Two-Loop DGLAP Splitting Functions from Light Cone Perturbation Theory

T. Lappi, R. Paatelainen, M. Seppälä [2601.10374]

Mikko Seppälä
University of Helsinki & Helsinki Institute of Physics

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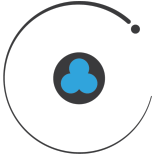

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Outline

1. Motivation: High-order Light Cone Perturbation Theory
2. DGLAP evolution in LCPT & leading order derivation
3. Main features of the two-loop calculation

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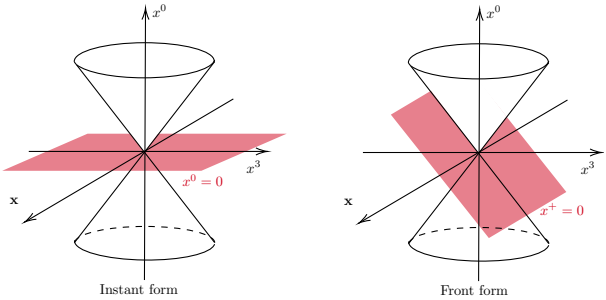
Quantum field theories on the light cone

- Light cone gauge $A^+ = 0$ important for clear partonic picture
  - only two physical gluon polarizations, no ghosts
  - However, results in spurious $1/k^+$ poles

- Solution to get rid of ambiguous poles: Quantize along lightlike time x^+
  $\rightarrow$ "new" dynamics

$$x^\mu = (x^+, x^-, \mathbf{x}),$$
$$x^\pm \equiv \frac{1}{\sqrt{2}}\left(x^0 \pm x^3\right), \quad \mathbf{x} \equiv \left(x^1, x^2\right)$$

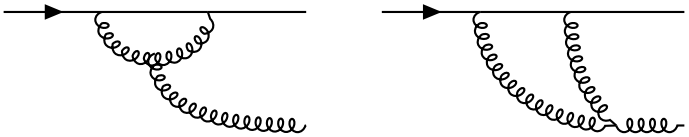


Light Cone Perturbation Theory (LCPT)

$=$ perturbation theory with field quantized along x^+

$\rightarrow$ Hamiltonian "old-school" time-ordered perturbation theory:

$$|\Psi\rangle_{\rm int} = |\Psi\rangle + \sum_{n_1} \frac{|n_1\rangle\langle n_1|H_{\rm int}|\Psi\rangle}{\mathrm{ED}_{1\Psi}} + \sum_{n_1,n_2} \frac{|n_2\rangle\langle n_2|H_{\rm int}|n_1\rangle\langle n_1|H_{\rm int}|\Psi\rangle}{\mathrm{ED}_{2\Psi}\mathrm{ED}_{1\Psi}} + \dots$$



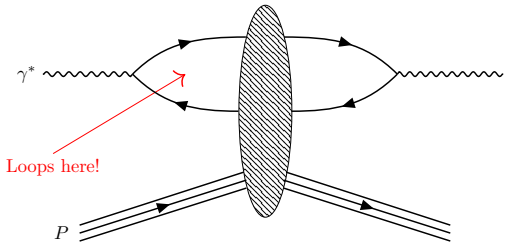
- Everything is on-shell, but Lorentz covariance not explicit (integrals in k^+ and $\mathbf{k}$)
- k^+ is strictly positive $\rightarrow$ no integration over spurious poles

LCPT at higher orders?

- Downside of LCPT: more difficult calculations
  $\rightarrow$ State of the art only one-loop level!

- If downsides of LCPT can be minimized, result is a **framework for systematic precision calculations with a transparent partonic interpretation**

- A necessary tool for high-order calculations for Color-Glass Condensate / saturation regime of DIS
  $\rightarrow$ **Precision physics for EIC**



Goal:

Completely new order in perturbative expansion in LCPT

Deriving DGLAP evolution from LCPT

Deriving (higher order) DGLAP evolution from LCPT is interesting for multiple reasons:

- Partonic interpretation clearer in LCPT
- Fundamental object for cross-checking validity of LCPT at higher orders
- Easy-ish multi-loop calculation
- Historically an important higher-order calculation with light cone gauge[1]

[1] Main ref: [Curci, Furmanski, Petronzio (1980)]

Deriving DGLAP evolution from LCPT

We computed[2] the NLO evolution of the non-singlet quark distribution
= two-loop qq and $q\bar{q}$ splitting functions

$$P^-(\alpha_s, x) = P^{-,(0)}(x) + \frac{\alpha_s}{2\pi} P^{-,(1)}(x) + \ldots, \qquad P^- \equiv P_{qq} - P_{q\bar{q}}$$

Goals:

- Systematic treatment of PDF and derivation of DGLAP evolution in LCPT
- First two-loop gauge theory calculation in LCPT

[2] T. Lappi, R. Paatelainen, M. Seppälä [2601.10374]

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PDF definition

$$f_{i/h}(\mu, x) \equiv \frac{1}{4\pi x} \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\langle h(p^+)|\, a_i^\dagger(xp^+, \mathbf{k}) a_i(xp^+, \mathbf{k})\, |h(p^+)\rangle}{\langle h(p^+)|h(p^+)\rangle}$$

- Expectation value of parton number operator (with fixed x) in hadronic state
- This is the bare PDF: transverse integral results in $1/\varepsilon$-poles $\rightarrow$ renormalization of PDF
- General hadronic state:

$$|h\rangle = \sum_n \psi_n |n\rangle_{\text{int}}$$

non-perturbative wave function

interacting n-particle state

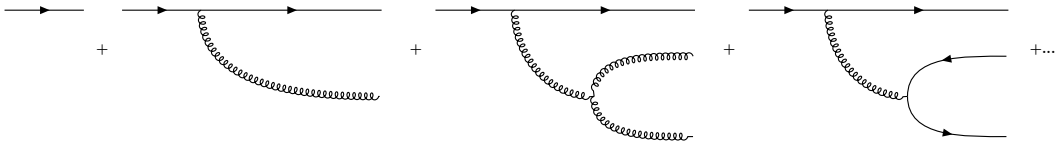
- But evolution is independent of the hadron, so choose for example $|h\rangle = |q\rangle_{\text{int}}$

Fock space expansion

In LCPT, we expand interacting states in terms of Fock states (partons):

$$|q\rangle_{\text{int}} = \sqrt{Z_q}\Big[|q\rangle + \psi^{q\to qg} \otimes |qg\rangle + \psi^{q\to qgg} \otimes |qgg\rangle + \psi^{q\to qq\bar{q}} \otimes |qq\bar{q}\rangle + \dots\Big]$$

This defines the **light cone wave functions** ψ, which are what we actually compute



Quark-in-quark PDF

To evaluate a PDF, we “measure” bare partons in the Fock space expansion of an interacting state, and evaluate the probabilities (wave functions) to find those partons

$$
\begin{aligned}
f_{q/q} &\sim \int_{\mathbf{k}} {}_{\text{int}}\langle q|b^\dagger b|q\rangle_{\text{int}} \\
&\sim \int_{\mathbf{k}} Z_q \Big[\langle q|b^\dagger b|q\rangle + |\psi^{q\to qg}|^2 \langle qg|b^\dagger b|qg\rangle \\
&\qquad + |\psi^{q\to qgg}|^2 \langle qgg|b^\dagger b|qgg\rangle + \left|\psi^{q\to qq\bar{q}}\right|^2 \langle qq\bar{q}|b^\dagger b|qq\bar{q}\rangle + \dots \Big]
\end{aligned}
$$

Matrix elements are just ladder operator algebra:
e.g. $\langle q|b^\dagger b|q\rangle = \langle 0|bb^\dagger bb^\dagger|0\rangle$, and $\{b(\vec{k}), b^\dagger(\vec{p})\} = 2k^+(2\pi)^{D-1}\delta^{(D-1)}(\vec{k}-\vec{p})$

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Leading order quark-in-quark PDF

Expanding in the strong coupling, at leading order we have:

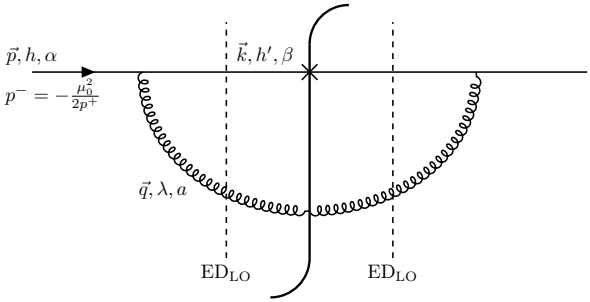
$$f_{q/q}(x) = \delta(1-x) + g^2 \left[Z_q^{(1)}(p^+)\delta(1-x) + \frac{1}{16\pi(p^+)^2 x(1-x)} \bar{\mu}^{4-D} \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \left| \psi_{(0)}^{q\to qg}(\vec{p}, \vec{k}, \vec{p}-\vec{k}) \right|^2 \right]$$

The wave function squared $\left|\psi_{(0)}^{q\to qg}\right|^2$ is evaluated with diagrammatic rules

Results straightforwardly in

$$f_{q/q}(x) = \delta(1-x) + \frac{\alpha_\mathrm{s}}{2\pi} \left[\frac{1}{\varepsilon} P_{qq}^{(0)}(x) + C_{qq}^{(0)} \right] + \mathcal{O}(\alpha_\mathrm{s}^2),$$

where $P_{qq}^{(0)}(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right]$



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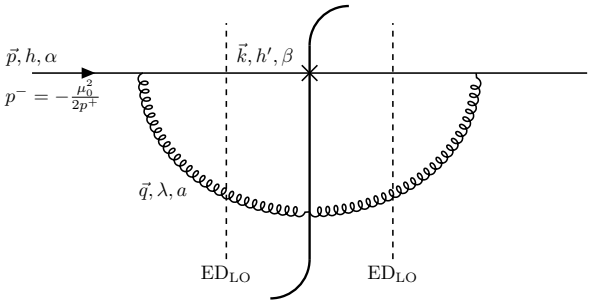

Splitting functions emerge naturally from squares of wave functions!

Leading order DGLAP evolution

Short recap of the derivation:

Start from operational definition of PDF
$\rightarrow$ Fock space expansion
$\rightarrow$ find probabilities = light cone wave functions (diagrammatic LCPT rules)
$\rightarrow$ divergent integral over phase space
$\rightarrow$ DGLAP (details of renormalization in our paper [2601.10374])

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Next-to-leading-order evolution

- To decouple evolution from gluons, consider non-singlet PDF

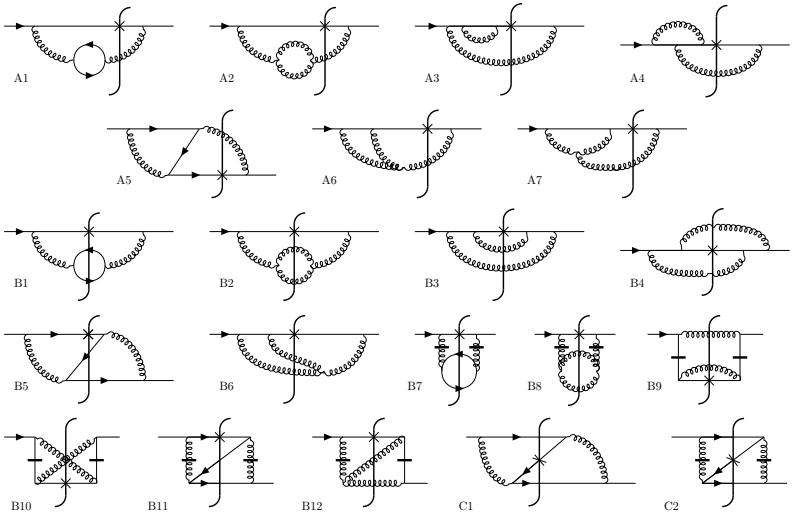
$$f^- \equiv f_{q/q} - f_{\bar{q}/q} \sim \langle q|b^\dagger b - d^\dagger d|q\rangle$$

- We now need more terms from the Fock space

$$|q\rangle_{\text{int}} = \sqrt{Z_q}\Big[|q\rangle + \underbrace{\psi^{q\to qg}}_{\text{1-loop}}\otimes|qg\rangle + \underbrace{\psi^{q\to qgg}}_{\text{tree-level}}\otimes|qgg\rangle + \underbrace{\psi^{q\to qq\bar{q}}}_{\text{tree-level}}\otimes|qq\bar{q}\rangle + \dots\Big]$$

- Then proceed as in the LO case

Two-loop diagrams



Automatization of two-loop diagrams

- Typical two-loop diagram expression:

$$\mathcal{I} \sim \int \mathrm{d}\xi \int \frac{\mathrm{d}^{D-2}\mathbf{l}}{(2\pi)^{D-2}} \frac{1}{\mathbf{l}^2 + \Delta_0 \mu^2} \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\mathcal{N}}{\mathbf{k}^2 \left[\mathbf{k}^2 + \Delta_1 (\mathbf{k} + \Delta_2 \mathbf{l})^2\right] \left[(\mathbf{k} + \Delta_3 \mathbf{l})^2 + \Delta_4 \mathbf{l}^2\right]}$$

  where Δ_i are functions of longitudinal momentum fractions (x, ξ) only, and $\mathcal{N}$ is the Dirac structure

- Need systematic computation procedure
- Standard multiloop automation tools work, but needed to be rebuilt for LCPT (Dirac algebra, tensor reduction, master integrals,...)

Final result

In the end, we get exactly the same result as the original two-loop covariant calculation [Curci, Furmanski, Petronzio (1980)], but for the first time in LCPT

$\rightarrow$ Everything works as should!

$$
\begin{aligned}
P_{qq}^{V,(1)}(x) &= C_F^2\left\{\left[\frac{3}{8}-\frac{\pi^2}{2}+6\zeta(3)\right]\delta(1-x)-\left[2\ln x\ln(1-x)+\frac{3}{2}\ln x\right]\frac{1+x^2}{(1-x)_+}-\left(\frac{3}{2}+\frac{7}{2}x\right)\ln x-\frac{1}{2}(1+x)\ln^2 x-5(1-x)\right\}\\
&\quad+C_FC_A\left\{\left[\frac{17}{24}+\frac{11\pi^2}{18}-3\zeta(3)\right]\delta(1-x)+\left[\frac{1}{2}\ln^2 x+\frac{11}{6}\ln x+\frac{67}{18}-\frac{\pi^2}{6}\right]\frac{1+x^2}{(1-x)_+}+(1+x)\ln x+\frac{20}{3}(1-x)\right\}\\
&\quad+C_FN_FT_F\left\{-\left[\frac{1}{6}+\frac{2\pi^2}{9}\right]\delta(1-x)-\left[\frac{2}{3}\ln x+\frac{10}{9}\right]\frac{1+x^2}{(1-x)_+}-\frac{4}{3}(1-x)\right\},\\
P_{q\bar{q}}^{V,(1)}(x) &= C_F\left(C_F-\frac{C_A}{2}\right)\left\{\left[-4\mathrm{Li}_2(-x)+\ln^2 x-4\ln x\ln(1+x)-\frac{\pi^2}{3}\right]\frac{1+x^2}{1+x}+2(1+x)\ln x+4(1-x)\right\}.
\end{aligned}
$$

Conclusions

Conclusions

- Systematic presentation of DGLAP evolution in LCPT (no ambiguities)
- Important steps towards automated high-order computations
- **First two-loop gauge theory LCPT calculation** $\rightarrow$ LCPT works as should!

Future work

- Systematic computation methodology can be applied to other processes in LCPT
- Application to other distributions?
- Full automation $\rightarrow$ precision physics for EIC

Thank you!

Backup slides

Backup: Renormalization of PDF

- Bare PDF (UV divergent):

$$f_{i/h}(\mu, x) \equiv \frac{1}{4\pi x} \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{\langle h(p^+)|\, a_i^\dagger(xp^+, \mathbf{k}) a_i(xp^+, \mathbf{k})\, |h(p^+)\rangle}{\langle h(p^+)|h(p^+)\rangle}$$

  Here μ is the *regularization scale* (appears with coupling: $\mu^\varepsilon g$)

- Renormalization of PDF means that we can find a matrix Z such that

$$f^R_{i/h}(\mu_F) = Z_{ii'}(\mu^2/\mu_F^2) \otimes f^0_{i'/h}(\mu)$$

  is UV-finite

- Here Z kills dependence on μ but introduces *factorization scale* μ_F
  $\rightarrow$ **Dependence on μ^2/μ_F^2 is what relates UV-divergences of PDF to resummation of large collinear logs of cross sections!**

Backup: Renormalization of PDF

- DGLAP equation is the renormalization group equation for f^R:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\ln\mu_F^2} f^R_{i/h}(\mu_F) &= \left(\frac{\mathrm{d}}{\mathrm{d}\ln\mu_F^2} Z_{ii'}(\mu^2/\mu_F^2)\right) \otimes f^0_{i'/h}(\mu) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}\ln\mu_F^2} Z_{ii'}(\mu^2/\mu_F^2)\right) \otimes Z^{-1}_{i'j}(\mu^2/\mu_F^2) \otimes f^R_{j/h}(\mu_F) \\ &\equiv P_{ij} \otimes f^R_{j/h}(\mu_F)\end{aligned}$$

- This defines the splitting functions P_{ij}
- The renormalization is chosen in the conventional way

$$Z(\mu^2/\mu_F^2, x) = \delta(1-x) - \left(\frac{\alpha_\mathrm{s}}{2\pi}\right)\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{\mu_F^2}\right) P^{(0)}(x) + \mathcal{O}(\alpha_s^2)$$

- For practical calculations, set $\mu = \mu_F = \mu_\mathrm{phys}$ and only look at $1/\epsilon$-pole

Backup: Algorithm for computing diagrams

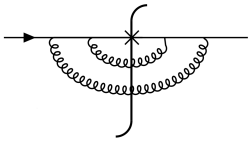
1. **Numerator**: $\mathcal{N} \sim [\bar{u}_h(p) \not{\epsilon}_\lambda(q) u_h(k)]^4$
   - Light cone spinors, polarizations etc... typically hard to automatize
   - BUT! Dirac structure written in helicity basis applicable to dim.reg. (see [1711.08207])
2. **Reduction of tensor integrals to scalar ones**:
   - e.g.:

$$\bar{\mu}^{4-D} \int \frac{\mathrm{d}^{D-2}\mathbf{k}}{(2\pi)^{D-2}} \frac{k^i k^j}{\mathbf{k}^2 \left[\mathbf{k}^2 + \Delta_1 \left(\mathbf{k} + \Delta_2 \mathbf{l}\right)^2\right] \left[\left(\mathbf{k} + \Delta_3 \mathbf{l}\right)^2 + \Delta_4 \mathbf{l}^2\right]} = l^i l^j C_{21} + \delta^{ij} C_{22}$$

   - Standard procedure, but now in $D-2$
3. **Evaluate scalar integrals**
   - Somewhat standard, but weird denominators due to Δ_i-weights
4. **Expansion in ϵ and integration over longitudinal momentum**

Backup: Final details for two-loop splitting functions

- One-loop UV sub-divergences are absorbed into renormalized coupling
- One-loop collinear sub-divergence is cancelled by counterterm from matching bare and renormalized PDF



- Contribution from two-loop renormalization constant:

$$Z_q^{(2)} \sim \int_0^{1-\alpha} \mathrm{d}z P^{-,(1)}(z)$$