Inclusive semi-leptonic decays of heavy mesons on the lattice: From D to B

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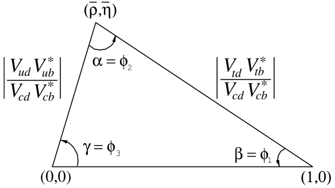
Flavour physics and the CKM matrix

- Quark flavour physics is an active area of research focusing on indirect search for signal of physics beyond the Standard Model (SM)
- Quark flavours mixing is parametrised by the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix
- CKM matrix elements determined matching calculations with measurements
- Precise determination of CKM matrix elements is challenging

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$



[PDG, CKM review]

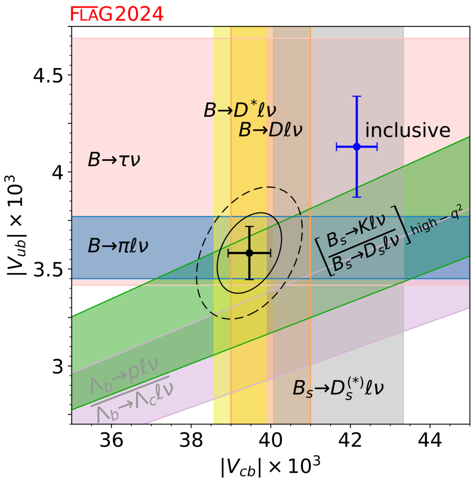
A long standing tension . . .

There is a perisitent tension between the **inclusive** and **exclusive** determination of the CKM matrix elements $|V_{cb}|$ and $|V_{ub}|$:

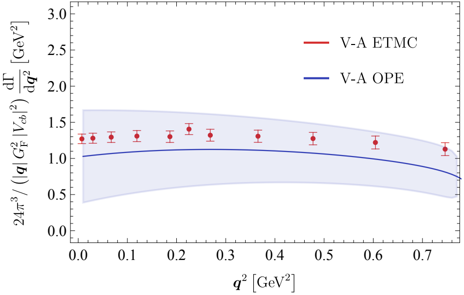
	$\|V_{cb}\|$	
Inclusive	$(42.16 \pm 0.51) \cdot 10^{-3}$	OPE
Exclusive	$(39.46 \pm 0.53) \cdot 10^{-3}$	LQCD

	$\|V_{ub}\|$	
Inclusive	$(4.13 \pm 0.26) \cdot 10^{-3}$	OPE
Exclusive	$(3.60 \pm 0.14) \cdot 10^{-3}$	LQCD

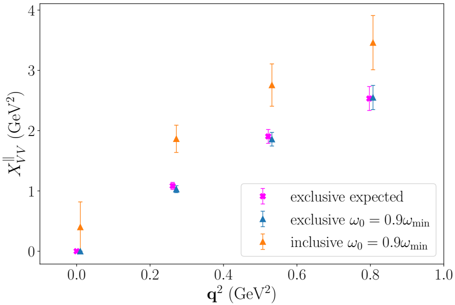
[FLAG '24]



Preliminary studies



[Gambino, AS *et al.* JHEP, 2203.11762]



[Barone *et al.* JHEP, 2305.14092]

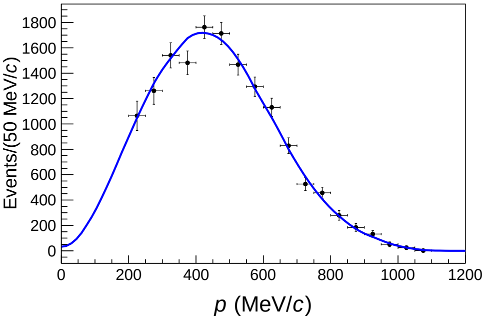
- Unphysically light b-quark
- $L \to \infty$ and $a \to 0$ limits missing
- Comparison only to the OPE

- b-quark close to physical
- $L \to \infty$ and $a \to 0$ limits missing
- Comparison with exclusive decay

Inclusive semileptonic decays of D-meson

$$\Gamma_{\text{semi-lep.}} = 8.27(21) \times 10^{-14}\text{GeV}(2.5\%)$$

- Simulating the b-quark relativistically on the lattice is challenging due to severe discretisation effects
- c-quarks can be simulated relativistically
- Experimental precision of $D_s \to Xl\nu$ can be reached with state-of-the-art lattice QCD simulations



[BES III]

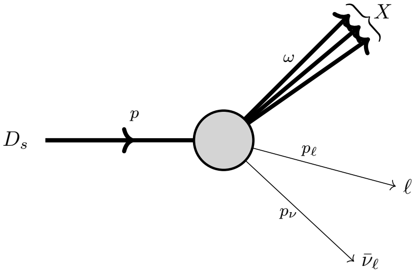
Opportunity for a complete and phenomenologically relevant calculation!

Inclusive semileptonic decays of heavy mesons

$$\frac{d\Gamma}{d\omega^2 d\omega^0 dE_\ell} = \frac{G_F^2 |V_{fg}|^2}{(2\pi)^3} L_{\mu\nu} W^{\mu\nu}$$

$$W_{\mu\nu}(\omega_0, \boldsymbol{\omega}) = \frac{(2\pi)^3}{2m_{D_s}} \langle D_s(\mathbf{0}) | J_\mu^\dagger(0) \delta(\hat{H} - \omega_0) \delta^3(\hat{\boldsymbol{P}} + \boldsymbol{\omega}) J_\nu | D_s(\mathbf{0}) \rangle$$

$\Rightarrow$ encapsulate the **non-perturbative** QCD contribution



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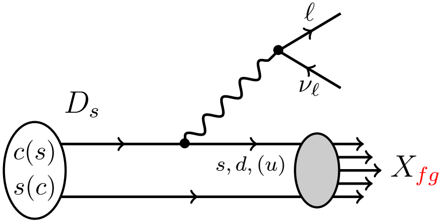

$$\Gamma = G_F^2 \Big(|V_{cd}|^2 \Gamma_{cd} + |V_{cs}|^2 \Gamma_{cs} |V_{us}|^2 \Gamma_{su} \Big)$$

Calculation strategy [Gambino, Hashimoto, **PRL**, 2005.13730]

After performing E_ℓ integration, we work in the D_s-meson rest frame and write in compact form:

$$\frac{24\pi^3}{G_F^2|V_{fg}|^2}\frac{d\Gamma_{fg}}{d\boldsymbol{\omega}^2} = \sum_{l=0}^{2} |\boldsymbol{\omega}|^{3-l} \int_{\omega_0^{\min}}^{\omega_0^{\max}} d\omega_0 \; (\omega_0^{\max} - \omega_0)^l Z^{(l)}(\omega_0, \boldsymbol{\omega}^2)$$

with

$$\omega_0^{\min} = \sqrt{m_{D_s}^2 + \boldsymbol{\omega}^2}, \quad \omega_0^{\max} = m_{D_s} - |\boldsymbol{\omega}|$$

where $Z^{(l)}(\omega_0, \boldsymbol{\omega}^2)$ is a linear combinations of $W_{\mu\nu}$:

$$Z^{(0)} = W^{00} + \sum_{i,j=1}^{3} \frac{\omega^i}{\sqrt{\boldsymbol{\omega}^2}}\frac{\omega^j}{\sqrt{\boldsymbol{\omega}^2}} W^{ij} + \frac{\omega^i}{\sqrt{\boldsymbol{\omega}^2}}(W^{0i} + W^{i0})$$

$W_{\mu\nu}$ vanishes for $\omega_0 < \omega_0^{\min}$, so we introduce the kernel

$$\Theta^{(l)}(\omega_0^{\max} - \omega_0) = (\omega_0^{\max} - \omega_0)^l \theta(\omega_0^{\max} - \omega_0)$$

so that:

$$\frac{24\pi^3}{G_F^2 |V_{fg}|^2} \frac{d\Gamma_{fg}}{d\boldsymbol{\omega}^2} = \sum_{l=0}^{2} |\boldsymbol{\omega}|^{3-l} Z^{(l)}(\boldsymbol{\omega}^2) \qquad Z^{(l)}(\boldsymbol{\omega}^2) = \int_0^\infty d\omega_0 \; \Theta^{(l)}(\omega_0^{\max} - \omega_0) \; Z^{(l)}(\omega_0, \boldsymbol{\omega}^2)$$

➜ In order to compute the inclusive decay rate we need to solve the integral over ω_0.
Can we do this using lattice QCD correlators?

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Bonus: Lepton energy moments

In a similar way we can define the first **lepton energy moment**

$$\frac{dM^{n}_{fg}}{d\boldsymbol{\omega}^2} = \int d\omega_0 \int dE_l E_l^n \frac{d\Gamma_{fg}}{d\omega_0 d\boldsymbol{\omega}^2 dE_l}$$

$$\Downarrow$$

$$\frac{1}{\overline{M}^1} \frac{dM^{1}_{fg}}{d\boldsymbol{\omega}^2} = \sum_{l=0}^{3} |\boldsymbol{\omega}|^{4-l} \int_0^\infty d\omega_0 \Theta^{(l)}(\omega_0^{\max} - \omega_0) Z^{(l)}_{n=1}(\omega_0, \boldsymbol{\omega}^2)$$

$$\frac{1}{\overline{M}^2} \frac{dM^{2}_{fg}}{d\boldsymbol{\omega}^2} = \sum_{l=0}^{4} |\boldsymbol{\omega}|^{5-l} \int_0^\infty d\omega_0 \Theta^{(l)}(\omega_0^{\max} - \omega_0) Z^{(l)}_{n=2}(\omega_0, \boldsymbol{\omega}^2)$$

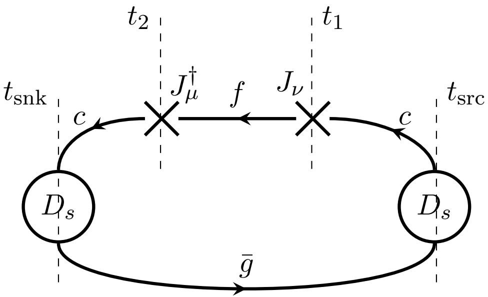
Decay rates from Euclidean correlators

On the lattice we can compute the forward-scattering matrix element from the 4-point function:

$$G_{\mu\nu}(t_2 - t_1, \boldsymbol{\omega}) = e^{-m_{D_s}|t_2-t_1|} \int d^3x \, \frac{e^{i\boldsymbol{\omega}\cdot\boldsymbol{x}}}{2m_{D_s}} \langle D_s(\mathbf{0})|J_\mu^\dagger(\boldsymbol{x}, t_2) J_\nu(\mathbf{0}, t_1)|D_s(\mathbf{0})\rangle$$

where:

$$G_{\mu\nu}(t_2 - t_1; \boldsymbol{\omega}) = \lim_{\substack{t_{\rm snk}\to+\infty \\ t_{\rm src}\to-\infty}} \frac{C_{\mu\nu}(t_{\rm snk}, t_2, t_1, t_{\rm src})}{C(t_{\rm snk} - t_2)C(t_1 - t_{\rm src})}$$



$C_{\mu\nu}(t_{\rm snk}, t_2, t_1, t_{\rm src})$

[Hashimoto, *PTEP*, 1703.01881 - Hansen, Meyer, Robaina, **PRD**, 1704.08993 - Gambino, Hashimoto, **PRL**, 2005.13730]

We can find a connection between lattice correlators and the hadronic tensor:

$$\begin{aligned}
G_{\mu\nu}(t_2 - t_1, \boldsymbol{\omega}) &= e^{-m_{fg}|t_2-t_1|} \int d^3x e^{-i\boldsymbol{\omega}\cdot\boldsymbol{x}} \langle D(0)|J_\mu^\dagger(\boldsymbol{x}, t_2) J_\nu(\boldsymbol{0}, t_1)|D(0)\rangle \\
&= \int d^3x \ \langle D(0)|J_\mu^\dagger(0) e^{-\hat{H}|t_2-t_1|+i(\hat{\boldsymbol{P}}-\boldsymbol{\omega})\cdot\boldsymbol{x}} J_\nu(0)|D(0)\rangle \\
&= \langle D(0)|J_\mu^\dagger(0) e^{-\hat{H}|t_2-t_1|} (2\pi)^3 \delta^3(\hat{\boldsymbol{P}} - \boldsymbol{\omega}) J_\nu(0)|D(0)\rangle \\
&= \int d\omega_0 \ e^{-\omega_0|t_2-t_1|} \langle D(0)|J_\mu^\dagger(0) \delta(\hat{H} - \omega_0)(2\pi)^3 \delta^3(\hat{\boldsymbol{P}} - \boldsymbol{\omega}) J_\nu(0)|D(0)\rangle
\end{aligned}$$

hence:

$$\boxed{G_{\mu\nu}(t; \boldsymbol{\omega}) = \int_0^\infty d\omega_0 \ W_{\mu\nu}(\omega_0, \boldsymbol{\omega}) \ e^{-\omega_0 t}}$$

or

$$G^{(l)}(t; \boldsymbol{\omega}) = \int_0^\infty d\omega_0 \ Z^{(l)}(\omega_0, \boldsymbol{\omega}) \ e^{-\omega_0 t}$$

Spectral densities & inverse problem

The problem of extracting $W_{\mu\nu}(\omega_0, \boldsymbol{\omega}^2)$ from 4-point correlators is equivalent to extracting $\rho(\omega)$ from 2-point correlators.

$$C(t) = \int_0^\infty d\omega \; \rho_L(\omega) \; K(\omega, t)$$

It requires solving an inverse problem which is ill-posed for lattice QCD correlators

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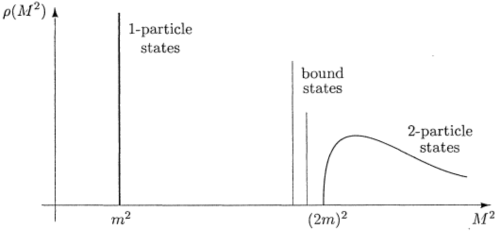

Figure 7.2. The spectral function $\rho(M^2)$ for a typical interacting field theory. The one-particle states contribute a delta function at m^2 (the square of the particle's mass). Multiparticle states have a continuous spectrum beginning at $(2m)^2$. There may also be bound states.

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It requires solving an inverse problem which is ill-posed for lattice QCD correlators

This issue has been investigated for a long time especially in the context of **finite temperature** lattice QCD simulation

Currently there are several approaches to tackle the inverse problem:

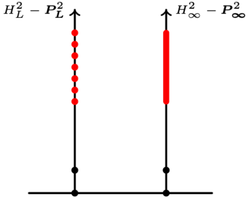
- Backus-Gilbert
- Chebyshev Polynomials
- Bayesian inference (MEM, BR)
- Machine Learning/Neural Network, Gaussan Processes

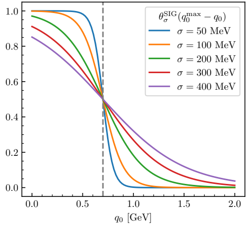
Smeared kernels

- Due to finite L, the spectral density contained in LQCD correlators is a **distribution**
- To be physically relevant, we need to convolute ρ with a **smeared** kernel
- Smearing is needed to deal with $\mathcal{C}^\infty$ kernels

$$\Theta(1-x) \mapsto \Theta(1-x)^{\mathsf{s}}_{\sigma} = \frac{1}{1+e^{-(1-x)/\sigma}}$$

$$\mapsto \Theta(1-x)^{\mathsf{e}}_{\sigma} = \frac{1+\mathrm{erf}(\frac{1-x}{\sigma})}{2}$$





Spectral reconstruction

Stone-Weierstrass **theorem** admits exact polynomial expression for Schwartz kernels

$$K(E,\omega) = \sum_{t=0}^{\infty} g_t(E) e^{-\omega t}.$$

Assuming knowledge of the exact correlators at infinite discrete times $(t = a\tau)$

$$C(a\tau) = \int_0^{\infty} d\omega \; e^{-a\tau\omega} \rho_L(\omega) \qquad\qquad \tau = 0, 1, \dots, \infty$$

It gives the <u>exact</u> **model independent** solution

$$\rho(E) = \int_0^{\infty} d\omega \; \rho_L(\omega) K(\omega, E) = \sum_{t=0}^{\infty} g_t(E) \int_0^{\infty} d\omega \; \rho_L(\omega) e^{-\omega t}$$
$$= \sum_{t=0}^{\infty} g_t(E) C(a\tau)$$

In **practice**

- LQCD correlators are affected by statistical noise
- Number of time-slices is finite: $\tau = 0, 1, \ldots, T$

Choosing an optimal finite set of $g_t(E)$ we can at most write

$$\hat{\rho}(E) \simeq \sum_{t=1}^{T} g_t(E) C(t) + \delta\rho$$

with

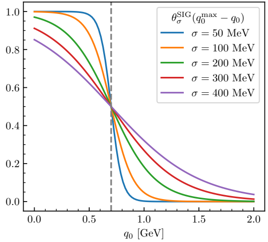
$$K(E,\omega) = \sum_{t=0}^{T} g_t(E) e^{-\omega t}$$

Integration kernel

Using $K_\sigma = \Theta_\sigma^{(l)}(\omega_0 - \omega_0^{\max})$:

$$G^{(l)}(t, \boldsymbol{\omega}) = \int_0^\infty d\omega_0 \; Z^{(l)}(\omega_0, \boldsymbol{\omega}) e^{-\omega_0 t} \qquad \Theta_\sigma^{(l)}(\omega_0 - \omega_0^{\max}) = \sum_t^{t_{\max}} g_t(\sigma, \omega_0^{\max}) e^{-a\omega_0 t}$$

$$\begin{aligned}\widehat{Z}_{\sigma,L}^{(l)}(\boldsymbol{\omega}^2) &= \int_0^\infty d\omega_0 \; \Theta_\sigma(\omega_0 - \omega_0^{\max}) Z_L^{(l)}(\omega_0, \boldsymbol{\omega}^2) \\ &= \sum_t g_t(\sigma, \omega_0^{\max}) \boxed{\int_0^\infty d\omega_0 \; Z_L^{(l)}(\omega_0, \boldsymbol{\omega}^2) \; e^{-a\omega_0 t}} \\ &= \sum_t g_t(\sigma, \omega_0^{\max}) G^{(l)}(t, \boldsymbol{\omega}^2)\end{aligned}$$



smeared kernel

$$Z^{(l)}(\boldsymbol{\omega}^2) = \lim_{\sigma \to 0} \lim_{a \to 0} \lim_{L \to \infty} \widehat{Z}_{\sigma,L}^{(l)}(\boldsymbol{\omega}^2, L)$$

Workflow

We have a procedure to calculate inclusive rates directly from lattice correlators!

i. Calculate 4-point functions on the lattice $G^{(l)}(t, \boldsymbol{\omega}^2)$

ii. Approximate smeared kernel $\Theta_\sigma = \sum_t g_t(\sigma, \omega_0^{\max}) e^{-\omega_0 t}$ to find g_t

iii. Calculate $\sum_t g_t(\sigma, \omega_0^{\max})\ G^{(l)}$ effectively implementing phase space integration

iv. Take the limits $\lim_{L\to\infty}$ and $\lim_{\sigma\to 0}$, in this order

I. ETMC: lattice setup and ensembles

[PRD 107, 074506 (2023) PRD 98, 054518 (2018) PRD 104, 074520 (2021)]

ID	$L^3 \times T$	a [fm]	L [fm]
B48	$48^3 \times 96$	0.07951	3.82
B64	$64^3 \times 128$	0.07951	5.09
B96	$96^3 \times 192$	0.07951	7.63
C80	$80^3 \times 160$	0.06816	5.45
D96	$96^3 \times 192$	0.05688	5.46
E112	$112^3 \times 224$	0.04891	5.47

- $Nf = 2+1+1$ dynamical Wilson-clover twisted-mass quarks ($\mathcal{O}(a)$ improved)
- $M_\pi \sim 140 MeV$
- Two regularisations for valence quarks: twisted-mass (tm) and Osterwalder-Seiler (OS)

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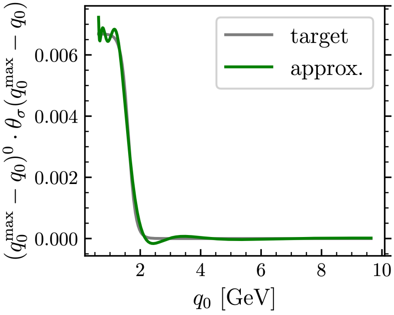
Backus-Gilbert regularisation

Hansen, Lupo, Tantalo **PRD**, 1903.06476

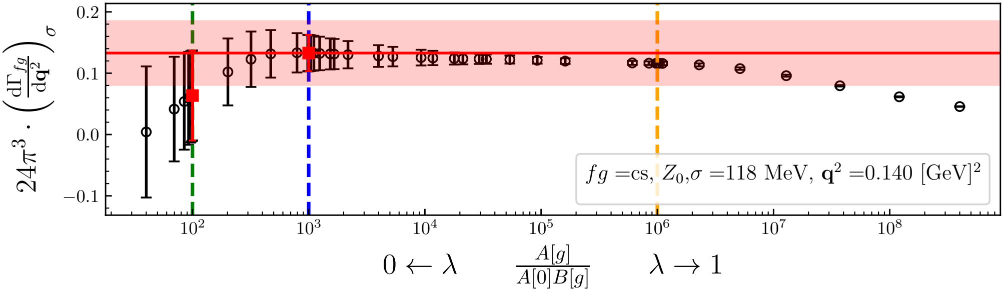
$$W_\lambda[g] = (1-\lambda)\frac{A[g]}{A[0]} + \lambda B[g], \quad \left.\frac{\partial W_\lambda[g]}{\partial g_t}\right|_{g_t = g_t^\lambda} = 0$$

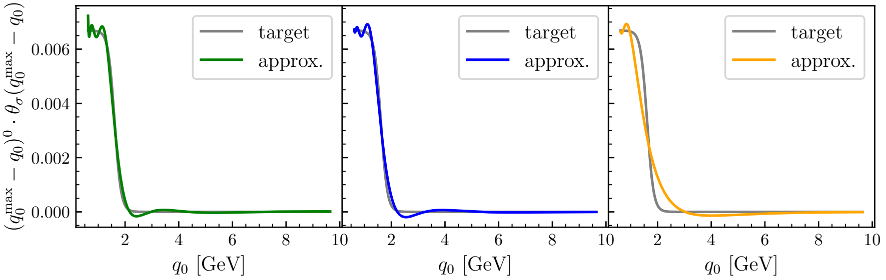
$$A[g] = \int_{E_0}^{\infty} d\omega_0 \left\{ \Theta_\sigma^{(l)} - \sum_{t=1}^{t_{max}} g_t e^{-a\omega_0 t} \right\}^2$$

$$B[g] = \sum_{t,t'=1}^{t_{max}} g_t g_{t'} \frac{\mathrm{Cov}[G^{(l)}(at), G^{(l)}(at')]}{[G^{(l)}(0)]^2}$$



Stability analysis Bulava, Hansen M.T., Hansen M.W., Patella, Tantalo, *JHEP*, 2111.12774





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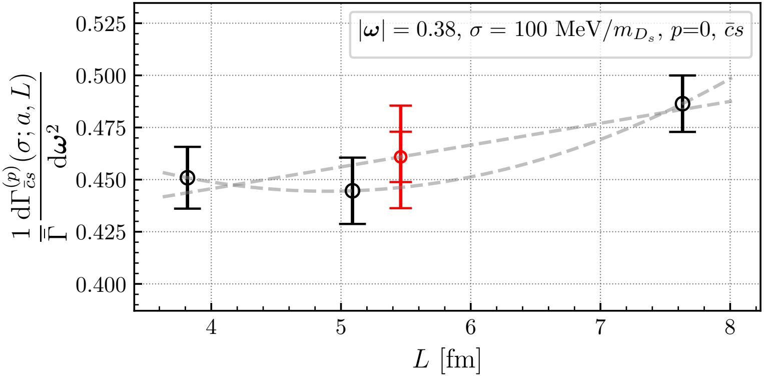
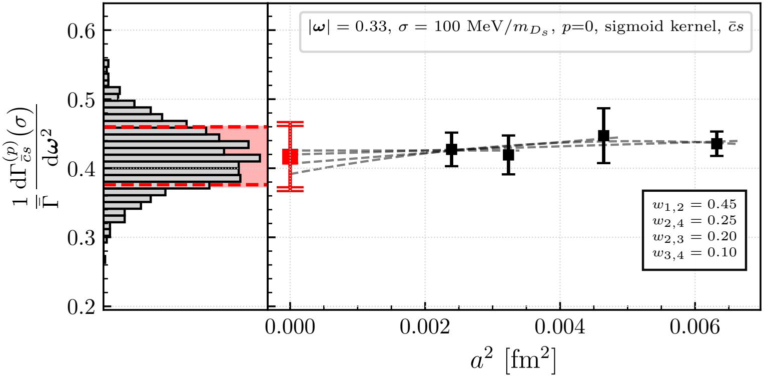
IV. Finite size effects



- Interpolation at reference volume $L \sim 5.46$ fm for ensemble B
- Three volumes to check volume dependence
- In all cases, very mild volume dependence
- FSE systematic error calculated as the maximum spread among different volumes

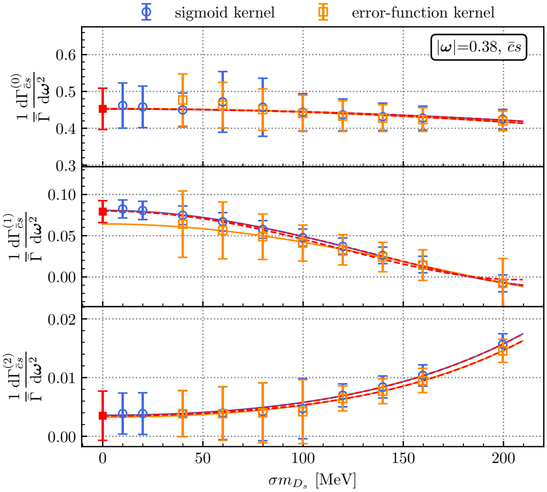
IV. Continuum limit



- Bayesian Akaike information criterion with constant, linear and quadratic fits
- Significant lattice artifacts absent in most cases

IV. Extrapolate to $\sigma \to 0$ [Evangelista *et al.* **PRD**, 2308.03125]

The theoretical asymptotic expansion for small σ captures well the data behaviour



$$\Delta Z_\sigma^{(l)} = \int_0^\infty d\omega_0 \; x^l [\theta_\sigma(x) - \theta(x)]\rho(\omega_0)$$

- l=0,1: $\Delta Z_\sigma = \mathcal{O}(\sigma^2)+$ even powers
- l=2: $\Delta Z_\sigma = \mathcal{O}(\sigma^4)+$ even powers

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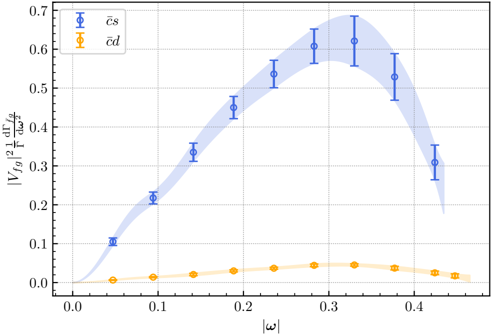
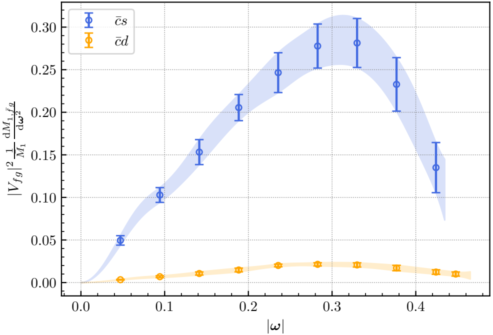
iv. Take the limit $\lim_{L\to\infty}$ and then $\lim_{\sigma\to 0}$, in this order ✓

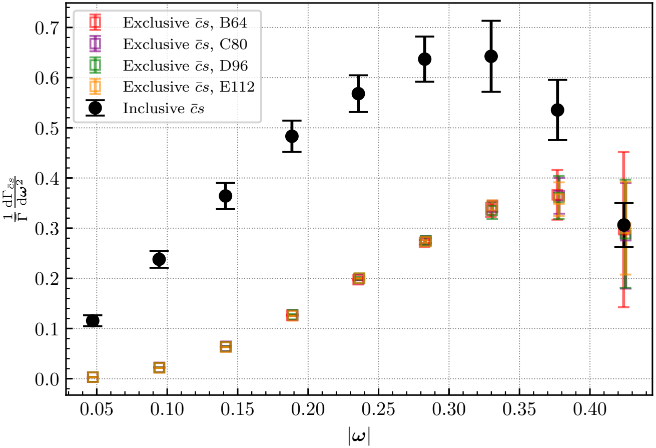
Differential rate & lepton moment





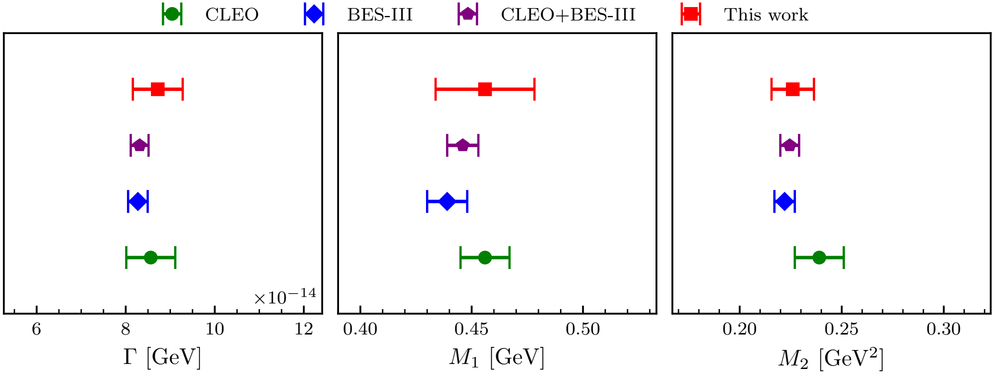
- Spline cubic interpolation + trapezoid integration
- cd channel is Cabibbo suppressed, su has a tiny phase space
- Disconnected contribution (weak annihilation) is compatible with zero within errors

Exclusive vs Inclusive

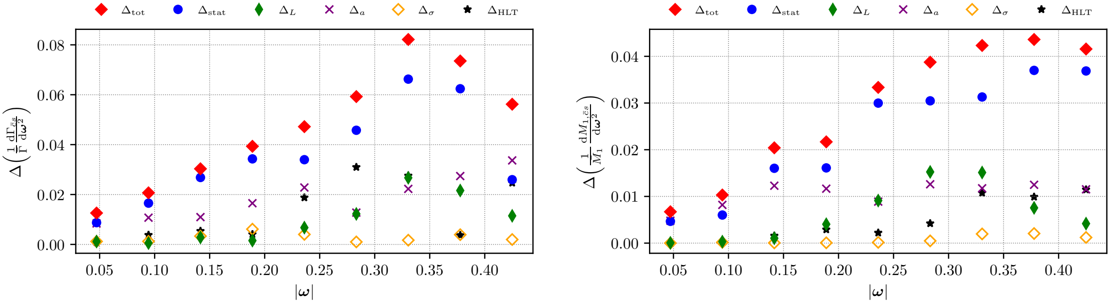


- We compute the exclusive contribution extracting $|f_+(\omega^2)|^2$ from lattice correlators in the dominant channel
- Inclusive result clearly larger than exclusive in the bulk of the phase space
- At the endpoint exclusive and inclusive are compatible within errors

Comparison with experiment



Error budget



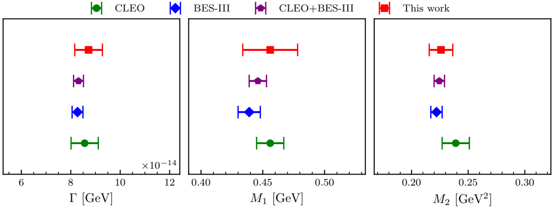
- All results are mainly dominated by statistical errors
- Result is systematically improvble by increasing statistics of lattice correlators

Going to the B

- The strategy adopted to calculate inclusive semi-leptonic decays of the B is the "ratio method" [ETMC, *JHEP*, 0909.3187]
- It consists in building approriate ratios of heavy-light meson masses at several values of heavy quark masses around the charm
- physical b is reached through an interpolation to the asymptotic point of infinitely heavy m_h
- Work just started, with open challenges in producing correlators with $m_h \gg m_c$

Conclusions

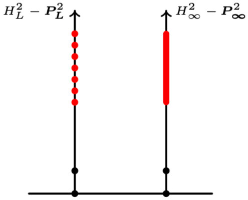
- **Inclusive semi-leptonic decays are now accessible with lattice QCD!**
- Papers on `arXiv.org` [De Santis *et al.*, 2504.06064 and 2504.06063] and sent to journal
- Excellent agreement between lattice QCD and experiment
- Precision of lattice result close to experiment and systematically improvable
- Extension to B-physics just started



BACKUP SLIDES

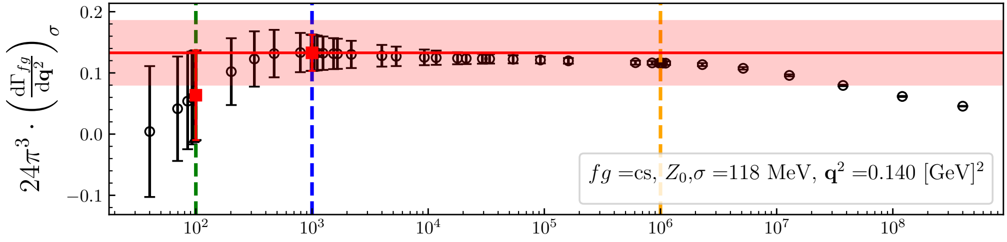
Comments on the limits

$$Z^{(l)}(\boldsymbol{\omega}^2) = \lim_{\sigma\to 0}\lim_{a\to 0}\lim_{L\to\infty} \widehat{Z}^{(l)}_{\sigma,L}(\boldsymbol{\omega}^2)$$



- Smearing allow to replace $\sum_n \delta_n$ with a smooth function. This is necessary to perform a meaningful infinite volume limit.
  [Hansen,Meyer,Robaina, **PRD**, 1704.08993]
- $\sigma \to 0$ is not strictly necessary, **if** we want to compare with an experimental result which can be equally smeared as our spectral density.
  [Hansen,Lupo,Tantalo, **PRD**, 1903.06476]

Systematic errors



$0 \leftarrow \lambda \qquad \frac{A[g]}{A[0]B[g]} \qquad \lambda \to 1$

pull varaible to assess systematic over statistical error

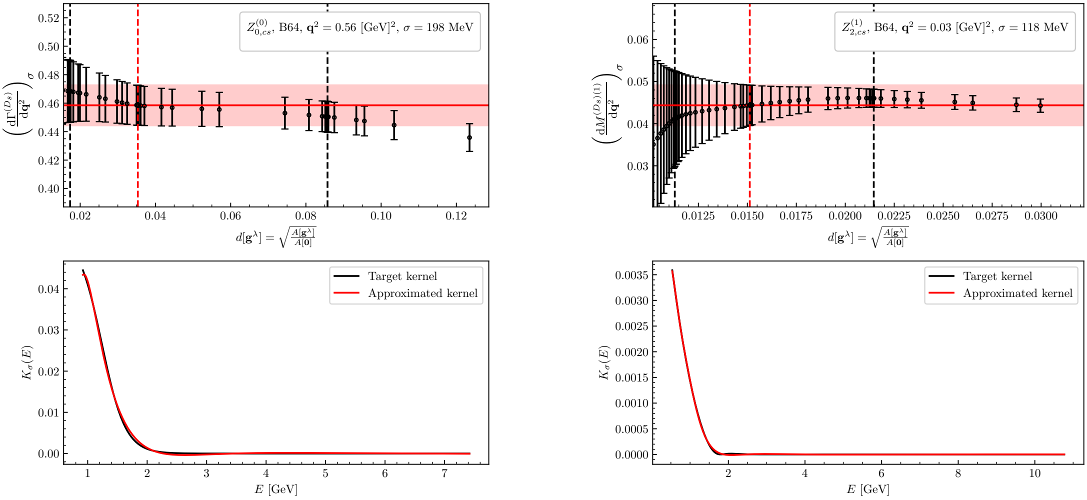
$$\rho_\star : \frac{A[\boldsymbol{g}]}{A[\mathbf{0}]B[\boldsymbol{g}]} = 10^3 \quad \text{plateaux}$$

$$\rho_{\star\star} : \frac{A[\boldsymbol{g}]}{A[\mathbf{0}]B[\boldsymbol{g}]} = 10^2 \quad \text{systematic}$$

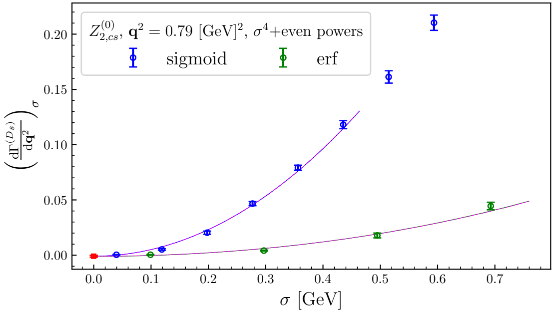
$$P^{HLT} = \frac{\rho_\star - \rho_{\star\star}}{\sqrt{\delta\rho_\star^2 + \delta\rho_{\star\star}^2}}$$

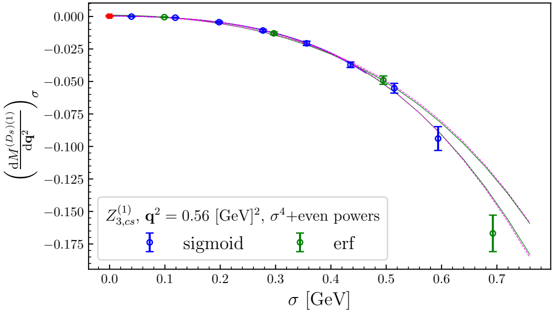
$$\Delta^{sys} = |\rho_\star - \rho_{\star\star}| \, \mathrm{erf}\left(\frac{P^{HLT}}{\sqrt{2}}\right)$$

Stability Z^1 and Z^2

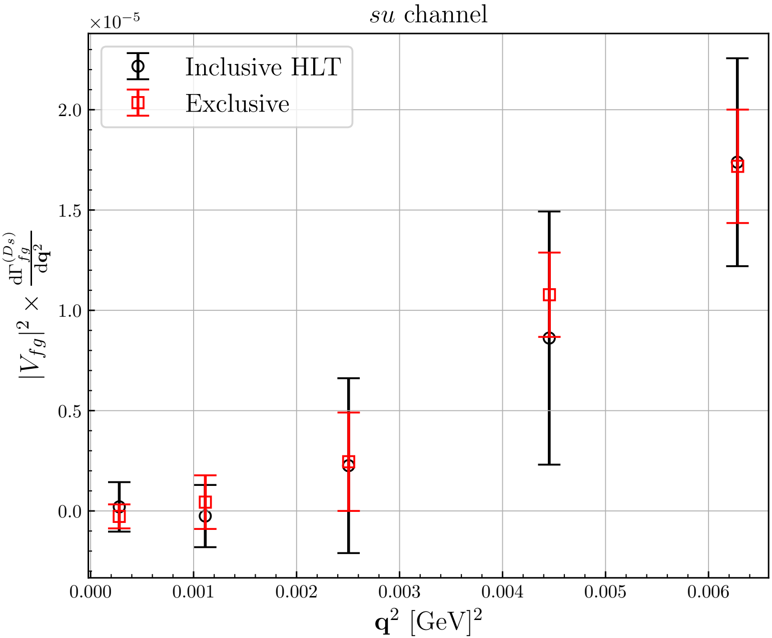


Extrapolation $Z^{(2)}$ and $Z_1^{(3)}$





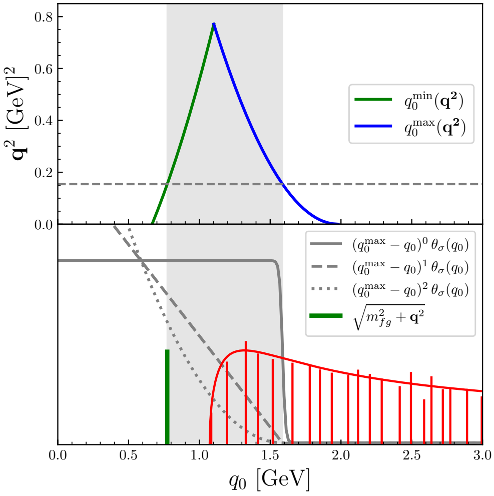
Exclusive vs Inclusive su

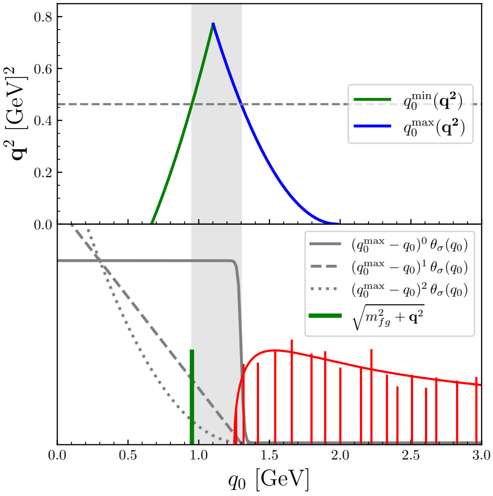


The final hadron phase space

$$\omega_0 \in \left[\sqrt{m_{fg}^2 + \boldsymbol{\omega}^2}, m_{D_s} - |\boldsymbol{\omega}|\right]$$

m_{fg}^2 lightest mass in the spectrum



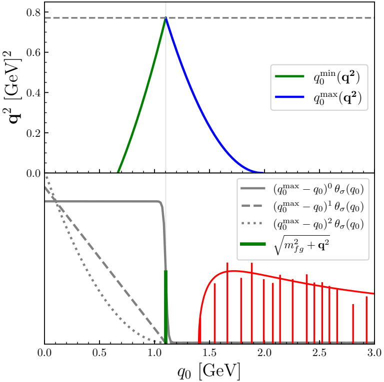


Asymptotic expansion for small σ [Evangelista *et al.* **PRD**, 2308.03125]

$$\Delta\rho_\sigma = \int_0^\infty d\omega_0 \; x^n [\theta_\sigma(x) - \theta(x)]\rho(\omega_0)$$

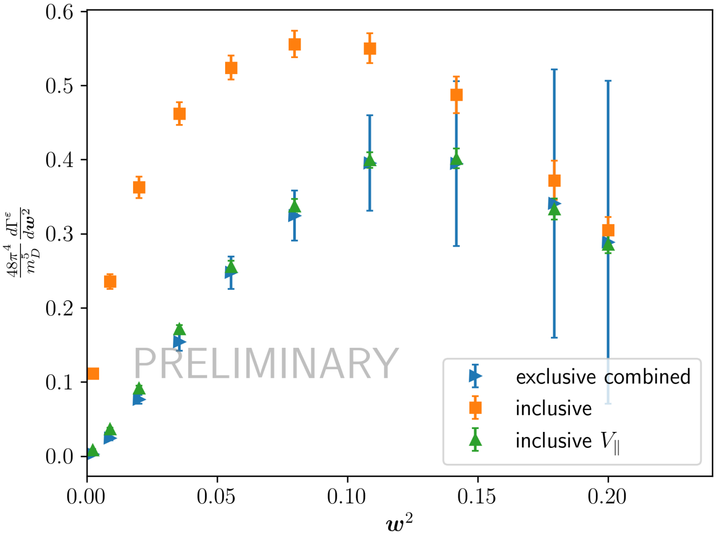
$x = \omega_0^{max} - \omega_0$

- If $\rho(\omega_0)$ is **regular** at ω_0^{max}
  - $n = 0, 1$ $\Delta\rho_\sigma = \mathcal{O}(\sigma^2) +$ even powers
  - $n = 2$ $\Delta\rho_\sigma = \mathcal{O}(\sigma^4) +$ even powers
- If $\rho(\omega_0) = Z \cdot \delta(\omega_0 - \omega_0^{max}) + \dots$
  - $n = 0$ $\Delta\rho_\sigma = \frac{1}{2} Z$!
  - $n > 0$ $\Delta\rho_\sigma = 0$

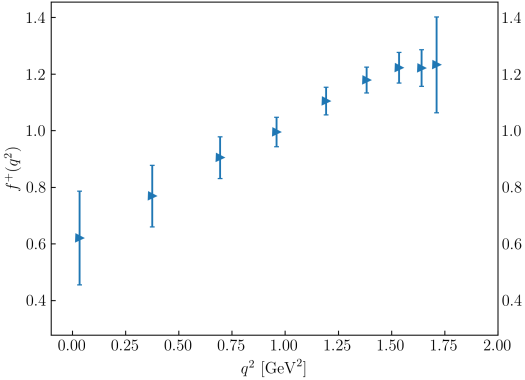


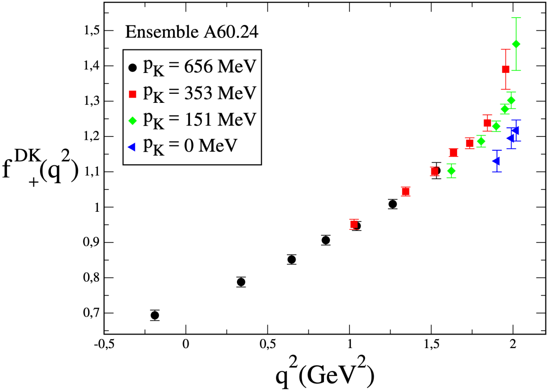
! Decay rate is not vanishing at $\omega^2_{\max}$

Comparison with exclusive

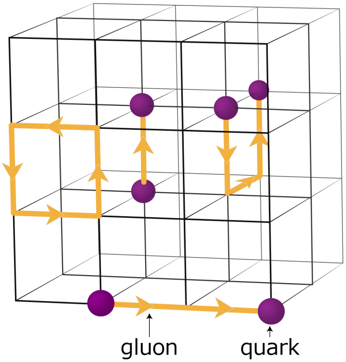


f_+ for cs channel





Lattice QCD



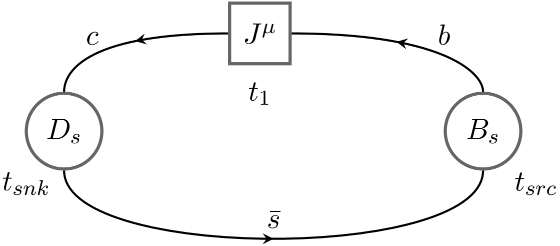
- It is a systematically improvable method
- QCD action is discretised on a lattice with spacing a which is the UV cut-off of the theory
- Euclidean $4D$ spacetime in a finite volume $L^3 \times T$
- Euclidean action is interpreted as the Boltzmann weight in the path integral

$$\langle \mathcal{O} \rangle = \mathcal{Z}^{-1} \int \mathcal{D}[U] e^{-S_E[U]} \mathcal{O}[U] \approx \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}[U_i]$$

Lattice QCD and exclusive semileptonic decays

Lattice QCD has been extremely succesful in determining $|V_{cb}|$ to a very high precision through the precise calculation of form factors, $e.g.$:

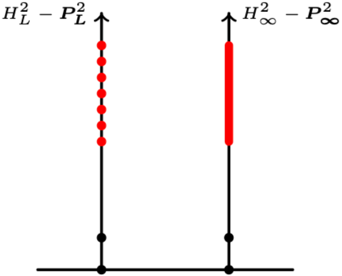
$$\frac{\langle D_s(p')|V^\mu|B_s(p)\rangle}{\sqrt{M_{B_s}M_{D_s}}} = h_+(w)(v+v')^\mu + h_-(w)(v-v')^\mu$$



where the vector current matrix element can be extracted from fits of 2- and 3-point correlation functions and can be used to calculate the relevant form factor.

Lattice QCD and semileptonic B-decays

The study of exclusive semileptonic decays on the lattice requires information only about the **ground state** of the daughter meson, which is something which is easily accessible in lattice QCD.



➜ Things are different for inclusive semileptonics decays, where we require a sum over all final states $\sum X_c$

Tensor Decomposition

Accoding to Lorentz invariance and time-reversal symmetry, the Hadronic Tensor can be decomposed as follows

$$W^{\mu\nu}(p,q) = -g^{\mu\nu}W_1(w,\boldsymbol{q}^2) + \frac{p^\mu p^\nu}{m_{B_s}^2}W_2(w,\boldsymbol{q}^2) - \frac{i\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta}{m_{B_s}^2}W_3(w,\boldsymbol{q}^2) + \frac{q^\mu q^\nu}{m_{B_s}^2}W_4(w,\boldsymbol{q}^2) + \frac{p^\mu q^\nu + p^\nu q^\mu}{m_{B_s}^2}W_5(w,\boldsymbol{q}^2)$$

For convenience we will redefine these components w.r.t. a different basis:

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{q}}{\sqrt{\boldsymbol{q}^2}} \qquad \boldsymbol{\varepsilon}^{(a)} \cdot \hat{\boldsymbol{n}} = 0 \qquad \boldsymbol{\varepsilon}^{(a)} \cdot \boldsymbol{\varepsilon}^{(b)} = \delta^{ab}$$

$$Y^{(1)} = -\sum_{a=1}^{2} \sum_{i,j=1}^{3} \varepsilon_i^{(a)} \varepsilon_j^{(a)} W^{ij}$$

$$Y^{(4)} = \sum_{i=1}^{3} \hat{n}^i (W^{0i} + W^{i0})$$

$$Y^{(2)} = W^{00}$$

$$Y^{(5)} = \frac{i}{2} \sum_{i,j,k=1}^{3} \epsilon^{ijk} \hat{n}^k W^{ij}$$

$$Y^{(3)} = \sum_{i,j=1}^{3} \hat{n}^i \hat{n}^j W^{ij}$$

Decay rate structure functions

$$W^{(0)} = Y^{(2)} + Y^{(3)} - Y^{(4)}$$

$$W^{(1)} = 2Y^{(3)} - 2Y^{(1)} - Y^{(4)}$$

$$W^{(2)} = Y^{(3)} - Y^{(1)}$$

Lattice Gauge Theory

LGT and so LQCD is the only **ab initio** non-perturbative approach available which has proven to be extremely successful

It was proposed in 1974 by Kenneth G. Wilson who was trying to develop a gauge invariant method to study the confinement of quarks using numerical thechniques

The lattice approach allows to estimate the path integral of the theory numerically using Monte Carlo calculations

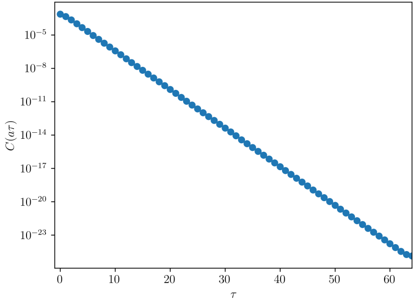
The lattice formulation takes inspiration from statistical field theory and the interplay between has always been crucial

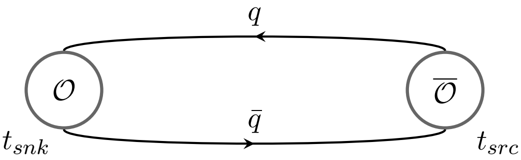
K. G. Wilson
[The Nobel Foundation]

Lattice correlation functions

One of the simples observables are 2-point correlation functions

$$C(t) = \langle \mathcal{O}(t)\overline{\mathcal{O}}(0)\rangle = \sum_k \langle 0|\hat{\mathcal{O}}|k\rangle\langle k|\hat{\mathcal{O}}|0\rangle e^{-tE_k}$$





$|V_{cb}|$ puzzle

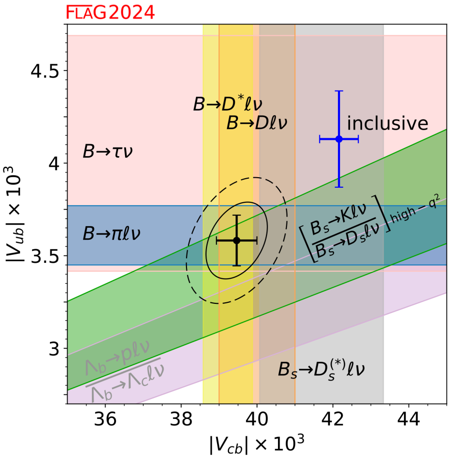
The PDG avarage of $|V_{cb}|$ is: $(41.1 \pm 1.2) \cdot 10^{-3}$

Even though it is unlikely to signal new physics, understanding the $|V_{cb}|$ puzzle is important because:

i. signal something not yet understood in exclusive or inclusive analysis with possible implications affecting $R(D^*)$

ii. limited accuracy of $|V_{cb}|$ affects FCNC studies in an important way

[Gambino, Jung, Shacht, *Phy. Lett. B*, 1905.08209]



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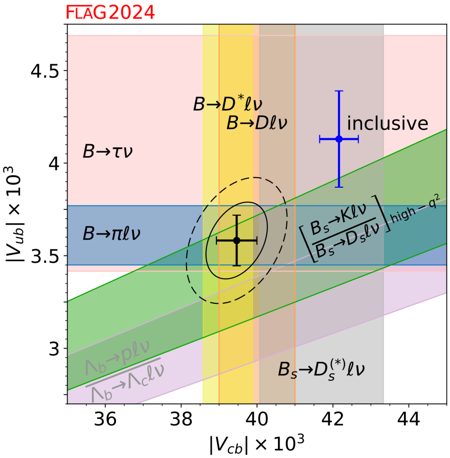
➜ **Can we study inclusive decays on the lattice?**

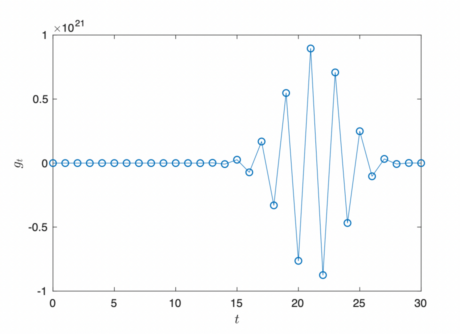


Backus-Gilbert regularisation

Hansen, Lupo, Tantalo, **PRD**, 1903.06476

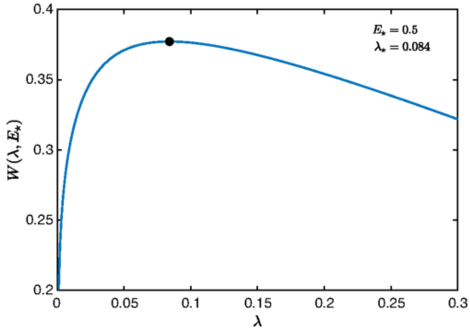
$$A_\alpha[g] = \int_{E_0}^{\infty} dw \; e^{\alpha a \omega} \left\{ \Theta_\sigma^{(l)} - \sum_{t=1}^{t_{max}} g_t e^{-awt} \right\}^2$$

- A_α is a family of weigthed L_2-norms
- $E_0 > m_{lightest}$



[Hansen, Lupo, Tantalo, **PRD**, 1903.06476]

Finding $\lambda_\star$



[Hansen, Lupo, Tantalo, **PRD**, 1903.06476]

$$\left.\frac{\partial W(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_\star} = 0$$

- small λ: good approximation **but** large statistical errors (inverse problem)
- large λ: bad approximation **but** small statistical errors (eccessive regularisation)
- $\lambda_\star$: optimal balance between systematic and statistical errors