Few-nucleon systems within EFT-pionless framework

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October 3, 2025

Finanziato dall'Unione europea NextGenerationEU
Ministero dell'Università e della Ricerca
Italiadomani PIANO NAZIONALE DI RIPRESA E RESILIENZA
Università di Pisa

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Low energy EFT contact (pionless) interaction

- EFT potential (LO, NLO, N3LO)
- pionless, contact interaction
- fitted to deuteron binding energy and two-body scattering observables up to 15 MeV
- local
- has a 3-nucleon LO term

Operator structure (2N interaction): $v = \sum_{l=1}^{15} v^l(r)\, \hat{O}^l_{15}$, with

$$\hat{O}^l_{15} \rightarrow 1,\, \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2,\, \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2,\, (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2),\, S_{12},\, S_{12}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2),$$
$$\mathrm{L} \cdot \mathrm{S},\, (\mathrm{L} \cdot \mathrm{S})(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2),\, (\mathrm{L} \cdot \mathrm{S})^2,\, \mathrm{L}^2,\, \mathrm{L}^2(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2),$$
$$T_{12},\, (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\, T_{12},\, S_{12}\, T_{12},\, (\mathrm{L} \cdot \mathrm{S})\, T_{12}\,.$$

R. Schiavilla *et al.*, Phys. Rev. C **103**, 054003 (2021)

Some information

The potential

1. has been developed in momentum space;
2. two isospin-dependent gaussian cutoff have been applied;
3. has been trasformed in coordinate space.

The cutoffs

$$\tilde{C}(k) = \sum_{T=0}^{1} e^{-R_T^2 k^2/4}\, P_T^\tau \quad \Rightarrow \quad C(r) = \sum_{T=0}^{1} C_T(r)\, P_T^\tau$$

→ P_0^τ (P_1^τ) is the $T=0$ ($T=1$) projector operator

→ $C_T(r) = \dfrac{1}{\pi^{3/2} R_T^3} e^{-r^2/R_T^2}$

Projecting the potential

To use it and (further) study the potential → projection on ST basis
Example (NLO) for $v^{ST}(r) \equiv \langle ST|v(r)|ST\rangle$

$$v^{00}(r) = \tilde{C}_1\left(-C_0^{(2)}(r) - \frac{2}{r}C_0^{(1)}(r)\right)$$

$$v^{10}(r) = C_{10}C_0(r) + \tilde{C}_2\left(-C_0^{(2)}(r) - \frac{2}{r}C_0^{(1)}(r)\right) + \\ + \tilde{C}_5\left(-C_0^{(2)} + \frac{1}{r}C_0^{(1)}(r)\right)S_{12} - C_7\frac{1}{r}C_0^{(1)}(r)\mathrm{L}\cdot\mathrm{S}$$

$$v^{01}(r) = C_{01}C_1(r) + \tilde{C}_3\left(-C_1^{(2)}(r) - \frac{2}{r}C_1^{(1)}(r)\right) + C_0^{\mathrm{IT}}C_1(r)T_{12}$$

$$v^{11}(r) = \tilde{C}_4\left(-C_1^{(2)}(r) - \frac{2}{r}C_1^{(1)}(r)\right) + \tilde{C}_6\left(-C_1^{(2)} + \frac{1}{r}C_1^{(1)}(r)\right)S_{12} \\ - C_7\frac{1}{r}C_1^{(1)}(r)\mathrm{L}\cdot\mathrm{S} + C_0^{\mathrm{IT}}C_1(r)T_{12}$$

$p-d$ scattering observables

Ingredients:

- potential model optimized (cutoff fitted) at N3LO;
- 2N and 2N+3N interactions used (cutoff 3N $R_3 = 1.5$ fm).

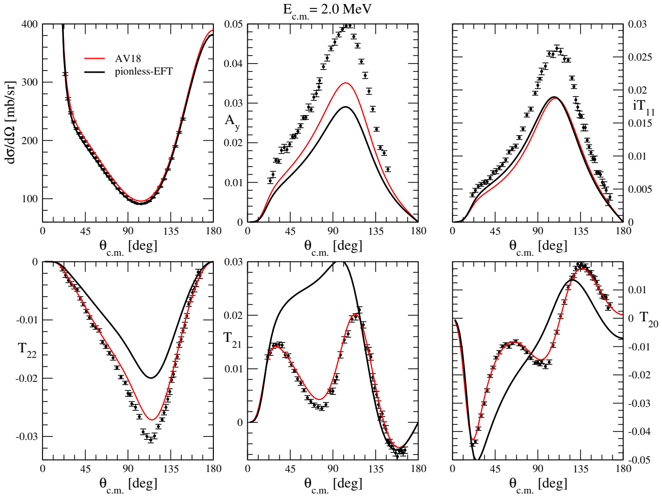
Evaluated phase shifts/mixing angles:

- at $E_{\mathrm{c.m.}} = 2$ MeV;
- up to $L = 6$;
- for $J^\pi = \{\frac{1}{2}^+, \frac{3}{2}^+, \dots, \frac{15}{2}^+\}$ and $J^\pi = \{\frac{1}{2}^-, \frac{3}{2}^-, \dots, \frac{13}{2}^-\}$

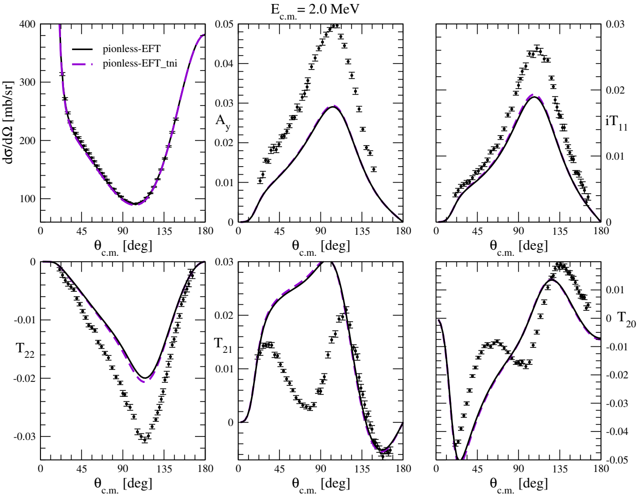
$\Rightarrow$ Observables:

- the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$;
- the spin asymmetry A_y;
- the tensor analyzing power iT_{11}, T_{20}, T_{21} and T_{22}.

Comparing observables between the AV18 and the Pionless-EFT pionless 2N potentials



Comparing observables between Pionless-EFT pionless 2N and Pionless-EFT pionless 2N+3N ($R_3 = 1.5$ fm)



Observations

- the best models of this potential $\rightarrow$ very bad results;
- cross section well reproduced $\rightarrow$ S-wave results good;
- 3N force doesn't contribute much.

Need for

- best model reproducing P and (?) D-waves;
- a reference to re-fit the potential low-energy constants (LECs)

Choice of the observables to fit to

Since

- AV18 $\rightarrow$ optimal results;
- lack of low-energy data for P-waves;

then

- we chose **low-energy observables evaluated with the AV18 potential** as "data";
- investigated more deeply the P-waves sector;
- eliminated the spin-orbit term from the deuteron channel $\rightarrow$ cutoff effect;
- focused on NLO;
- started from the NLO optimized potential.

What did we fit

Scattering $np \to$ phase-shifts δ_ℓ at low energy obey

$$k^{2\ell+1}\cot\delta_\ell \simeq -\frac{1}{a_\ell} + \frac{1}{2} r_\ell k^2 + \sum_{i>2} b_\ell k^{2i} \,,$$

where

- $a_\ell \to$ scattering length/volume;
- $r_\ell \to$ effective range.

For coupled channels the mixing angles ϵ_J

$$\epsilon_J \simeq \epsilon_0 \, k^{\ell_1+\ell_2} \,.$$

We focused at NLO up to terms $\propto k^2$, fitting a_ℓ, r_ℓ and $\epsilon_{J=1}$.

Potentials projections and channels

Potential projection	channels
$v^{00}(r)$	$^1P_1$, $^1F_3$, ...
$v^{10}(r)$	$^3S_1 - {}^3D_1$, $^3D_2$, ...
$v^{01}(r)$	$^1S_0$, $^1D_2$, ...
$v^{11}(r)$	$^3P_0$, $^3P_1$, $^3P_2 - {}^3F_2$, ...

- the channel $^1S_0$ is well reproduced $\rightarrow$ v^{01} not modified;
- v^{00} depends on an independent variable $\rightarrow$ first to be fitted;
- v^{10} fitted to reproduce also the deuteron binding energy, removed the spin-orbit term;
- v^{11} needs to fit $^3P_0$, $^3P_2$ and $^3P_2$ (coupled) channels $\rightarrow$ 3* LECs to fit.

Fitting the v^{00} potential projection

The fit was done using

$$y = ax^2 + bx + c\,,$$

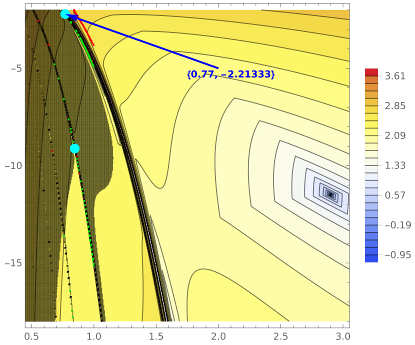
with $y = k^3 \cot\delta$ and $x = k^2$. Therefore

$$a_\ell = -\frac{1}{c} \qquad \text{and} \qquad r_\ell = 2b\,.$$

Methods:

- $\{R_0,\ \tilde{C}_1\}$ grid;
- Pounders code from PETSc framework.

The $\{R_0,\ \tilde{C}_1\}$ grid



- **Black** points (shaded): "small" values for a
- Red points: good values for b
- Green points: good values for c
- **Cyan** points: common good values for b and c

The PETSc framework

Starting from the best point of the grid:
$\{0.77\ \mathrm{fm}, -2.21333\ \mathrm{MeV}^4\} \rightarrow$ **minimize the value**

$$\chi^2 = (a - a_{18})^2 w_a + (b - b_{18})^2 w_b + (c - c_{18})^2 w_c$$

with

$$w_a = 1\,, \qquad w_b = 200 \qquad \text{and} \qquad w_c = 150$$

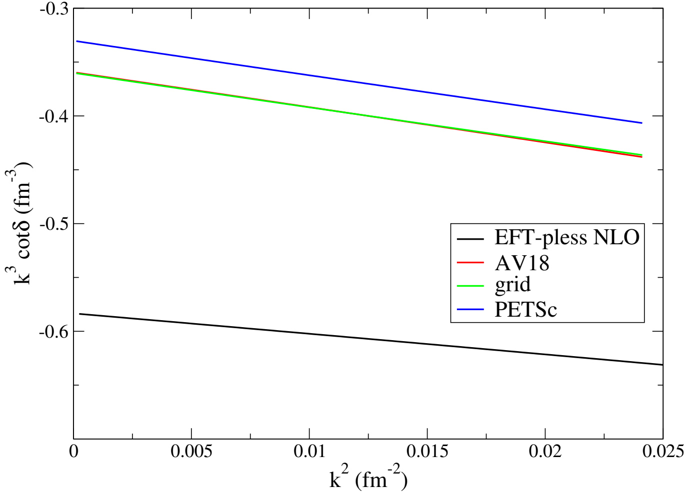
+ **hard limits** on a, b and c.

Found $\Rightarrow \{0.76178\ \mathrm{fm}, -2.14454\ \mathrm{MeV}^4\}$

Comparison

potential		$^1P_1$
EFTπ-less NLO	a	1.7141
AV18		2.7838
EFTπ-less grid		2.7774
EFTπ-less PETSc		3.0285
EFTπ-less NLO	r_e	-3.7482
AV18		-6.4960
EFTπ-less grid		-6.4492
EFTπ-less PETSc		-6.4960

Comparison (2)



Modifying and fitting the v^{10} potential projection

The formula for v^{10} was modified to

$$v^{10} = C_{10} C_0(r) P_0^L + \tilde{C}_2 \left(-C_0^{(2)}(r) - \frac{2}{r} C_0^{(1)}(r) \right) + \\ + \tilde{C}_5 \left(-C_0^{(2)} + \frac{1}{r} C_0^{(1)}(r) \right) S_{12} - \cancel{C_7 \frac{1}{r} C_0^{(1)}(r) \mathrm{L} \cdot \mathrm{S}}$$

$\mathrm{L} \cdot \mathrm{S} \rightarrow$ **disregarded as a cutoff effect**

Refitted to

- **deuteron binding energy**
- $^3S_1$ **scattering length** (AV18 potential)
- $^3S_1$ **phase shifts** and ϵ_1 **mixing angles** (1/5/10 MeV Granada database)

Fitting the v^{11} potential projection

In this case we used four methods:

- grid + PETSc (C_7 fixed to article value);
- PETSc, focused on scattering length;
- PETSc, focused on effective range;
- PETSc;

to reproduce the aforementioned observables for ${}^3P_0$, ${}^3P_1$ and ${}^3P_2$.

Notice: 4 LECs / 6 observables $\rightarrow$ too many observables.
Notice: cannot well reproduce all effective ranges and scattering lengths at the same time!

Comparison

potential		$^3P_0$	$^3P_1$	$^3P_2$
EFTπ-less NLO		-2.1422	1.3333	-0.3536
AV18		-2.5162	1.5277	-0.2928
EFTπ-less grid	a	-2.9050	1.3493	-0.7220
EFTπ-less a_J		-2.5162	1.5277	-0.2928
EFTπ-less r_J		-2.8300	1.6517	-0.1565
EFTπ-less ar_J		-2.7512	1.6449	-0.1632
EFTπ-less NLO		1.4354	-4.3470	6.40
AV18		3.7680	-8.5800	9.39
EFTπ-less grid	r_e	3.7971	-8.5230	15.86
EFTπ-less a_J		3.8050	-7.9104	21.31
EFTπ-less r_J		3.7690	-8.5800	9.39
EFTπ-less ar_J		3.9152	-8.6002	9.14

Fitted potential sets

Because for the $S = 0$ and $T = 0$ we have two potentials

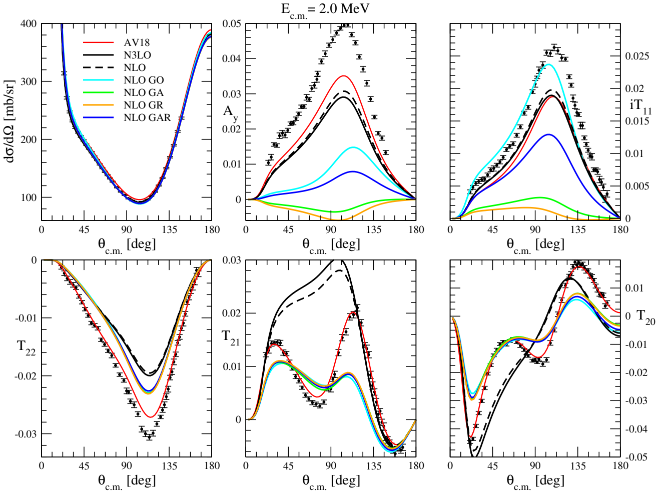
- grid potential (G);
- PETSc potential (P);

and four for the $S = 1$ and $T = 1$

- originally fitted months ago (O);
- a_J fitted (A);
- r_J fitted (R);
- a_j and r_J fitted (AR);

we end up with **8** sets of potentials: GO, GA, GR, GAR, PO, PA, PR, PAR.

Preliminary results in the 3-body *pd* scattering observables



Conclusions and Outlook

Conclusions:

- differential cross section well reproduced $\rightarrow$ S-waves well behaved
- A_y and $i\,T_{11}$ poorly reproduced: P-waves need readjustments
- T_{2x} badly reproduced: bad D-waves?

$\Rightarrow$ Results with the new set of potentials solve a little: **need for N3LO** and **fit higher waves**?

Thank you for your attention

Spares

The nuclear potential (LO, NLO, N3LO)

The nuclear potential is a pionless EFT potential with 15 operator components. The 15 operator components of the potential are divided into 11 charge independent components (CI) and four charge dependent components (CD), therefore

$$v^{NUC} = v^{CI} + v^{CD}$$

with

$$v^{CI} = \sum_{l=1}^{11} v^{l}(r)\, O_{12}^{l}\,, \qquad v^{CD} = \sum_{l=12}^{15} v^{l}(r)\, O_{12}^{l}$$

and

$$O_{12}^{l} \to c,\, \tau,\, \sigma,\, \sigma\tau,\, t,\, t\tau,\, b,\, b\tau,\, bb,\, q,\, q\sigma,\, T,\, \sigma T,\, tT,\, bT\,.$$

It is fitted to the deuteron binding energy, the *np* scattering lengths, the effective radii in the singlet and triplet channels and the *np* and *pp* scattering data.

The 3N-nuclear interaction

In order to study the 3N bound state and the core interaction in a 3N scattering state we add a **3-nucleon interaction** to the previous potential.

The potential is at NLO and has the form

$$V_{\mathrm{LO}}^{3N} = c_E \frac{1}{\Lambda_\chi f_\pi^4} \frac{1}{\pi^3 R_3^6} \sum_{\mathrm{cyclic}\ ijk} e^{-(r_{ij}^2 + r_{jk}^2)/R_3^2} ,$$

where $\Lambda_\chi = 1\ \mathrm{GeV}$ is the **breaking scale of the theory** and $f_\pi = 92.4\ \mathrm{MeV}$ is the **pion decay constant**.

The LEC c_E is fitted from a single three-nucleon data point once R_3 has been chosen.

The author chooses several R_3 in the range between 1 fm and 2.5 fm and evaluates c_E accordingly.

Variational principle for bound states

The variational principle says that the matrix element of the Hamiltonian minus the energy evaluated between the states is stationary with respect to the expansion constants, *i.e.*

$$\frac{\partial}{\partial c_{\alpha nl}} \langle \Psi_3 | H - E | \Psi_3 \rangle = 0$$

This can be expressed as an eigensystem that can be solved numerically.
Using this method and for now only the $2N$ interaction I evaluated the binding energy of ^{3}H and ^{3}He, reproducing part of TABLE X in Phys. Rev. C **103**, 054003 (2021), where the potential is given.

The Kohn variational principle

The Kohn variational principle states that the functional

$$\left[{}^J R_{\alpha'\alpha}\right] \equiv {}^J R_{\alpha'\alpha} - 2\,\mu\,\langle\psi_{\alpha'}|H - E|\psi_\alpha\rangle$$

is stationary to all the parameters $c^{R/I}_{\alpha nl}$ and ${}^J R_{\alpha'\alpha}$, *i.e.*

$$\frac{\partial\left[{}^J R_{\alpha'\alpha}\right]}{\partial c^{R/I}_{\alpha nl}} = \frac{\partial\left[{}^J R_{\alpha'\alpha}\right]}{\partial R_{\beta'\beta}} = 0\,,$$

where ${}^J R_{\alpha'\alpha}$ and $c^{R/I}_{\alpha nl}$ are the parameters used to expand the channel α of the wave function (ψ_α).

Once one has all the parameters it is possible to obtain the S-matrix and therefore the **phase-shifts** δ_ℓ and the **mixing angles** ε_i.

^{3}H and ^{3}He, numerical methods

To evaluate the wave function and the $3N$ observables for ^{3}H and ^{3}He we decompose the wave function using **the hyperspherical harmonics**.
More than that the final wave function must be **totally antisymmetric** with respect to the exchange of two nuclei, so we write it as

$$\Psi_3 = \sum_{p=1}^{3} \psi(\boldsymbol{x}_1^{(p)}, \mathrm{x}_2^{(p)})\,,$$

where $\mathrm{x}_i^{(p)}$ are the two relative Jacobi coordinates in the system where the particle with index p is the "spectator". The expansion can be done as

$$\psi(\boldsymbol{x}_1^{(p)}, \mathrm{x}_2^{(p)}) = \sum_{\alpha=1}^{N_c} \sum_{n=N_\alpha^0}^{N_\alpha} \sum_{l=0}^{M} c_{\alpha n l}\, f_l(\rho)\, B_{n\alpha}^{\mathrm{HH}}(\hat{\mathrm{x}}_1^{(p)}, \hat{\mathrm{x}}_2^{(p)}, \varphi^{(p)})$$

- Kohn variational principle

$N - d$ scattering state

To evaluate the phase-shifts and mixing angles of the states of the $p - d$ elastic scattering we write the wave function as

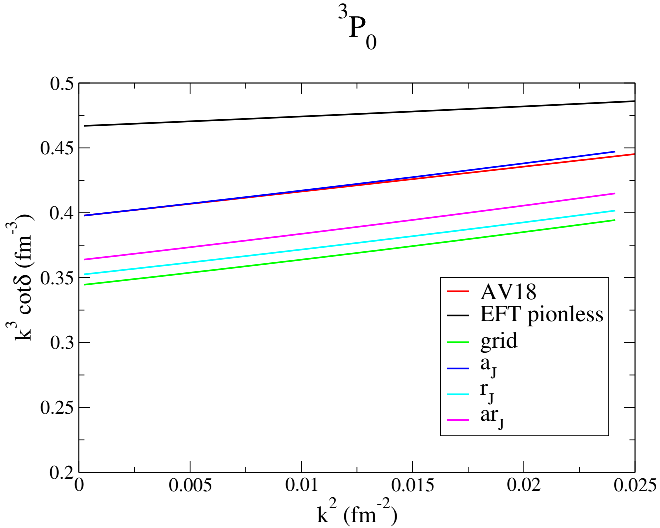
$$\Psi_3 = \Psi_3^{(c)} + \Psi_3^{(a)}$$

where (c) $((a))$ stands for the core (asymptotic) part of the wave function.

Ingredients:

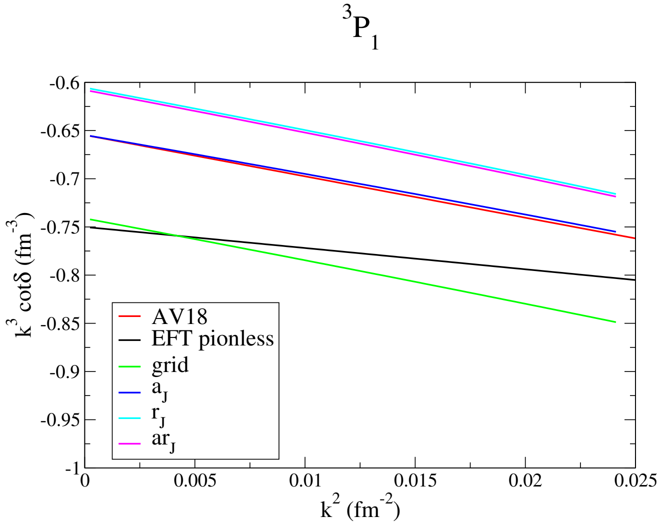
- the NN and $3N$ potential with the standard Coulomb interaction (to be consistent with the scattering code);
- the deuteron wave function $\psi_d(r)$;
- a variational method suited for scattering states (Kohn variational principle).

$^3P_0$ comparison



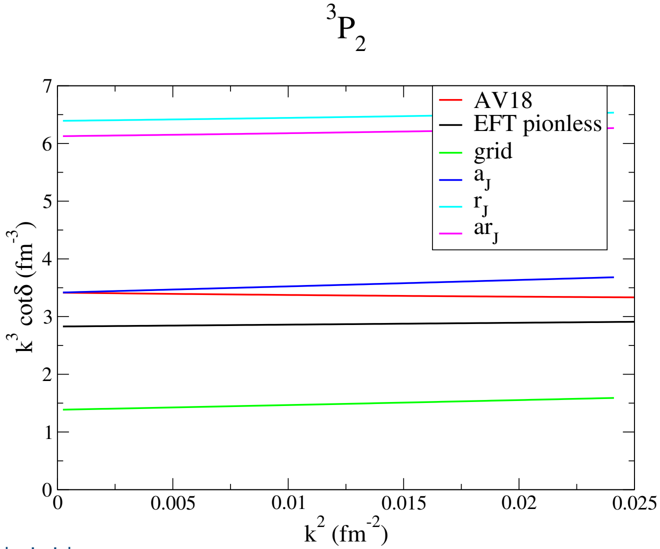
- Kohn variational principle

$^3P_1$ comparison



- Kohn variational principle

$^3P_2$ comparison



- Kohn variational principle