Higher power contributions in inclusive semileptonic B decays

Gael Finauri

Semileptonic B decays at the junction of experiment and theory

Torino - 12 June 2025

based on GF 2501.09090

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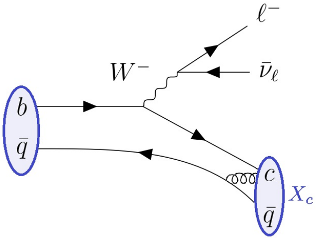

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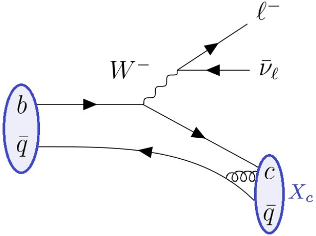

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is computed through a power expansion in $\Lambda_{\text{QCD}}/m_b \sim 0.1$

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$$f(m_b, m_c, \ldots) = f^{\text{LP}} + f^{\text{NLP},\pi} \frac{\mu_\pi^2}{m_b^2} + f^{\text{NLP},G} \frac{\mu_G^2}{m_b^2} + f^{\text{NNLP},D} \frac{\rho_D^3}{m_b^3} + f^{\text{NNLP},LS} \frac{\rho_{LS}^3}{m_b^3} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^4}{m_b^4}\right)$$

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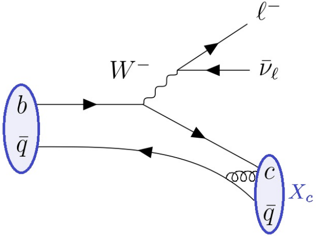

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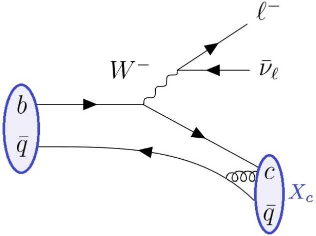

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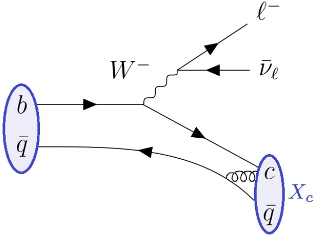

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WARNING: non-pert. parameters also need to be extracted from data

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Experimentally we have access to the moments of the spectrum (Belle, Belle II, BaBar, ...)

$$M_{ijk} \equiv \int dE_\ell dq^2 dm_X^2 \, (E_\ell)^i (q^2)^j (m_X^2)^k \frac{d^3\Gamma}{dE_\ell dq^2 dm_X^2}$$

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So far only existed fits to (E_ℓ, m_X^2) moments [Bordone et al. '21] or q^2 separately [Bernlochner et al. '22].
First combined (E_ℓ, m_X^2, q^2) fit in [GF, Gambino '23]

Theory State of the Art in $\bar{B} \to X_c \ell \bar{\nu}_\ell$

	dE_ℓ	dm_X^2	dq^2	Γ
1	α_s^2 [Melnikov 2008] [Pak, Czarnecki 2008]	α_s^2	α_s^2 [Fael, Herren 2024]	α_s^3 [Fael, Schönwald, Steinhauser 2020]
$1/m_b^2$	α_s [Alberti, Ewerth, Gambino, Nandi 2012, 2013]	α_s	α_s	α_s
$1/m_b^3$	1 [Gremm, Kapustin 1997]	1	α_s [Mannel, Moreno Pivovarov 2021]	α_s [Mannel, Pivovarov 2019]
$1/m_b^{4,5}$ $1/(m_b^3 m_c^2)$	1 [Mannel, Turczyk, Uraltsev 2010]	1	1 [Mannel, Milutin, Vos 2023]	1 [Mannel, Turczyk, Uraltsev 2010]

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Updated in GF 2501.09090
will be the focus of this talk

Highlights of the Calculation

$$\frac{d^3\Gamma}{dE_\ell dE_{\bar{\nu}_\ell} dq^2} = \frac{|V_{cb}|^2 G_F^2}{4\pi^3 m_b} \theta(4E_\ell E_{\bar{\nu}_\ell} - q^2) \left[2q^2 W_1 + (4E_\ell E_{\bar{\nu}_\ell} - q^2) W_2 + \frac{2q^2}{m_b}(E_\ell - E_{\bar{\nu}_\ell}) W_3 \right]$$

in terms of structure functions W_i (encoding hadronic physics)

$$m_b W^{\mu\nu} = -g^{\mu\nu} W_1 + v^\mu v^\nu W_2 + i\epsilon^{\mu\nu\rho\sigma} v_\rho \frac{q_\sigma}{m_b} W_3 + \frac{q^\mu q^\nu}{m_b^2} W_4 + (v^\mu q^\nu + v^\nu q^\mu) \frac{W_5}{m_b}$$

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at tree level ($p^\mu = m_b v^\mu - q^\mu$)

$$W^{\mu\nu} = \sum_{n=0}^{\infty} \frac{\delta^{(n)}(p^2 - m_c^2)}{n!} \mathrm{Tr}\left\{\gamma^\nu P_L M^{(n)}_{\mu_1\ldots\mu_n}(v) P_R \gamma^\mu (\not{p} + m_c)\left[\prod_{k=1}^{n} \gamma^{\mu_k}(\not{p} + m_c)\right]\right\}$$

with the Dirac matrix

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once $M^{(n)}_{\mu_1\ldots\mu_n}(v)$ is determined:  $\longrightarrow$ 

Strategy for Determining $M^{(n)}_{\mu_1 \dots \mu_n}(v)$

We look for $n \leq 5$, neglecting corrections of $\mathcal{O}(\Lambda^6_{\text{QCD}}/m_b^6)$

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- $\boxed{n = 2}$: 2 pars

$$2m_B\mu_\pi^2 = -\langle \bar{B}|\bar{b}_v iD_\perp^\mu iD_{\perp\mu} b_v|\bar{B}\rangle\,,$$

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- $\boxed{n = 3}$: 2 pars

$$2m_B\rho_D^3 = \frac{1}{2}\langle\bar{B}|\bar{b}_v\left[iD_\perp^\mu, \left[iv\cdot D, iD_{\perp\mu}\right]\right]b_v|\bar{B}\rangle\,,$$

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Proliferation of Parameters...

- $\boxed{n=4}$: 9 pars

$$2m_B m_1 = \frac{1}{3}\langle \bar{B}|\bar{b}_v\, iD_\rho iD_\sigma iD_\xi iD_\delta\, b_v|\bar{B}\rangle \left(g_\perp^{\rho\sigma} g_\perp^{\xi\delta} + g_\perp^{\rho\xi} g_\perp^{\sigma\delta} + g_\perp^{\rho\delta} g_\perp^{\sigma\xi}\right)$$

$$2m_B m_2 = \langle \bar{B}|\bar{b}_v\, [iD_\perp^\mu, iv\cdot D][iv\cdot D, iD_{\perp\mu}]\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_3 = \langle \bar{B}|\bar{b}_v\, [iD_\perp^\mu, iD_\perp^\nu][iD_{\perp\mu}, iD_{\perp\nu}]\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_4 = \langle \bar{B}|\bar{b}_v \left\{iD_\perp^\mu, \left[iD_\perp^\nu, [iD_{\perp\nu}, iD_{\perp\mu}]\right]\right\} b_v|\bar{B}\rangle\,,$$

$$2m_B m_5 = \langle \bar{B}|\bar{b}_v\, [iD_\perp^\mu, iv\cdot D][iv\cdot D, iD_\perp^\nu](-i\sigma_{\mu\nu})\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_6 = \langle \bar{B}|\bar{b}_v\, [iD_\perp^\rho, iD_\perp^\mu][iD_\perp^\nu, iD_{\perp\rho}](-i\sigma_{\mu\nu})\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_7 = \langle \bar{B}|\bar{b}_v \left\{\{iD_\perp^\mu, iD_\perp^\rho\}, [iD_{\perp\rho}, iD_\perp^\nu]\right\}(-i\sigma_{\mu\nu})\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_8 = \langle \bar{B}|\bar{b}_v \left\{\{iD_\perp^\rho, iD_{\perp\rho}\}, [iD_\perp^\mu, iD_\perp^\nu]\right\}(-i\sigma_{\mu\nu})\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_9 = \langle \bar{B}|\bar{b}_v \left[iD_\perp^\nu, \left[iD_\perp^\rho, [iD_\perp^\mu, iD_{\perp\rho}]\right]\right](-i\sigma_{\mu\nu})\, b_v|\bar{B}\rangle\,.$$

Proliferation of Parameters...

- $\boxed{n=4}$: 9 pars

$$2m_B m_1 = \frac{1}{3}\langle \bar{B}|\bar{b}_v\, iD_\rho iD_\sigma iD_\xi iD_\delta\, b_v|\bar{B}\rangle \left(g_\perp^{\rho\sigma} g_\perp^{\xi\delta} + g_\perp^{\rho\xi} g_\perp^{\sigma\delta} + g_\perp^{\rho\delta} g_\perp^{\sigma\xi}\right)$$

$$2m_B m_2 = \langle \bar{B}|\bar{b}_v\, [iD_\perp^\mu, iv\cdot D][iv\cdot D, iD_{\perp\mu}]\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_3 = \langle \bar{B}|\bar{b}_v\, [iD_\perp^\mu, iD_\perp^\nu][iD_{\perp\mu}, iD_{\perp\nu}]\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_4 = \langle \bar{B}|\bar{b}_v\, \left\{iD_\perp^\mu, \left[iD_\perp^\nu, [iD_{\perp\nu}, iD_{\perp\mu}]\right]\right\} b_v|\bar{B}\rangle\,,$$

$$2m_B m_5 = \langle \bar{B}|\bar{b}_v\, [iD_\perp^\mu, iv\cdot D][iv\cdot D, iD_\perp^\nu](-i\sigma_{\mu\nu})\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_6 = \langle \bar{B}|\bar{b}_v\, [iD_\perp^\rho, iD_\perp^\mu][iD_\perp^\nu, iD_{\perp\rho}](-i\sigma_{\mu\nu})\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_7 = \langle \bar{B}|\bar{b}_v\, \left\{\{iD_\perp^\mu, iD_\perp^\rho\}, [iD_{\perp\rho}, iD_\perp^\nu]\right\}(-i\sigma_{\mu\nu})\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_8 = \langle \bar{B}|\bar{b}_v\, \left\{\{iD_\perp^\rho, iD_{\perp\rho}\}, [iD_\perp^\mu, iD_\perp^\nu]\right\}(-i\sigma_{\mu\nu})\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_9 = \langle \bar{B}|\bar{b}_v\, \left[iD_\perp^\nu, \left[iD_\perp^\rho, [iD_\perp^\mu, iD_{\perp\rho}]\right]\right](-i\sigma_{\mu\nu})\, b_v|\bar{B}\rangle\,.$$

- $\boxed{n=5}$: 18 pars

$$2m_B r_1 = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, (iv\cdot D)^3\, iD^\rho\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_2 = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, (iv\cdot D)\, iD^\rho\, iD_\sigma\, iD^\sigma\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_3 = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, (iv\cdot D)\, iD_\sigma\, iD^\rho\, iD^\sigma\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_4 = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, (iv\cdot D)\, iD_\sigma\, iD^\sigma\, iD^\rho\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_5 = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, iD^\rho\, (iv\cdot D)\, iD_\sigma\, iD^\sigma\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_6 = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, iD_\sigma\, (iv\cdot D)\, iD^\sigma\, iD^\rho\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_7 = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, iD_\sigma\, (iv\cdot D)\, iD^\rho\, iD^\sigma\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_8 = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\mu\, (iv\cdot D)^3\, iD_\nu\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_9 = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\mu\, (iv\cdot D)\, iD_\nu\, iD_\rho\, iD^\rho\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_{10} = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, (iv\cdot D)\, iD^\rho\, iD_\mu\, iD_\nu\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_{11} = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, (iv\cdot D)\, iD_\mu\, iD^\rho\, iD_\nu\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_{12} = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\mu\, (iv\cdot D)\, iD_\rho\, iD_\nu\, iD^\rho\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_{13} = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, (iv\cdot D)\, iD_\mu\, iD_\nu\, iD^\rho\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_{14} = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\mu\, (iv\cdot D)\, iD_\rho\, iD^\rho\, iD_\nu\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_{15} = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\mu\, iD_\nu\, (iv\cdot D)\, iD_\rho\, iD^\rho\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_{16} = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, iD_\mu\, (iv\cdot D)\, iD_\nu\, iD^\rho\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_{17} = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\mu\, iD_\rho\, (iv\cdot D)\, iD^\rho\, iD_\nu\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,,$$

$$2m_B r_{18} = \mathrm{Re}\Big\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, iD_\mu\, (iv\cdot D)\, iD^\rho\, iD_\nu\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\Big\}\,.$$

Proliferation of Parameters...

- $\boxed{n=4}$: 9 pars

$$2m_B m_1 = \frac{1}{3}\langle \bar{B}|\bar{b}_v\, iD_\rho iD_\sigma iD_\xi iD_\delta\, b_v|\bar{B}\rangle \left(g_\perp^{\rho\sigma} g_\perp^{\xi\delta} + g_\perp^{\rho\xi} g_\perp^{\sigma\delta} + g_\perp^{\rho\delta} g_\perp^{\sigma\xi}\right)$$

$$2m_B m_2 = \langle \bar{B}|\bar{b}_v\, [iD_\perp^\mu, iv\cdot D][iv\cdot D, iD_{\perp\mu}]\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_3 = \langle \bar{B}|\bar{b}_v\, [iD_\perp^\mu, iD_\perp^\nu][iD_{\perp\mu}, iD_{\perp\nu}]\, b_v|\bar{B}\rangle\,,$$

$$2m_B m_4 = \langle \bar{B}|\bar{b}_v \left\{iD_\perp^\mu, \left[iD_\perp^\nu, [iD_{\perp\nu}, iD_{\perp\mu}]\right]\right\} b_v|\bar{B}\rangle\,,$$

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- $\boxed{n=5}$: 18 pars

$$2m_B r_1 = \mathrm{Re}\left\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, (iv\cdot D)^3\, iD^\rho\, b_v|\bar{B}\rangle\right\},$$

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$$2m_B r_4 = \mathrm{Re}\left\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, (iv\cdot D)\, iD_\sigma\, iD^\sigma\, iD^\rho\, b_v|\bar{B}\rangle\right\},$$

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$$2m_B r_6 = \mathrm{Re}\left\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, iD_\sigma\, (iv\cdot D)\, iD^\sigma\, iD^\rho\, b_v|\bar{B}\rangle\right\},$$

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$$2m_B r_8 = \mathrm{Re}\left\{\langle \bar{B}|\bar{b}_v\, iD_\mu\, (iv\cdot D)^3\, iD_\nu\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\right\},$$

$$2m_B r_9 = \mathrm{Re}\left\{\langle \bar{B}|\bar{b}_v\, iD_\mu\, (iv\cdot D)\, iD_\nu\, iD_\rho\, iD^\rho\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\right\},$$

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$$2m_B r_{17} = \mathrm{Re}\left\{\langle \bar{B}|\bar{b}_v\, iD_\mu\, iD_\rho\, (iv\cdot D)\, iD^\rho\, iD_\nu\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\right\},$$

$$2m_B r_{18} = \mathrm{Re}\left\{\langle \bar{B}|\bar{b}_v\, iD_\rho\, iD_\mu\, (iv\cdot D)\, iD^\rho\, iD_\nu\, (-i\sigma^{\mu\nu})\, b_v|\bar{B}\rangle\right\}.$$

Lowest-Lying State Saturation Ansatz (LLSA)

Euristic method to estimate the HQE parameters starting from μ_π^2, μ_G^2 and $\epsilon_j \sim \mathcal{O}(\Lambda_{\text{QCD}})$

[Heinonen, Mannel '14]

$$\rho_D^3 = \frac{1}{3}\epsilon_{1/2}(\mu_\pi^2 - \mu_G^2) + \frac{1}{3}\epsilon_{3/2}(2\mu_\pi^2 + \mu_G^2)\,,$$

$$\rho_{\text{LS}}^3 = \frac{2}{3}\epsilon_{1/2}(\mu_\pi^2 - \mu_G^2) - \frac{1}{3}\epsilon_{3/2}(2\mu_\pi^2 + \mu_G^2)\,,$$

$\Rightarrow \epsilon_{1/2} = 379.5$ MeV, $\epsilon_{3/2} = 388.8$ MeV

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$\Rightarrow \epsilon_{1/2} = 379.5$ MeV, $\epsilon_{3/2} = 388.8$ MeV

$$m_1 = \frac{5}{9}\mu_\pi^4\,, \qquad m_2 = -\frac{\epsilon_{1/2}^2}{3}(\mu_\pi^2 - \mu_G^2) - \frac{\epsilon_{3/2}^2}{3}(2\mu_\pi^2 + \mu_G^2)$$

$$m_3 = -\frac{2}{3}\mu_G^4\,, \qquad m_4 = \mu_G^4 + \frac{4}{3}\mu_\pi^4\,,$$

$$m_5 = -\frac{2}{3}\epsilon_{1/2}^2(\mu_\pi^2 - \mu_G^2) + \frac{\epsilon_{3/2}^2}{3}(2\mu_\pi^2 + \mu_G^2)\,, \qquad m_6 = \frac{2}{3}\mu_G^4\,,$$

$$m_7 = -\frac{8}{3}\mu_\pi^2\mu_G^2\,, \qquad m_8 = -8\mu_\pi^2\mu_G^2\,,$$

$$m_9 = \mu_G^4 - \frac{10}{3}\mu_\pi^2\mu_G^2\,,$$

$$r_1 = \frac{\epsilon_{1/2}^2}{3}(\rho_D^3 + \rho_{\rm LS}^3) + \frac{\epsilon_{3/2}^2}{3}(2\rho_D^3 - \rho_{\rm LS}^3)\,, \qquad r_2 = -\mu_\pi^2\rho_D^3\,,$$

$$r_3 = -\frac{1}{6}\mu_G^2\rho_{\rm LS}^3 - \frac{1}{3}\mu_\pi^2\rho_D^3\,, \qquad r_4 = \frac{1}{6}\mu_G^2\rho_{\rm LS}^3 - \frac{1}{3}\mu_\pi^2\rho_D^3\,,$$

$$r_5 = 0\,, \qquad r_6 = 0\,,$$

$$r_7 = 0\,, \qquad r_8 = \frac{2}{3}\epsilon_{1/2}^2(\rho_D^3 + \rho_{\rm LS}^3) - \frac{\epsilon_{3/2}^2}{3}(2\rho_D^3 - \rho_{\rm LS}^3)$$

$$r_9 = -\mu_\pi^2\rho_{\rm LS}^3\,, \qquad r_{10} = \mu_G^2\rho_D^3\,,$$

$$r_{11} = \frac{1}{6}(2\mu_\pi^2 - \mu_G^2)\rho_{\rm LS}^3 + \frac{1}{3}\mu_G^2\rho_D^3\,, \qquad r_{12} = -\frac{1}{6}(2\mu_\pi^2 + \mu_G^2)\rho_{\rm LS}^3 - \frac{1}{3}\mu_G^2\rho_D^3\,,$$

$$r_{13} = \frac{1}{6}(2\mu_\pi^2 + \mu_G^2)\rho_{\rm LS}^3 - \frac{1}{3}\mu_G^2\rho_D^3\,, \qquad r_{14} = -\frac{1}{6}(2\mu_\pi^2 - \mu_G^2)\rho_{\rm LS}^3 + \frac{1}{3}\mu_G^2\rho_D^3\,,$$

$$r_{15} = 0\,, \qquad r_{16} = 0\,,$$

$$r_{17} = 0\,, \qquad r_{18} = 0\,.$$

Lowest-Lying State Saturation Ansatz (LLSA)

Euristic method to estimate the HQE parameters starting from μ_π^2, μ_G^2 and $\epsilon_j \sim \mathcal{O}(\Lambda_{\rm QCD})$

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$$\rho_{\rm LS}^3 = \frac{2}{3}\epsilon_{1/2}(\mu_\pi^2 - \mu_G^2) - \frac{1}{3}\epsilon_{3/2}(2\mu_\pi^2 + \mu_G^2)\,,$$

$\Rightarrow \epsilon_{1/2} = 379.5$ MeV, $\epsilon_{3/2} = 388.8$ MeV

$$m_1 = \frac{5}{9}\mu_\pi^4\,, \qquad m_2 = -\frac{\epsilon_{1/2}^2}{3}(\mu_\pi^2 - \mu_G^2) - \frac{\epsilon_{3/2}^2}{3}(2\mu_\pi^2 + \mu_G^2)$$
$$m_3 = -\frac{2}{3}\mu_G^4\,, \qquad m_4 = \mu_G^4 + \frac{4}{3}\mu_\pi^4\,,$$
$$m_5 = -\frac{2}{3}\epsilon_{1/2}^2(\mu_\pi^2 - \mu_G^2) + \frac{\epsilon_{3/2}^2}{3}(2\mu_\pi^2 + \mu_G^2)\,, \qquad m_6 = \frac{2}{3}\mu_G^4\,,$$
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$$m_9 = \mu_G^4 - \frac{10}{3}\mu_\pi^2\mu_G^2\,,$$

$m_1 = 0.115 \pm 0.069\,,$	$m_2 = -0.068 \pm 0.041\,,$	$m_3 = -0.055 \pm 0.033\,,$
$m_4 = 0.36 \pm 0.21\,,$	$m_5 = 0.044 \pm 0.027\,,$	$m_6 = 0.055 \pm 0.033\,,$
$m_7 = -0.35 \pm 0.21\,,$	$m_8 = -1.05 \pm 0.63\,,$	$m_9 = -0.35 \pm 0.21\,.$

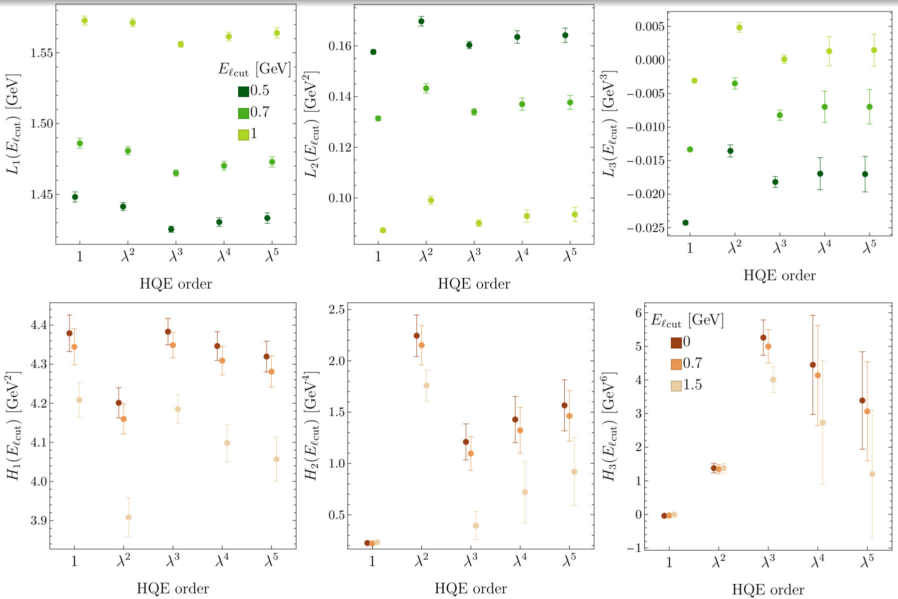
$$r_1 = \frac{\epsilon_{1/2}^2}{3}(\rho_D^3 + \rho_{\rm LS}^3) + \frac{\epsilon_{3/2}^2}{3}(2\rho_D^3 - \rho_{\rm LS}^3)\,, \qquad r_2 = -\mu_\pi^2\rho_D^3\,,$$
$$r_3 = -\frac{1}{6}\mu_G^2\rho_{\rm LS}^3 - \frac{1}{3}\mu_\pi^2\rho_D^3\,, \qquad r_4 = \frac{1}{6}\mu_G^2\rho_{\rm LS}^3 - \frac{1}{3}\mu_\pi^2\rho_D^3\,,$$
$$r_5 = 0\,, \qquad r_6 = 0\,,$$
$$r_7 = 0\,, \qquad r_8 = \frac{2}{3}\epsilon_{1/2}^2(\rho_D^3 + \rho_{\rm LS}^3) - \frac{\epsilon_{3/2}^2}{3}(2\rho_D^3 - \rho_{\rm LS}^3)$$
$$r_9 = -\mu_\pi^2\rho_{\rm LS}^3\,, \qquad r_{10} = \mu_G^2\rho_D^3\,,$$
$$r_{11} = \frac{1}{6}(2\mu_\pi^2 - \mu_G^2)\rho_{\rm LS}^3 + \frac{1}{3}\mu_G^2\rho_D^3\,, \qquad r_{12} = -\frac{1}{6}(2\mu_\pi^2 + \mu_G^2)\rho_{\rm LS}^3 - \frac{1}{3}\mu_G^2\rho_D^3\,,$$
$$r_{13} = \frac{1}{6}(2\mu_\pi^2 + \mu_G^2)\rho_{\rm LS}^3 - \frac{1}{3}\mu_G^2\rho_D^3\,, \qquad r_{14} = -\frac{1}{6}(2\mu_\pi^2 - \mu_G^2)\rho_{\rm LS}^3 + \frac{1}{3}\mu_G^2\rho_D^3\,,$$
$$r_{15} = 0\,, \qquad r_{16} = 0\,,$$
$$r_{17} = 0\,, \qquad r_{18} = 0\,.$$

$r_1 = 0.026 \pm 0.016\,,$	$r_2 = -0.080 \pm 0.048\,,$	$r_3 = -0.021 \pm 0.013\,,$
$r_4 = -0.032 \pm 0.019\,,$	$r_5 = 0 \pm 0.01\,,$	$r_6 = 0 \pm 0.01\,,$
$r_7 = 0 \pm 0.01\,,$	$r_8 = -0.029 \pm 0.018\,,$	$r_9 = 0.051 \pm 0.031\,,$
$r_{10} = 0.051 \pm 0.030\,,$	$r_{11} = 0.005 \pm 0.010\,,$	$r_{12} = 0.006 \pm 0.010\,,$
$r_{13} = -0.039 \pm 0.024\,,$	$r_{14} = 0.029 \pm 0.017\,,$	$r_{15} = 0 \pm 0.01\,,$
$r_{16} = 0 \pm 0.01\,,$	$r_{17} = 0 \pm 0.01\,,$	$r_{18} = 0 \pm 0.01\,.$

assign 60% relative uncertainty (uncorrelated)

For the $r_i = 0$ assign uncertainty: $\mathcal{O}(\Lambda_{\rm QCD}^5) \simeq (400\ \text{MeV})^5 \simeq 0.01\ \text{GeV}^5$

Central Moments Order by Order in Λ_{QCD}/m_b

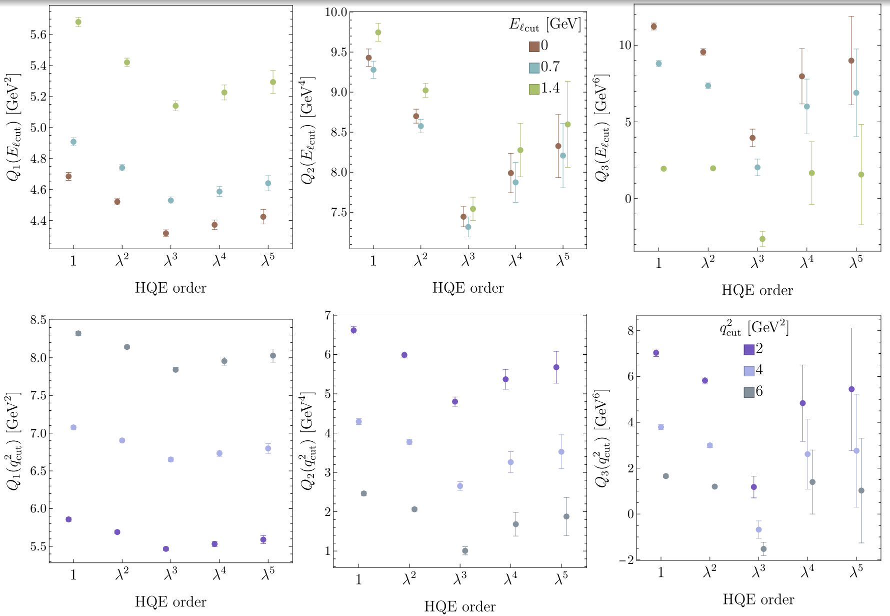


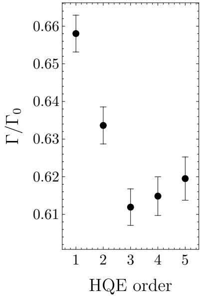
theoretical uncertainties not included

(only parametric)

$\lambda = \Lambda_{\text{QCD}}/m_b$

Central Moments Order by Order in Λ_{QCD}/m_b





theoretical uncertainties not included

(only parametric)

$\lambda = \Lambda_{\text{QCD}}/m_b$

CLEO 2004: Puzzle or Just Bad Data?

only $Q_i(E_{\ell\,\text{cut}})$ measurement: [CLEO 2004]

$$\langle q^2\rangle(E_\ell > 1~\text{GeV}) = 4.89 \pm 0.14~\text{GeV}^2\,,$$
$$\langle q^2\rangle(E_\ell > 1.5~\text{GeV}) = 5.29 \pm 0.12~\text{GeV}^2\,,$$
$$\langle (q^2 - \langle q^2\rangle)^2\rangle(E_\ell > 1~\text{GeV}) = 2.852 \pm 0.047~\text{GeV}^4\,,$$
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CLEO 2004: Puzzle or Just Bad Data?

only $Q_i(E_{\ell\text{cut}})$ measurement: [CLEO 2004]

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compare with $Q_i(q^2_{\text{cut}})$ at q^2_{cut} min: [Belle II 2022]

$$\langle q^2\rangle(q^2 > 1.5\text{ GeV}^2) = 5.16 \pm 0.11\text{ GeV}^2\,,$$
$$\langle(q^2 - \langle q^2\rangle)^2\rangle(q^2 > 1.5\text{ GeV}^2) = 5.97 \pm 0.24\text{ GeV}^4\,,$$

as $Q_i(q^2_{\text{cut}} = 0) = Q_i(E_{\ell\text{cut}} = 0)$

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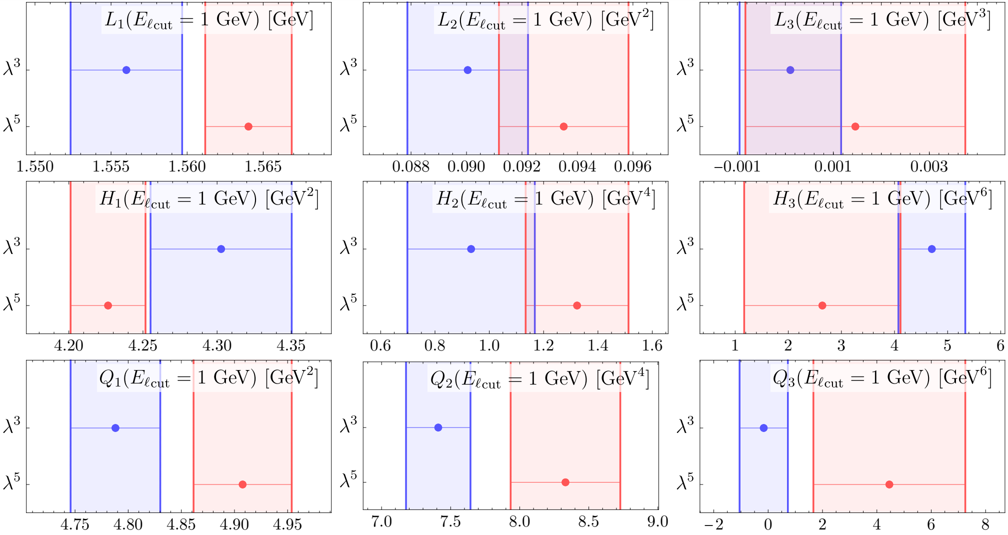
Theoretical Uncertainties



Up to $\mathcal{O}(\Lambda^3_{\mathrm{QCD}}/m_b^3)$ with theoretical uncertainties / Up to $\mathcal{O}(\Lambda^5_{\mathrm{QCD}}/m_b^5)$ with LLSA errors

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Thank You!

Backup Slides

Kinetic Scheme

To avoid renormalon ambiguities and badly converging perturbative series:

on-shell $\rightarrow$ kinetic scheme ($\mu_k = 1$ GeV, $\alpha_s^{(4)}(m_b) = 0.2185$)

[Bigi, Shifman, Uraltsev, Vainshtein '97]

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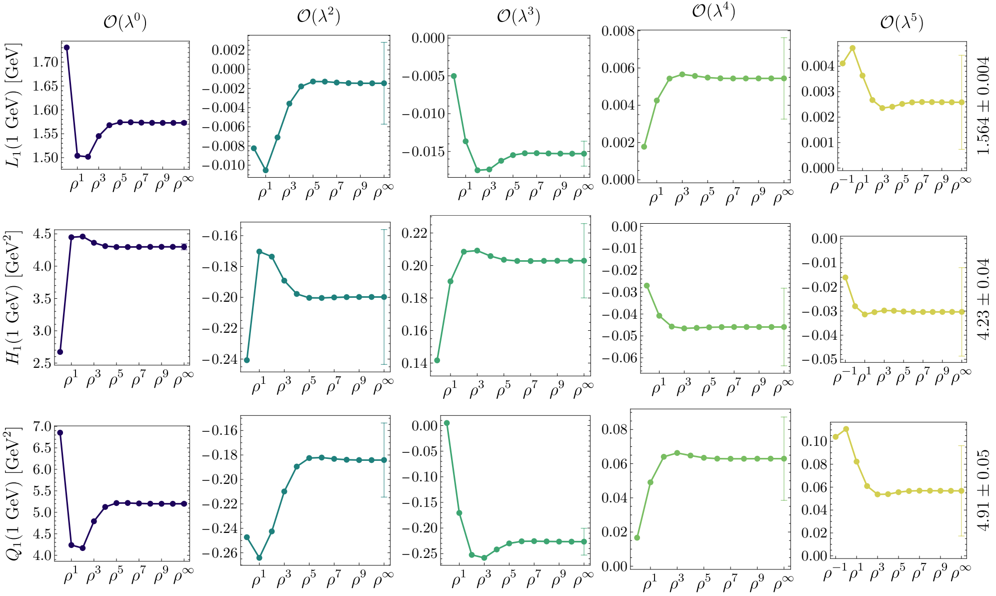
- m_c: on-shell $\rightarrow$ $\overline{\text{MS}}$ at μ_c

$$m_c^{\text{OS}} = m_c(2\text{ GeV})\big(1 + 0.18_{\alpha_s} + 0.14_{\alpha_s^2}\big) = m_c(3\text{ GeV})\big(1 + 0.25_{\alpha_s} + 0.18_{\alpha_s^2}\big)$$

Expansion in $\rho = m_c^2/m_b^2$ (First Moments)



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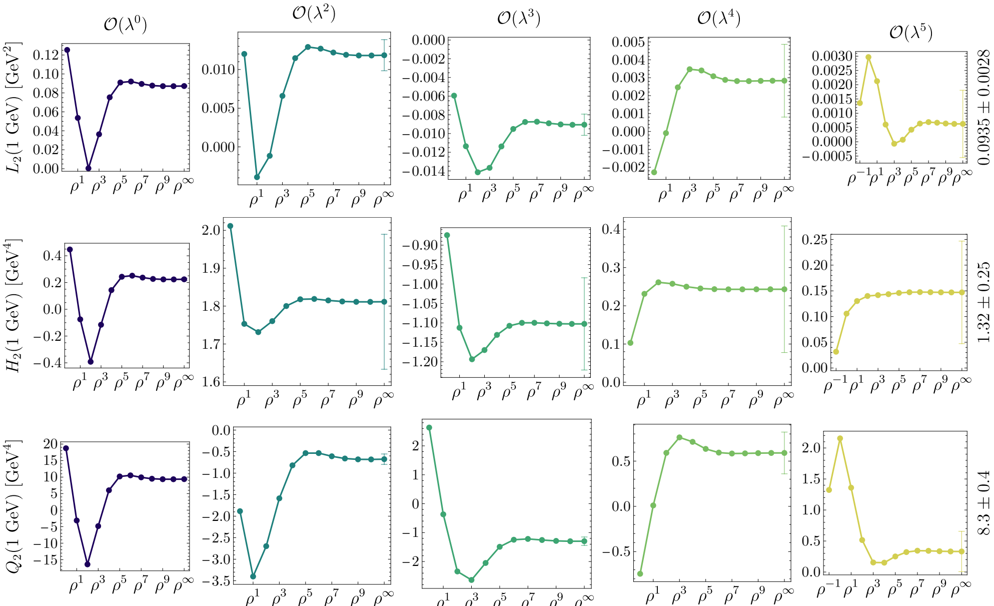

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