Toward CT PDF analyses at N3LO in QCD: ACOT-χ GMVFN scheme for DIS and DGLAP evolution @N3LO in QCD

Marco Guzzi

with

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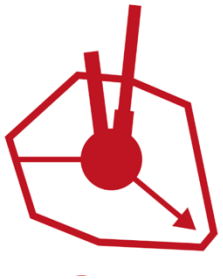


BOLOGNA, ITALY 4-8 MAY 2026

33rd INTERNATIONAL WORKSHOP ON DEEP INELASTIC SCATTERING AND RELATED SUBJECTS

Outline/Motivations

Ongoing efforts of the CTEQ–TEA group toward global PDF analyses at N^3LO

- Recent progress in the 4-loop DGLAP kernel calc. (Gehrmann et al. 2604.09534)
- Recent N^3LO DGLAP evol. code `Candia-v2` (M.G. Hampson, 2512.22667, PRD 2026)
- Operator Matrix Elements (OMEs) at 3-loop accomplished (Ablinger et al. 2510.02175).
- DIS Wilson Coeff. at $\mathcal{O}(\alpha_s^3)$ in the $Q^2 \gg m^2$ limit accomplished (Ablinger et al. 2509.16124)
- DIS Wilson Coeff. at $\mathcal{O}(\alpha_s^3)$ in full kinematics still beyond reach.

With these ingredients available, we explore:

- ACOT-χ GMVFN scheme for deep-inelastic scattering (DIS) at partial next-to-next-to-next-to-leading order (pN^3LO) in QCD.
- Features of current approximations in $F_{2,c}$ with the goal of improving the description of the DIS data

DIS is the backbone of global PDF analyses

The DIS structure Functions

$$F(x,Q) = \sum_{i=1}^{N_f^{fs}} e_i^2 \sum_{a=0}^{N_f} \int_x^1 \frac{d\xi}{\xi} C_{i,a}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \frac{m_h}{\mu}, \alpha_s(\mu)\right) \Phi_{a/p}(\xi,\mu)$$

$$\equiv \sum_{i=1}^{N_f^{fs}} e_i^2 \sum_{a=0}^{N_f} \left[C_{i,a} \otimes \Phi_{a/p}\right](x,Q),$$

To determine $C_{i,a}$ we calculate auxiliary structure functions for scattering on an initial-state parton *b*

$$F(e+b \to e+X) \equiv \sum_{i=1}^{N_f^{fs}} e_i^2 F_{i,b}$$

$C_{i,a}$ are infrared-safe parts of $F_{i,b}$. They enter convolutions together with parton-level PDFs $\Phi_{a/b}(\xi,\mu)$ for splittings $b \to a$

$$F_{i,b}(\hat{x},Q) = \sum_{a=0}^{N_f} \left[C_{i,a} \otimes \Phi_{a/b}\right](\hat{x},Q)$$

The ACOT recipe

Expanding perturbatively all the ingredients

$$F_{i,b}(x) = F_{i,b}^{(0)}(x) + a_s\, F_{i,b}^{(1)}(x) + a_s^2\, F_{i,b}^{(2)}(x) + a_s^3\, F_{i,b}^{(3)}(x) + \ldots,$$
$$C_{i,a}(\widehat{x}) = C_{i,a}^{(0)}(\widehat{x}) + a_s\, C_{i,a}^{(1)}(\widehat{x}) + a_s^2 C_{i,a}^{(2)}(\widehat{x}) + a_s^3 C_{i,a}^{(3)}(\widehat{x}) + \ldots,$$
$$\Phi_{a/b}(\xi) = \delta_{ab}\delta(1-\xi) + a_s\, A_{ab}^{(1)}(\xi) + a_s^2 A_{ab}^{(2)}(\xi) + a_s^3 A_{ab}^{(3)}(\xi) + \ldots.$$

and substituting back in the equation,
we can collect equal powers of α_s

$$C_{i,b}^{(0)}(\widehat{x}) = F_{i,b}^{(0)}(\widehat{x}),$$
$$C_{i,b}^{(1)}(\widehat{x}) = F_{i,b}^{(1)}(\widehat{x}) - \left[C_{i,a}^{(0)} \otimes A_{ab}^{(1)}\right](\widehat{x}),$$
$$C_{i,b}^{(2)}(\widehat{x}) = F_{i,b}^{(2)}(\widehat{x}) - \left[C_{i,a}^{(0)} \otimes A_{ab}^{(2)}\right](\widehat{x}) - \left[C_{i,a}^{(1)} \otimes A_{ab}^{(1)}\right](\widehat{x})$$
$$C_{i,b}^{(3)}(\widehat{x}) = F_{i,b}^{(3)}(\widehat{x}) - \left[C_{i,a}^{(0)} \otimes A_{ab}^{(3)}\right](\widehat{x}) - \left[C_{i,a}^{(1)} \otimes A_{ab}^{(2)}\right](\widehat{x}) - \left[C_{i,a}^{(2)} \otimes A_{ab}^{(1)}\right](\widehat{x})$$

Procedure rooted in Collins' factorization theorem:

- Collins PRD58 1998
- MG, Lai, Nadolsky, Yuan, PRD86 2012

Operator Matrix Elements (OMEs)

The functions $A_{i,j}^{(n)}$ (OMEs) can take two different forms depending on the masses of partons i and j.

$$A_{ij}^{(n)}(\xi,\mu^2) = \sum_{l=1}^{n}\left(\frac{1}{\epsilon}\right)^l P_{ij}^{(n,l)}(\xi) + \sum_{l=0}^{n}\ln^l\left(\frac{\mu^2}{\mu_{\rm IR}^2}\right) P_{ij}^{\prime(n,l)}(\xi)$$ massless i, j

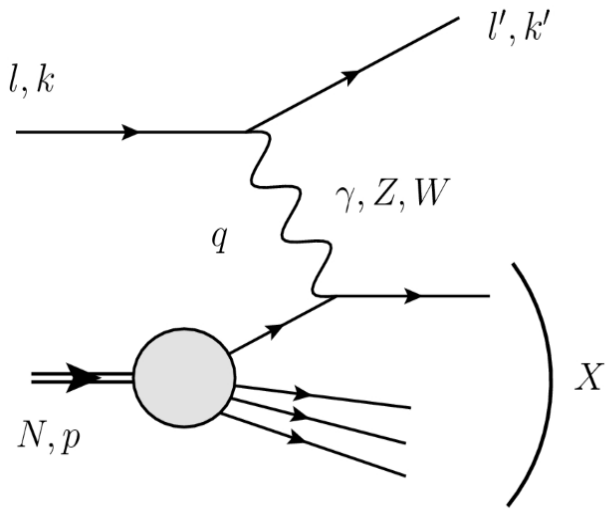
$$A_{Qj}^{(n)}\left(\xi, \frac{\mu^2}{m_Q^2}\right) = \sum_{l=0}^{n}\ln^l\left(\frac{\mu^2}{m_Q^2}\right) a_{Qj}^{(n,l)}(\xi)$$

For massive partons
(at 3-loop -> Blümlein et al.)

Infrared (IR) safe SFs are obtained by subtracting massless parton PDFs:

$$\widehat{F}_{i,b}^{(k)}\left(\widehat{x}, \frac{Q^2}{\mu^2}\right) = F_{i,b}^{(k)}\left(\widehat{x}, \frac{Q^2}{\mu_{IR}^2}, \frac{1}{\epsilon}\right) - \sum_{p=0}^{k}\left[C_{i,a}^{(p)} \otimes A_{ab}^{(k-p)}\right]\left(\widehat{x}, \frac{\mu^2}{\mu_{IR}^2}, \frac{1}{\epsilon}\right)$$

DIS Inclusive Structure Functions



$$F = \sum_{l=1}^{N_l} F_l + F_h$$

Virtual photon coupling to light quark

$$F_l(x,Q) = e_l^2 \sum_a \left[C_{l,a} \otimes \Phi_{a/p}\right](x,Q)$$

Virtual photon coupling to heavy quark

$$F_h(x,Q) = e_h^2 \sum_a \left[C_{h,a} \otimes \Phi_{a/p}\right](x,Q)$$

Here we are concerned with F_h at $\mathcal{O}(\alpha_s^3)$ which is relevant for $F_{2,c/b}$. The ACOT recipe at N3LO gives

$$\begin{aligned} a_s^3 F_h^{(3)} =& a_s^3 \left(\widehat{F}_{h,g}^{(3)} - c_{h,h}^{(0)} \otimes A_{hg}^{(3)} - c_{h,h}^{(1)} \otimes A_{hg}^{(2)} - c_{h,h}^{(2)} \otimes A_{hg}^{(1)}\right) \otimes g \\ &+ a_s^3 \left(\widehat{F}_{h,l}^{(3)} - c_{h,h}^{(0)} \otimes A_{hl}^{(3)} - c_{h,h}^{\mathrm{NS}(1)} \otimes A_{hl}^{(2)}\right) \otimes \Sigma + a_s^2 c_{h,h}^{(2)} \otimes f_h \end{aligned}$$

$$c_{hh}^{(2)} = c_{hh}^{\mathrm{NS}(2)} + c_{hh}^{\mathrm{PS}(2)}$$

Why is full N3LO theory is important?

- Precise and accurate theory predictions are sensitive to heavy-flavor dynamics and rely on delicate cancellation patterns that must in principle be realized with ``Swiss-clock'' precision

Cancellation patterns and current pN^3LO approx

★ $\hat{F}_{h,g}^{(3)}$ represents IR-safe flavor creation (FC) contributions at $\mathcal{O}(\alpha_s^3)$ which incorporate the full mass dependence
No complete results available in the full kinematics.

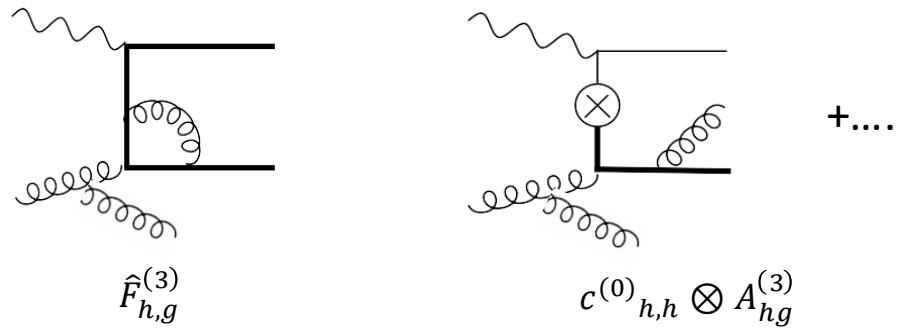
★ $-c_{h,h}^{(0)} \otimes A_{hg}^{(3)} - c_{h,h}^{(1)} \otimes A_{hg}^{(2)} - c_{h,h}^{(2)} \otimes A_{hg}^{(1)}$

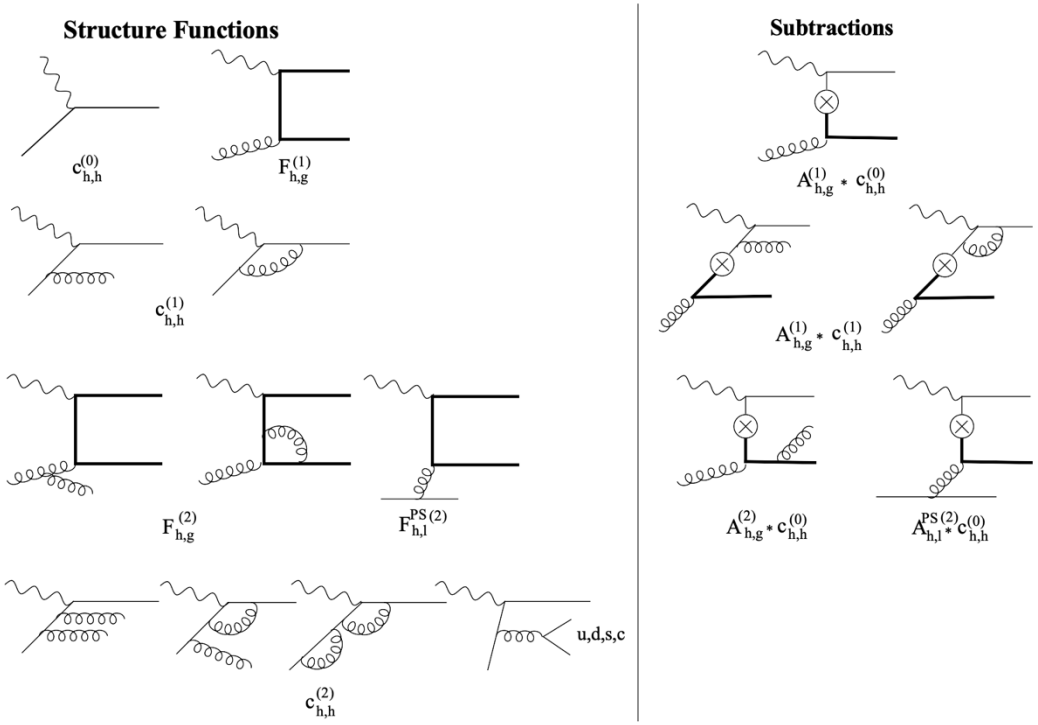
subtraction terms (Sub) to cancel the singular behavior of the FC terms in the high-scale ($Q^2 \gg m^2$) limit

★ $c_{h,h}^{(2)}$ represents the flavor excitation (FE) terms

The realization of the ACOT scheme at each perturbative order relies on delicate cancellations between FC, subtraction terms and FE contributions

For the FC terms at $\mathcal{O}(\alpha_s^3)$ approximated expressions have been obtained that match three different kinematic regions





S-ACOT-χ NNLO: M.G., Lai, Nadolsky, Yuan PRD 2012

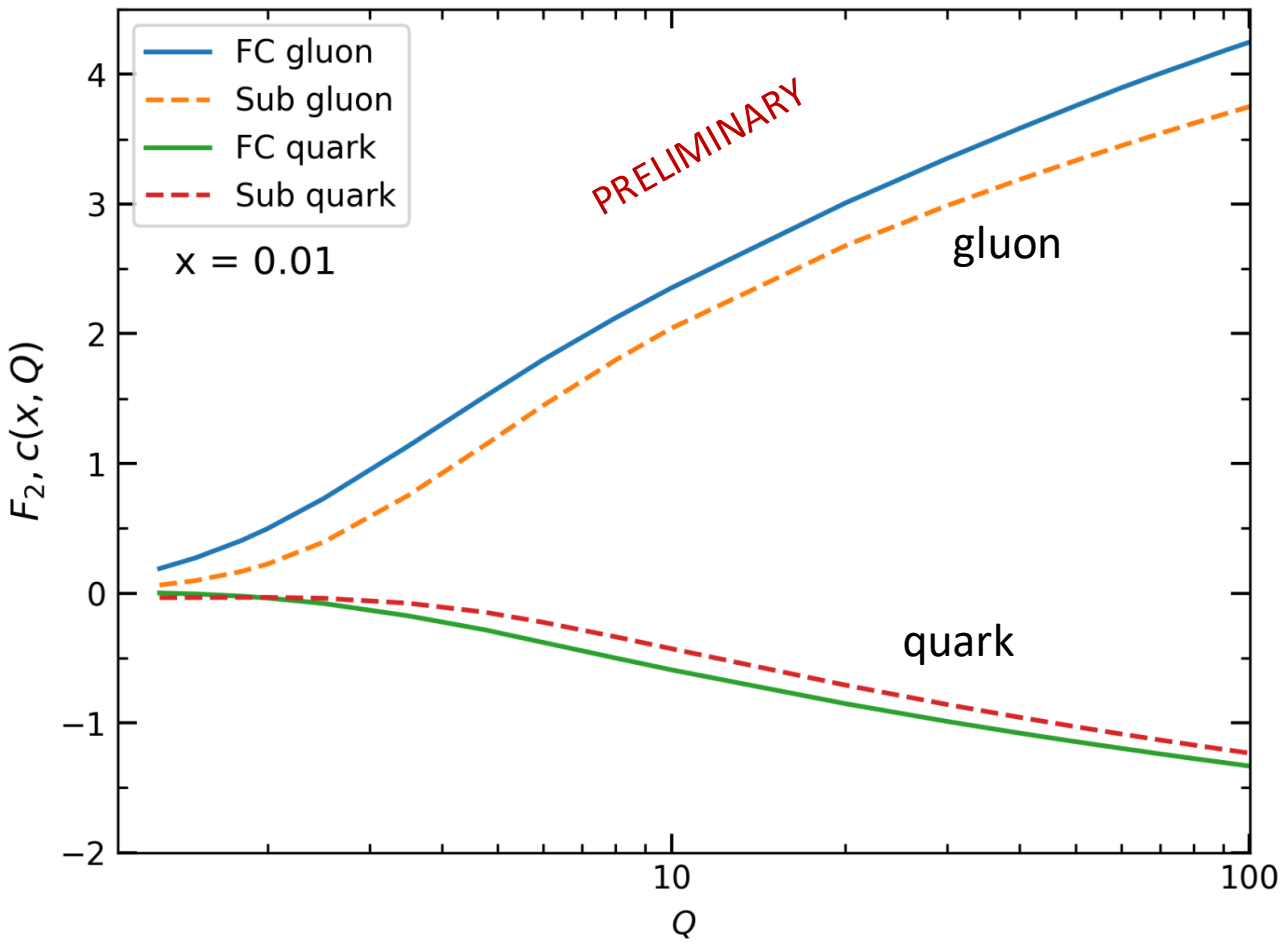
Cancellation patterns @ O(α_s^2)

- NNLO (α_s^2) contributions to $F_{2,c}$

$$C_{h,g}^{(2)} = \widehat{F}_{h,g}^{(2)} - A_{hg}^{(2)} - c_{h,h}^{(1)} \otimes A_{hg}^{(1)};$$

$$C_{h,l}^{(2)} = \widehat{F}_{h,l}^{PS,(2)} - A_{hl}^{PS,(2)}$$

α_s^2 contributions only

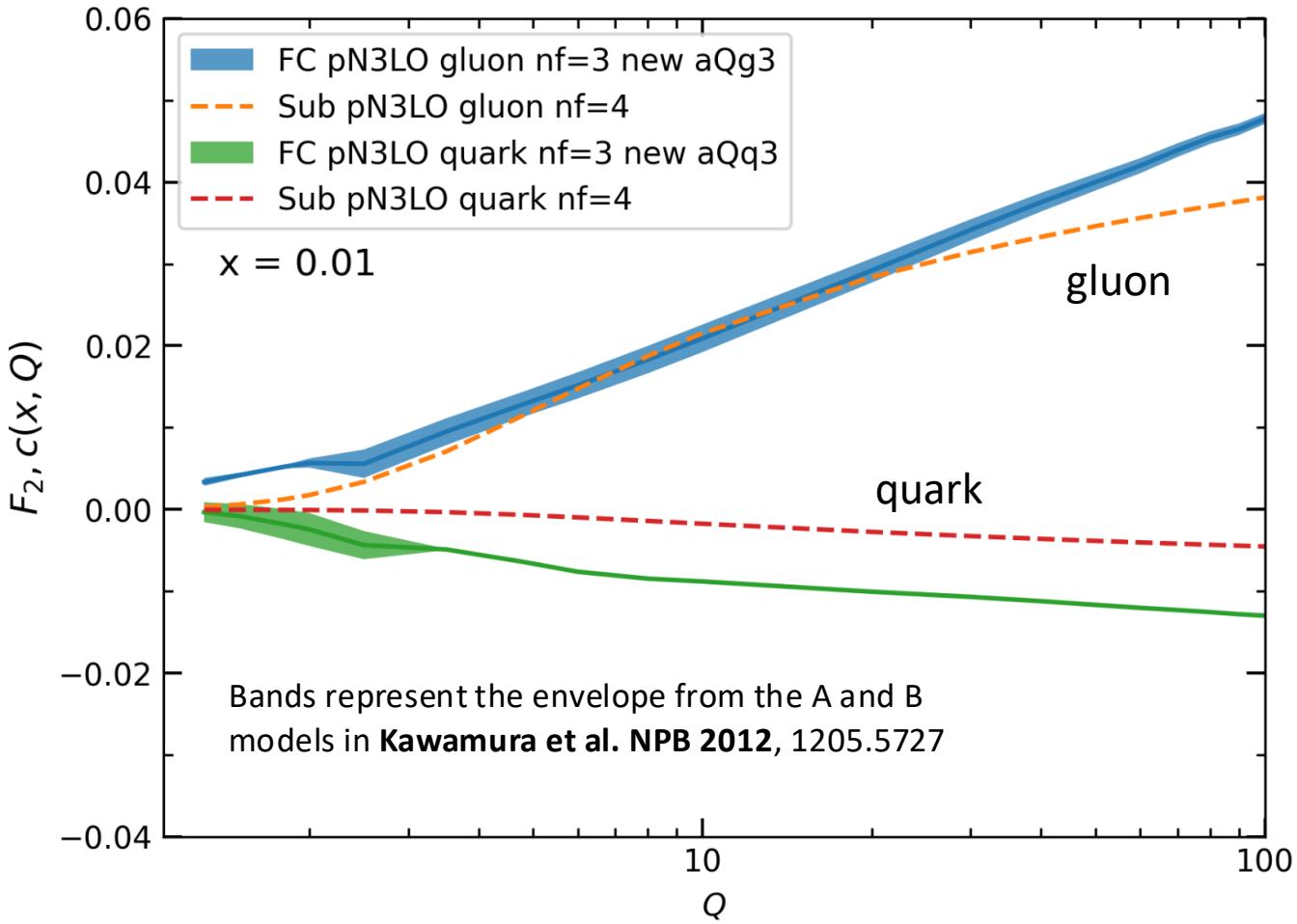


Reproduce the result of the CTEQ fitting code at O(α_s^2)

Cancellation patterns @ O(α_s^3)

- pN^3LO (α_s^3) contributions to $F_{2,c}$

α_s^3 contributions only



$$a_s^3 F_h^{(3)} = a_s^3 \left(\widehat{F}_{h,g}^{(3)} - c_{h,h}^{(0)} \otimes A_{hg}^{(3)} - c_{h,h}^{(1)} \otimes A_{hg}^{(2)} - c_{h,h}^{(2)} \otimes A_{hg}^{(1)} \right) \otimes g$$
$$+ a_s^3 \left(\widehat{F}_{h,l}^{(3)} - c_{h,h}^{(0)} \otimes A_{hl}^{(3)} - c_{h,h}^{\mathrm{NS}(1)} \otimes A_{hl}^{(2)} \right) \otimes \Sigma + a_s^2 c_{h,h}^{(2)} \otimes f_h$$

New $a_{Qg}^{(3),0}(x)$ and $a_{Q\mathrm{q,ps}}^{(3),0}(x)$ (asym.) from https://gitlab.com/libome/libome -> OMEs repo. (Behring, Blümlein et al. 2510.02175)

DIS Wilson Coefficients in the current CT $\mathrm{p}N^3LO$

The current CTEQ approximation for $\widehat{F}_{h,g}^{(3)}$, relies on the work of **Kawamura et al. 1205.5727.**

The perturbative expansion in terms of kinematic variables is

$$c_{k,i}(\eta,\xi,\mu^2) = \sum_{j=0}^{\infty}(4\pi\alpha_s)^j c_{k,i}^{(j)}(\eta,\xi,\mu^2) = \sum_{j=0}^{\infty}(4\pi\alpha_s)^j \sum_{l=0}^{j} c_{k,i}^{(j,\ell)}(\eta,\xi)\ln^{\ell}\frac{\mu^2}{m^2}$$

Wilson Coeff.

$$\eta = \frac{s}{4m^2} - 1 \quad , \qquad \xi = \frac{Q^2}{m^2} \quad , \quad \beta = \sqrt{1-4m^2/s} \quad , \qquad \rho = 4m^2/s$$

Approx obtained by matching kinematic limits with dumping functions

$$c_{2,g}^{(2)} \simeq c_{2,g}^{(2)\,\mathrm{thr}} + (1-f(\xi))\,\beta^k c_{2,g}^{(2)\,\mathrm{asm}} + f(\xi)\,\beta^3\left(-c_{2,g}^{(2)\mathrm{LLx}}\frac{\ln\eta}{\ln x} + c_{2,g}^{(2)\mathrm{NLL}}\frac{\eta^\gamma}{C+\eta^\gamma}\right)$$

needed at $\mathcal{O}(\alpha_s^3)$

best known piece from OMEs computation (Behring, et al. 2510.02175)

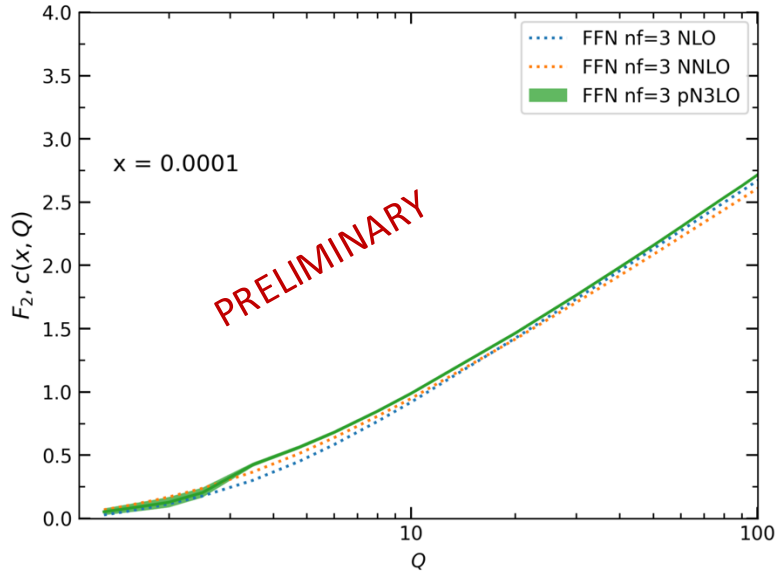
Dumping function

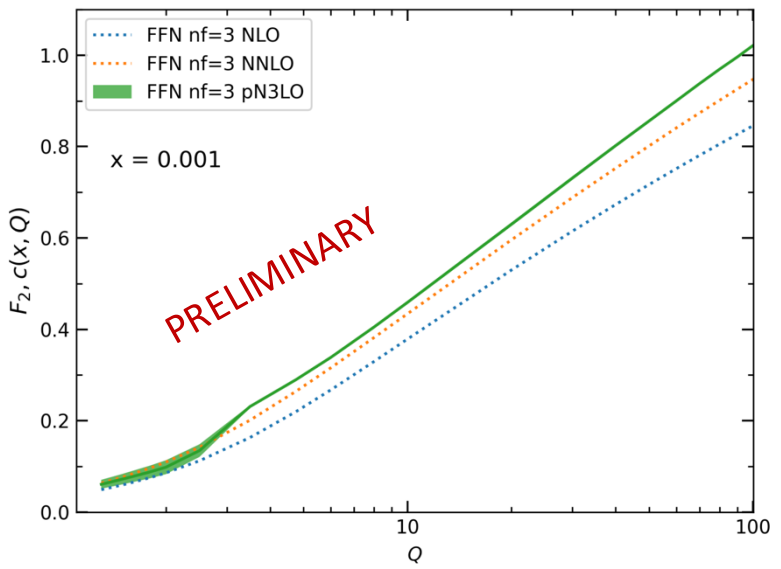
$$f(\xi) = \frac{1}{1+e^{2(\xi-4)}}$$

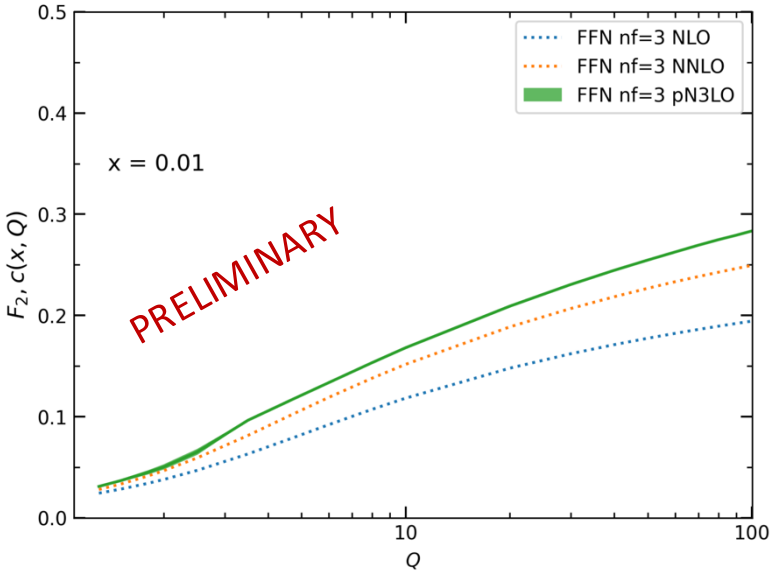
- $s \cong 4\,m^2$ Threshold
- $s \gg 4\,m^2$ High-energy regime $s = Q^2(1/x-1) \to \infty \Longrightarrow x \to 0$ (small x limit)
- $Q^2 \gg m^2$ High-scale (asymptotic) limit

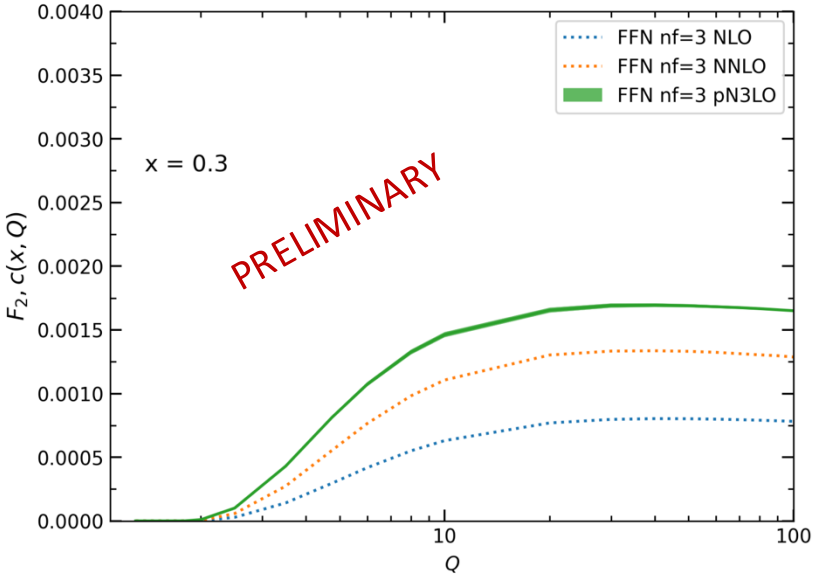
See also Laurenti's Thesis 2401.12139

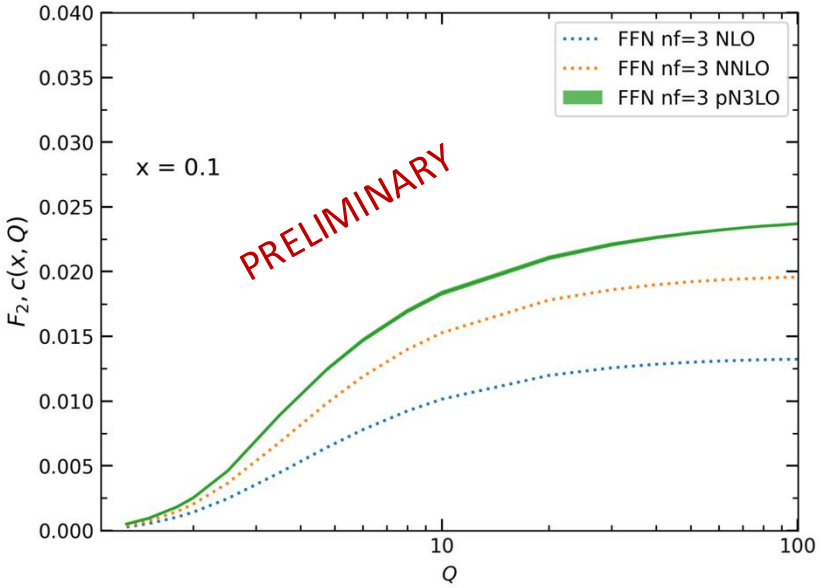
Current CTEQ pN^3LO for F2c in FFN











CT25NNLO PDFs at Q_0 evolved with N3LO Hoppet-v2

Green band: pN3LO uncertainty from matching to the small-x limit

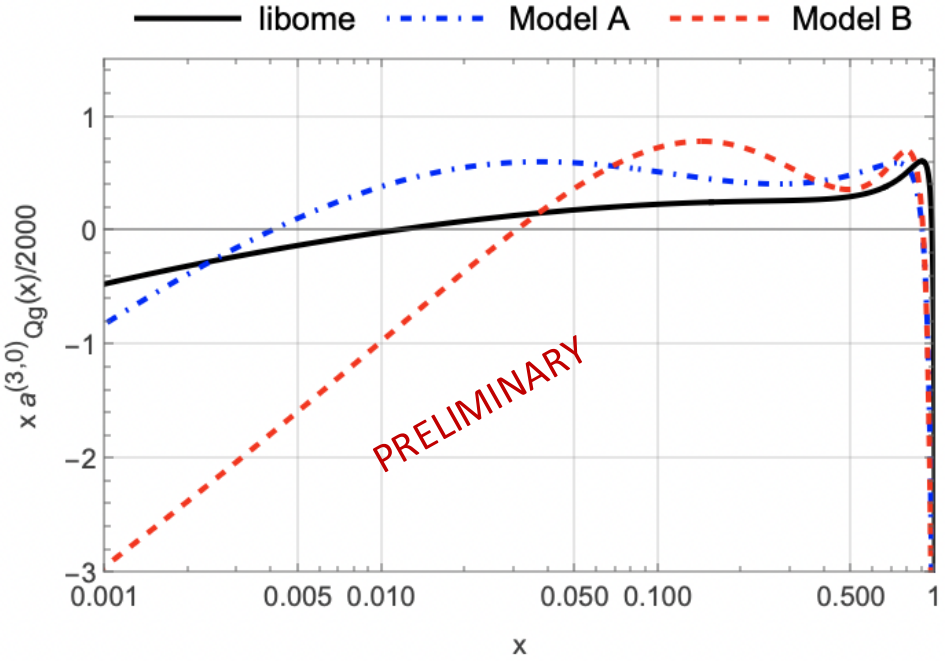
These contributions are now in `DragonWell`, a C++ platform for precision CT PDF fits (Yao Fu)

Asymptotic contributions at pN^3LO

In the $Q^2 \gg m^2$ limit:

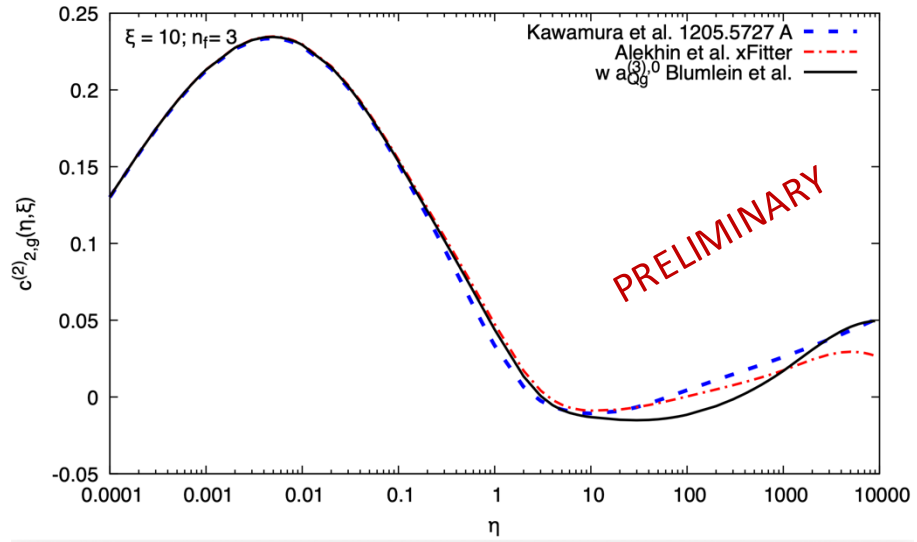
$$H_{2,g}^{(3)}(x,Q^2,m^2) = \text{const}\,[\ldots] + L_Q[\ldots] + L_Q^2[\ldots,\text{nf}+1] + L_Q^3[\ldots,\text{nf}+1]$$
$$+L_\mu[\ldots] + L_\mu^2[\ldots] + L_Q L_\mu\,[\ldots] + L_Q^2 L_\mu\,[\ldots] + L_\mu^2 L_Q\,[\ldots]$$
$$+ a_{Qg}^{(3)0}(x) + c_{2,g}^{(3)}(x, n_f+1)/(n_f+1)$$

State-of-the-art $a_{Qg}^{(3,0)}(x)$ from libome* differs from the previous approximations in Kawamura et al. 1205.5727



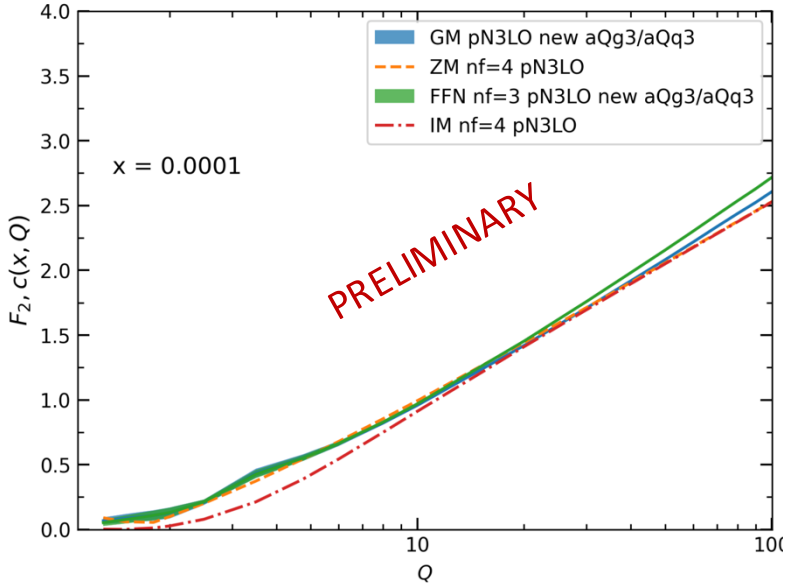
New results for the $Q^2 \gg m^2$ limit are provided by Ablinger, Blümlein et al. in 2509.1624 (`WILS3HQF2`)

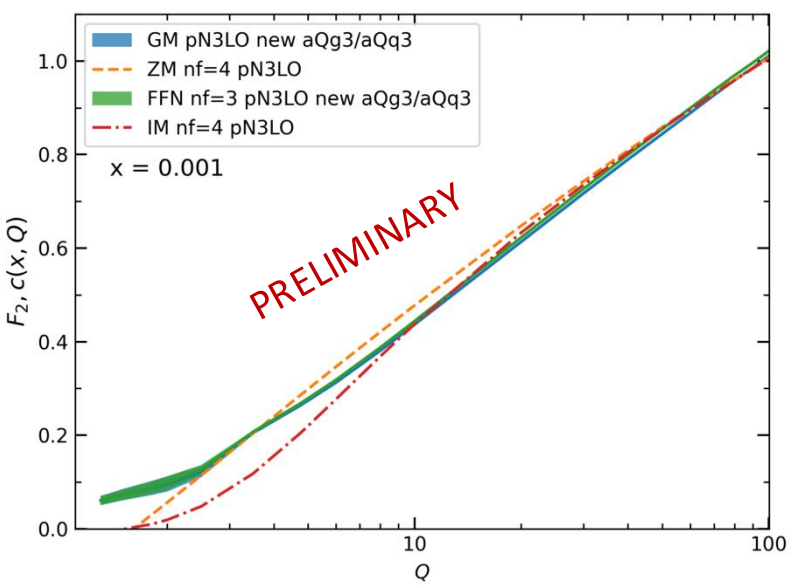
$$H_{2,g} = a_s\left[A_{Qg}^{(1)} + \tilde{C}_{2,g}\right] + a_s^2\left[A_{Qg}^{(2)} + A_{Qg}^{(1)}C_{2,q}^{\text{NS},(1)} + A_{gg,Q}^{(1)}\tilde{C}_{2,g}^{(1)} + \tilde{C}_{2,g}^{(2)}\right] + a_s^3\left[A_{Qg}^{(3)} + A_{Qg}^{(2)}C_{2,q}^{\text{NS},(1)}\right.$$
$$\left.+A_{gg,Q}^{(2)}\tilde{C}_{2,g}^{(1)} + A_{Qg}^{(1)}\left(C_{2,q}^{\text{NS},(2)} + \tilde{C}_{2,q}^{\text{PS},(2)}\right) + A_{gg,Q}^{(1)}\tilde{C}_{2,g}^{(2)} + \tilde{C}_{2,g}^{(3)}\right]$$

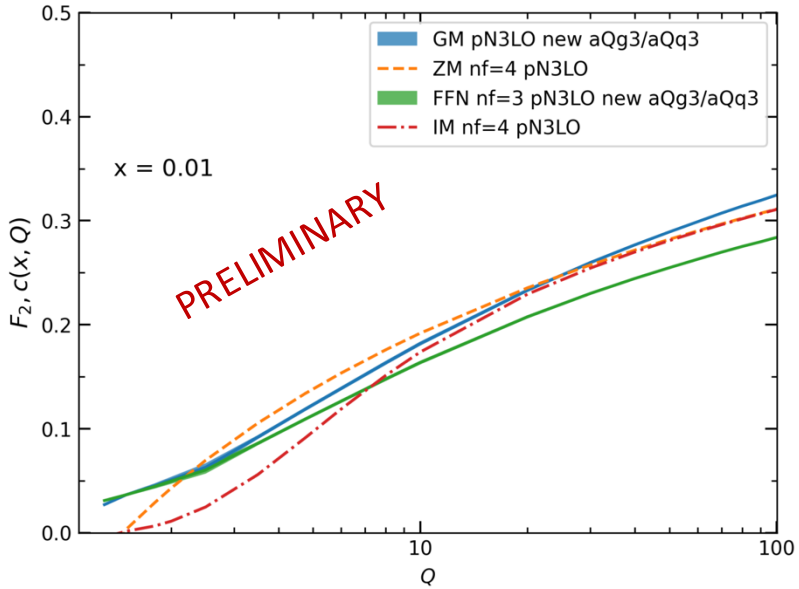


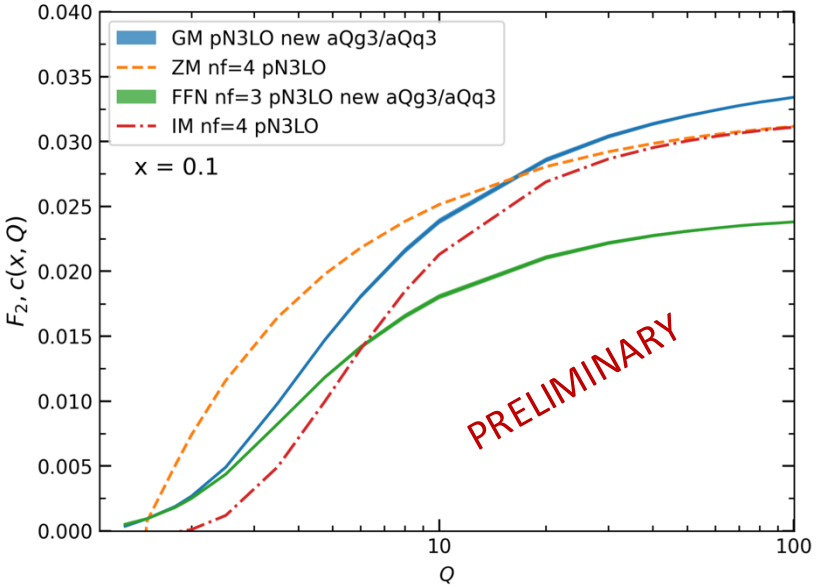
* https://gitlab.com/libome/libome -> OMEs repository (Behring, Blümlein et al. 2510.02175)

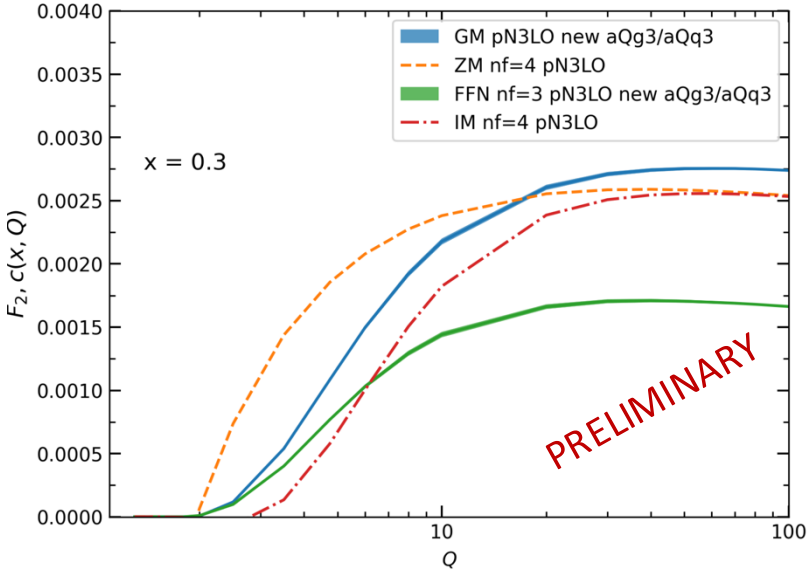
Current CT pN^3LO for $F_{2,c}$











These predictions are obtained with CT25NNLO PDFs at Q_0 and evolved at N3LO with HOPPET-v2

IM = Intermediate Mass scheme: Implementation of $SU(N_f)$ flavor structure at N3LO in the CTEQ fitting code by B. Wang and K. Xie.

- https://scholar.smu.edu/hum_sci_physics_etds/7 (2019). Keping Xie PhD Thesis
- Applications of QCD factorization in multi-scale hadronic scattering, SMU-HEP-15-14 (2015). Bowen Wang, Ph.D. Thesis

Subtraction & Residual PDFs

$$SUB = c_{hh}^{(0)} \otimes \overbrace{\left(a_s A_{hg}^{(1)} + a_s^2 A_{hg}^{(2)} + a_s^3 A_{hg}^{(3)}\right) \otimes g + c_{hh}^{(0)} \otimes \left(a_s^2 A_{hl}^{(2),\mathrm{PS}} + a_s^3 A_{hl}^{(3),\mathrm{PS}}\right) \otimes \Sigma + \cdots}^{\tilde{f}_{\mathrm{h}}^{\mathrm{N3LO}}}$$

$$FE = c_{hh}^{(0)} \otimes \left(a_s f_h^{(1)} + a_s^2 f_h^{(2)} + a_s^3 f_h^{(3)} + \cdots\right) = c_{hh}^{(0)} \otimes f_h + \cdots$$ ← This is the *h*-PDF from N3LO DGLAP

$$\tilde{f}_{\mathrm{h}}^{\mathrm{N3LO}} = \tilde{f}_h^{(1)} + \tilde{f}_h^{(2)} + \tilde{f}_h^{(3)}; \qquad \delta f_h^{\mathrm{N3LO}} = f_h - \tilde{f}_h^{\mathrm{N3LO}}$$

Subtraction PDF — Residual PDF ⟶ Provided by us a LHPDF grids

The subtraction terms can be grouped together with the FE contributions, as they share the same Wilson Coeff. functions which are convoluted with the corresponding HQ PDFs

$$F_{2,h} \approx \mathrm{FC} - \mathrm{SUB} + \mathrm{FE} = \mathrm{FC} + c_{hh}^{(0)} \otimes \delta f_h^{\mathrm{N3LO}} + \cdots$$

PP collisions: MG, Nadolsky, Reina, Wackeroth, Xie, PRD 2024
Higgs production: Biello, Sankar, Xie, MG, Gauld, Nadolsky, Wieseman, Zanderighi (In preparation)

Part II

The x-space Candia-v2 algorithm to N^3LO in QCD for solving the DGLAP evolution

DGLAP evolution at $\mathrm{N^3LO}$ in QCD

Recent progress in the analytical computation of the DGLAP splitting function at N3LO in QCD and what included in Our code

- Falcioni, Herzog, Moch, Pelloni, Vogt, PLB (2025) 2410.08089: Pgg @aN3LO
- Falcioni, Herzog, Moch, Pelloni, Vogt, PLB (2024) 2404.09701: Pgq @aN3LO
- Falcioni, Herzog, Moch, Vogt, PLB (2023) 2307.04158: Pqg @aN3LO
- Falcioni, Herzog, Moch, Vogt, PLB (2023) 2302.07593: Pqq @aN3LO
- Moch, Ruijl, Ueda, Vermaseren, Vogt, JHEP 10 (2017) 1707.08315: First iteration on approximate P_{ij}^{N3LO}
- Kniehl, Moch, Velizhanin, Vogt, PRL (2025) 2505.09381 $P_{NS}^{\pm}$ @N3LO → **nf-part exact**

Since last Fall:

- Fast access to OMEs via precise local overlapping series expansions with the LIBOME library
  Ablinger, Behring, Blümlein, De Freitas, von Manteuffel, Schneider, Schönwald (2025) arXiv:2510.02175 [hep-ph]
- Non-zero (but small) contribution to heavy-quark asymmetry OMEs $A_{Qq}^{PS,s,(3)}$
  - Behring, Blümlein, De Freitas, von Manteuffel, Schneider, Schönwald (2025) arXiv:2512.13508 [hep-ph]
- **Gehrmann, von Manteuffel, Sotnikov, Yang, 2604.09534 → exact expressions for $P_{NS}^{\pm}$ @ N3LO**

DGLAP evolution codes

After the first iteration on the aN3LO splitting functions computation, global PDF fits @aN3LO performed:

1. MSHT20 aN3LO PDFs, J. McGowan, T. Cridge, L. A. Harland-Lang and R. S. Thorne, EPJC (2023) 2207.04739
2. NNPDF4.0 aN3LO PDFs, The NNPDF Collaboration EPJC (2024)
3. xFitter -> See Fra's Talk

DGLAP evolution codes (LO, NLO, NNLO) **& methods**

- PEGASUS, (Mellin Space; U-matrix), Vogt, CPC (2005) hep-ph/0408244
- CANDIA, (x-space, recursion relations), Cafarella, Corianò, M.G., CPC (2008) 0803.0462 [hep-ph]
- HOPPET, (x-space, brute force, Runge-Kutta), Salam, Rojo, CPC (2009) 0804.3755 [hep-ph]
- QCDNum, (x-space fast evol. + spline interpolation) M. Botje, CPC (2011), 1005.1481 [hep-ph]

More recently (@ aN3LO):

- HOPPETv2, (x-space, brute force, Runge-Kutta) Karlberg, Nason, Salam, Zanderighi, Dreyer, 2510.09310 [hep-ph]
- EKO, (Mellin space, interpolation), Candido, Hekhorn, Magni, EPJC (2022) [2202.02338]
- Apfel & Apfel++, (x-space, interpolation, Runge-Kutta), Bertone, Carrazza, Rojo, CPC (2014) [1310.1394].
- CANDIA-v2, Hampson, M.G., PRD (2026) [2512.22667]

N3LO Benchmarking

A Benchmarking of QCD Evolution at aN3LO: Cooper-Sarkar et. al. 2406.16188 [hep-ph]

Candia's algorithm (now in C++)

- PDF evolution; insight into analytical structure of DGLAP solutions.

- Recursion relations from an ansatz in x-space where:

  - Exact expansions (equivalent to exact DGLAP solutions as series) in the NS sector
  - Logarithmic expansions for both the NS and Singlet sectors

U-matrix method (Mellin space) ⟺ logarithmic expansions in x-space

(*Cafarella, Corianò Guzzi, NPB* 2006; *Guzzi*, hep-ph/0612355; *Cafarella, Corianò, Guzzi*, CPC 2008).

It extends a method outlined by Rossi (PRD 1984) at LO, and by Da Luz Vieira and Storrow at NLO (ZPC 1991), in their studies of photon PDFs.

Candia's method in a nutshell

- Non singlet (scalar equations)

$$\frac{\partial q_{\rm NS}(x,\alpha_s)}{\partial \alpha_s} = \frac{P^{\pm,v}_{\rm NS}}{\beta(\alpha_s)} \otimes q_{\rm NS}(x,\alpha_s)$$

Solutions from two types of ansatz:

1. Logarithmic expansion ($\kappa^{\rm th}$ order)

$$f_{N^{\kappa'}LO}(x,\alpha_s)\big|_{O(\alpha_s^{\kappa'})} = \sum_{n=0}^{\infty}\left(A_n^0(x)+\alpha_s A_n^1(x)+\alpha_s^2 A_n^2(x)+\ldots+\alpha_s^{\kappa'} A_n^{\kappa'}(x)\right)\left[\ln\left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)\right]^n$$

2. Exact expansion

$$f(x,Q^2) = \left(\sum_{n=0}^{\infty}\frac{a(x)^n}{n!}L^n\right)_{\otimes}\left(\sum_{m=0}^{\infty}\frac{b(x)^m}{m!}M^m\right)_{\otimes}\left(\sum_{\ell=0}^{\infty}\frac{c(x)^\ell}{\ell!}Q^\ell\right)_{\otimes}\left(\sum_{k=0}^{\infty}\frac{d(x)^k}{k!}P^k\right)_{\otimes}\ldots f(x,Q_0^2)$$

Generalizable to all orders in QCD. $L^n, M^m, Q^l, \ldots$ are scalar functions of α_s and α_0, but do not depend on x.

- Singlet (matrix equation)

$$\frac{\partial}{\partial \alpha_s}\begin{pmatrix} q^{(+)} \\ g \end{pmatrix} = \frac{1}{\beta(\alpha_s)}\begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q^{(+)} \\ g \end{pmatrix}$$

Solutions from logarithmic expansions:

$$\mathbf{f}(x,Q^2) = \sum_{n=0}^{\infty}\left\{\left[\sum_{i=0}^{\kappa}\left(\alpha_s(Q^2)\right)^i \frac{\mathbf{S}_n^i(x)}{n!}\right]\ln^n\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right\}$$

Cafarella, Corianò, Guzzi, NPB (2006) for NNLO
C. H., Guzzi PRD 2026, for N^3LO

What's new: Exact power-series expansion for the NS sector at N3LO and new N3LO recursions for the Singlet

The key idea is to observe that in the NS case, the solution has the following structure

$$f(x,\alpha_s) = \left(\frac{\alpha_s}{\alpha_0}\right)^{-\frac{2}{\beta_0}P^{(0)}} \otimes e^{\mathcal{F}(\alpha_s,\alpha_0,P^{(0)},P^{(1)},P^{(2)},P^{(3)}\ldots,\beta_0,\ldots,\beta_3)} \otimes f(x,\alpha_0)$$

We devise an ansatz in x-space which

$$f(x,Q^2) = \left(\sum_{n=0}^{\infty}\frac{a(x)^n}{n!}\mathcal{L}^n\right)_{\otimes}\left(\sum_{m=0}^{\infty}\frac{b(x)^m}{m!}\mathcal{M}^m\right)_{\otimes}\left(\sum_{\ell=0}^{\infty}\frac{c(x)^\ell}{\ell!}\mathcal{Q}^\ell\right)_{\otimes}\left(\sum_{k=0}^{\infty}\frac{d(x)^k}{k!}\mathcal{P}^k\right)_{\otimes} f(x,Q_0^2)$$

which generates 4 (very cumbersome) recursion relations to determine the coefficients $a(x)$, $b(x)$, …. or $D^s_{t,m,n}(x)$ in

$$f^{\mathrm{N^3LO}}(x,Q^2) = \sum_{s=0}^{\infty}\sum_{t=0}^{s}\sum_{m=0}^{t}\sum_{n=0}^{m}\frac{D^s_{t,m,n}(x)}{n!(m-n)!(t-m)!(s-t)!}\mathcal{L}^n\mathcal{M}^{m-n}\mathcal{Q}^{t-m}\mathcal{P}^{s-t}$$

where we simply rearranged the series and defined

$$D^s_{t,m,n}(x) = a_n(x)\otimes b_{m-n}(x)\otimes c_{t-m}(x)\otimes d_{s-t}(x)\otimes f(x,Q_0^2).$$

What's new: Exact expansions for the NS sector at N3LO and new N3LO recursions for the Singlet

$$\mathcal{L} = \ln\left(\frac{a_s}{a_0}\right),$$
$$\mathcal{M} = \ln\left(\frac{a_s^2 + b a_s + c}{a_0^2 + b a_0 + c}\right),$$
$$\mathcal{Q} = \ln\left(\frac{a_s - r_1}{a_0 - r_1}\right),$$
$$\mathcal{P} = \frac{1}{\sqrt{-b^2+c}} \tan^{-1}\left(\frac{(a_s - a_0)\sqrt{-b^2+c}}{a_s b + a_0(2a_s + b) + 2c}\right)$$

L, M, Q, are P are hard-coded scalar functions that depends on the perturbative order and are directly read off from the analytical Mellin-space solutions (no numeric calc. needed). $D^0_{0,0,0} = f_0$ specifies the initial conditions.

This algorithm is exact up to N3LO. It's the $P^{(3)}$ splitting function that is approximate and leads to approximate N3LO evolved PDFs.

Recursion relations at N3LO

$$D^s_{t,m,n} = -\frac{2}{\beta_0}[P^{(0)} \otimes D^{s-1}_{t-1,m-1,n-1}], \qquad \gamma \equiv r_1^2 + \bar{b} r_1 + \bar{c},$$

$$\begin{aligned}\beta_3 \gamma D^s_{t,m,n} &= \frac{1}{2}\left[16\pi^2\beta_1 + 4\pi r_1 \beta_2 - (c + b r_1)\beta_3\right] D^s_{t,m,n+1} \\ &+ 32\pi^2[P^{(1)} \otimes D^{s-1}_{t-1,m-1,n}] + 16\pi r_1 [P^{(2)} \otimes D^{s-1}_{t-1,m-1,n}] - 8(c + b r_1)[P^{(3)} \otimes D^{s-1}_{t-1,m-1,n}]\end{aligned}$$

$$\begin{aligned}\beta_3 \gamma D^s_{t,m,n} &= (-16\pi^2\beta_1 - 4\pi r_1\beta_2 - r_1^2\beta_3) D^s_{t,m+1,n+1} \\ &- 64\pi^2[P^{(1)} \otimes D^{s-1}_{t-1,m,n}] - 32\pi r_1[P^{(2)} \otimes D^{s-1}_{t-1,m,n}] - 16 r_1^2 [P^{(3)} \otimes D^{s-1}_{t-1,m,n}],\end{aligned}$$

$$\begin{aligned}\beta_3 \gamma D^s_{t,m,n} &= -2b\beta_3\gamma D^s_{t+1,m+1,n+1} + \left(32\pi^2(b + r_1)\beta_1 - 8\pi c\beta_2 - 2c r_1 \beta_3\right) D^s_{t+1,m+1,n+1} \\ &+ 128\pi^2(b + r_1)[P^{(1)} \otimes D^{s-1}_{t,m,n}] - 64\pi c[P^{(2)} \otimes D^{s-1}_{t,m,n}] - 32 c r_1 [P^{(3)} \otimes D^{s-1}_{t,m,n}],\end{aligned}$$

What's new: Exact expansions for the NS sector at N3LO and new N3LO recursions for the Singlet

At LO, there is nothing to truncate in the DGLAP equation, so the logarithmic expansion ansatz valid in the singlet and NS sectors is given by an exact solution with the following recursion relation:

$$\mathbf{S}^0_{n+1} = -\frac{2}{\beta_0}\left[\hat{\mathbf{P}}^{(0)} \otimes \mathbf{S}^0_n\right](x).$$

- At higher orders there is the inclusion of the truncation index κ, giving, at N3LO, the following for $i \geq 3$:

$$\begin{aligned}
\mathbf{S}^i_{n+1}(x) = &-\frac{\beta_1}{4\pi\beta_0}\mathbf{S}^{i-1}_{n+1}(x) - \frac{\beta_2}{16\pi^2\beta_0}\mathbf{S}^{i-2}_{n+1}(x) - \frac{\beta_3}{64\pi^3\beta_0}\mathbf{S}^{i-3}_{n+1}(x) \\
&- i\mathbf{S}^i_n(x) - (i-1)\frac{\beta_1}{4\pi\beta_0}\mathbf{S}^{i-1}_n(x) - (i-2)\frac{\beta_2}{16\pi^2\beta_0}\mathbf{S}^{i-2}_n(x) - (i-3)\frac{\beta_3}{64\pi^3\beta_0}\mathbf{S}^{i-3}_n(x) \\
&- \frac{2}{\beta_0}[\mathbf{P}^{(0)} \otimes \mathbf{S}^i_n](x) - \frac{1}{\pi\beta_0}[\mathbf{P}^{(1)} \otimes \mathbf{S}^{i-1}_n](x) - \frac{1}{2\pi^2\beta_0}[\mathbf{P}^{(2)} \otimes \mathbf{S}^{i-2}_n](x) \\
&- \frac{1}{4\pi^3\beta_0}[\mathbf{P}^{(3)} \otimes \mathbf{S}^{i-3}_n](x).
\end{aligned}$$

- The $i < 3$ pieces are given by the respective lower order contributions that follow a similar structure as above
- These recursion relations generate truncated solutions at arbitrary order κ that are equivalent to those obtained using the U-matrix method (*Cafarella, Corianò, Guzzi, NPB* 2006; *Guzzi*, hep-ph/0612355; *Cafarella, Corianò, Guzzi*, CPC 2008)

CANDIA-V2 Results & Comparisons

Candia-v2 N3LO results for different flavors

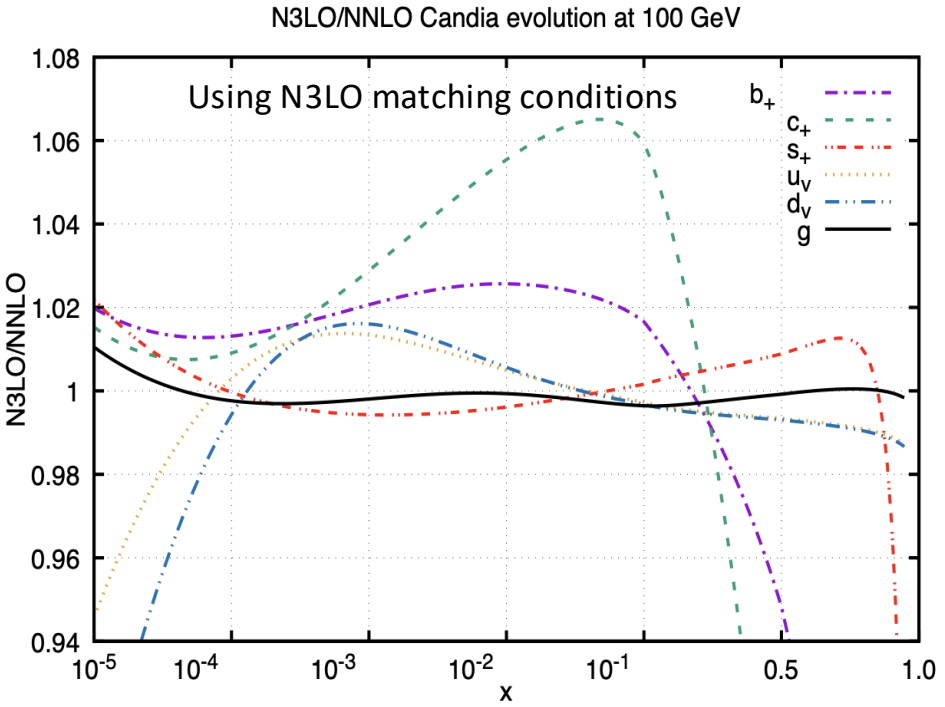
$n_f = 3 \ldots 5,\ \mu_f^2 = 10^4\ \mathrm{GeV}^2$								
x	xuv	xdv	xL_-	xL_+	xs_+	xc_+	xb_+	xg
$\mu_r^2 = 1.0\mu_f^2$								
1e−5	3.017e−3	1.750e−3	1.281e−4	3.546e+1	1.706e+1	1.612e+1	1.315e+1	2.224e+2
1e−4	1.406e−2	8.228e−3	4.883e−4	1.561e+1	7.272e+0	6.785e+0	5.329e+0	8.859e+1
1e−3	6.082e−2	3.507e−2	1.763e−3	6.382e+0	2.779e+0	2.520e+0	1.851e+0	3.034e+1
1e−2	2.336e−1	1.307e−1	5.822e−3	2.267e+0	8.542e−1	7.045e−1	4.623e−1	7.786e+0
1e−1	5.485e−1	2.695e−1	9.996e−3	3.845e−1	1.125e−1	6.830e−2	3.790e−2	8.496e−1
3e−1	3.444e−1	1.276e−1	2.946e−3	3.457e−2	8.887e−3	3.966e−3	2.085e−3	7.870e−2
5e−1	1.179e−1	3.060e−2	3.653e−4	2.321e−3	5.681e−4	2.019e−4	1.138e−4	7.634e−3
7e−1	1.933e−2	2.965e−3	1.285e−5	5.242e−5	1.266e−5	3.402e−6	2.496e−6	3.709e−4
9e−1	3.315e−4	1.674e−5	8.104e−9	2.520e−8	6.644e−9	7.614e−10	1.432e−9	1.172e−6

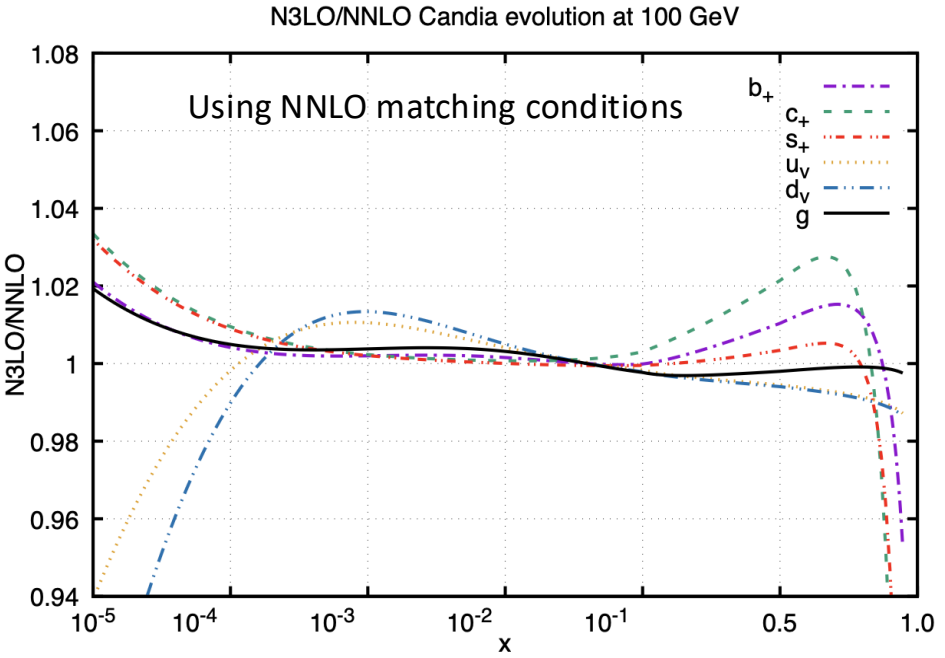
%Error difference: Candia-v2 vs Hoppet-v2

$n_f = 3 \ldots 5,\ \mu_f^2 = 10^4\ \mathrm{GeV}^2$											
x	g	xu	xd	xs	xc	xb	$x\bar{u}$	$x\bar{d}$	$x\bar{s}$	$x\bar{c}$	$x\bar{b}$
$\mu_r^2 = 1.0\mu_f^2$ **per mille level agreement**											
1e−5	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1e−4	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1e−3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1e−2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
1e−1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
3e−1	0.00%	0.00%	0.00%	0.00%	0.01%	0.00%	0.00%	0.00%	0.00%	0.01%	0.00%
5e−1	0.00%	0.00%	0.00%	0.00%	0.02%	0.01%	0.01%	0.00%	0.00%	0.02%	0.01%
7e−1	0.00%	0.00%	0.00%	0.00%	0.06%	0.03%	0.04%	0.00%	0.00%	0.06%	0.03%
9e−1	0.01%	0.00%	0.00%	0.01%	0.00%	0.02%	0.32%	0.01%	0.01%	0.00%	0.02%

With Les Houches PDFs initial conditions 2406.16188 [hep-ph].

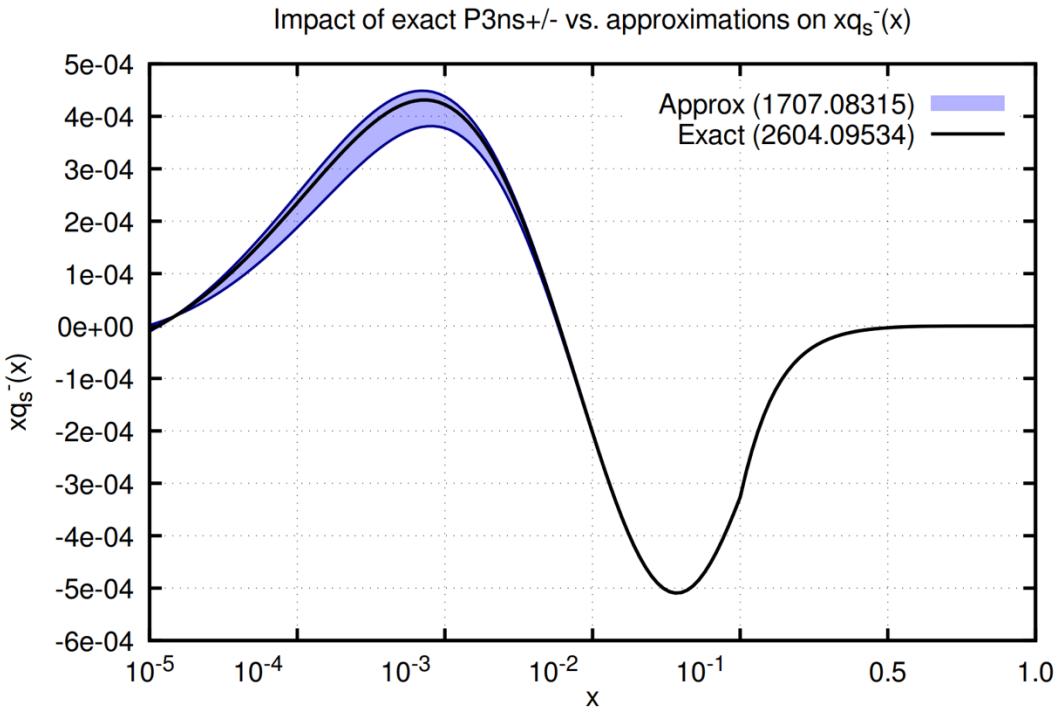
Casey Hampson, undergraduate student at KSU has done most of the work on `Candia-v2` .





Impact from new $P_{NS}^{(3)\pm}$ from Gehrmann et al. 2604.09534

$n_f = 3 \ldots 5\,,\ \mu_f^2 = 10^4\ \mathrm{GeV}^2$								
x	xuv	xdv	xL_-	xL_+	xs_+	xc_+	xb_+	xg
$\mu_r^2 = 1.0\mu_f^2$								
1e−5	9.22e-2%	1.72e-2%	3.92e+0%	2.16e-4%	1.26e-4%	2.62e-4%	1.18e-4%	0.00%
1e−4	9.42e-2%	1.66e-1%	2.23e+0%	2.11e-4%	2.98e-4%	4.52e-4%	1.41e-4%	0.00%
1e−3	5.30e-3%	2.59e-2%	6.10e-1%	1.84e-4%	3.79e-4%	4.54e-4%	1.12e-4%	0.00%
1e−2	3.86e-4%	1.23e-4%	1.47e-3%	1.13e-4%	1.56e-4%	2.30e-4%	7.42e-5%	0.00%
1e−1	2.89e-5%	1.67e-5%	7.13e-4%	6.61e-5%	1.96e-5%	3.80e-5%	6.86e-6%	0.00%
3e−1	1.04e-5%	1.15e-5%	5.88e-5%	4.40e-5%	1.81e-4%	4.19e-4%	1.17e-4%	0.00%
5e−1	5.28e-6%	4.21e-6%	6.21e-4%	6.91e-5%	5.51e-4%	1.54e-3%	2.71e-4%	0.00%
7e−1	2.64e-5%	1.50e-5%	4.62e-3%	6.53e-3%	7.46e-3%	2.77e-2%	9.43e-3%	0.00%
9e−1	1.39e-4%	1.69e-4%	1.12e-1%	9.64e-1%	1.69e+0%	1.57e+1%	1.82e+0%	0.00%



Summary & Conclusions

- Work is in progress toward CT PDF analyses at N3LO in QCD
- DIS cross section is the most important contribution for advancing from CT NNLO to CT N3LO PDFs
- CT default GMVFN scheme, ACOT-χ, now available at pN3LO within current approximations
- On going work on ACOT-χ @pN3LO in DIS with subtraction & residual PDFs
- We showed predictions for $F_{2,c}$ at pN3LO in QCD
- N3LO SFs are now in `Dragonwell`, a C++ platform for precision CT PDF fits.
- CT PDF fits at N3LO: coming soon!
- `Candia-v2` x-space recursion relations at N3LO
- `Candia-v2 C++` code now publicly available (Hampson, M.G., PRD2026)

BACKUP

The Candia method: main idea

LO Ansatz: $f(x,Q^2)^{LO} = \sum_{n=0}^{\infty} \frac{A_n(x)}{n!} \left[\ln\left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)\right]^n$

LO DGLAP

$$\frac{\partial f(x,Q^2)}{\partial \alpha_s} = -\frac{\left(\frac{\alpha_s}{2\pi}\right) P^{(0)}(x)}{\frac{\beta_0}{4\pi}\alpha_s^2} \otimes f(x,Q^2)$$

Recursion relation in the x-space

$$A_{n+1} = -\frac{2}{\beta_0} P^{(0)} \otimes A_n \qquad A_n(x) = \left(-\frac{2}{\beta_0} P^{(0)}\right)^n f(x,\alpha_s(Q_0^2)), \qquad A_0 = f(x,\alpha_s(Q_0^2))$$

Substituting A_n back in the ansatz gives $f(x,\alpha_s) = \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \left[-\frac{2P^{(0)}(x)}{\beta_0}\right]^n \log^n\left(\frac{\alpha_s}{\alpha_0}\right) \right\}_{\otimes} f(x,\alpha_0)$

Which generates the LO solution once it is resummed

$$f(x,\alpha_s) = \left(\frac{\alpha_s}{\alpha_0}\right)^{-\frac{2P^{(0)}(x)}{\beta_0}} \otimes f(x,\alpha_0) = \exp\left\{-\frac{2P^{(0)}(x)}{\beta_0}\log\left(\frac{\alpha_s}{\alpha_0}\right)\right\}_{\otimes} f(x,\alpha_0)$$

This is generalized to all orders: $\mathbf{f}(x,Q^2) = \sum_{n=0}^{\infty} \left\{ \left[\sum_{i=0}^{\kappa} \left(\alpha_s(Q^2)\right)^i \frac{\mathbf{S}_n^i(x)}{n!}\right] \ln^n \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \right\}$ → multiple recursion relations

What's new: Computational Improvements in CANDIA-v2 vs. CANDIA(-v1)

- **Multi-threading**:
  - The independent nature of the non-singlet PDFs allows them to be evolved concurrently
  - Thread creation/management very straightforward with **`std::thread`**
  - The singlet sector cannot do this: but the expensive convolutions/loops can be parallelized
    and even vectorized with C++ execution policies (requires Intel TBB library, available on all systems)

- **Reduced memory footprint**: recursive nature requires information from only the previous (outermost) iteration
  CANDIA-v1 allocated memory for *all*, rather than just two

- **Smarter grid setup and convolution routine**:
  - Grid is split up into three segments [1e-5,0.1], [0.1, 0.9], [0.9, 1.0]
  - Each segment is mapped to [0,1] logarithmically, linearly, and quadratically, respectively,
    with the points in the quadratic interval mapped more densely to 1.0,
    then a standard Gauss-Legendre quadrature routine is used
  - Reduces the number of total grid points from >1500 to <200 for equivalent, if not better, accuracy

- **More readability** in many routines, cleaner API for end users, platform-independent build system Cmake
  (though LHAPDF/GSL dependencies still require UNIX-like systems)