J\varphi} \tag{3}$$

Performing a Legendre transform of $W[J]$ we obtain the *effective action* Γ:

$$\Gamma[\phi] = \sup_J \left(\int J\phi - W[J] \right) \tag{4}$$

At $J = J_{\text{sup}}$, we get

$$\frac{\delta}{\delta J(x)} \left(\int J\phi - W[J] \right) = 0$$

$$\Rightarrow \quad \phi = \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J} = \langle \varphi \rangle_J \tag{5}$$

Functional renormalization group

Flow equation

A versatile approach to the computation of Γ is based on Renormalization Group RG concepts.
We are looking for an interpolating action Γ_k, the *effective average action*, such that

$$\Gamma_{k\to\Lambda} = S_{bare}\,, \quad \Gamma_{k\to 0} = \Gamma \tag{6}$$

This can be constructed through the definition of the IR regulated functional

$$e^{W_k[J]} \equiv Z_k[J] := \exp\left(-\Delta S_k\left[\frac{\delta}{\delta J}\right]\right) Z[J] =$$

$$= \int_\Lambda \mathcal{D}\varphi\, e^{-S[\varphi]-\Delta S_k[\varphi]-\int J\varphi} \tag{7}$$

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where ΔS_K is a regulator therm of the form

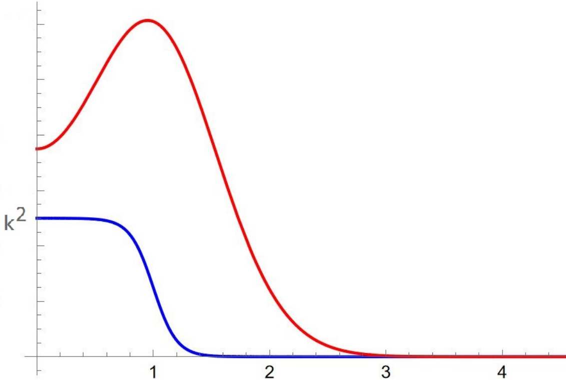
$$\Delta S_k[\varphi] = \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \varphi(-q) R_k(q) \varphi(q) \tag{8}$$

The regulator function $R_k(q)$ should satisfy

$$\lim_{q^2/k^2 \to 0} R_k > 0 \tag{9}$$

$$\lim_{k^2/q^2 \to 0} R_k(q) = 0 \tag{10}$$

$$\lim_{k^2 \to \Lambda^2 \to \infty} R_k(q) \to \infty \tag{11}$$



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We introduce the RG-time t, using the abbreviations

$$t = \ln \frac{k}{\Lambda}, \qquad \partial_t = k \frac{d}{dk} \tag{12}$$

Keeping the source J fixed, i.e. k independent, we obtain

$$\partial_t W_k[J] = -\frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \partial_t R_k(q) G_k(q) - \partial_t \Delta S_k[\phi] \tag{13}$$

Here, we have defined the *connected propagator*

$$G_k(p) = \left(\frac{\delta^2 W_k}{\delta J \delta J} \right)(p) = \langle \varphi(-p) \varphi(p) \rangle - \langle \varphi(-p) \rangle \langle \varphi(p) \rangle \tag{14}$$

We define the interpolating effective action Γ_k

$$\Gamma_k[\phi] = \sup_J \left(\int J\phi - W_k[J] \right) - \Delta S_k[\phi] \tag{15}$$

At $J = J_{\text{sup}}$:

$$\phi(x) = \langle \varphi(x) \rangle_J = \frac{\delta W_k[J]}{\delta J(x)} \tag{16}$$

Computing the functional derivative of Eq. (15) with respect to ϕ we get

$$J(x) = \frac{\delta \Gamma_k[\phi]}{\delta \phi(x)} + (R_k \phi)(x) \tag{17}$$

From this, we deduce:

$$\frac{\delta J(x)}{\delta \phi(y)} = \frac{\delta^2 \Gamma_k[\phi]}{\delta \phi(x) \delta \phi(y)} + R_k(x, y) \tag{18}$$

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We obtain from Eq. (16):

$$\frac{\delta\phi(y)}{\delta J(x')} = \frac{\delta^2 W_k[J]}{\delta J(x')\delta J(y)} \equiv G_k(y - x') \tag{19}$$

This implies the important identity

$$\mathbb{I} = (\Gamma^{(2)} + R_k)G_k \tag{20}$$

Here, we have introduced the notation

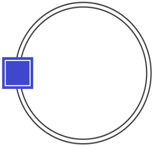
$$\Gamma_k^{(n)}(x_1, \cdots x_n) = \frac{\delta^n \Gamma_k[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)} \tag{21}$$

Finally we can derive the *flow equation* for Γ_k for fixed ϕ and at $J = J_{\text{sup}}$:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\partial_t R_k(\Gamma^{(2)}[\phi] + R_k)^{-1}\right] \tag{22}$$

Functional renormalization group

$$\Gamma_k[\phi] = \sup_J \left(\int J\phi - W_k[J] \right) - \Delta S_k[\phi] \qquad \partial_t \Gamma_k[\phi] = \frac{1}{2}\, \partial_t R_k$$



We can derive the *flow equation* for Γ_k:

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathsf{Tr} \left[\partial_t R_k (\Gamma_k^{(2)}[\phi] + R_k)^{-1} \right] \tag{23}$$

where

$$t = -\ln \frac{k}{\Lambda} \qquad \partial_t = -k \frac{d}{dk}$$

We need some ansatz to solve the flow equation. We will use the *derivative expansion*.

[1] C. Wetterich, Phys. Lett. B 301 (1993) 90-94.
[2] K. G. Wilson, Phys. Rev. B 4, (1971) 3174, Phys. Rev. B 4, (1971) 3184.
[3] J. Berges, N. Tetradis, C. Wetterich, Phys.Rept. 363 (2002) 223-386.

We can notice that:

- The flow equation is a functional differential equation for Γ_k;
- We may define QFT based on the flow equation.
- The purpose of the regulator is twofold;
- The solution of the flow equation corresponds to an RG trajectory in *theory space*;
- The variation of the trajectory with respect to R_k reflects the RG scheme dependence of a non-universal quantity, but the final point on the trajectory is independent of R_k;
- Perturbation theory can immediately be re-derived from the flow equation, for instance, imposing the loop expansion on Γ_k, $\Gamma_k = S + \hbar \Gamma_k^{1-loop}$.

Various systematic approximation exist which can be summarized under the label of *method of truncations*.
A first example for such an approximation scheme is the vertex expansion, which now reads

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^D x_1 \cdots d^D x_n \, \Gamma_k^{(n)}(x_1 \cdots x_n) \, \phi(x_1) \cdots \phi(x_n) \tag{24}$$

As a second example, let us introduce the *operator expansion*

$$\Gamma_k = \int d^D x \left[V_k(\phi) + \frac{1}{2} Z_k(\phi) (\partial_\mu \phi)^2 + \mathcal{O}(\partial^4) \right] \tag{25}$$

where, for instance, $V_k(\phi)$ corresponds to the effective potential.

Flow equation O(N) LPA

As ansatz for Γ_k we will use the LPA:

$$\Gamma_k[\vec{\phi}] = \int_x \left\{ \frac{1}{2}(\partial_\mu \phi_a)^2 + V_k(t,\rho) \right\} \tag{26}$$

where $V_k(t,\rho)$ is the effective potential.
Then we compute the two point functions $\Gamma_k^{(2)}$:

$$\Gamma_{k,ab}^{(2)}(t,\rho,p) = [p^2 + V_k'(t,\rho)]\delta_{ab} + 2\rho V_k''(t,\rho)\delta_{aN}\delta_{bN} \tag{27}$$

with

$$V_k'(t,\rho) = \partial_\rho V_k(t,\rho) \qquad \text{and} \qquad V_k''(t,\rho) = \partial_\rho V_k'(t,\rho)$$

As regulator, we chose the Litim Regulator

$$R_k(p) = (k^2 - p^2)\Theta(k^2 - p^2) \tag{28}$$

[6] Daniel F. Litim, Phys. Rev. D 64, 105007 (2001).

Inserting Eq.s (27)-(28) into (23) we obtain the flow equation for $V_k(t,\rho)$

$$\partial_t V_k(t,\rho) = -A_d\, k^{d+2}\left(\frac{N-1}{k^2+V_k'(t,\rho)} + \frac{1}{k^2+V_k'(t,\rho)+2\rho V_k''(t,\rho)}\right) \tag{29}$$

with $A_d = \frac{\Omega_d}{(2\pi)^d d}$, and $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$.

It is preferable to formulate the problem in terms of the field $\sigma = \sqrt{2\rho}$.

We thus rewrite the flow equation as:

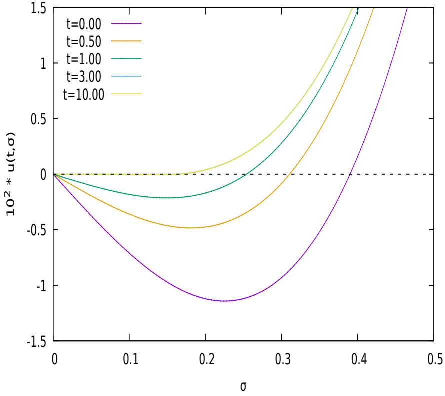
$$\partial_t V_k(t,\sigma) = -A_d\, k^{d+2}\left(\frac{N-1}{k^2+\frac{1}{\sigma}\partial_\sigma V_k(t,\sigma)} + \frac{1}{k^2+\partial^2_{\sigma\sigma}V_k(t,\sigma)}\right) \tag{30}$$

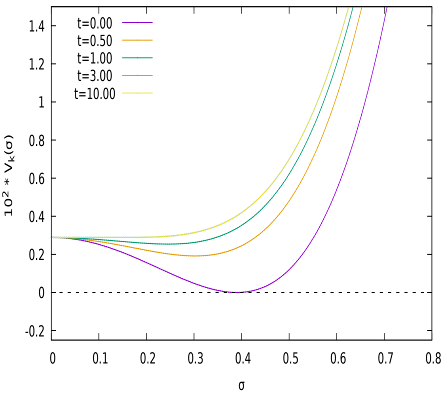
We introduce the derivative of the potential as new variable

$$u(t,\sigma) = \partial_\sigma V_k(t,\sigma), \qquad u'(t,\sigma) = \partial_\sigma u(t,\sigma) \tag{31}$$

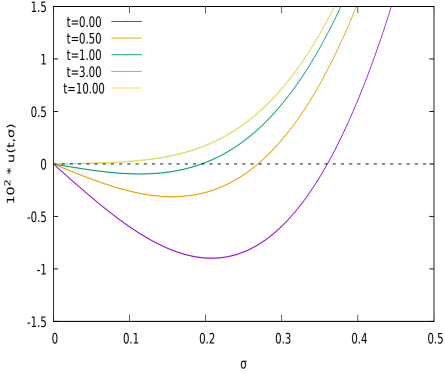
Other plots O(N)

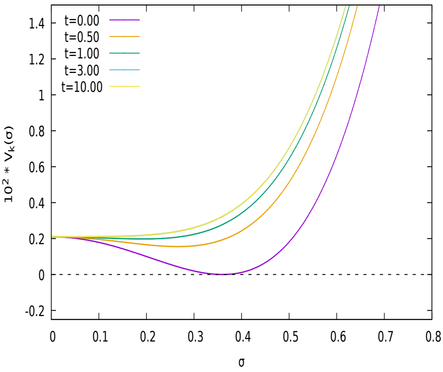
Different t broken phase



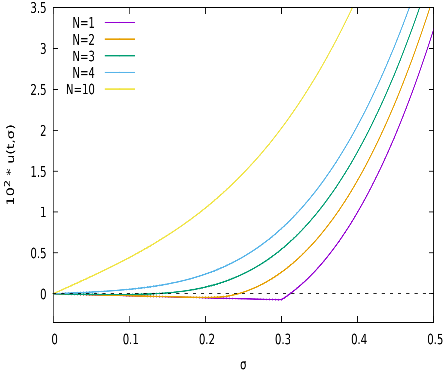


Different t symmetric phase

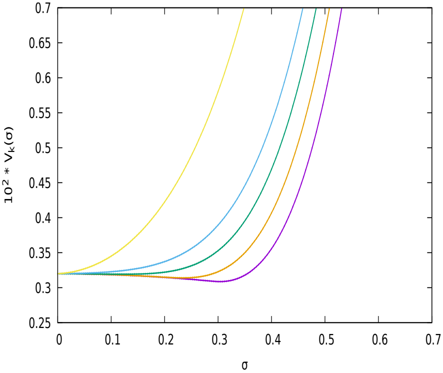




Different N

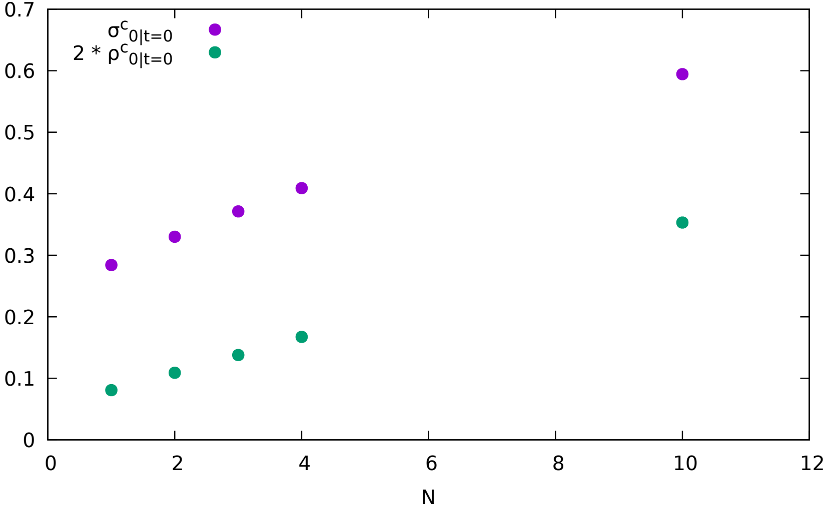


N=1
N=2
N=3
N=4
N=10
10² * u(t,σ)
σ
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5



10² * V_k(σ)
σ
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

Scaling of ρ_0^c vs N



Testing the method: numerical precision and error estimates

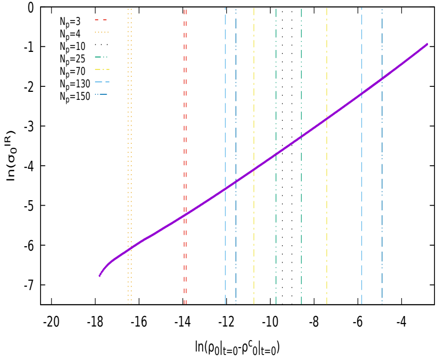
Numerical precision and error estimates

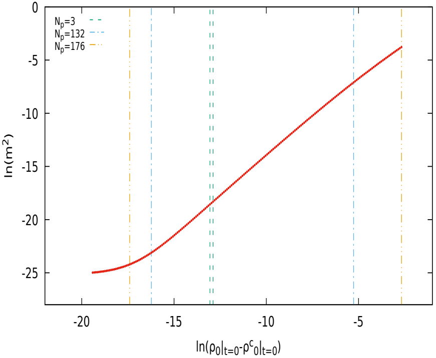
We use critical exponents as a test ground for the capabilities and limitations of our method.
We analize the following contribution to the error:

- general fit errors;
- error on the determination of the curvature mass in the IR;
- error on the determination of the position of the minimum in the IR σ_0^{IR}.

General fit error

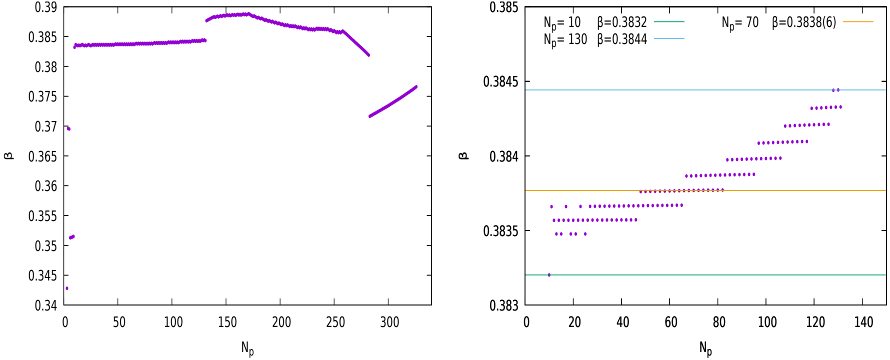
- Using linear regression to extract the critical exponent is not enough!
- The fitting region influences the value of the critical exponents.



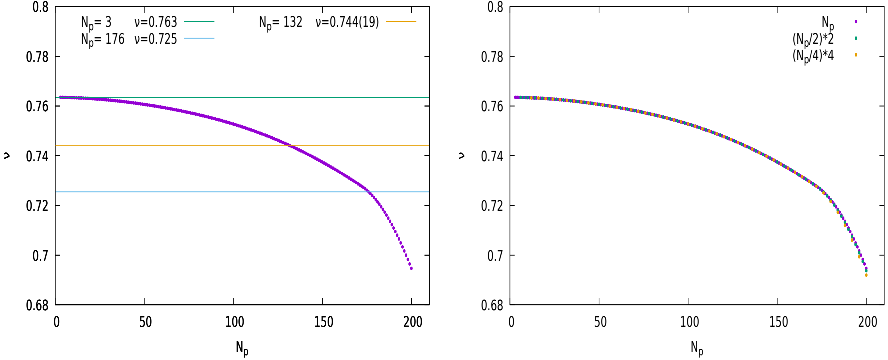


General fit error

- Ideally, the critical exponent should be independent of the fitting region (perfect scaling behavior);
- We choose the region in which the critical exponent is "less dependent" on the fitting region.



General fit error

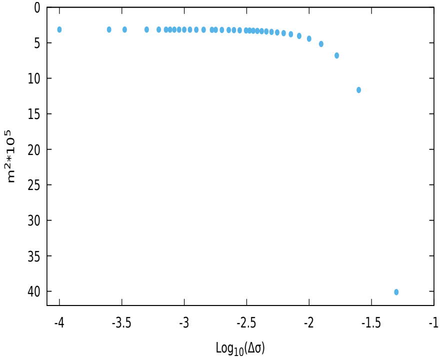


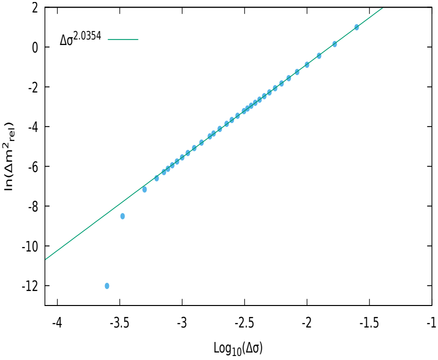
- The fitting error is independent of the number density.
- The Fitting error is more relevant for the critical exponent ν.

Error on m^2

- Numerical method is used $\Rightarrow$ discretization error;
- We look for a region for which m^2 is independent of $\Delta\sigma$;
- We can lower the discretization error lowering $\Delta\sigma$;
- No further error on IR extrapolation.

$$\Delta m^2_{rel}(\Delta\sigma) = \left|1 - \frac{m^2(\Delta\sigma)}{m^2(\Delta\sigma_{min})}\right|$$

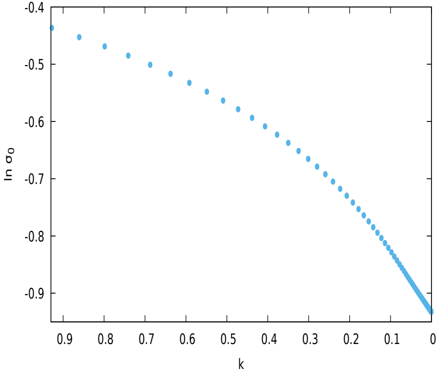


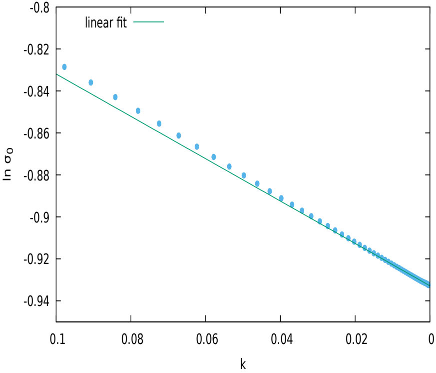


Error on σ_0^{IR}

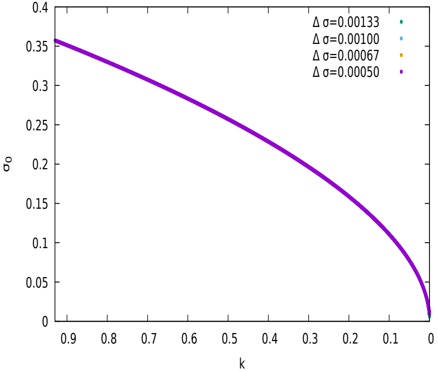
- In the broken phase, the flow equation becomes stiff $\Rightarrow$ it is not possible to go arbitrarily far in the IR.
- We can determine the position of the minimum σ_0^{IR} via exponential extrapolation:

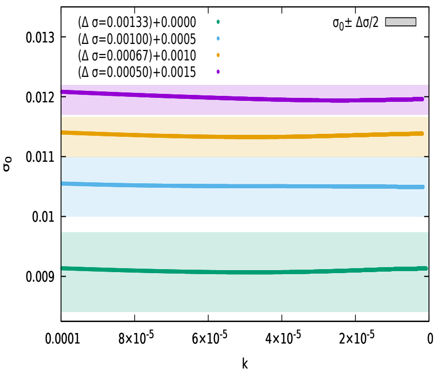
$$\ln \sigma_0 = a\,k + b\,, \qquad \Longleftrightarrow \qquad \sigma_0 = e^b\,e^{ak}\,.$$





- Even after the extrapolation the uncertainty on σ_0^{IR} is still limited from below by $\Delta\sigma$.
- One can determine whether σ_0 lies to the left or the right of the cell center, such that the uncertainty is $\Delta\sigma_0^{\mathrm{IR}} \lesssim \Delta\sigma/2$.





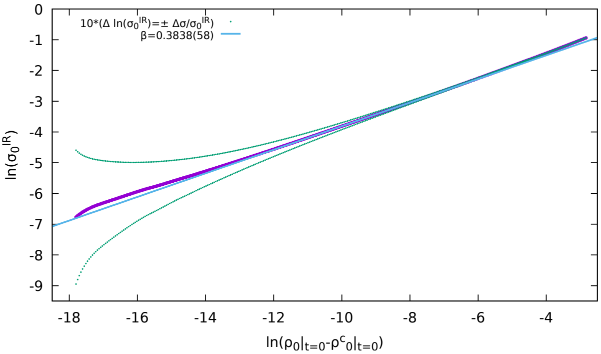
Error on σ_0^{IR}

- The error on $\log \sigma_0^{\mathrm{IR}}$ depends on σ_0^{IR} itself:

$$\Delta \ln \sigma_0^{\mathrm{IR}} = \frac{\Delta \sigma_0^{\mathrm{IR}}}{\sigma_0^{\mathrm{IR}}} = \frac{\Delta \sigma}{2\sigma_0^{\mathrm{IR}}} \, .$$

- We use the errors on the left and right edge of the fitting region

$$\Delta \beta_{extr} = \frac{\Delta \ln \sigma_0^{\mathrm{IR},left} + \Delta \ln \sigma_0^{\mathrm{IR},right}}{\Delta \varrho_{fit}} \, ,$$



Error values

- Using typical values encountered during the calculations, i.e., $\Delta\varrho_{fit} \simeq 3.5$, $\Delta\sigma = 0.0005$ one finds

$$\Delta\beta_{extr} \simeq 0.006 \; .$$

- This contribution is significantly larger than the ones coming from the fit error,

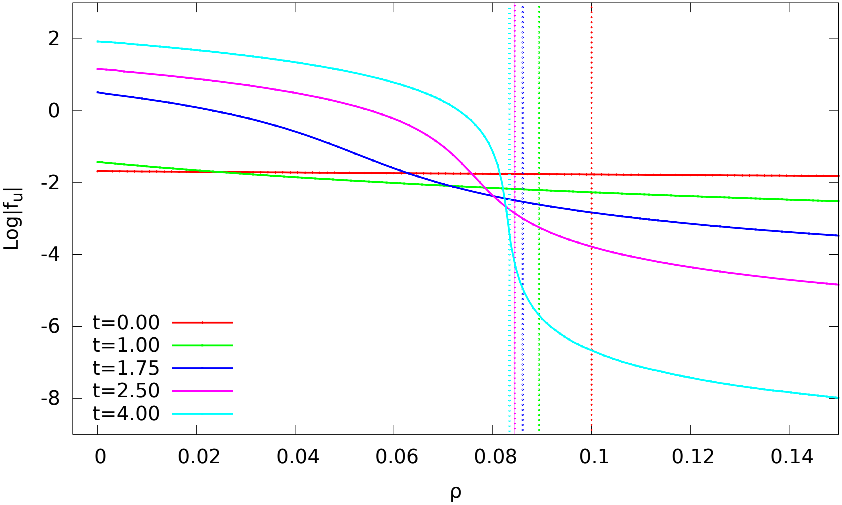
$$\Delta\beta_{fitting\,region} \sim 0.0006 \qquad \Delta\beta_{fit} \sim 0.0002 \; .$$

- For the critical exponent ν, the dominant contribution arises from the choice of the fitting region:

$$\Delta\nu_{fitting\,region} \sim 0.02$$

$$\Delta\nu_{fit} \sim 0.0002 \qquad \Delta\nu_{m^2} \sim 0.00005$$

Local speed


2
0
-2
-4
-6
-8
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
ρ
Log|f_u|
t=0.00
t=1.00
t=1.75
t=2.50
t=4.00

Flow equation O(N) LPA'

In addition to the effective potential $V_k(t,\rho)$, the effective action includes a field renormalization factor Z_k (in the LPA $Z_k = 1$) which is field independent:

$$\Gamma_k[\vec{\phi}\,] = \int d^D x \left[\frac{1}{2} Z_k (\partial_\mu \phi_a)^2 + V_k(t,\rho)\right] \tag{32}$$

One can then define a "running" anomalous dimension

$$\eta_k = -k\,\partial_k \ln Z_k = \partial_t \ln Z_k \tag{33}$$

At the critical point one has

$$\lim_{k\to 0} \eta_k \equiv \eta \tag{34}$$

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We need to compute the 2-point function. We choose a simple background field

$$\vec{\phi} = (0, \cdots, 0, \sigma)$$

$$\Gamma^{(2)}_{k,ab}(p, \vec{\phi}) = \frac{\phi_a \phi_b}{2\rho} \Gamma^{(2)}_{k,L}(p, \sigma) + \left(\delta_{ab} - \frac{\phi_a \phi_b}{2\rho} \right) \Gamma^{(2)}_{k,T}(p, \sigma) \tag{35}$$

where

$$\Gamma^{(2)}_{k,L}(p, \sigma) = Z_k p^2 + \partial^2_{\sigma\sigma} V_k(t, \sigma) \tag{36}$$

$$\Gamma^{(2)}_{k,T}(p, \sigma) = Z_k p^2 + \frac{1}{\sigma} \partial_\sigma V_k(t, \sigma) \tag{37}$$

Analogously, we can introduce the *longitudinal* and *transverse propagators*:

$$G_{k,\alpha}(p, \sigma) = \left[\Gamma^{(2)}_{k,\alpha}(p, \sigma) + R_k(p) \right]^{-1} \qquad \alpha = L, T \tag{38}$$

Now we just need to plug Eq (47) into the flow equation :

$$\partial_t V_k(t,\sigma) = \frac{1}{2}\int_q \partial_t R_k(q)\left[G_{k,L}(q,\sigma) + (N-1)G_{k,T}(q,\sigma)\right] =$$

$$= \frac{1}{2}\int_q \partial_t R_k(q)\left[\frac{1}{Z_k q^2 + \frac{1}{\sigma}\partial_\sigma V_k(t,\sigma) + R_k(q)} + \right.$$

$$\left. \frac{N-1}{Z_k q^2 + \partial^2_{\sigma\sigma} V_k(t,\sigma) + R_k(q)}\right] \tag{39}$$

As regulator we choose:

$$R_k(p) = \alpha Z_k (k^2 - p^2)\Theta(k^2 - p^2) \tag{40}$$

with $\alpha \sim 1$ as a free parameter.

We obtain the flow equation for the effective potential:

$$\partial_t V_k(t,\sigma,\eta_k) = \frac{\alpha}{4\pi^2(1-\alpha)} k^3 \Bigg\{ (N-1)\Big\{ \left[\eta_k\,(1+m_1)-2\right] \times \left(1-\sqrt{m_1}\arctan\frac{1}{\sqrt{m_1}}\right) - \frac{1}{3}\eta_k \Big\} + \Big\{ \left[\eta_k\,(1+m_2)-2\right] \times \left(1-\sqrt{m_2}\arctan\frac{1}{\sqrt{m_2}}\right) - \frac{1}{3}\eta_k \Big\} \Bigg\} \tag{41}$$

where:

$$m_1(Z_k,\partial_\sigma V_k,\sigma) = \frac{\alpha}{1-\alpha} + \frac{\partial_\sigma V_k}{Z_k\,\sigma\,k^2\,(1-\alpha)} \tag{42}$$

$$m_2(Z_k,\partial^2_{\sigma\sigma} V_k) = \frac{\alpha}{1-\alpha} + \frac{\partial^2_{\sigma\sigma} V}{Z_k\,k^2\,(1-\alpha)} \tag{43}$$

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We introduce the conserved quantity

$$u(t, \sigma, \eta_k) = \partial_\sigma V_k(t, \sigma, \eta_k) \tag{44}$$

and

$$u'(t, \sigma, \eta_k) = \partial_\sigma u(t, \sigma, \eta_k) = \partial^2_{\sigma\sigma} V_k(t, \sigma, \eta_k) \tag{45}$$

Thus we can rewrite

$$m_1(Z_k, u, \sigma) = \frac{\alpha}{1-\alpha} + \frac{u}{Z_k\,\sigma\,k^2\,(1-\alpha)} \tag{46}$$

$$m_2(Z_k, u') = \frac{\alpha}{1-\alpha} + \frac{u'}{Z_k\,k^2\,(1-\alpha)} \tag{47}$$

Flow equation O(N) LPA'

We can now define the convection flux

$$f(k,u,\sigma,\eta_k) = \frac{\alpha}{4\pi^2(1-\alpha)}(N-1)k^3 \times$$

$$\times \left\{ \left[2-\eta_k\left(1+m_1\right)\right]\left(1-\sqrt{m_1}\arctan\frac{1}{\sqrt{m_1}}\right)+\frac{1}{3}\eta_k \right\} \tag{48}$$

and the diffusion flux

$$g(k,u',\eta_k) = \frac{\alpha}{4\pi(1-\alpha)}k^3 \times$$

$$\times \left\{ \left[\eta_k\left(1+m_2\right)-2\right]\left(1-\sqrt{m_2}\arctan\frac{1}{\sqrt{m_2}}\right)-\frac{1}{3}\eta_k \right\} \tag{49}$$

We take a derivative of the flow equation with respect to σ:

$$\partial_t u(t,\sigma,\eta_k) + \partial_\sigma f(t,u,\sigma,\eta_k) = \partial_\sigma g(t,u',\eta_k) \tag{50}$$

Flow equation O(N) LPA'

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From the definition of the effective action we derive that

$$Z_k = \lim_{p\to 0} \frac{\partial}{\partial p^2} \Gamma^{(2)}_{k,T}(p, \sigma_{0,k}) \tag{51}$$

Trivially we can obtain an equation for $\partial_t Z_k$ just taking the derivative w.r.t RG time t:

$$\partial_t Z_k = \lim_{p\to 0} \frac{\partial}{\partial p^2} \partial_t \Gamma^{(2)}_{k,T}(p, \sigma_{0,k}) \tag{52}$$

Now we can easily exploit the flow equation for Γ_k to obtain a flow equation for $\Gamma^{(2)}_k$:

$$\partial_t \Gamma^{(2)}_k[\vec{\phi}\,] = \frac{1}{2} \mathrm{Tr} \left[\partial_t R_k \frac{\delta^2}{\delta\phi_a \delta\phi_b} (\Gamma^{(2)}_k[\vec{\phi}\,] + R_k)^{-1} \right] \tag{53}$$

Flow equation O(N) LPA'

Computing the derivatives, projecting onto the transverse direction and summing over the indices one obtains

$$\partial_t Z_k = 4A_d \partial^2_{\rho\rho} V_k(\rho_{0,k}, t) \tilde{\partial}_t \int_0^\infty dq \; q^{d+1} \partial_{q^2} G_{k,L}(q, \rho_{0,k}) \, \partial_{q^2} G_{k,T}(q, \rho_{0,k}) \tag{54}$$

where the symbol $\tilde{\partial}_t$ indicates that the time derivative acts only on the t dependence of R_k.

It is convenient to introduce the following dimensionless quantities:

$$\tilde{\rho} = Z_k k^{2-d} \rho \qquad \tilde{V}_k(\tilde{\rho}, t) = k^{-d} V_k(\rho, t) \tag{55}$$

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We will also use the dimensionless form for the regulator:

$$R_k(p^2) = Z_k p^2 r(y) \tag{56}$$

with $y = p^2/k^2$. We will also use the shorthand notations

$$r' = \frac{dr(y)}{dy} \qquad r'' = \frac{d^2r(y)}{dy^2}$$

In particular we will use the regulator thus

$$r(y) = \alpha \left(\frac{1-y}{y}\right) \theta(1-y) \tag{57}$$

In this way, we can easily put the flow equation for η in a dimensionless form:

$$\eta_k = 4A_d\tilde{\rho}_{0,k}\,(\partial^2_{\tilde{\rho}\tilde{\rho}}\tilde{V}_k(\tilde{\rho}_{0,k}))^2\,m^d_{22}(2\tilde{\rho}_{0,k}\partial^2_{\tilde{\rho}\tilde{\rho}}\tilde{V}_k(\tilde{\rho}_{0,k}),\eta_k) \tag{58}$$

where we have defined the threshold function

$$m^d_{22}(w,\eta) = \int_0^\infty dy\,y^{d/2}\frac{1+r+yr'}{P(w)^2P(0)^2}\left\{y(\eta r+2yr')(1+r+yr')\times \right.$$

$$\left. \times\left[\frac{1}{P(w)}+\frac{1}{P(0)}\right]-\eta r-(\eta+4)yr'-2y^2r''\right\} \tag{59}$$

and

$$P(w) = y(1+r)+w$$

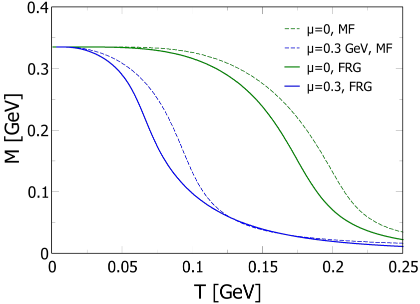
In the particular choice we made for the regulator we get:

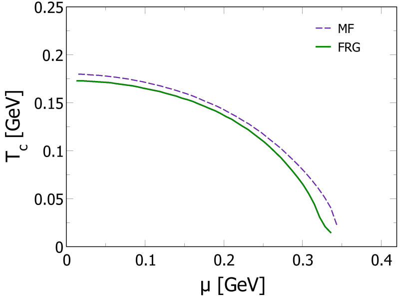
$$m_{22}^{d}(w,\eta) = \int_0^1 dy\, y^{d/2} \frac{(1-\alpha)\alpha}{P(w)^2 P(0)^2} \Big\{ \left[\eta(1-y)-2\right] \times$$

$$eta3 \times (1-\alpha)\left[\frac{1}{P(w)} + \frac{1}{P(0)}\right] + \eta \Bigg\} \tag{60}$$

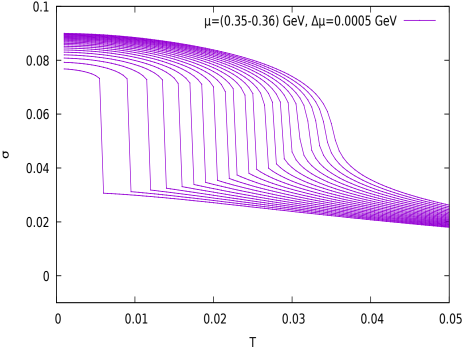
Quark meson model

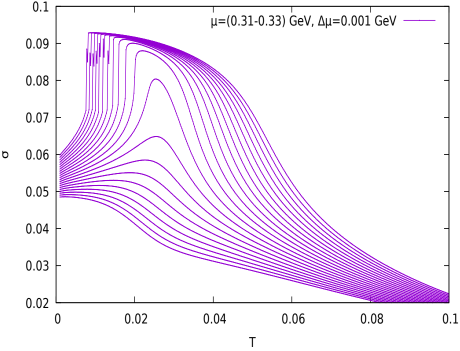
MF vs FRG finite pion mass



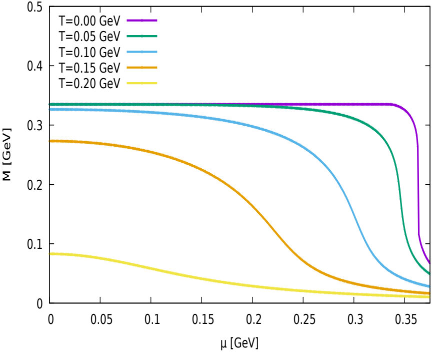


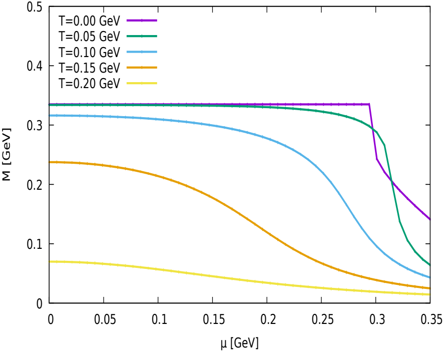
M_ψ as a function of T MF vs FRG finite pion mass



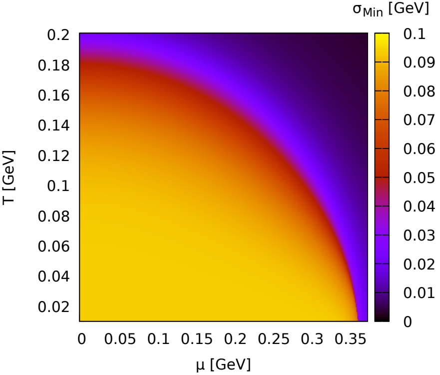


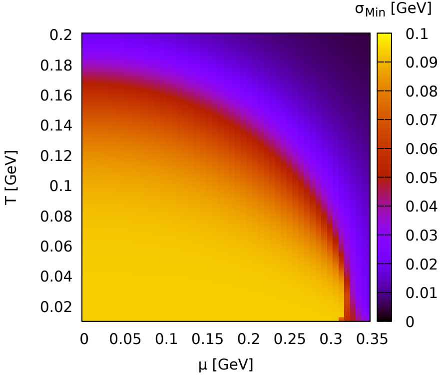
M_ψ as a function of μ MF vs FRG finite pion mass



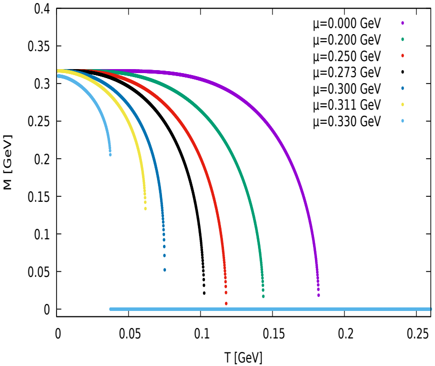


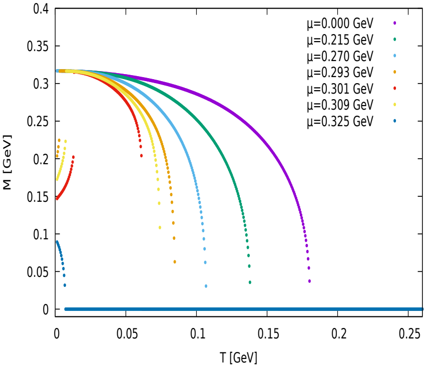
Phase diagram MF vs FRG finite pion mass



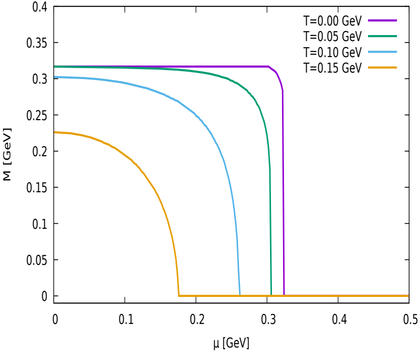


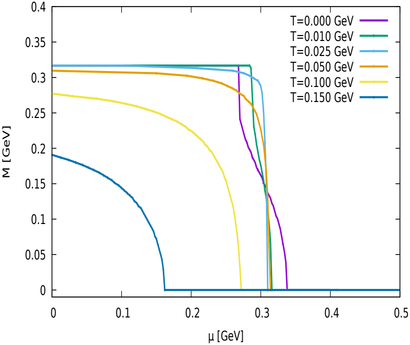
M_ψ as a function of T, MF vs FRG chiral limit



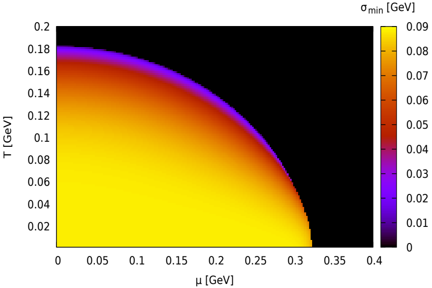


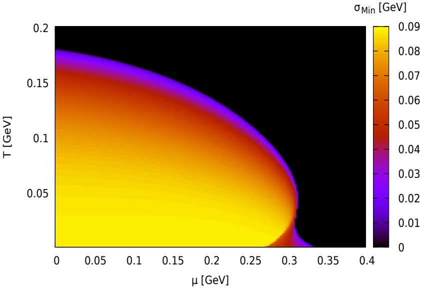
M_ψ as a function of μ MF vs FRG chiral limit

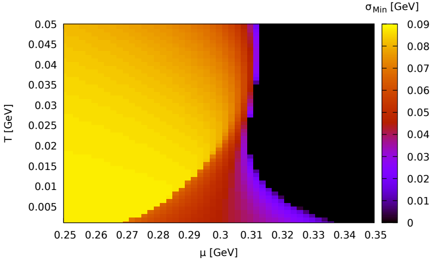




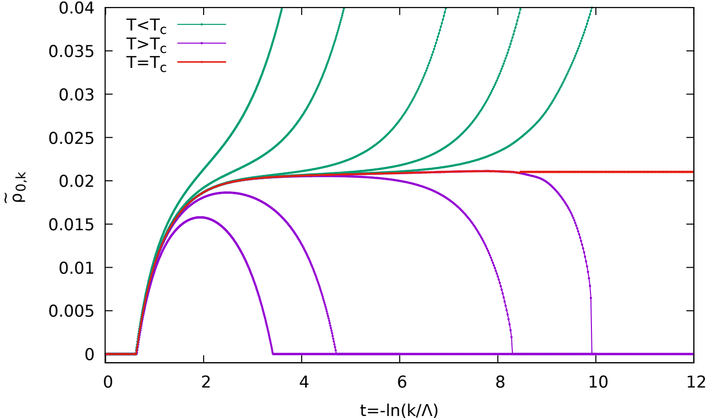
Phase diagram MF vs FRG chiral limit







Scaling of $\tilde{\rho}_{0,k}$ in the QM model



Finite quark mass

- A mass term for fermions would be:

$$\mathcal{L}_m = -m^2 \bar{\Psi}\Psi \ . \tag{61}$$

- In order to mimic the presence of a finite current quark mass we use a term

$$\mathcal{L}_m = -c\sigma \tag{62}$$

- The total Lagrangian then reads:

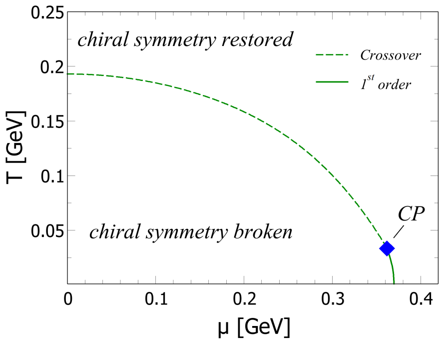
$$\mathcal{L}_{QM}^{E} = \bar{\psi}(\gamma_\mu \partial^\mu + h(\sigma + i\gamma^5 \vec{\tau}\cdot\vec{\pi}))\psi + \frac{1}{2}(\partial_\mu \Phi)^2 + U(\Phi^2) - c\sigma \ . \tag{63}$$

- The O(4) symmetry is also explicitly broken by the term $-c\sigma$, and the spontaneous symmetry-breaking pattern is not exact.
- Pions turn into massive pseudo-Goldstone mesons, acquiring a finite mass given by

$$M_\pi^2 = \frac{\partial^2 \left(U(\Phi^2) - c\sigma\right)}{\partial \pi^2}\bigg|_{\langle\sigma\rangle = f_\pi} = \frac{c}{f_\pi} \ . \tag{64}$$

- $\langle\sigma\rangle$ never truly vanishes and the O(4) symmetry is never restored, even though it can be considered approximately restored when $\langle\sigma\rangle \ll f_\pi$.

- Phase diagram:
  - Crossover;
  - 1st-order phase transition;
  - critical endpoint.



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$$\partial_t U_t(\sigma) = \frac{1}{8\pi^2} \int_0^\infty d\,q\; q^4 \left\{ \partial_t r_t^B(q) \left[\frac{3}{E_{k,\pi}(q)} \coth\left(\frac{E_{k,\pi}(q)}{2T} \right) + \frac{1}{E_{k,\sigma}(q)} \coth\left(\frac{E_{k,\sigma}(q)}{2T} \right) \right] \right.$$

$$\left. -2\,N_c \left(1 + r_t^F(x)\right) \partial_t r_t^F(x)\, \frac{1}{E_{k,\Psi}(q)} \left[1 - n_F\left(\frac{E_{k,\Psi}(q) + \mu}{T} \right) - n_F\left(\frac{E_{k,\Psi}(q) - \mu}{T} \right) \right] \right\} .$$

For the Litim-litim combination we get:

$$\partial_t U_k(\sigma) = -\frac{k^5}{12\pi^2} \left\{ \left[\frac{3}{E_{k,\pi}} \coth\left(\frac{E_{k,\pi}}{2T} \right) + \frac{1}{E_{k,\sigma}} \coth\left(\frac{E_{k,\sigma}}{2T} \right) \right] \right.$$

$$\left. -4N_c \frac{1}{E_{k,\Psi}} \left[\tanh\left(\frac{E_{k,\psi} - \mu}{2T} \right) + \tanh\left(\frac{E_{k,\psi} + \mu}{2T} \right) \right] \right\}, \qquad (65)$$

Quark-Meson model regulator dependency: mean-field

- We start from mean-field $\Rightarrow$ phase diagram is regulator-independent.
- We introduce the following rescaling transformation:

$$\sigma \to \sqrt{N_c}\sigma \qquad U_k(\sigma) \to N_c U_k(\sigma) \qquad u_k(\sigma) \to \sqrt{N_c} u_k(\sigma)$$

- And get the following rescaled flow equation:

$$\partial_t u_k(\sigma) + \frac{1}{\sqrt{N_c}} \partial_{u_k} f_k(\sigma, u_k(\sigma)) u_k'(\sigma) = \frac{1}{\sqrt{N_c}} \partial_\sigma g_k(u_k'(\sigma)) + \partial_\sigma S_k(\sigma)$$

- We can now take the $N_c \to \infty$ limit such that only the source term contributes

$$\partial_t u_k(\sigma) = \partial_\sigma S_k(\sigma)$$

Quark-Meson model regulator dependence: Mean-Field Approximation

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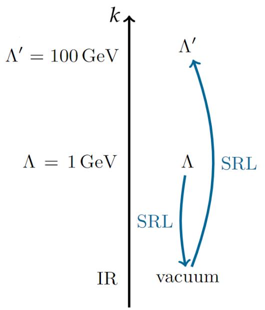
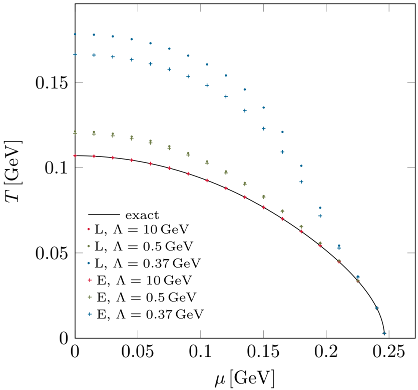
- We start from Mean-Field (MF) Approximation ⇒ No feedback of the effective potential in the Wetterich flow equation (equivalent to $N_c \to \infty$).
- Only the source term contributes

$$\partial_t u_k(\sigma) = \partial_\sigma S_k(\sigma)$$

- We use RG-consistency to determine the proper initial condition.
- As expected, the MF phase diagram is regulator independent for sufficiently large Λ.





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- We can consider a different regulator for fermions: Fermi-surface regulator

$$R_{k,F}(p,\mu) = (-\mu - |\vec{p}|)r_- P_- \gamma_0 + (-\mu + |\vec{p}|)r_+ P_+ \gamma_0$$

with

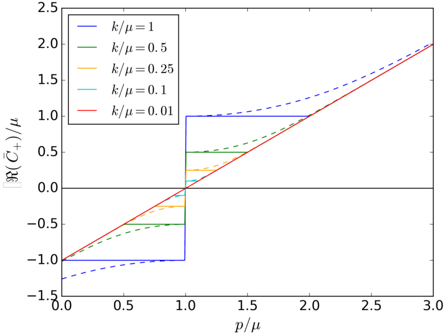
$$r_\pm = r(x_\pm) \qquad x_\pm k^2 = (-\mu \pm |\vec{p}|)^2 \qquad P_\pm = \frac{1}{2i}\left(i\gamma_0 \pm \frac{\vec{p}}{|\vec{p}|}\right)\gamma_0$$

such that the kinetic operator of the fermionic action can be written as

$$T = C_- P_- \gamma_0 + C_+ P_+ \gamma_0 + h\sigma$$
$$C_\mp = ip_o + (-\mu \mp |\vec{p}|)$$

- the regularized inverse propagator is

$$T + R_{k,F} = \bar{C}_- P_- \gamma_0 + \bar{C}_+ P_+ \gamma_0 + h\sigma$$
$$\bar{C}_\pm = ip_o + (-\mu \mp |\vec{p}|)(1 + r_\mp)$$



[9] J. Braun et al., Renormalization Group Studies of Dense Relativistic Systems (2020)

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- The inversion of the previous expression leads to:

$$(T + R_{k,F})^{-1} = \frac{\bar{C}_- P_- \gamma_0 + \bar{C}_+ P_+ \gamma_0 + h\sigma}{\bar{C}_+ \bar{C}_- + (h\sigma)^2}$$

- After some manipulation we obtain the following flow equation

$$\partial_t u_k(\sigma) = -\partial_t \left(4h^2\sigma \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\tilde{E}} \left[1 - n_f\left(\frac{\tilde{E} - \tilde{\mu}}{T}\right) - n_f\left(\frac{\tilde{E} + \tilde{\mu}}{T}\right) \right] \right)$$

with

$$\tilde{E} = \sqrt{\left(\frac{\omega_+ - \omega_-}{2}\right)^2 + (h\sigma)^2} \qquad \tilde{\mu} = \frac{\omega_+ + \omega_-}{2} \qquad \omega_\pm = (\mu \mp p)(1 + r_\pm)$$

- How do we solve this? $\Rightarrow$ μ-dependent initial condition.
  - Solve the mean-field flow equation $\Rightarrow$ RG-consistency.
  - Integrate mean-field flow equation $\Rightarrow$ fixing infrared physics.

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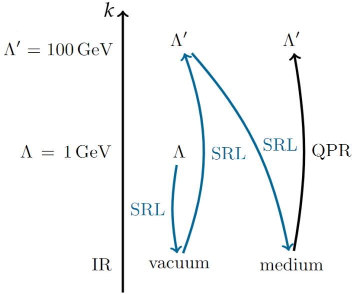
- RG consistency $\Rightarrow \Gamma$ independent of cut-off choice.

$$\Lambda \frac{d\Gamma}{d\Lambda} = 0$$

- RG consistency can be used to determine an μ-dependent initial condition for the flow equation.



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- "Analytical" solution ⇒ perform explicit t integration of equation 66.

$$u_{UV}(\sigma,\mu) - u_{IR}(\sigma,T,\mu) = K(\sigma,T,\mu,k=\Lambda) - K(\sigma,T,\mu,k=0)$$

$$K(\sigma,T,\mu,k) = -4h^2\sigma \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\tilde{E}}\left[1 - n_f\left(\frac{\tilde{E}-\tilde{\mu}}{T}\right) - n_f\left(\frac{\tilde{E}+\tilde{\mu}}{T}\right)\right]$$

- Now we suppose that the initial condition has the following shape:

$$u_{UV}(\sigma,\mu) = m(\mu)\,\sigma + \lambda(\mu)\,\sigma^3$$

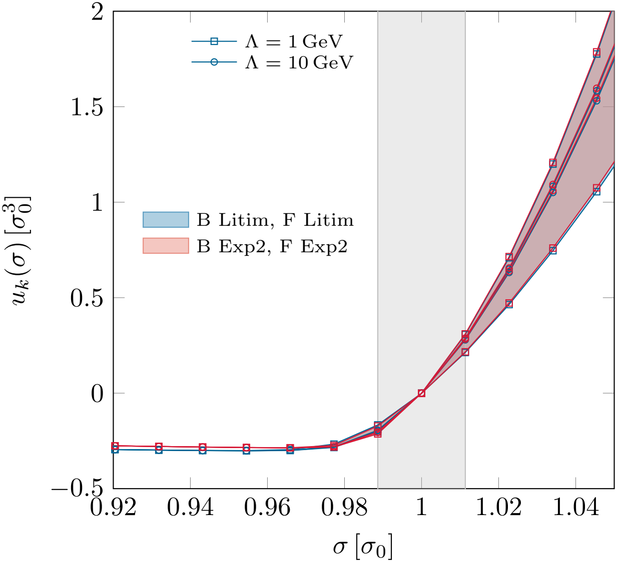
- Assuming the knowledge at $T=0$ of

$$\sigma_0(\mu) \qquad \text{and} \qquad M_\sigma^2(\sigma_0(\mu),\mu)$$

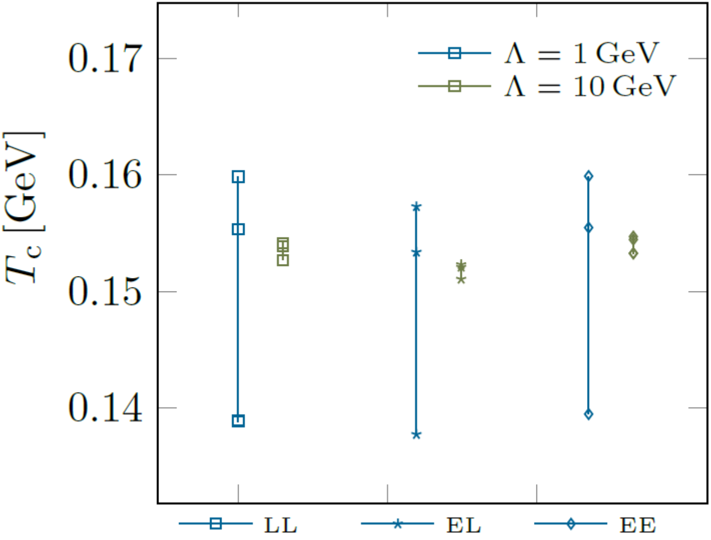
- we can determine the value of $m(\mu)$ and $\lambda(\mu)$ using

$$u_{IR}(\sigma_0(\mu),\mu) = 0 \qquad u'_{IR}(\sigma_0(\mu),\mu) = M_\sigma^2(\sigma_0(\mu),\mu)$$

Local enclosure of the potential



Critical temperature



Thermodynamic geometry

Thermodynamic geometry: the concept

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- An equilibrium state for a thermodynamic system is characterized by the pair $(\beta = 1/T\,,\, \gamma = -\mu/T)$.
- Key idea: we consider the (β, γ)-space as a two-dimensional manifold.
- We introduce a distance in this space

$$dl^2 = g_{\beta\beta} d\beta d\beta + 2 g_{\beta\gamma} d\beta d\gamma + g_{\gamma\gamma} d\gamma d\gamma \,, \tag{66}$$

where the metric tensor is

$$g_{ij} = \frac{\partial^2 \log \mathcal{Z}}{\partial \beta^i \partial \beta^j} = \frac{\partial^2 \phi}{\partial \beta^i \partial \beta^j} \equiv \phi_{,ij} \,, \tag{67}$$

with $\phi = \beta P$, $P = -\Omega$, Ω denotes the grand canonical thermodynamic potential density, $\beta^1 = \beta$ and $\beta^2 = \gamma$.

- Given these, we can construct the metric determinant

$$g = g_{\beta\beta} g_{\gamma\gamma} - g^2_{\beta\gamma} \tag{68}$$

Thermodynamic geometry: the concept

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- Thermodynamic stability requires that $g_{\beta\beta} > 0$, $g > 0 \Rightarrow d\ell^2 > 0$.
- $g = 0$ corresponds to a phase boundary.
- g_{ij} measures fluctuations of the observables to β^i, β^j:

$$V\phi_{,\beta\beta} = \langle (U - \langle U \rangle)^2 \rangle \,, \tag{69}$$
$$V\phi_{,\beta\gamma} = \langle (U - \langle U \rangle) \rangle \langle (N - \langle N \rangle) \rangle \,, \tag{70}$$
$$V\phi_{,\gamma\gamma} = \langle (N - \langle N \rangle)^2 \rangle \,, \tag{71}$$

  with U internal energy, N particle number, V volume of the system.
- One can define the Riemann tensor as

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}, \tag{72}$$

  with the Christoffel symbols

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right). \tag{73}$$

- Ricci tensor $R_{ij} = R^k_{ikj}$, and scalar curvature $R = R^i_i$. Within thermodynamic geometry, R is called the *thermodynamic curvature*.

Thermodynamic geometry: the concept

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- For our two-dimensional manifold there is only one independent component of the Riemann tensor $R = 2R_{1212}/g$, so we have:

$$R = -\frac{1}{2\,g^2}\begin{vmatrix} \phi_{,\beta\beta} & \phi_{,\beta\gamma} & \phi_{,\gamma\gamma} \\ \phi_{,\beta\beta\beta} & \phi_{,\beta\beta\gamma} & \phi_{,\beta\gamma\gamma} \\ \phi_{,\beta\beta\gamma} & \phi_{,\beta\gamma\gamma} & \phi_{,\gamma\gamma\gamma} \end{vmatrix}, \quad (74)$$

- The curvature diverges on a phase boundary $g \to 0$.
- R depends on the second- and third-order moments of the thermodynamic variables $\Rightarrow$information about the fluctuation of the physical quantities.
- Close to a second-order phase transition $|R| \propto \xi^3 \Rightarrow$ information on the correlation volume.
- R can convey details about the nature of the interaction:
  - $R > 0$ indicates an attractive interaction;
  - $R < 0$ corresponds to a repulsive one.
- These interactions include also the statistical attraction and repulsion in phase space :
  1. $R < 0$ for an ideal Fermi gas;
  2. $R > 0$ for an ideal Bose Gas;
  3. $R = 0$ for an ideal classical gas.

Thermodynamic geometry: FRG setup

- Thermodynamic quantities are extracted from the grand canonical thermodynamic potential:
$$\Omega(T,\mu) \equiv U_{k=0}(\sigma=\sigma_0;T,\mu) - c\sigma_0 \ . \tag{75}$$
- The solution $u(\sigma;T,\mu)$ has to be integrated in the σ variable $\Rightarrow$ the effective potential defined up to an arbitrary σ-independent but T and μ-dependent integration constant
$$U(\sigma;T,\mu) = \int_{\bar{\sigma}}^{\sigma} d\sigma' u(\sigma;T,\mu) + U(\bar{\sigma};T,\mu) \ , \tag{76}$$
where $\bar{\sigma} \in [0, \sigma_{\text{max}}]$ is an arbitrary grid point.
- In order to obtain the correct thermodynamic properties, we calculate this constant using the flow equation for the effective potential $\partial_t U_k(\sigma=\bar{\sigma})$.
- We choose a quartic potential for the initial condition $U_\Lambda(\sigma) = \frac{m_{UV}^2}{2}\sigma^2 + \frac{\lambda_{UV}}{4}\sigma^4$.
- The finite cutoff Λ cuts out thermal modes with $2\pi T > \Lambda$
- One expects the fermionic degrees of freedom to be relevant at higher temperature, thus
$$U_\Lambda(\sigma) \rightarrow U_\Lambda(\sigma) + U_\Lambda^\infty(\sigma) \ , \qquad U_\Lambda^\infty(\sigma) = \int_\infty^\Lambda S_k(\sigma) dk \tag{77}$$

QM model Thermodynamics

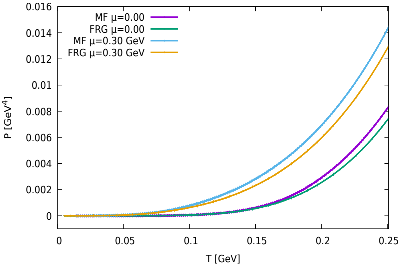
- The pressure is given by

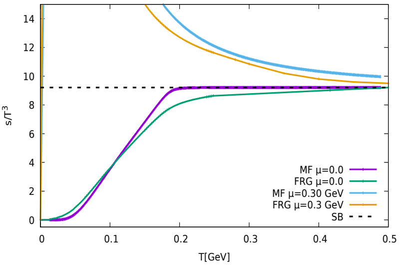
$$P(T,\mu) = -\Omega(T,\mu) + \Omega(0,0) \ , \tag{78}$$

- The entropy density is defined as

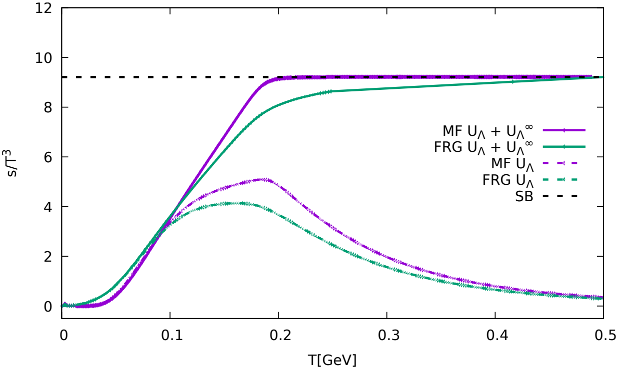
$$s = \frac{\partial P(T,\mu)}{\partial T} \ . \tag{79}$$

- In the QM model, at high temperature the main contribution to the pressure comes from the (nearly) massless quarks, while both the pions and the sigma bosons are massive due to the (partial) restoration of chiral symmetry and so effectively decouple. $\Rightarrow$ SB limit.

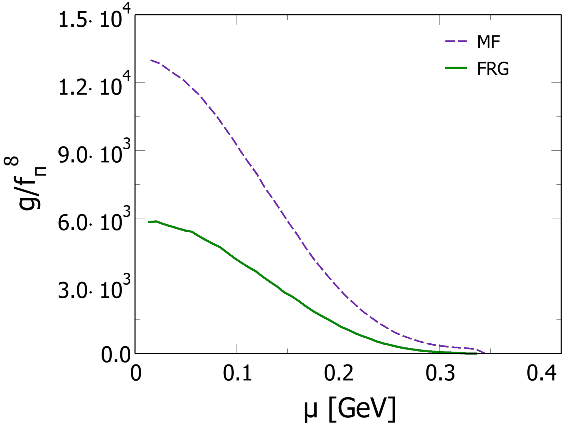




- Importance of the UV correction for the correct calculation of thermodynamic quantities at high temperatures.



- $g(T = T_c)$ decreases as we get closer to the critical point;
- FRG reaches criticality at lower chemical potentials then MF.



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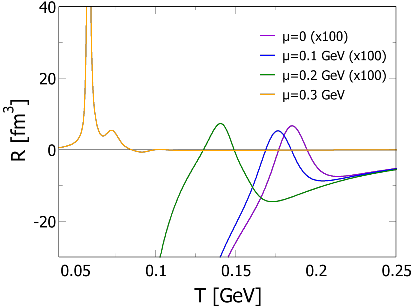
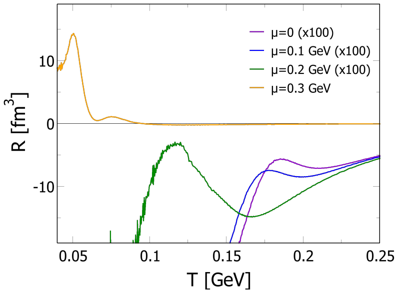
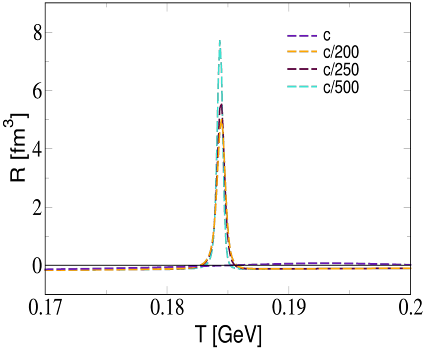
- R exhibits a peak in the around the pseudo-critical temperature;
- The inclusion of fluctuation lowers the magnitude of the peak;
- Both MF and FRG show a positive peak as criticality is approached;
- Multiple peak structure due to higher order momenta.

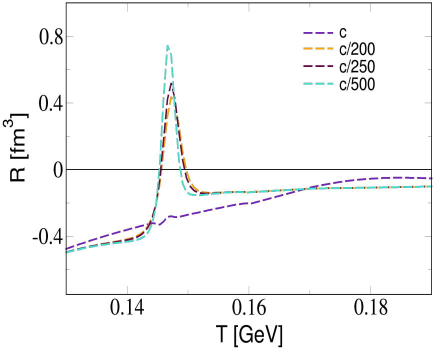




Thermodynamic geometry: results, chiral limit

- Chiral limit is challenging ⇒ we artificially lower c;
- Both MF and FRG show a peak around the critical temperature which becomes sharper towards the chiral limit;
- For Both MF and FRG the peak is positive ⇒ Statistical attraction;
- Fluctuations lower the magnitude of the peak;
- Qualitative behavior of R independent on the approximation.





Kurganov and Tadmor scheme

Kurganov and Tadmore scheme

Finite volume schemes are methods for solving non-linear advection–diffusion equations. In particular conservation laws are expressed in the form

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) = 0 \tag{80}$$

while convection-diffusion equations are represented by

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) = \frac{\partial}{\partial x}Q[u(x,t), u_x(x,t)] \tag{81}$$

where:

- $u(x,t) = (u_1(x,t), \cdots, u_N(x,t))$ is an N-component vector of conserved quantities in the d spatial variables $x = (x_1, \cdots, x_d)$;
- $f(u) = (f_1, \cdots, f_d)$ is the advection flux;
- $Q(u, u_x)$, or in general $Q(u, \nabla_x u) = (Q_1, \cdots, Q_d)$, is a diffusion flux.

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The computational domain V is discretized in the so called *control volumes* $I_x \times [t^n, t^n + \Delta t]$, where $|I_x| = |I|/N_x$ is the spatial control volume of width Δx centered around the point x obtained sampling I with N_x grid points, and t^n is the n-th point in which T is discretized.
The next step to build a FVM is to consider the average $\bar{u}$ of the solution u in the interval I_x:

$$\bar{u}(x,t) = \frac{1}{|I_x|} \int_{I_x} u(\xi,t)\, d\xi, \qquad I_x = \left\{ \xi : |\xi - x| \leq \frac{\Delta x}{2} \right\} . \tag{82}$$

Once we defined the averages, we discretize the partial differential equations, transforming them into algebraic equations by integrating them over the control volume $I_x \times [t^n, t^n + \Delta t]$ and we get the exact equation:

$$\begin{aligned} \bar{u}(x, t^n + \Delta t) = \bar{u}(x, t^n) - \frac{1}{\Delta x} \Bigg[& \int_{t^n}^{t^n + \Delta t} f\left(u\left(x + \frac{\Delta x}{2}, \tau \right) \right) d\tau \\ & - \int_{t^n}^{t^n + \Delta t} f\left(u\left(x - \frac{\Delta x}{2}, \tau \right) \right) d\tau \Bigg] . \end{aligned} \tag{83}$$

Kurganov and Tadmor scheme

- Sampling at $x = x_j$, at each time step t^n we obtain the algebraic system for $\bar{u}_j^{n+1} = \bar{u}(x_j, t^n + \Delta t)$, assuming that we already know the solution at the previous time step $\bar{u}_j^n = \bar{u}(x_j, t^n)$.
- An evaluation of the fluxes at the volume interfaces is needed.
- We will denote with x_j the grid points of the mesh, located in the middle of the control interval, and with $x_{j+1/2} = x_j + \Delta x/2$ the cell interfaces where the fluxes have to be evaluated.
- Since in FVM the information on the solution of the PDE is stored in the cell averages, one has no access to the value of the solution at the cell interface, which we will label as $\bar{u}^n_{j+1/2} = \bar{u}(x_j + \Delta x/2, t^n)$.
- Thus, in this case a *reconstruction* of $\bar{u}^n_{j+1/2}$ is needed, as a function of the cell averages $\bar{u}_j^n$. The way in which this reconstruction is performed differentiates the various FVM.
- Central schemes are based on sampling at the interfacing breakpoints, $x = x_{j\pm 1/2}$,

$$\bar{u}^{n+1}_{j+1/2} = \bar{u}^n_{j+1/2} - \frac{1}{\Delta x}\left[\int_t^{t+\Delta t} f(u_{j+1}, \tau) d\tau - \int_t^{t+\Delta t} f(u_j, \tau) d\tau\right] . \qquad (84)$$

Kurganov and Tadmor scheme

The main idea in the Kurganov and Tadmor central scheme is to average the nonsmooth parts of the computed solution over smaller cells of variable size.
We need to estimate the local speed of propagation at the cell boundaries: the upper bound is denoted by $a^n_{j+1/2}$ and given by

$$a^n_{j+1/2} = \max_{u \in \mathcal{C}(u^-_{j+1/2}, u^+_{j+1/2})} \rho\left(\frac{\partial f}{\partial u}(u)\right) \tag{85}$$

where

- ρ denotes the spectral radius of the flux Jacobian,
- $u^-_{j+1/2} = u^n_{j+1} - \frac{\Delta x}{2}(u_x)^n_{j+1}$ and $u^+_{j+1/2} = u^n_j + \frac{\Delta x}{2}(u_x)^n_j$ are the correspondent left and right intermediate values of $\hat{u}(x,t)$ at $x_{j+1/2} = x_j + \Delta x/2$,
- $\mathcal{C}(u^-_{j+1/2}, u^+_{j+1/2})$ is a curve in phase space connecting $u^-_{j+1/2}$ and $u^+_{j+1/2}$.

The derivatives are reconstructed through the *minmod limiter*:

$$(u_x)^n_j = \text{minmod}\left(\frac{\bar{u}^n_j - \bar{u}^n_{j-1}}{\Delta x}, \frac{\bar{u}^n_{j+1} - \bar{u}^n_j}{\Delta x}\right), \tag{86}$$

with $\text{minmod}(a,b) = 1/2[\text{sgn}(a) + \text{sgn}(b)] \cdot \min(|a|,|b|)$

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Kurganov and Tadmor scheme is then constructed in the following steps.

- We assume we have already computed the piecewise-linear solution at time level t^n, based on the cell averages $\bar{u}_j^n$

$$u(x,t^n) \approx \hat{u}(x,t^n) = \sum_j \bar{u}_j^n + (u_x)_j^n (x - x_j)\mathbb{I}_{[x_{j-1/2},\, x_{j+1/2}]} \tag{87}$$

- We compute the new cell averages $w_{j+1/2}^{n+1}$ and w_j^{n+1} at t^{n+1} in the following way:

$$w_{j+1/2}^{n+1} = \frac{1}{\Delta x_{j+1/2}} \int_{x_{j+1/2,l}^n}^{x_{j+1/2,r}^n} u(\xi, t^{n+1}) = \frac{u_j^n + u_{j+1}^n}{2} + \frac{\Delta x - a_{j+1/2}^n \Delta t}{4} \times$$

$$((u_x)_j^n - (u_x)_{j+1}^n) - \frac{1}{2a_{j+1/2}^n}[f(u_{j+1/2,r}^{n+1/2}) - f(u_{j+1/2,l}^{n+1/2})] \tag{88}$$

Kurganov and Tadmor scheme

$$w_j^{n+1} = \frac{1}{\Delta x_j} \int_{x^n_{j-1/2,r}}^{x^n_{j+1/2,l}} u(\xi, t^{n+1}) = u_j^n + \frac{\Delta t}{2}(a^n_{j-1/2} - a^n_{j+1/2})(u_x)^n_j$$

$$- \frac{\lambda}{1 - \lambda(a^n_{j-1/2} + a^n_{j+1/2})}[f(u^{n+1/2}_{j+1/2,r}) - f(u^{n+1/2}_{j+1/2,l})] \tag{89}$$

with

$$x^n_{j+1/2,r} = x_{j+1/2} + a^n_{j+1/2}\Delta t \qquad x^n_{j+1/2,l} = x_{j+1/2} - a^n_{j+1/2}\Delta t \tag{90}$$

$$\Delta x_{j+1/2} = x^n_{j+1/2,r} - x^n_{j+1/2,l} \qquad \Delta x_j = x^n_{j+1/2,l} - x^n_{j-1/2,r} \tag{91}$$

Kurganov and Tadmor scheme

$$u^{n+1/2}_{j+1/2,l} = u^n_{j+1/2,l} - \frac{\Delta t}{2} f(u^n_{j+1/2,l})_x \tag{92}$$

$$u^n_{j+1/2,l} = u^n_j + \Delta x (u_x)^n_j \left(\frac{1}{2} - \lambda a^n_{j+1/2}\right) \tag{93}$$

$$u^{n+1/2}_{j+1/2,r} = u^n_{j+1/2,r} - \frac{\Delta t}{2} f(u^n_{j+1/2,r})_x \tag{94}$$

$$u^n_{j+1/2,r} = u^n_j - \Delta x (u_x)^n_{j+1} \left(\frac{1}{2} - \lambda a^n_{j+1/2}\right) \tag{95}$$

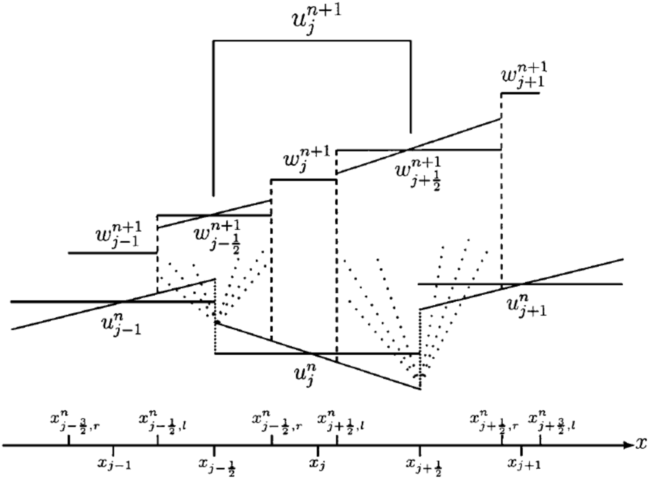
$$\lambda = \frac{\Delta t}{\Delta x} \tag{96}$$

- We consider the piecewise-linear reconstruction over the nonuniform cells at $t = t^{n+1}$

$$\hat{w}(x, t^{n+1}) = \sum_j \{[w_{j+1/2}^{n+1} + (u_x)_{j+1/2}^{n+1}(x - x_{j+1/2})]\mathbb{I}_{[x_{j+1/2,l}^n, x_{j+1/2,r}^n]} + w_j^{n+1}\mathbb{I}_{[x_{j-1/2,r}^n, x_{j+1/2,l}^n]}\} \tag{97}$$

- We project its averages back onto the original uniform grid.

$$u_j^{n+1} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \hat{w}(\xi, t^{n+1})d\xi = \lambda a_{j-1/2}^n w_{j-1/2}^{n+1} + [1 - \lambda(a_{j-1/2}^n + a_{j+1/2}^n)]w_j^{n+1} + \lambda a_{j+1/2}^n w_{j+1/2}^{n+1} + \frac{\Delta x}{2}[(\lambda a_{j-1/2}^n)^2(u_x)_{j-1/2}^{n+1} - (\lambda a_{j+1/2}^n)^2(u_x)_{j+1/2}^{n+1}] \tag{98}$$


u_j^{n+1}
w_{j+1}^{n+1}
w_j^{n+1}
$w_{j+\frac{1}{2}}^{n+1}$
w_{j-1}^{n+1}
$w_{j-\frac{1}{2}}^{n+1}$
u_{j+1}^n
u_{j-1}^n
u_j^n
$x_{j-\frac{3}{2},r}^n$
$x_{j-\frac{1}{2},l}^n$
$x_{j-\frac{1}{2},r}^n$
$x_{j+\frac{1}{2},l}^n$
$x_{j+\frac{1}{2},r}^n$
$x_{j+\frac{3}{2},l}^n$
x
x_{j-1}
$x_{j-\frac{1}{2}}$
x_j
$x_{j+\frac{1}{2}}$
x_{j+1}

Kurganov and Tadmor scheme

Semi-discrete reduction: $\Delta t \to 0$,$\lambda \to 0$.

$$\frac{d}{dt}u_j(t) = -\frac{(f(u^+_{j+1/2}(t) + f(u^-_{j+1/2}(t)) - (f(u^+_{j-1/2}(t) + f(u^-_{j-1/2}(t))}{2\Delta x}$$

$$+\frac{1}{2\Delta x}\{a_{j+1/2}[u^+_{j+1/2}(t) - u^-_{j+1/2}(t)] - a_{j-1/2}[u^+_{j-1/2}(t) - u^-_{j-1/2}(t)]\} \quad (99)$$

In this reduction the maximal local speed $a_{j+1/2}(t)$ takes the form

$$a^n_{j+1/2}(t) = \max\left\{\rho\left(\frac{\partial f}{\partial u}(u^-_{j+1/2}(t))\right), \rho\left(\frac{\partial f}{\partial u}(u^+_{j+1/2}(t))\right)\right\} \quad (100)$$

Conservative form:

$$\frac{d}{dt}u_j(t) = -\frac{H_{j+1/2}(t) - H_{j-1/2}(t)}{\Delta x} \tag{101}$$

with the numerical flux

$$H_{j+1/2}(t) = \frac{f(u^+_{j+1/2}(t)) + f(u^-_{j+1/2}(t))}{2} - \frac{a_{j+1/2}(t)}{2}[u^+_{j+1/2}(t) - u^-_{j+1/2}(t)] \tag{102}$$

Kurganov and Tadmor second-order semi-discrete scheme, can be easily applied to one-dimensional convection–diffusion equations

$$\frac{d}{dt}u_j(t) = -\frac{H_{j+1/2}(t) - H_{j-1/2}(t)}{\Delta x} + \frac{P_{j+1/2}(t) - P_{j-1/2}(t)}{\Delta x} \tag{103}$$

Kurganov and Tadmor scheme

with $P_{j+1/2}(t)$ is a reasonable approximation of the diffusion flux

$$P_{j+1/2}(t) = \frac{1}{2}\left[Q\left(u_j(t), \frac{u_{j+1}(t) - u_j(t)}{\Delta x}\right) + Q\left(u_{j+1}(t), \frac{u_{j+1}(t) - u_j(t)}{\Delta x}\right)\right]$$

Key features of Kurganov and Tadmor scheme

- semplicity, since no spectral decomposition of the flux f is needed;
- second order precision in Δx;
- sharp resolution of discontinuities;
- stability, since semi-discrete formulation allows to treat easily small Δt.

Kurganov and Tadmore scheme

Example 1: linear steady shock.
Let us consider the following problem

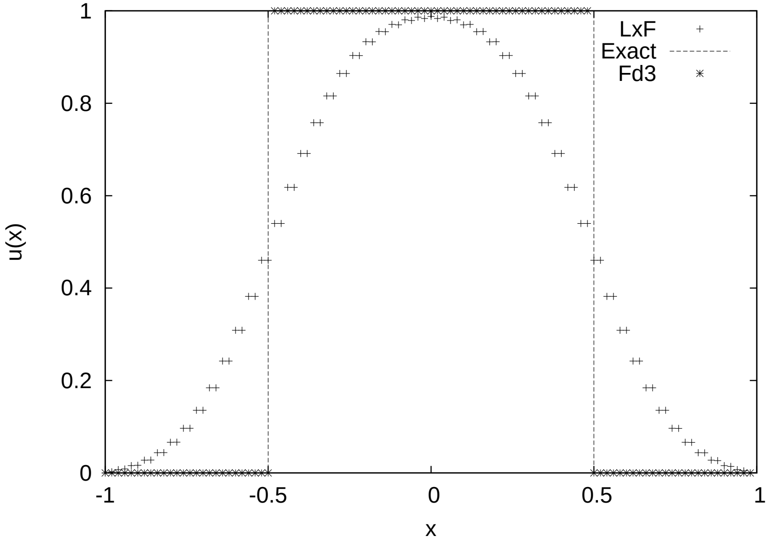
$$f(u) = 0 \tag{104}$$

$$u_t = 0 \tag{105}$$

subject to the discontinuous initial data

$$u(x,0) = \begin{cases} 1 & -0.5 < x < 0.5 \\ 0 & \text{otherwise} \end{cases} \tag{106}$$

Kurganov and Tadmor scheme



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Example 2: inviscid Burgers' equation
Let us consider the following problem

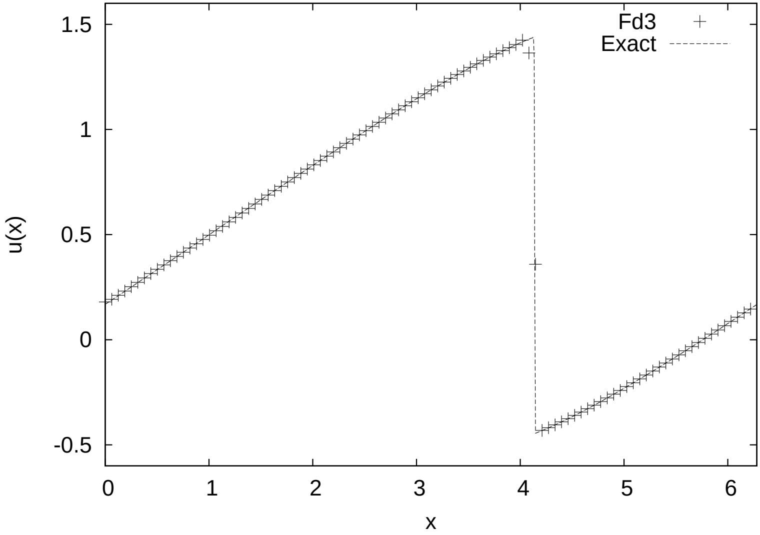
$$f(u) = \frac{u^2}{2} \tag{107}$$

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{108}$$

with a smooth periodic initial data

$$u(x,0) = 0.5 + \sin x. \tag{109}$$

Kurganov and Tadmor scheme



$O(N)$ Model

$O(N)$ Model

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The theory describes N scalar fields $\phi_a(x)$ with $a = 1, \cdots, N$.
The associated bare action at the cutoff scale $k = \Lambda$ is

$$S_{k=\Lambda}[\vec{\phi}] = \int d^d x \left\{ \frac{1}{2}(\partial_\mu \phi_a)^2 + V_{k=\Lambda}(\rho) \right\}$$

$$V_{k=\Lambda}(\rho) = \frac{\lambda}{4}(\rho - \rho_0)^2 , \qquad \rho = \frac{1}{2}\phi^a \phi_a , \qquad \sigma = \sqrt{2\rho}$$

The model exhibits a spontaneous symmetry breaking of the $O(N)$-group down to $O(N-1)$, which is restored at sufficiently high temperature via a second-order phase transition.

Motivation:

- Universality: close to phase transition UV details do not count $\Rightarrow$ chiral phase transition in 2-flavour QCD;
- Easy to solve, well known results, good framework for numerical tests.

[4] Gell-Mann M and Levy M 1960 Nuovo Cimento 16 705–26.
[5] Adrian Koenigstein, Martin J. Steil, Nicolas Wink, Eduardo Grossi, Jens Braun, Michael Buballa, and Dirk H. Rischke, Numerical fluid dynamics for FRG flow equations: Zero-dimensional QFTs as numerical test cases - Part I: The $O(N)$ model.

O(N) model: flow equation in the Local Potential Approximation (LPA)

O(N) model: LPA flow equation

Within the LPA, one obtains the flow equation for the effective potential:

$$\partial_t V_k(t,\sigma) = -A_d\, k^{d+2} \left(\frac{N-1}{k^2 + \frac{1}{\sigma}\partial_\sigma V_k(t,\sigma)} + \frac{1}{k^2 + \partial^2_{\sigma\sigma} V_k(t,\sigma)} \right)$$

We introduce the derivative of the potential as new variable

$$u(t,\sigma) = \partial_\sigma V_k(t,\sigma), \qquad u'(t,\sigma) = \partial_\sigma u(t,\sigma)$$

We can now introduce the advection and the diffusion fluxes

$$f(t,\sigma,u) = A_d\, k^{d+2} \frac{N-1}{k^2 + \frac{1}{\sigma} u(t,\sigma)} \qquad g(t,u') = -A_d\, k^{d+2} \frac{1}{k^2 + u'(t,\sigma)}$$

Taking the derivative of the flow equation with respect to σ we obtain

$$\partial_t u(t,\sigma) + \partial_\sigma f(t,\sigma,u) = \partial_\sigma g(t,u')$$

which is an *advection-diffusion* equation for $u(t,\sigma)$.

[5] Adrian Koenigstein, Martin J. Steil, Nicolas Wink, Eduardo Grossi, Jens Braun, Michael Buballa, and Dirk H. Rischke, Numerical fluid dynamics for FRG flow equations: Zero-dimensional QFTs as numerical test cases - Part I: The $O(N)$ model.
[6]E. Grossi and N. Wink (2019), arXiv:1903.09503.

O(N) model: LPA flow equation

Advection contribution

$$\partial_t u(t,\sigma) + \partial_u f(t,\sigma,u)\, u'(t,\sigma) + \partial_\sigma f(t,\sigma,u) = 0$$

Advection coefficient

$$\partial_u f(t,\sigma,u) = -A_d\, k^{d+2} \frac{N-1}{\sigma[k^2 + \frac{1}{\sigma} u(t,\sigma)]^2} < 0 \qquad \forall \sigma > 0$$

Diffusion contribution:

$$\partial_t u(t,\sigma) = \partial_{u'} g(t,u')\, u''(t,\sigma)$$

Diffusion coefficient

$$\partial_{u'} g(t,\sigma') = A_d\, k^{d+2} \frac{1}{[k^2 + u'(t,\sigma)]^2} > 0$$

We used Kurganov-Tadmor scheme (FV scheme, second order in Δx... [8]).

[7] Alexander Kurganov and Eitan Tadmor, Journal of Computational Physics 160, 241 (2000).

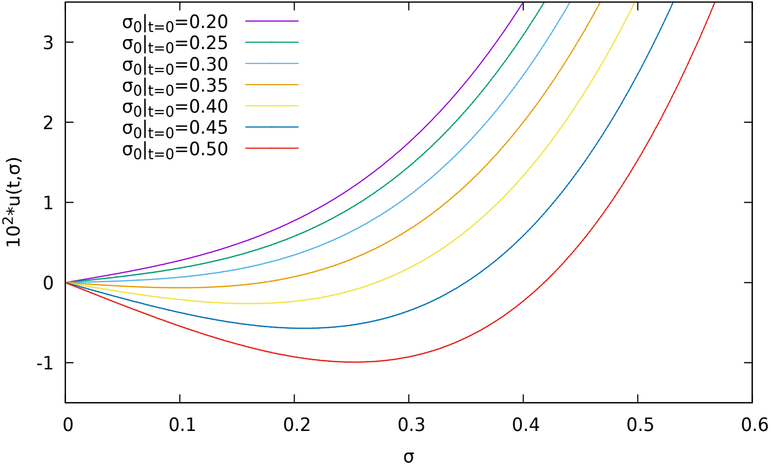


Figure: Effective potential derivative $u(t,\sigma)$ at $t=1$ for $N=3$ in LPA.

O(N) model: LPA flow equation

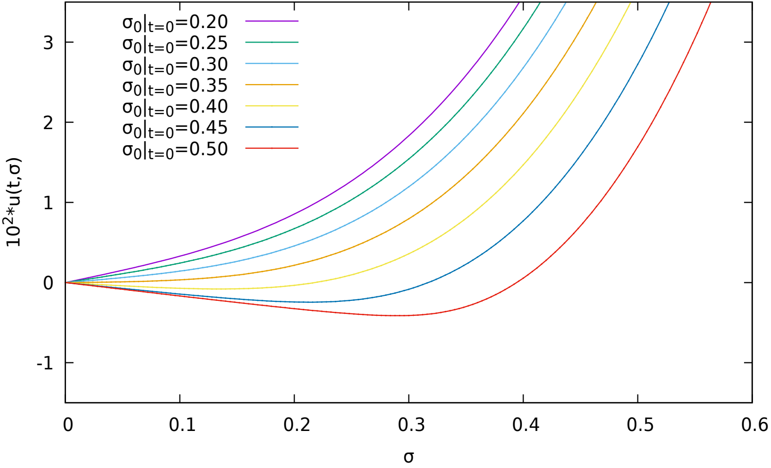


Figure: Effective potential derivative $u(t, \sigma)$ at $t = 2$ for $N = 3$ in LPA.

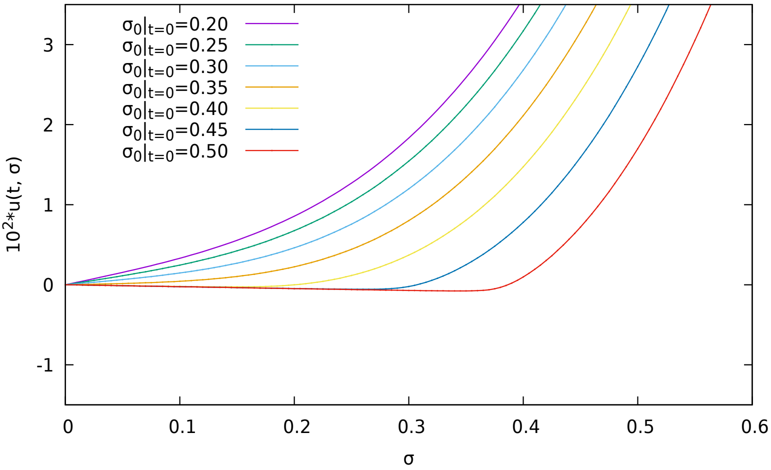


Figure: Effective potential derivative $u(t, \sigma)$ at $t = 3$ for $N = 3$ in LPA.

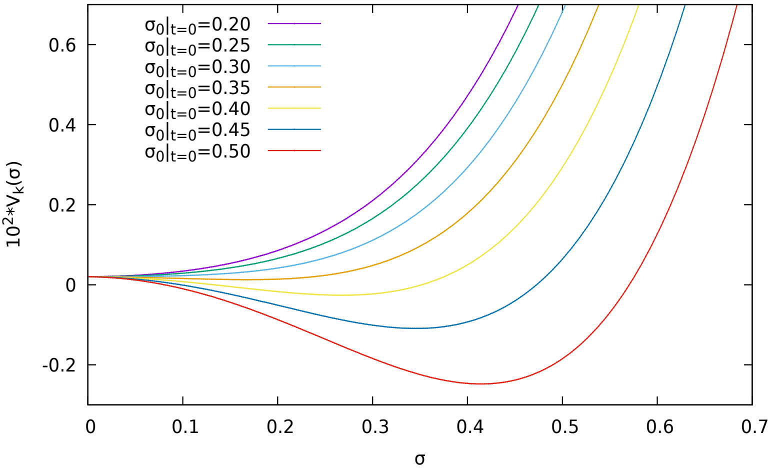


Figure: Effective potential $V_k(\sigma)$ at $t = 1$ for $N = 3$ in LPA.

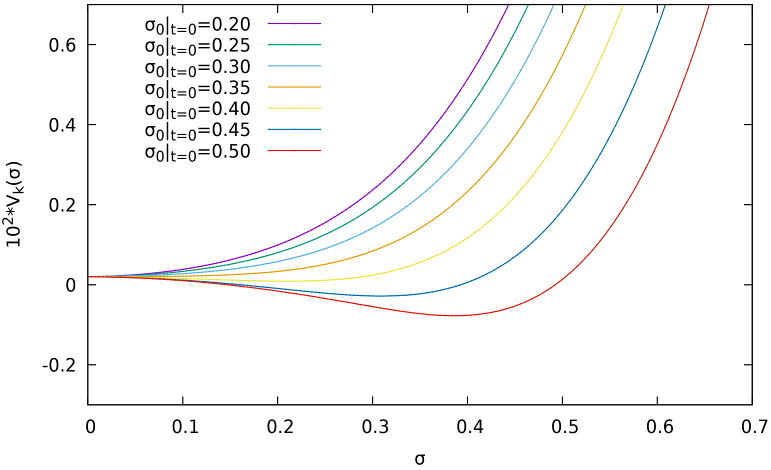


Figure: Effective potential $V_k(\sigma)$ at $t = 2$ for $N = 3$ in LPA.

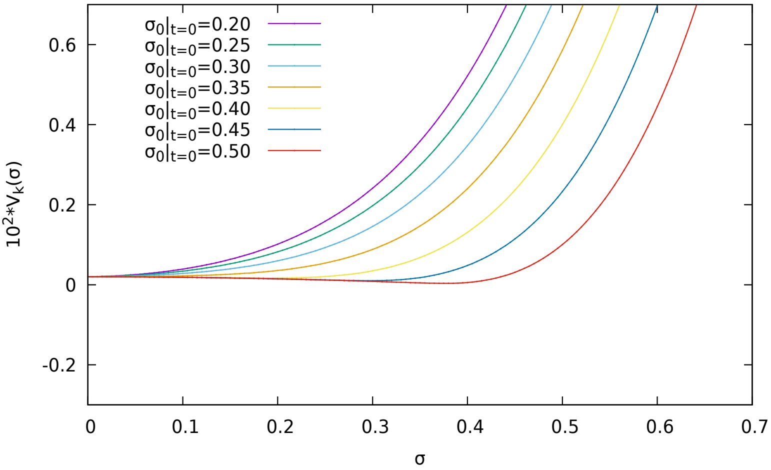


Figure: Effective potential $V_k(\sigma)$ at $t = 3$ for $N = 3$ in LPA.

O(N) model: critical behavior

O(N) model: critical $\rho_0|_{t=0}$

- Dimensional reduction phenomenon: close to criticality we can use the $T = 0$ $d = 3$ flow equations.

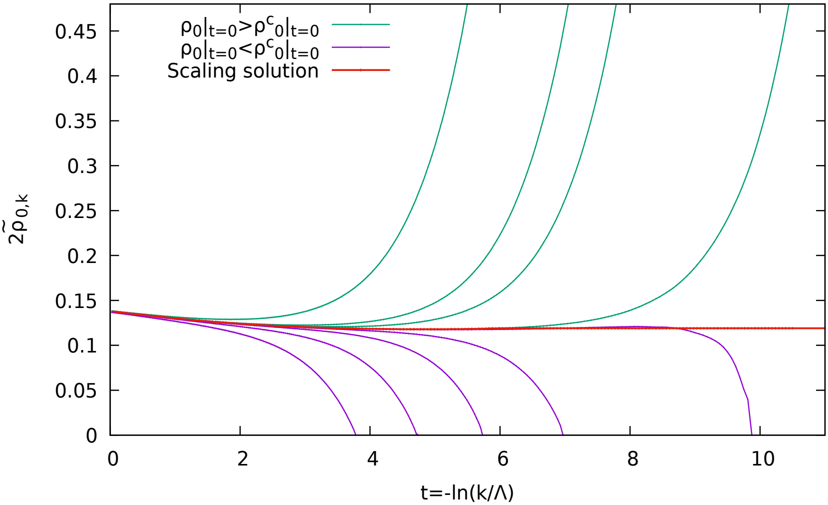
$\sigma_0|_{IR}$ is the *order parameter*, $\rho_0|_{t=0}$ plays the role of the temperature :

- if $\rho_0|_{t=0} > \rho_0^c|_{t=0}$ $(T < T_c) \Rightarrow$ broken phase $\sigma_0|_{IR} > 0$;
- if $\rho_0|_{t=0} < \rho_0^c|_{t=0}$ $(T > T_c) \Rightarrow$ symmetric phase $\sigma_0|_{IR} = 0$.

We fix λ to an arbitrary value $\lambda = 0.5$ and adjust $\rho_0|_{t=0}$ at $t = 0$ in order to find the *scaling solution*.

- broken phase $(\rho_0|_{t=0} > \rho_0^c|_{t=0})$: $\rho_{0,k} \to \rho_0|_{IR} > 0$
  $\Rightarrow\ \tilde{\rho}_{0,k} = \frac{\rho_0}{k} \to +\infty$ as $k \to 0$;
- symmetric phase $(\rho_0|_{t=0} < \rho_0^c|_{t=0})$: $\rho_{0,k} \to \rho_0|_{IR} = 0$
  $\Rightarrow\ \tilde{\rho}_{0,k} = \frac{\rho_0}{k} \to 0$ as $k \to 0$;

O(N) model: critical $\rho_0|_{t=0}$



Critical exponent ν

The *correlation length* ξ diverges close to criticality as

$$\xi(\rho_0|_{t=0}) \sim \left(\big|\rho_0|_{t=0} - \rho_0^c|_{t=0}\big|\right)^{-\nu}$$

We consider the renormalized mass m^2

$$m^2 = \lim_{k\to 0} u'(\sigma = 0, k) = \lim_{k\to 0} V''(\sigma = 0, k) = \frac{1}{\xi^2}$$

Thus

$$m^2 \sim \left(\big|\rho_0|_{t=0} - \rho_0^c|_{t=0}\big|\right)^{2\nu}$$

We can obtain ν from

$$\ln m^2 = 2\nu \ln\left(\big|\rho_0|_{t=0} - \rho_0^c|_{t=0}\big|\right) + const$$

O(N) model: critical exponent ν

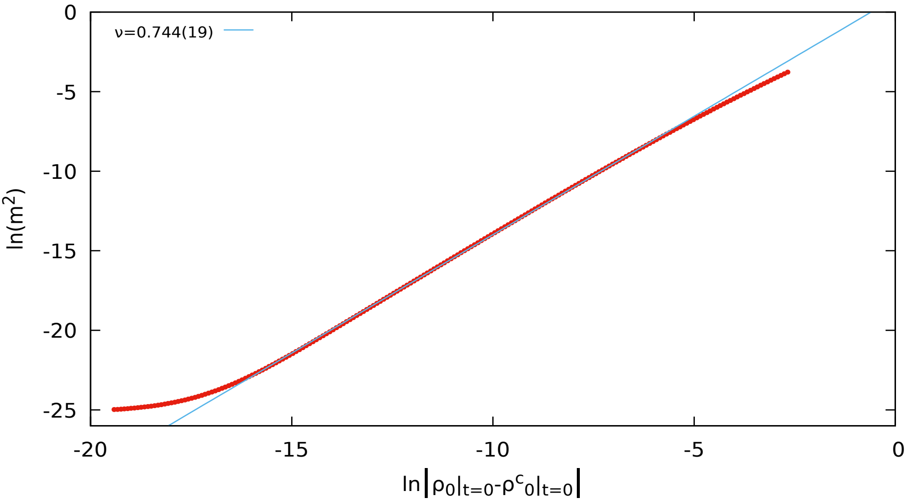


Figure: Critical exponent ν for $N = 3$ in LPA.

Critical exponent β

Close to criticality, the order parameter $\sigma_0|_{IR}$ is described by the following behaviour

$$\sigma_0|_{IR} = \begin{cases} 0 & \rho_0|_{t=0} < \rho_0^c|_{t=0} \\ \sim (\rho_0|_{t=0} - \rho_0^c|_{t=0})^\beta & \rho_0|_{t=0} > \rho_0^c|_{t=0} \end{cases}$$

So we can extract β from

$$\ln \sigma_0|_{IR} = \beta \ln(\rho_0|_{t=0} - \rho_0^c|_{t=0}) + const$$

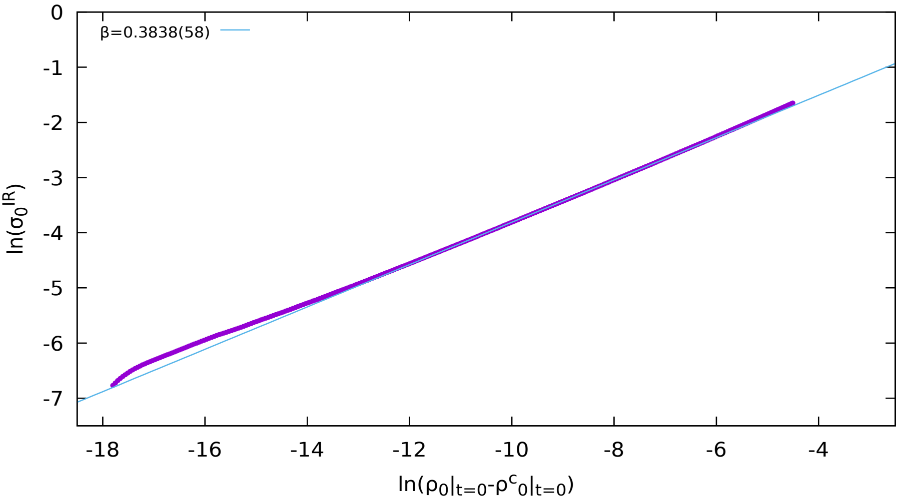


Figure: Critical exponent β for $N = 3$ in LPA.

O(N) model: results critical exponents