Solving the homogenous Bethe-Salpeter equation with a quantum annealer

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Introduction to the physics problem

Our aim is to apply a variational method, tailored to a Quantum Annealer, for solving the **homogeneous Bethe-Salpeter equation (hBSE)** in Minkowski momentum-space
[E.E Salpeter and H.A. Bethe, Phys. Rev. 84, 1232 (195)]

$$\phi_b(k,p) = G_0^{(12)}(k,p) \int \frac{d^4k'}{(2\pi)^4} i\mathscr{K}(k,k',p)\phi_b(k',p), \text{ with } p = p_1 + p_2 \text{ and } k = \frac{p_1 - p_2}{2}$$

In **relativistic quantum-field-theory**, this equation has the same role in the **bound state** description that the **Schrödinger equation** has in non **relativistic quantum mechanics**

We consider the hBSE describing a **bound system** composed by **two massive scalars** interacting through the **exchange of a massive boson.**

The hBSE in this case can be written as a **Non Symmetric Generalized Eigenvalue Problem (GEVP)** adopting a standard discretization method*

***T.Frederico, G.Salmè, and M.Viviani, Phys. Rev. D 89, 016010 (2014)**

hBSE as a Generalized Eigenvalue Problem (GEVP)

[T.Frederico, G.Salmè and M.Viviani, Phys. Rev. D 89, 016010 (2014)]

$$A \ \mathbf{v}_i = \boxed{\lambda_i} B \ \mathbf{v}_i$$

- The eigenvalues are $\lambda_i = \dfrac{1}{\alpha_i}$, where $\alpha_i = \dfrac{g_i^2}{32\pi^2}$,with g_i the **coupling constant** of the interaction.
- We are interested **only** in the in the eigenpair corresponding to the **largest, real and positive eigenvalue** λ_n, since $\lambda_n = \dfrac{1}{\alpha_n}$
- α_n is the **minimal coupling constant** that allows the existence of a **bound system** with a given mass $M = 2m - B$

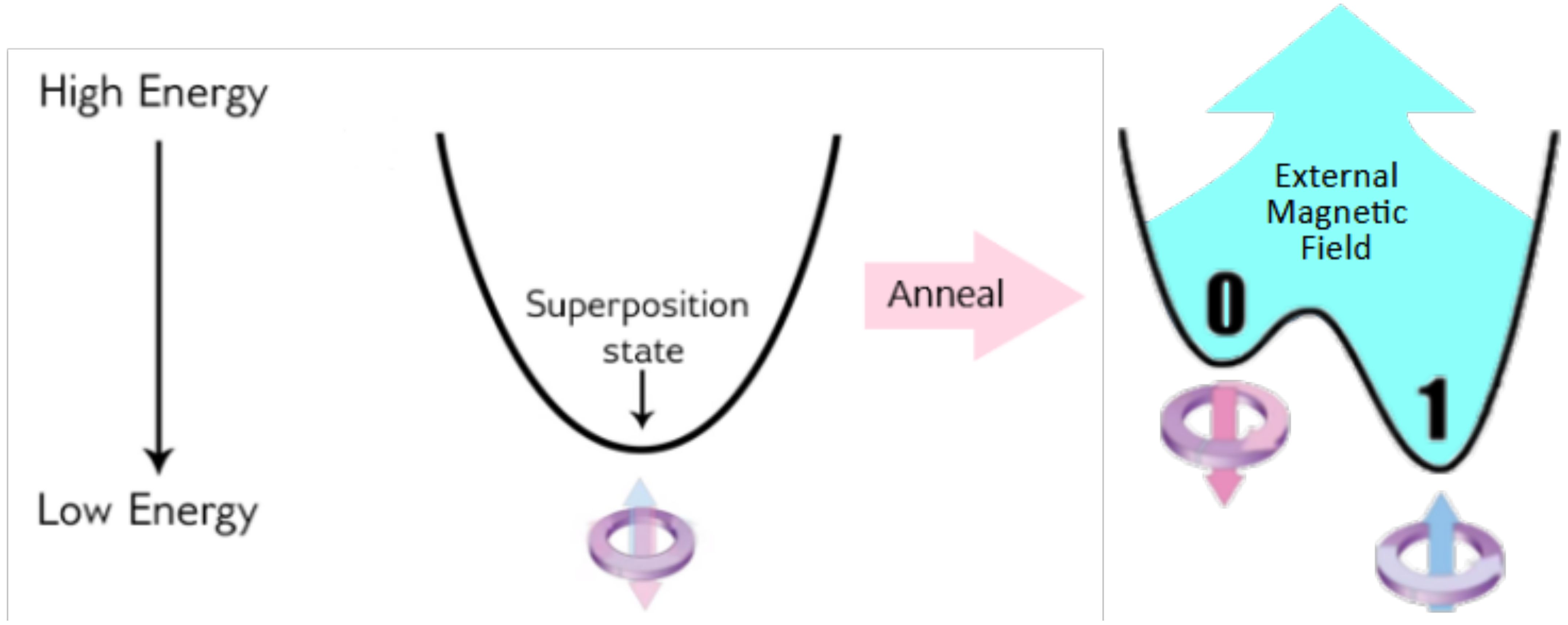
We used $(n \times n)$ **matrices** obtained by the **discretization of the hBSE** with binding energy $\dfrac{B}{m} = 1.0$ and with an exchanged boson of mass $\dfrac{\mu}{m} = 0.15$

Quantum Annealing

We used both **Quantum Annealing (QA)** and **Simulated Annealing (SA)**

Quantum Annealing: Adiabatic transformation from tunneling Hamiltonian to problem Hamiltonian.

The qubits are initially in the ground state of the tunneling Hamiltonian. After the annealing, the qubits collapse into the ground state of the Ising Hamiltonian $\hat{H}_p$.

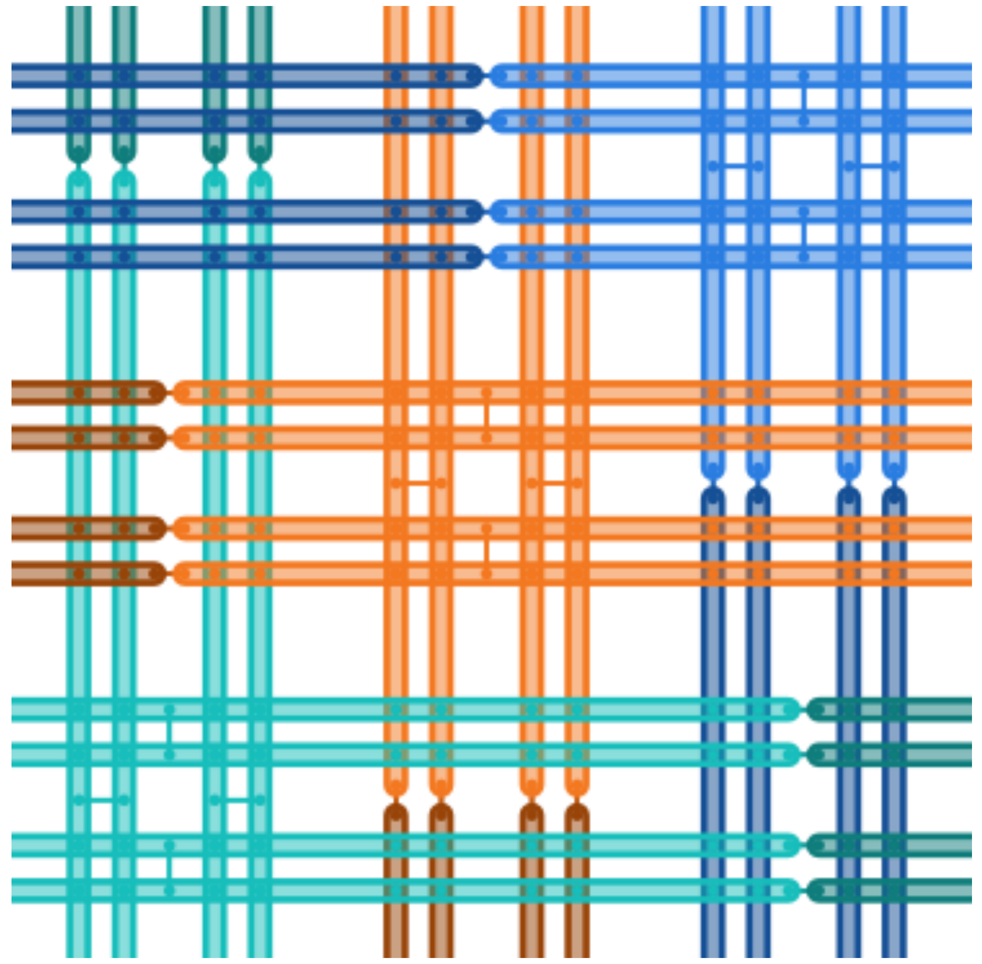


$$\boxed{\mathbf{x} = \arg\min_{\mathbf{v}\in C_N} \mathbf{v}^T Q \mathbf{v}}$$: Quantum Uncostrained Binary Optimization (**QUBO**)

Advantage 4.1

Thanks to the **D-Wave-Cineca** agreement, as part of an international project approved by Q@TN(INFN-UNITN-FBK-CNR), we used the QA **Advantage 4.1** provided by **D-Wave Systems**

The transverse-Ising Hamiltonian that represents the problem to minimize is **encoded** into the target topology (**PEGASUS**) through an **embedding procedure**, with a heuristic algorithm provided by the D-Wave software



To properly translate the problem on the topology of the hardware, the embedding needs to represent the logical qubits of the original problem with a **larger number** of physical qubits

With the **Simulated Annealing (SA)**, the original problem can be directly solved without the an embedding procedure

Formulate a QUBO problem

Goal: Write the **GEVP** as a problem **suitable** by the QA

First step: write the non symmetric **GEVP** as a **variational problem***

$$f(A, B, \mathbf{v}, \tilde{\lambda}) = \mathbf{v}^T[A - \tilde{\lambda}B]^T[A - \tilde{\lambda}B]\mathbf{v} \geq 0$$

The **objective function (OF)** is minimized by all the eigenvectors $\mathbf{v}$:

$$f(A, B, \mathbf{v}, \tilde{\lambda}) = 0 \Rightarrow \lambda^R(\mathbf{v}) = \mathbf{v}^T \frac{AB^T + B^T A}{2} \mathbf{v}$$

$\lambda(\mathbf{v}) = \lambda^R(\mathbf{v}) \pm i\lambda^I(\mathbf{v}) \Rightarrow$ We want $\lambda^I(\mathbf{v}) = 0$

Second step: decompose the B matrix to transform the GEVP into a **non symmetric eigenvalue problem** with the standard **LDL** decomposition

$$CL^T\mathbf{v}_i = \lambda_i L^T \mathbf{v}_i \,, \;\; \mathbf{w}_i = L^T \mathbf{v}_i$$

$$f(C, \mathbf{w}, \tilde{\lambda}) = \mathbf{w}^T[C - \tilde{\lambda}I]^T[C - \tilde{\lambda}I]\mathbf{w} = \mathbf{w}^T S(\tilde{\lambda})\mathbf{w} \geq 0$$

We have to exploit the **non singularity** of B

***S. Alliney, F. Laudiero and M. Savoia, Applied mathematical modelling 16, 148 (1992)**

Formulate a QUBO problem

The **QA** is designed to deal with transverse-Ising model and **QUBO** problems in an hypercube $C_{n,b} = [0,1]^{n*b}$. After a **single annealing cycle** the QA returns:

$$\boxed{\mathbf{x} = \arg\min_{\mathbf{x}\in C_{n,b}} \mathbf{x}^T Q \mathbf{x}}$$

We need to approximate our **quadratic form** into a **QUBO** one, rewriting the matrices elements in a **binary basis**

$$\mathbf{w} \simeq P^T\mathbf{x} \Rightarrow w_\alpha = -q_{b,\alpha} + \sum_{i=1}^{b-1} \frac{q_{i,\alpha}}{2^i} \quad P^T = diag(\mathbf{p}^T, \ldots, \mathbf{p}^T) \quad \mathbf{p} = (-1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2^{b-1}})^T$$

$$\mathbf{w} = (w_1, w_2, \ldots, w_n)^T \in [-1,1[^n \Rightarrow \mathbf{x} = (q_{1,1}, \ldots, q_{b,1}, \ldots, q_{1,n}, \ldots, q_{b,n})^T \in C_{n,b}$$

$$\boxed{f(C, \mathbf{w}, \tilde{\lambda}) = \mathbf{w}^T S(\tilde{\lambda})\mathbf{w} \simeq \mathbf{x}^T P S(\tilde{\lambda}) P^T \mathbf{x} = \mathbf{x}^T Q \mathbf{x}}$$

Gershgorin Theorem

$$\boxed{f(C, \mathbf{w}, \tilde{\lambda}) = \mathbf{w}^T[C - \tilde{\lambda}I]^T[C - \tilde{\lambda}I]\mathbf{w} = \mathbf{w}^T S(\tilde{\lambda})\mathbf{w} \geq 0}$$

We **must set** $\tilde{\lambda} \simeq \lambda_n$ if we want to obtain $\mathbf{w}_n$ from a QA cycle

We can guide the “guess phase” by using the **Gershgorin circle theorem***:

$$\boxed{\mathscr{C}_i : |y - c_{ii}| = \leq \sum_{j \neq i} |c_{ij}| = R_G(c_{ii})}$$

c_{ij} elements of C

If a circle $\mathscr{C}_j$ is **disconnected** from the others then it contains **one ad only one real eigenvalue**

If $\mathscr{C}_N$ is **disconnected** from the **others**, we can set $\tilde{\lambda} = c_{11} > c_{22} > c_{33}\ldots$ and we select **only the solutions** $\in \mathscr{C}_N$

* **R. S. Varga, “Gershgorin and his circles”, Vol. 36 (Springer Science & Business Media, 2010)**

Algorithm I: Guess Phase

$$\boxed{f(C, \mathbf{w}, \tilde{\lambda}) = \mathbf{w}^T S(\tilde{\lambda})\mathbf{w} \simeq \mathbf{x}^T P S(\tilde{\lambda}) P^T \mathbf{x} = \mathbf{x}^T Q \mathbf{x}}$$

After $\alpha = 1,\ldots, N_A^{GP}$ annealing cycles, the QA or the SA returns N_A^{GP} **binary vectors** $\{\mathbf{x}_\alpha\}$

We analyze the set $\{\mathbf{w}_\alpha, f(C, \mathbf{w}, \tilde{\lambda} = c_{11}), \lambda^R(\mathbf{w}_\alpha), \lambda^I(\mathbf{w}_\alpha)\}$

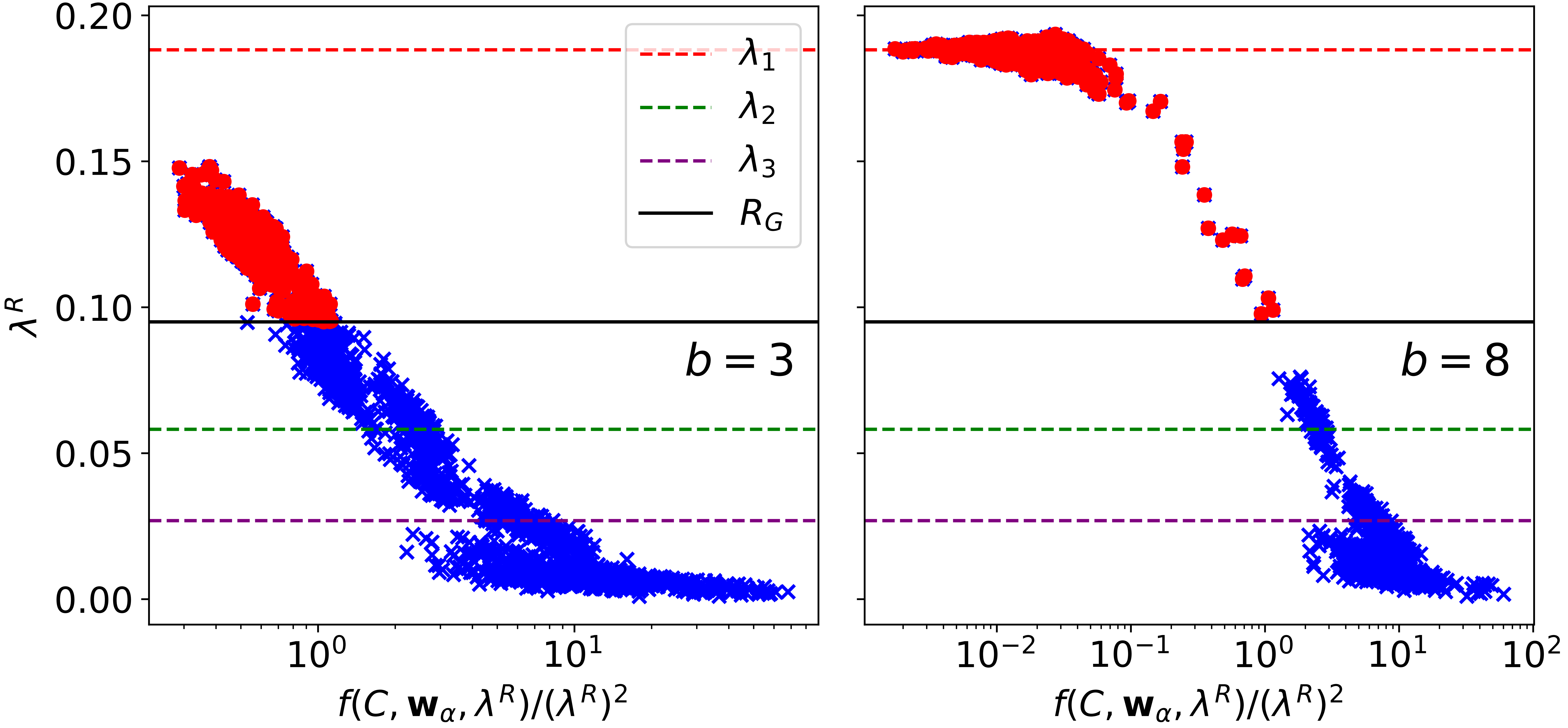
We eliminate the **solutions outside the disc** $\mathscr{C}_n$

Among the surviving solutions, we take the one that satisfies:

$$f_{best}^{GP} = \min_{\mathbf{w}_\alpha} f(A, \mathbf{w}_\alpha, \tilde{\lambda} = \lambda^R(\mathbf{w}_\alpha))$$

At the end of the **Guess Phase** the best eigenpair $(\mathbf{w}_{\alpha_{GP}}, \lambda^R(\mathbf{w}_{\alpha_{GP}}))$ is passed to the **Gradient-Descent phase** in order to improve the precision on that solution

F.F., A.Gnech, T.Frederico, F.Pederiva, M.Rinaldi, A.Roggero, G.Salmè, S.Scopetta and M.Viviani, *Phys.Rev.D* 110 (2024) 5, 056012



- Single run on the SA ($N_A^{GP} = 2000$) for a 32x32 matrix

Algorithm II: gradient-descent

At the **end of the GP** of the algorithm we find a solution $\mathbf{w}^{(z=0)} = \mathbf{w}_{\alpha_{GP}}$

Gradient-descent (GD): iterative algorithm that founds a new solution $\mathbf{w}^{(z)}$

$$\mathbf{w}^{(z)} = \mathbf{w}^{(z-1)} + \frac{1}{2^z}\delta(z)$$

The **OF** can be expanded around $\mathbf{w}^{(z-1)}$:

$$f(C, \mathbf{w}^{(z)}, \tilde{\lambda}) = f(C, \mathbf{w}^{(z-1)}, \tilde{\lambda}) + \boxed{\mathbf{w}^{(z-1)T} S(\tilde{\lambda})\delta(z) + \delta(z)^T \frac{S(\tilde{\lambda})}{2}\delta(z)}$$

New **OF**: $\hat{f}(C, \delta(z), \tilde{\lambda}) = \delta(z)^T \mathcal{Q}(z, \tilde{\lambda})\delta(z)$ $\qquad$ $\mathcal{Q}(z, \tilde{\lambda})_{ij} = \frac{1}{2}S(\tilde{\lambda})_{ij} + \delta_{ij}[\mathbf{w}^{(z-1)T} S(\tilde{\lambda})]_i$

Algorithm II: gradient-descent

$\lambda_{best} = \lambda(\mathbf{w}^{(z-1)})$ and $\mathbf{w}_{best} = \mathbf{w}^{(z-1)}$

At **each zoom step** z ($z = 1,\ldots,z_{max}$) an **inner loop is opened** $i^z = 1,2,\ldots$

The QA or the SA returns en **ensemble of** $\alpha_i = 1,\ldots,N_A^{GD}$ **qubits states** $\{\mathbf{x}_{\alpha_i}\}$

Among the ensemble we select the one with the minimal energy:

$$\hat{f}^{DP}_{best;i}(z) = \min_{\delta_{\alpha_i}(z)} \hat{f}(C, \mathbf{w}^z_{\alpha_i}, \tilde{\lambda} = \lambda_{best})$$

If $\lambda^R(\mathbf{w}^z_{best;i}) \geq \lambda_{best}$ we pass the **next zoom step** $z+1$. $i^z_{max} = i^z$

i^z_{max}: iterations needed to find the **best solution for each** z

Total annealing time: $T \propto N_A^{GD} \sum_{z=1}^{z_{max}} i^z_{max}$

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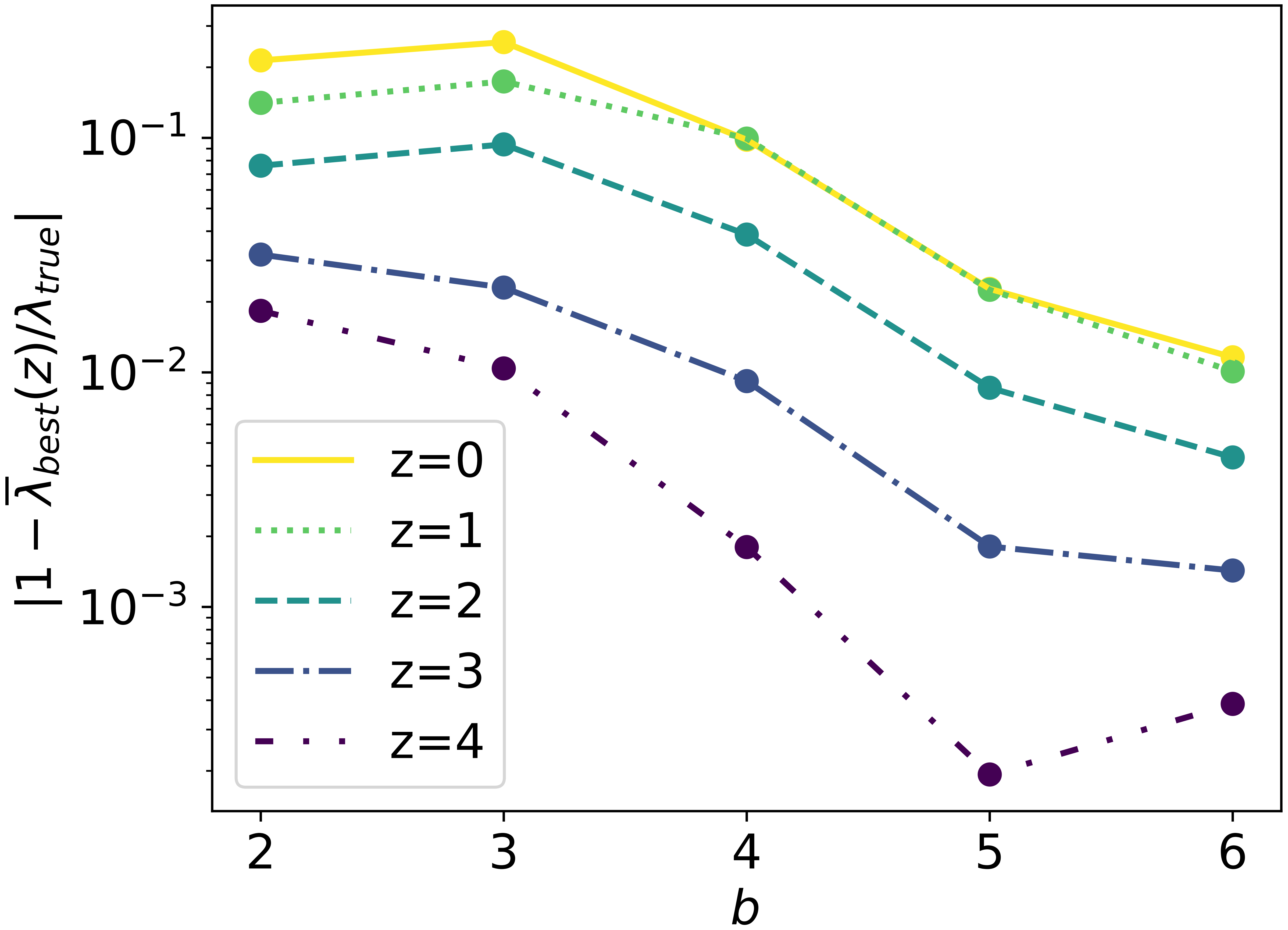

- **500 independent samples** on the SA ($N_A^{GP} = 200, N_A^{GD} = 20$) for a 32x32 matrix
- **Trade-off between b and z_{max}**, already found in the symmetric case *

* B. Krakoff, S. M. Mniszewski and C. F. A. Negre, arXiv:2104.11 (2021)

n_M	b	N_{run}	λ_{true}	$\bar{\lambda}_{best}$	$\bar{\lambda}^I_{best}/\bar{\lambda}_{best}$	$\overline{\lVert \mathbf{v}_{true} - \mathbf{v}_{best} \rVert}$
4	3	80	0.188026	$0.188012^{+1\cdot 10^{-5}}_{-7\cdot 10^{-6}}$	$0.00024^{+2\cdot 10^{-5}}_{-5\cdot 10^{-6}}$	$0.00024^{+3\cdot 10^{-5}}_{-2\cdot 10^{-5}}$
8	3	80	0.188204	$0.18820^{+2\cdot 10^{-5}}_{-2\cdot 10^{-5}}$	$0.0003^{+1\cdot 10^{-4}}_{-1\cdot 10^{-4}}$	$0.0003^{+1\cdot 10^{-4}}_{-1\cdot 10^{-4}}$
12	3	80	0.188203	$0.18821^{+2\cdot 10^{-5}}_{-2\cdot 10^{-5}}$	$0.0005^{+1\cdot 10^{-4}}_{-1\cdot 10^{-4}}$	$0.0006^{+2\cdot 10^{-4}}_{-1\cdot 10^{-4}}$
16	2	80	0.188203	$0.18820^{+4\cdot 10^{-5}}_{-3\cdot 10^{-5}}$	$0.0009^{+1\cdot 10^{-4}}_{-1\cdot 10^{-4}}$	$0.0011^{+2\cdot 10^{-4}}_{-2\cdot 10^{-4}}$
24	2	80	0.188225	$0.18822^{+5\cdot 10^{-5}}_{-4\cdot 10^{-5}}$	$0.0013^{+1\cdot 10^{-4}}_{-3\cdot 10^{-4}}$	$0.0015^{+3\cdot 10^{-4}}_{-2\cdot 10^{-4}}$
32	2	200	0.188225	$0.18823^{+4\cdot 10^{-5}}_{-3\cdot 10^{-5}}$	$0.0016^{+2\cdot 10^{-4}}_{-2\cdot 10^{-4}}$	$0.0018^{+4\cdot 10^{-4}}_{-3\cdot 10^{-4}}$

- $z_{max} = 9$
- **Euclidean distance** and magnitude of the **imaginary part** slightly increase with $n_M \times b$

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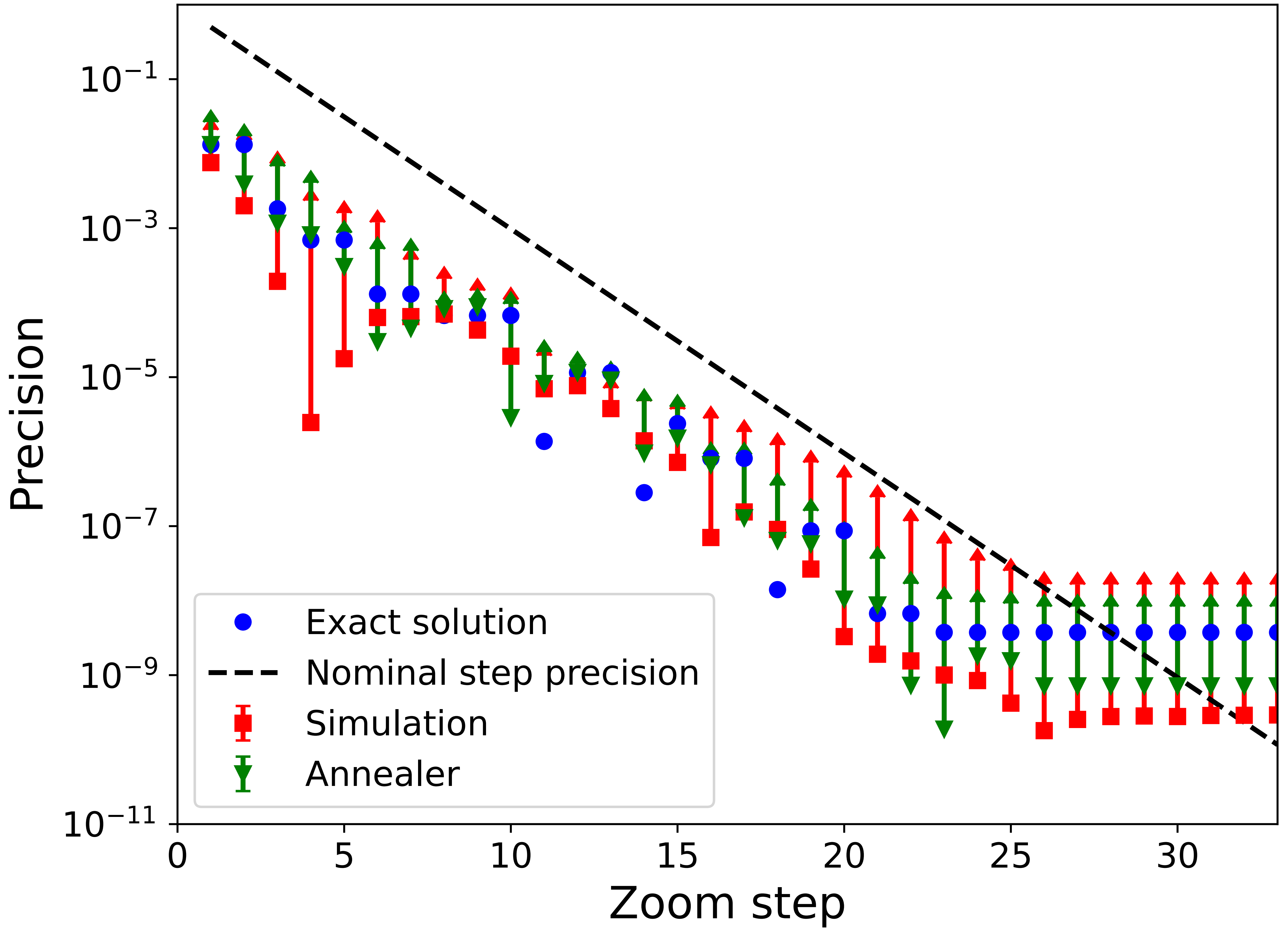

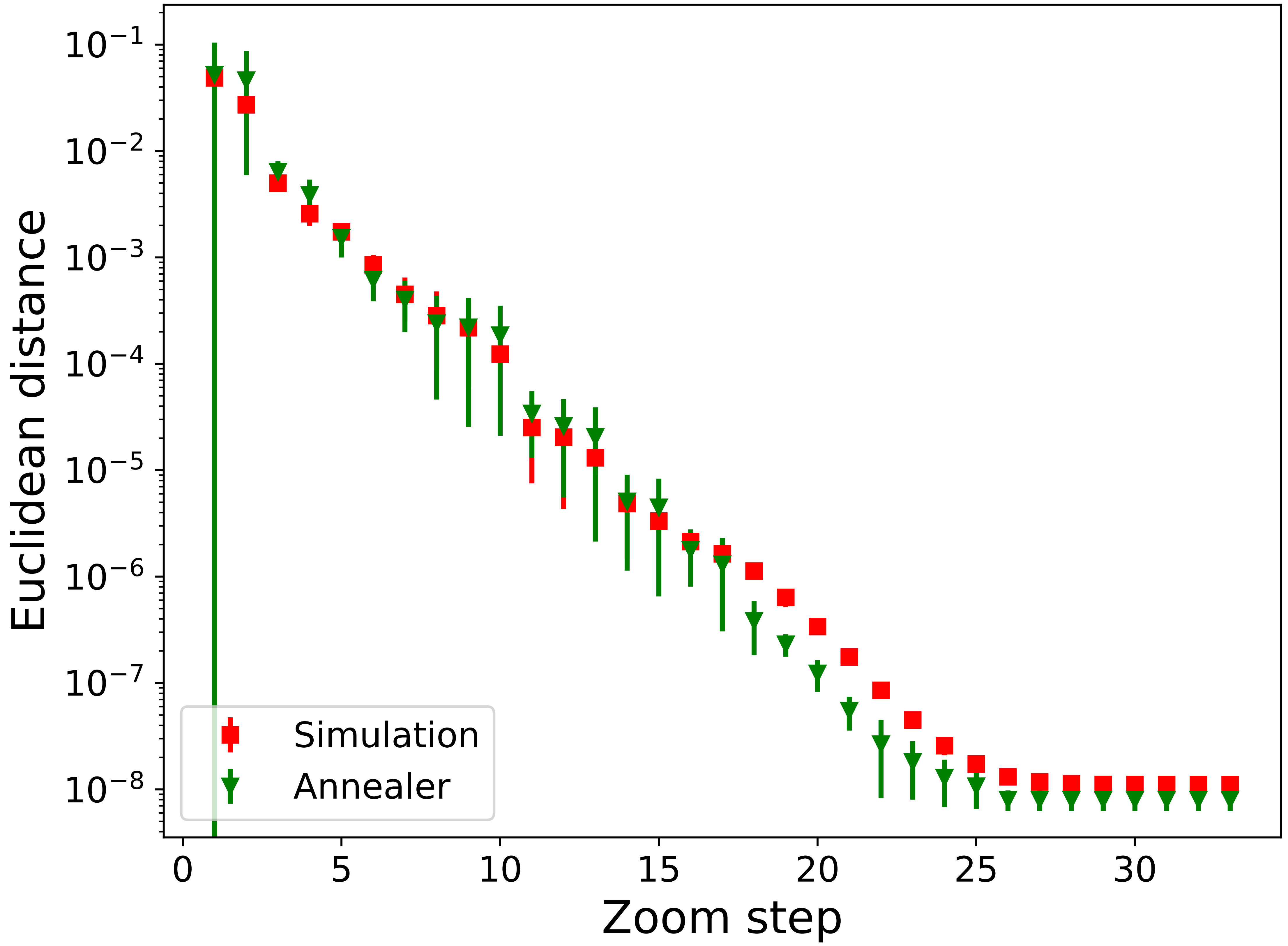

- 4×4 matrices with $b = 3$. $N_{run} = 1000$ on the SA and $N_{run} = 10$ on the QA
- **Actual** 0 of the **OF** $\simeq 10^{-16} \Rightarrow$ **Plateau** at 10^{-8}

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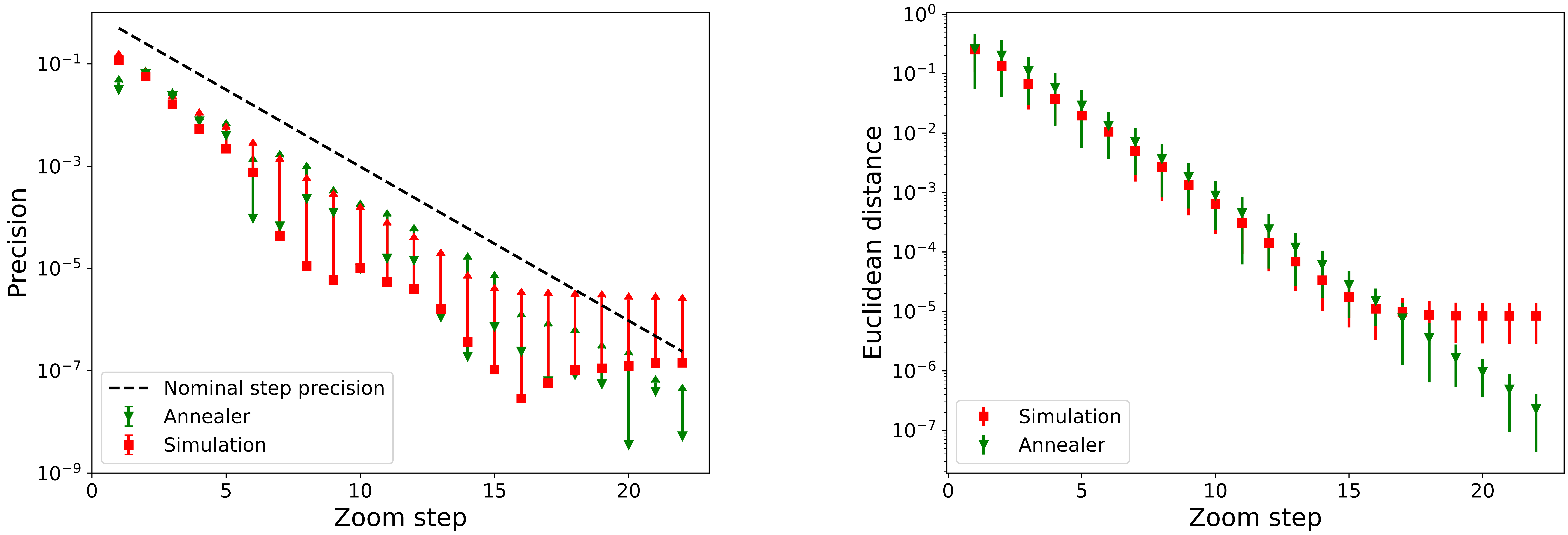

- 32×32 matrices with $b = 2$. $N_{run} = 1000$ on the SA and $N_{run} = 10$ on the QA
- **Lower SA**’s performance in managing large matrices

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$n_M \times b$	N_{run}	$T[ms]$	N_{qubits}
$12(4 \times 3)$	80	$18.5^{+0.9}_{-0.7}$	$24.1^{+0.9}_{-1.1}$
$24(8 \times 3)$	80	$21.7^{+1.3}_{-0.9}$	$81.3^{+2.1}_{-2.3}$
$32(16 \times 2)$	80	$23.0^{+1.4}_{-1.0}$	$140.9^{+1.9}_{-1.9}$
$36(12 \times 3)$	80	$23.0^{+1.4}_{-1.3}$	$177.6^{+6.4}_{-6.6}$
$48(24 \times 2)$	80	$23.0^{+1.0}_{-1.0}$	$306.6^{+11.8}_{-15.6}$
$64(32 \times 2)$	200	$23.6^{+1.2}_{-1.2}$	$529.8^{+30.6}_{-33.8}$

- **Total annealing time** T and **total number of physical qubits** N_{qubits}, averaged on N_{run}
- T is **slightly increasing** when the matrix dimension $n_M \times b$ increase
- Quadratic growth of N_{qubits} with the total dimension $n_M \times b$

Conclusions and perspectives

- A hybrid algorithm, suitable for a quantum annealer, was implemented to evaluate the **largest real eigenvalue and corresponding eigenvector** of a **GEVP** for the **discretization of the hBSE**
- Numerical results obtained by running our **two-phase algorithm** both on **Advantage 4.1** and a **SA**
- The results obtained by the **SA** established a **practical set of input parameters** (b, N_A^{GP}, N_A^{GD})
- We successfully approached the target eigenpair by **running the code on the D-Wave QA**, obtaining very encouraging results, up to a matrix with dimension $n = 32$ and $b = 2$
- **The next challenge** is to **improve the algorithm** in order to address the GEVP without exploiting the **non singularity of the symmetric matrix** B