# High-Energy DIS Beyond the Eikonal Approximation

Giovanni Antonio Chirilli

National Centre for Nuclear Research (NCBJ), Warsaw

DIS 2026

Bologna, Italy

4–8 May, 2026

# Motivation

- EIC physics requires a unified description of DIS across the full kinematic range: from the finite-$x_B$ partonic regime to the small-$x_B$ saturation/dipole regime.
- Two well-established but separate descriptions:
  - Finite $x_B$: non-local light-ray operators, collinear PDFs, helicity distributions, DGLAP.
  - Small $x_B$: Wilson lines, dipole picture, BK/B-JIMWLK evolution.
- Sub-eikonal corrections are the natural bridge: they encode the first energy-suppressed interactions beyond the eikonal dipole approximation and carry the quark/helicity operator content absent at leading power.
- Central question: How do the standard light-ray operators $q_f(x_B)$, $\Delta q_f(x_B)$ emerge explicitly from the Wilson-line framework at first sub-eikonal order?

# Two complementary papers

**Paper 1** G.A.C., arXiv:2603.23428 [hep-ph]

- Sub-eikonal DIS in shock-wave formalism $\Rightarrow$ standard quark and helicity operators at $x_B \neq 0$.
- *Non-commutativity* of small-$x_B$ limit and phase-space integration.
- Independent derivation from high-energy limit of non-local OPE.
- High-energy evolution of $Q_1^f$, $Q_5^S$, $Q_5^{NS}$ in dipole form; double-logarithmic analysis; Kirschner–Lipatov exponent with full $C_F$.

**Paper 2** G.A.C., arXiv:2603.30000 [hep-ph]

- Mixed-space (momentum/coordinate) formulation of sub-eikonal DIS.
- Sub-eikonal Feynman rules from LSZ reduction in the shock-wave background.
- Full sub-eikonal corrections to $F_L$, $F_T$, and the helicity asymmetry related to $g_1$, organized in dipole form.
- Divergence structure: $F_L$ finite; $F_T$ and $g_1$ carry single-log divergences absorbed by the one-loop operator evolution.

# References

In recent years, there has been a lot of activity in calculating and/or implementing sub-eikonal corrections:

Balitsky-Tarasov, (2014-2016) Armesto et.al (2014-2026); Altinoluk, Beuf et.al (2020-2026); Kovchegov et.al (2012-2025)

# Notation

$n_1 \cdot n_2 = 1$, $n_1^2 = n_2^2 = 0$; $\not{n}_1 = \gamma^-$, $\not{n}_2 = \gamma^+$.

Finite gauge link along $n_1$:

$$[x^+, y^+]_x \;=\; \mathrm{P}\exp\left\{ ig \int_{y^+}^{x^+} dw^+ A^-(w^+ n_1 + x_\perp) \right\}$$

Infinite Wilson line: $U_x \equiv [\infty n_1, -\infty n_1]_x$

Dipole operator (vanishes as $x_\perp \to y_\perp$):

$$\mathcal{U}_{xy} \;\equiv\; 1 - \frac{1}{N_c} \mathrm{Tr}\{U_x U_y^\dagger\}$$

High-energy power counting under longitudinal boost $\lambda \to \infty$ ($P^- \to \infty$):

$$\bar{\psi} t^a \gamma^- \psi \sim \lambda, \quad \bar{\psi} t^a \gamma^\perp \psi \sim \lambda^0, \quad \bar{\psi} t^a \gamma^+ \psi \sim \lambda^{-1}$$

# Physics question

## Small-$x_B$ / high-energy DIS

$$\gamma^* \to q\bar{q}$$

$$U(x_\perp) = P \exp\left[ig \int dx^+ A^-(x^+, x_\perp)\right]$$

Wilson lines, dipoles, BK/JIMWLK evolution.

## Finite-$x_B$ partonic DIS

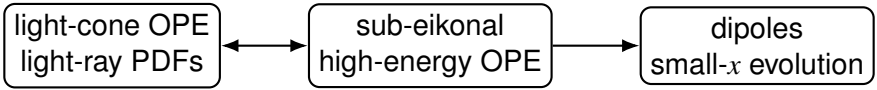
$$\bar{\psi}(0)\, \Gamma\, [0, z^+]\, \psi(z^+)$$

Light-ray operators, PDFs, helicity distributions, DGLAP.

## Question

Where do the finite-$x_B$ quark and helicity light-ray operators appear in the high-energy expansion?

# The bridge is sub-eikonal




- At eikonal accuracy the Wilson line is spin blind.
- Quark and helicity information first appears through sub-eikonal insertions.
- The same operator sector controls finite-$x_B$ light-ray distributions and the $x_B = 0$ high-energy evolution problem.

# Sub-eikonal diagrams

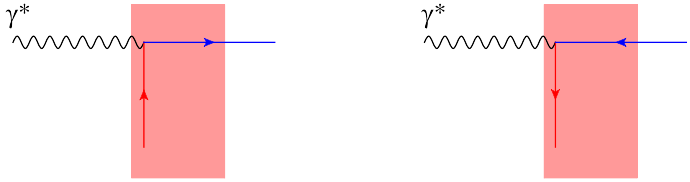
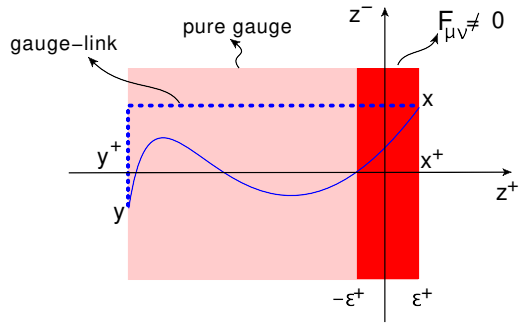



**Figure:** Transition amplitude $\gamma^*(q) \to q(k)$ (right) and $\gamma^*(q) \to \bar{q}(k)$ (left). Blue: quantum fields; red: classical background.

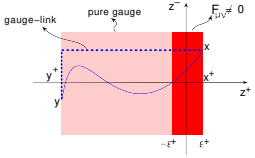
One endpoint of the quark propagator inside the shock-wave, one outside. This is the minimal configuration in which quark operator content enters beyond the strict eikonal dipole approximation.

# Quark propagator with one point in the shock-wave




G.A.C. (2019), JHEP 01, 118

# Quark propagator with one point in the shock-wave




$$\langle x| \frac{i}{\not{p} + i\epsilon} |y\rangle_{\text{Fig.}}$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \frac{i\ \not{k}}{k^+(k^2+i\epsilon)} e^{-ik\cdot(x-y)}$$

$$\times \left[\theta(x^+-y^+)\left([x^+,-\infty]_x\ \not{k}\gamma^+ - g\int_{-\infty}^{x^+} dz^+ \gamma^i\gamma^+ [x^+,z^+]_x F_i{}^-(z^+,x_\perp)[z^+,-\infty]_x\right)\right.$$

$$\left. +\theta(y^+-x^+)\left([x^+,+\infty]_x\ \not{k}\gamma^+ + g\int_{x^+}^{+\infty} dz^+ \gamma^i\gamma^+ [x^+,z^+]_x F_i{}^-(z^+,x_\perp)[z^+,+\infty]_x\right)\right]$$

G.A.C. (2019), JHEP 01, 118

# Differential cross-section: quark TMD

## Transverse photon polarization

$$\frac{d^2\sigma^{\gamma_T^* \to q}}{d^2k} = \frac{\alpha_{\rm em}}{s}\, e_f^2\, q_{1,f}(x_B, k_\perp).$$

Identifying the quark TMD at $x_B \neq 0$:

$$q_{1,f}(x_B, k_\perp) \equiv \frac{1}{4\pi}\int d\Delta^+ d^2r\, e^{-i(k,r)_\perp} e^{i\left(x_B + \frac{k_\perp^2}{s}\right)P^-\Delta^+} \times \langle N(P)|\bar{\psi}_f(0^+, 0_\perp)[0, \infty n_1]\gamma^-[\infty n_1, \Delta^+]_r \psi_f(\Delta^+, r_\perp)|N(P)\rangle$$

G. A. C. (2026)

Note: the kinematic factor $\exp\left(i\frac{k_\perp^2}{s}P^-\Delta^+\right)$ drops if the high-energy limit is taken *before* identifying the operator.

# Non-commutativity: small-$x_B$ vs. phase-space integration

**Naive:** *before* integrating over $k_\perp$, set

$$e^{-i(q^- - k_\perp^2/2q^+)(x^+ - y^+)} \to 1$$

$$\frac{1}{2\pi\delta(0)} \int \frac{d^4k}{(2\pi)^3}\, \delta(k^2)\theta(k^+) \left[ \left| \langle q|\gamma_T^*\rangle \right|^2 + \left| \langle \bar{q}|\gamma_T^*\rangle \right|^2 \right] \Big|_{\text{naive}}$$
$$= \sum_f e^2 e_f^2 \int d^2x\, dx^+ dy^+ \langle \bar{\psi}(y^+, x_\perp)\gamma^- [y^+, x^+]_x \psi(x^+, x_\perp)\rangle + O(\lambda^{-2})$$

$\Rightarrow$ Naive collinear pdf at $x_B = 0$ only:

$$q_f(x_B{=}0) = \frac{1}{4\pi} \int dx^+ \langle N(P)|\bar{\psi}_f(0^+, 0_\perp)\gamma^- [0^+, x^+]\psi_f(x^+, 0_\perp)|N(P)\rangle$$

Altinoluk, Beuf, Swaleha Mulani (2025);
at NLO: Tolga Altinoluk, Beuf, Jules Favrel, Michael Fucilla (2025)

# Non-commutativity: small-$x_B$ vs. phase-space integration

We can complete the phase-space integration without making any small-$x_B$ approximation in the integrand

$$\mathcal{M}^{T,LO}_{\text{Quark}} \equiv \frac{1}{2\pi\delta(0)} \int \bar{d}^4k \bar{\delta}(k^2)\theta(k^+)\left(\left|\langle q(k)|\gamma^*_T(q)\rangle_{\text{Fig.1}a}\right|^2 + \left|\langle q(k)|\gamma^*_T(q)\rangle_{\text{Fig.1}b}\right|^2\right)$$

$$= \sum_f e^2 e_f^2 \int d^2x d^2y\, e^{i(q,\Delta)_\perp} \int dx^+ dy^+ \frac{is}{4\pi P^- \Delta^+} e^{-iq^-\Delta^+ - i\frac{\Delta_\perp^2}{4P^-\Delta^+}s}$$

$$\times \bar{\psi}(y^+, y_\perp)[y^+, +\infty n_1]_y \not{n}_1 [+\infty n_1, x^+]_x \psi(x^+, x_\perp) \text{sign}(\Delta^+) + O(\lambda^{-2})$$

# Non-commutativity: small-$x_B$ vs. phase-space integration

**Correct:** complete phase-space integration *first*, then take high-energy limit
After integrating over $k_\perp$ at fixed $x_B$, the transverse kernel becomes:

$$\frac{is}{4\pi P^- \Delta^+} \exp\left\{ -i \frac{\Delta_\perp^2 s}{4P^- \Delta^+} \right\} \xrightarrow{s\to\infty} \delta^{(2)}(\Delta_\perp)$$

Operator becomes local in $\perp$, nonlocal in $x^+$ $\Rightarrow$ light-ray quark distributions at $x_B \neq 0$:

$$\begin{aligned} q_f(x_B) &= \frac{1}{4\pi}\int_0^{+\infty} dx^+ e^{ix_B P^- x^+} \langle N(P)|\hat{\bar{\psi}}_f(0^+,0_\perp)\gamma^-[0^+,x^+]\hat{\psi}_f(x^+,0_\perp)|N(P)\rangle \\ \bar{q}_f(x_B) &= -\frac{1}{4\pi}\int_0^{+\infty} dx^+ e^{-ix_B P^- x^+} \langle N(P)|\hat{\bar{\psi}}_f(x^+,0_\perp)\gamma^-[x^+,0^+]\hat{\psi}_f(0^+,0_\perp)|N(P)\rangle \end{aligned}$$

$\Rightarrow$ The small-$x_B$ limit and the phase-space integration do **not** commute.

G.A.C. (2026)

## Helicity distributions

The asymmetry $(\varepsilon_+^i \varepsilon_+^{*j} - \varepsilon_-^i \varepsilon_-^{*j})$ gives access to the helicity distribution.
Repeating the analysis with the correct phase-space ordering:

$$\sigma_A^{\gamma^* P} = \frac{2\pi}{s} \alpha_{\rm em} \sum_f e_f^2 \Big( \Delta q_f(x_B) + \Delta \bar{q}_f(x_B) \Big) + O(\lambda^{-2})$$

with helicity distributions at $x_B \neq 0$:

$$\begin{aligned}
\Delta q_f(x_B) &= \frac{1}{4\pi} \int_0^{+\infty} dx^+ e^{i x_B P^- x^+} \langle N(P) | \hat{\bar{\psi}}_f(0^+, 0_\perp) \gamma^5 \gamma^- [0^+, x^+] \hat{\psi}_f(x^+, 0_\perp) | N(P) \rangle \\
\Delta \bar{q}_f(x_B) &= -\frac{1}{4\pi} \int_0^{+\infty} dx^+ e^{-i x_B P^- x^+} \langle N(P) | \hat{\bar{\psi}}_f(x^+, 0_\perp) \gamma^5 \gamma^- [x^+, 0^+] \hat{\psi}_f(0^+, 0_\perp) | N(P) \rangle
\end{aligned}$$

The first sub-eikonal correction already reconstructs, in the inclusive limit, the standard nonlocal quark and helicity distributions at $x_B \neq 0$.

G.A.C. (2026)

# The two operations do not commute

$$\lim_{\text{HE}} \int d\Phi \neq \int d\Phi \lim_{\text{HE}}$$

## Approximate too early

Set the phase to one before the transverse phase-space integration:

$$e^{-i\left(q^- - k_\perp^2/(2q^+)\right)\Delta^+} \to 1.$$

Then one obtains only the naive collinear object at

$$x_B = 0.$$

## Correct inclusive limit

Complete the phase-space integration first and keep

$$e^{-iq^-\Delta^+} = e^{ix_B P^- \Delta^+}.$$

Then the finite-$x_B$ light-ray distributions are reconstructed.

# From non-local OPE to High-energy OPE

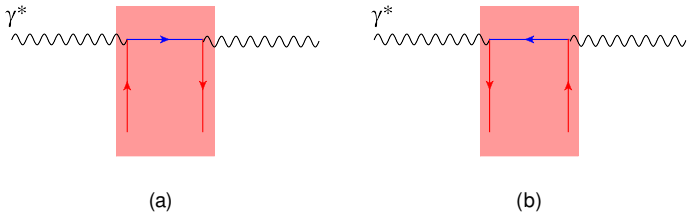



**Figure:** Diagrams contributing to $T_{\mu\nu}$

Starting point: the non-local OPE propagator in the background gluon field

$$\langle \mathrm{T}\psi_q(x)\bar{\psi}_q(y)\rangle_A = \frac{i(\not{x}-\not{y})}{2\pi^2[(x-y)^2-i\epsilon]^2}[x,y]_c + O((x-y)^{-2})$$

with *straight* gauge link $[x,y]_c = \mathrm{P}\exp\left\{ig\int_y^x dz^\mu A^c_\mu(z)\right\}$.

Under a large longitudinal boost, the straight link reduces to a light-cone Wilson line at fixed $x_\perp$.

# From non-local OPE to high-energy OPE

Consider transverse polarization; use the boost reduction $[x, y] \to [x^+, y^+]_x$

take $q^- = -\frac{Q^2}{2q^+} = -x_B P^-$, then integrate over $\Delta^-$:

$$\frac{1}{2}\sum_{\lambda=\pm 1} \varepsilon^i_\lambda \varepsilon^{*j}_\lambda T_{ij} = \lim_{\epsilon^+\to 0} \frac{q^+}{4\pi} \frac{\sum_f e_f^2}{\Delta^+} \int dx^+ dy^+ \theta(x^+ - y^+) d^2x d^2y\, e^{i\frac{q^+\Delta_\perp^2}{2\Delta^+} - i\frac{Q^2}{2q^+}\Delta^+}$$
$$\times \left[ \langle N(P)|\hat{\bar{\psi}}(x^+, x_\perp)\gamma^- [x^+, y^+]_x \hat{\psi}(y^+, y_\perp)|N(P)\rangle + \cdots \right] + O(\lambda^{-2})$$

High-energy limit: $\frac{-is}{4\pi P^- \Delta^+} \exp\left\{ i\frac{s\Delta_\perp^2}{4P^-\Delta^+} \right\} \xrightarrow{s\to\infty} \delta^{(2)}(\Delta_\perp)$

# From non-local OPE to high-energy OPE

$$\frac{1}{2}\sum_{\lambda=\pm1}\varepsilon^i_\lambda(q)\varepsilon^j_\lambda(q)\,\mathrm{Im}\,T_{ij}=\frac{2\pi}{s}\sum_f e_f^2\big(q_f(x_B)+\bar{q}_f(x_B)\big)+O(\lambda^{-2})$$

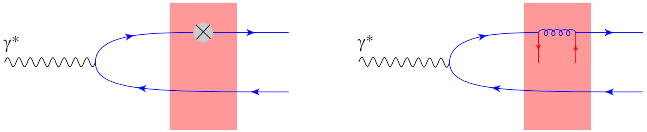
The optical theorem then gives:

$$\frac{1}{2}\sum_{\lambda=\pm1}\varepsilon^i_\lambda(q)\varepsilon^j_\lambda(q)W_{ij}=\frac{1}{2\pi}\frac{1}{4\pi\alpha_{\mathrm{em}}}\Big[4\pi\alpha_{\mathrm{em}}\sum_f e_f^2\big(q_f(x_B)+\bar{q}_f(x_B)\big)\Big]$$

Both methods—shock-wave and non-local OPE—yield the same result. This establishes an explicit *operator-level bridge* between the two formalisms.

G.A.C. (2026)

# Mixed-space formulation and sub-eikonal operator basis




The sub-eikonal quark propagator generates operator insertions localized at $z^+ = 0$:

Gluonic operators on the shock-wave:

- Chromomagnetic: $\mathcal{F}(z_\perp) = ig\frac{s}{4}\int dz^+ [\infty n_1, z^+]_z \, \epsilon^{ij} F_{ij}(z^+, z_\perp)[z^+, -\infty n_1]_z$ Parity odd
- Chromoelectric composite: $\mathcal{G}_2(z_\perp) \equiv \mathcal{F}_2(z_\perp) - \mathcal{F}_2'(z_\perp)$ (built from $F_i{}^-$) Parity even

Quark operators on the shock-wave:

$$Q_{ij}(x_\perp) = g^2\frac{s}{2}\int dz^+ \int^{z^+} dz'^+ \, [\infty n_1, z^+]_x t^a \psi(z^+, x_\perp)[z^+, z'^+]^{ab} \bar{\psi}(z'^+, x_\perp) t^b [z'^+, -\infty n_1]_{x\,ij}$$

All operators vanish as the dipole size $\to 0$ when placed in dipole combinations.

G.A.C. (2026)

# Sub-eikonal corrections: $F_L$ and $F_T$

Define dipole-type combinations (all vanish as $x_\perp \to y_\perp$):

Parity even $\quad \mathcal{Q}^f_{1,xy} \equiv Q^f_{1,x}\mathcal{U}_{xy}, \quad \Psi^f_{1,xy} \equiv \mathrm{Tr}\{\tilde{Q}^f_{1,x}(U^\dagger_x - U^\dagger_y)\}, \quad \mathcal{G}_2(x,y) \equiv \mathrm{Tr}\{(U^\dagger_x - U^\dagger_y)\mathcal{G}_{2,x}\}$

**$F_L$ at sub-eikonal order - finite, no UV divergence**

$$F_L = \frac{4Q^2N_c\alpha_{\rm em}}{\pi^3}\sum_f e_f^2 \int_0^1 dz\, z^2\bar{z}^2 \int d^2x d^2y\, \left|K_0(\bar{Q}|x-y|)\right|^2 \left[\mathcal{U}_{xy} + \frac{\sqrt{s/2}}{4\,z\,\bar{z}\,s^2N_c}\left(N_c\mathcal{Q}^f_{1\,xy} - \tfrac{1}{N_c}\Psi^f_{1\,xy} + 2\mathcal{G}_{2\,xy} + \text{h.c.}\right)\right]$$

**$F_T$ at sub-eikonal order - single-log UV divergence**

$$F_T = \frac{Q^2N_c\alpha_{\rm em}}{\pi^3}\sum_f e_f^2 \int_0^1 dz\, z\bar{z}(z^2+\bar{z}^2) \int d^2x d^2y\, \left|K_1(\bar{Q}|x-y|)\right|^2 \left[\mathcal{U}_{xy} + \frac{\sqrt{s/2}}{4\,z\,\bar{z}\,s^2N_c}\left(N_c\mathcal{Q}^f_{1xy} - \tfrac{1}{N_c}\Psi^f_{1\,xy} + 2\mathcal{G}_{2\,xy} + \text{h.c.}\right)\right]$$

Eikonal $\to$ known

Sub-eikonal $\to$ new G.A.C. (2026)

# Operators parity even

$$\tilde{Q}^{f}_{1ij}(x_\perp) \equiv g^2 \frac{s}{2} \int_{-\infty}^{+\infty} dz^+ \int_{-\infty}^{z^+} dz'^+ \left( [\infty n_1, z^+]_x \mathrm{tr}\{ \psi_f(z^+, x_\perp) \bar{\psi}_f(z'^+, x_\perp) \, \gamma^- \} [z'^+, -\infty n_1]_x \right)_{ij}$$

$$\mathcal{G}_2(z_\perp) \equiv i g^2 s \int_{-\infty}^{+\infty} d\omega^+ \int_{\omega^+}^{+\infty} d\omega'^+ (\omega^+ - \omega'^+) [\infty n_1, \omega'^+]_z F^{i-}(\omega'^+, z_\perp) [\omega'^+, \omega^+]_z F_i{}^-(\omega^+, z_\perp) [\omega^+, -\infty n_1]_z$$

# Sub-eikonal correction to $g_1$: Singlet case

Define the helicity dipole-type operators:

Parity odd $\quad \mathcal{Q}^f_{5\,xy} \equiv \mathcal{Q}^f_{5\,x}\,\mathcal{U}_{xy}, \quad \Psi^f_{5,xy} \equiv \mathrm{Tr}\{\tilde{\mathcal{Q}}^f_{5\,x}(U^\dagger_x - U^\dagger_y)\}, \quad \mathcal{F}_{xy} \equiv \mathrm{Tr}\{U^\dagger_x \mathcal{F}_y\}$

**$g_1$ at sub-eikonal order - single-log UV divergence**

$$g_1 = \frac{Q^2\alpha_{\rm em}}{\pi^3}\sum_f e_f^2 \int_0^1 dz\,(z-\bar{z})^2 \int d^2x d^2y |K_1(\bar{Q}|x-y|)|^2 \frac{\sqrt{s/2}}{s^2}\Big[2\mathcal{F}_{xy} + N_c\,\mathcal{Q}^f_{5\,xy} - \frac{1}{N_c}\Psi^f_{5\,xy} - \text{h.c.}\Big]$$

G.A.C. (2026)

- The gluon operator $\mathcal{F}(x,y)$ enters in the flavor-singlet sector only.
- The single-log divergences in $F_T$ and $g_1$ are precisely those generated by the one-loop rapidity evolution of $\mathcal{Q}^f_1$ and $\mathcal{Q}^f_5$ respectively.

Sub-eikonal corrections:

- Quark propagator in the background of quark and gluon fields at sub-eikonal level
- Gluon propagator in the background of quark and gluon fields at sub-eikonal level

G.A.C. (2019), JHEP 01, 118

# Operators parity odd

$$\tilde{Q}^f_{5ij}(x_\perp) \equiv g^2 \frac{s}{2} \int_{-\infty}^{\infty} dz^+ \int_{-\infty}^{z^+} dz'^+ \left( [\infty n_1, z^+]_x \mathrm{tr}\{\psi_f(z^+, x_\perp) \bar{\psi}_f(z'^+, x_\perp) \gamma^5 \gamma^-\} [z'^+, -\infty n_1]_x \right)_{ij}$$

$$\mathcal{F}(z_\perp) = ig \frac{s}{4} \int dz^+ [\infty n_1, z^+]_z \, \epsilon^{ij} F_{ij}(z^+, z_\perp) [z^+, -\infty n_1]_z$$

# Sub-eikonal correction to $g_1$: Non-singlet case

Define the helicity dipole-type operators:

$$Q_5^{\rm NS}(x,y) \equiv \sum_f c_f\, Q_5^f(x,y), \qquad \Psi_5^{\rm NS}(x,y) \equiv \sum_f c_f\, \Psi_5^f(x,y)$$

with coefficients $c_f$ satisfying $\sum_f c_f = 0$

**$g_1$ at sub-eikonal order - single-log UV divergence**

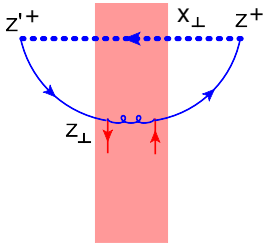
$$g_1^{\rm NS}(Q^2) = \frac{Q^2\,\alpha_{\rm em}}{\pi^3}\int_0^1 dz\,(z-\bar z)^2 \int d^2x\, d^2y\, \left|K_1(\bar Q|x-y|)\right|^2 \frac{\sqrt{s/2}}{s^2}\left(N_c\, Q_{5\,xy}^{\rm NS} - \frac{1}{N_c}\,\Psi_{5\,xy}^{\rm NS} - \text{h.c.}\right)$$

- In the non-singlet projection $g_1^{\rm NS}$, only $\mathcal{Q}_5^{\rm NS}$ and $\Psi_5^{\rm NS}$ remain.

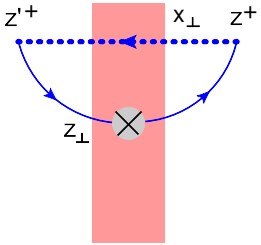
G.A.C. (2026)

# High-energy evolution in dipole form

Diagrams for the evolution of $Q_1$ and $Q_5$ operators




a)




b)

# High-energy evolution in dipole form

The $x_B = 0$ operators $Q_1^f(x_\perp)$, $Q_5^S(x_\perp)$, $Q_5^{\rm NS}(x_\perp)$ obey:

$$\begin{aligned}
\frac{d}{d\eta} Q_{1x}^f &= \frac{\alpha_s}{4\pi^2} \int \frac{d^2z}{(x-z)_\perp^2} \left[ 2C_F Q_{1z}^f - N_c Q_{1,zx}^f + \frac{1}{N_c} \Psi_{1,zx}^f \right] \\
\frac{d}{d\eta} Q_{5x}^S &= \frac{\alpha_s}{4\pi^2} \int \frac{d^2z}{(x-z)_\perp^2} \left[ 2C_F Q_{5z}^S - N_c Q_{5,zx}^S + \frac{1}{N_c} \Psi_{5,zx}^S + 2N_f \mathcal{F}_{xz} \right] \\
\frac{d}{d\eta} Q_{5x}^{\rm NS} &= \frac{\alpha_s}{4\pi^2} \int \frac{d^2z}{(x-z)_\perp^2} \left[ 2C_F Q_{5z}^{\rm NS} - N_c Q_{5,zx}^{\rm NS} + \frac{1}{N_c} \Psi_{5,zx}^{\rm NS} \right]
\end{aligned}$$

Key features:

- All bilocal combinations vanish as $x \to y \Rightarrow$ small-dipole behavior manifest.
- Singlet $Q_5^S$ mixes with the gluon operator $\mathcal{F}$; non-singlet $Q_5^{\rm NS}$ does not.
- The log divergences of the sub-eikonal dipole cross sections are absorbed by the $2C_F$ terms of these kernels.

G.A.C. (2021), JHEP 06, 096; G.A.C. (2026)

# Double-logarithmic approximation

Project onto the local ladder sector ($z_\perp \to x_\perp$; dipole combinations vanish):

$$\frac{d}{d\eta}\mathcal{Q}_1^f(x_\perp,\eta) = \frac{\bar{\alpha}_s C_F}{2\pi^2}\int \frac{d^2z}{(x-z)_\perp^2}\,\mathcal{Q}_1^f(z_\perp,\eta), \quad \bar{\alpha}_s \equiv \frac{\alpha_s C_F}{2\pi}$$

Same equation for $\mathcal{Q}_5^{\rm NS}$ (non-singlet).

**Independent $\perp$ phase space**

Introduce $\rho \equiv \ln(L_\perp^2/r_\perp^2)$ $\eta = \ln(1/x_B)$

$$\frac{\partial^2 \mathcal{Q}_1}{\partial\eta\,\partial\rho} = \bar{\alpha}_s\,\mathcal{Q}_1$$

Bessel solution:

$$\mathcal{Q}_1 = \mathcal{Q}_1^{(0)} I_0\big(2\sqrt{\bar{\alpha}_s\,\eta\rho}\big)$$

Resums

$$\left(\alpha_s \ln\frac{1}{x_B}\ln Q_\perp^2\right)^n$$

*mixed* longitudinal–transverse DLA.

**Longitudinally constrained $\perp$ phase space**

Kinematic constraint: $k_\perp^2 \lesssim \dfrac{k^+}{q^+}s = 2P^-k^+$

Hence $\rho_{\max}(k^+) \sim \ln\dfrac{2P^-k^+}{Q^2} = \ln\dfrac{k^+}{x_B q^+}$.

The second logarithm is converted into an energy logarithm:

$$\int_{x_B q^+}^{q^+}\frac{dk^+}{k^+}\ln\frac{k^+}{x_B q^+} = \frac{1}{2}\ln^2\frac{1}{x_B}$$

Evaluate the same mixed-DLA solution on the symmetric trajectory:

$$\rho \simeq \eta \;\Rightarrow\; \mathcal{Q}_1(\eta,\eta) \simeq \mathcal{Q}_1^{(0)}\frac{e^{2\sqrt{\bar{\alpha}_s}\,\eta}}{\sqrt{4\pi\sqrt{\bar{\alpha}_s}\,\eta}}.$$

Resums $\left(\alpha_s \ln^2\dfrac{1}{x_B}\right)^n$ genuine double log of energy

Fixed-coupling Kirschner–Lipatov exponent with full finite-$N_c$ color factor:

$$\Delta = 2\sqrt{\bar{\alpha}_s} = \sqrt{2\alpha_s C_F/\pi}$$

G.A.C. (2026)

# Conclusions

- Operator bridge: first sub-eikonal correction reconstructs, in the inclusive limit, the standard nonlocal quark and helicity distributions $q_f(x_B)$, $\Delta q_f(x_B)$ at $x_B \neq 0$. Established from two independent routes: shock-wave formalism and high-energy limit of the non-local OPE.
- Non-commutativity: small-$x_B$ limit and full phase-space integration do *not* commute. Taking the limit prematurely yields only the naive $x_B = 0$ operator.
- Sub-eikonal dipole structure functions: $F_L$, $F_T$, $g_1$ expressed in terms of dipole-type operators that vanish as the dipole size goes to zero. $F_L$ is UV finite; $F_T$ and $g_1$ carry single-log divergences absorbed by the one-loop operator evolution.
- Double-logarithmic evolution: in the local ladder sector, $Q_1^f$ and $Q_5^{\mathrm{NS}}$ obey a mixed DLA equation (Bessel solution). When transverse phase space is tied to longitudinal ordering, the second logarithm converts into an energy logarithm, recovering the fixed-coupling Kirschner–Lipatov exponent $\Delta = \sqrt{2\alpha_s C_F/\pi}$ with full $C_F$.

**Next steps:** closed evolution system for the full dipole operator basis; matching to Bartels–Ermolaev–Ryskin beyond the ladder; extension to finite-$x_B$ evolution and EIC phenomenology.

$$\mathcal{G}_2(z_\perp) \equiv ig^2 s \int_{-\infty}^{+\infty} d\omega^+ \int_{\omega^+}^{+\infty} d\omega'^+ (\omega^+ - \omega'^+)[\infty n_1, \omega'^+]_z F^{i-}(\omega'^+, z_\perp)[\omega'^+, \omega^+]_z F_i{}^-(\omega^+, z_\perp)[\omega^+, -\infty n_1]_z$$

$$G(z_\perp) \equiv \mathrm{Tr}\{\mathcal{G}_{2z} U_z^\dagger\} = \frac{i s g^2}{2} \int_{-\infty}^{+\infty} d\omega^+ \int_{\omega^+}^{+\infty} d\omega'^+ (\omega'^+ - \omega^+) F_i^{a,-}(\omega'^+, z_\perp)[\omega'^+, \omega^+]_z^{ab} F_i^{b,-}(\omega^+, z_\perp)$$

# Quark propagator at sub-eikonal level (background of gluon field)

G.A.C. (2019), JHEP 01, 118

$$\langle \mathrm{T}\{\psi(x)\bar{\psi}(y)\}\rangle_A = \left[\int_0^{+\infty}\frac{đ p^+}{4(p^+)^2}\theta(x^+-y^+) - \int_{-\infty}^{0}\frac{đ p^+}{4(p^+)^2}\theta(y^+-x^+)\right] e^{-ip^+(x^- - y^-)}\langle x_\perp| e^{-i\frac{\hat{p}_\perp^2}{2p^+}x^+}$$
$$\times\left\{\hat{\not{p}}\gamma^+[x^+,y^+]\hat{\not{p}} + \hat{\not{p}}\gamma^+\hat{\mathcal{O}}_1(x^+,y^+;p_\perp)\hat{\not{p}} + \hat{\not{p}}\gamma^+\frac{1}{2}\hat{\mathcal{O}}_2(x^+,y^+;p_\perp) - \frac{1}{2}\hat{\mathcal{O}}_2(x^+,y^+;p_\perp)\not{n}_2\hat{\not{p}}\right\} e^{i\frac{\hat{p}_\perp^2}{2p^+}y^+}|y_\perp\rangle$$
$$+O(\lambda^{-2})\,.$$

$$\hat{\mathcal{O}}_1(x^+,y^+;p_\perp) = \frac{ig}{2p^+}\int_{y^+}^{x^+}d\omega^+\Big([x^+,\omega^+]\frac{1}{2}\sigma^{ij}F_{ij}[\omega^+,y^+] + \{\hat{p}^i,[x^+,\omega^+]\ \omega^+ F_i - [\omega^+,y^+]\}$$
$$+g\int_{\omega^+}^{x^+}d\omega'^+\big(\omega^+-\omega'^+\big)[x^+,\omega'^+]F^{i-}[\omega'^+,\omega^+]\,F_i - [\omega^+,y^+]\Big)\,,$$

$$\hat{\mathcal{O}}_2(x^+,y^+;p_\perp) = \frac{ig}{2p^+}\int_{y^+}^{x^+}d\omega^+\Big[\{\hat{p}^k,[x^+,\omega^+]iF_{kj}\gamma^j[\omega^+,y^+]\} + [x^+,\omega^+]iF_{kj}\gamma^j(i\mathcal{D}^k[\omega^+,y^+]) - (i\mathcal{D}^k[x^+,\omega^+])iF_{kj}\gamma^j[\omega^+,y^+]$$
$$-[x^+,\omega^+]\,i\,F^{-+}(i\mathcal{D}_\perp[\omega^+,y^+]) + (i\mathcal{D}_\perp[x^+,\omega^+])\,i\,F^{-+}[\omega^+,y^+] + (\hat{p}^+\gamma^- - \hat{\not{p}}_\perp)[x^+,\omega^+]\,i\,F^{-+}[\omega^+,y^+]\Big]\,,$$

# Quark propagator in the background of quark fields

G.A.C. (2019), JHEP 01, 118

$$\langle \mathrm{T}\{\psi(x)\bar{\psi}(y)\}\rangle_{\psi,\bar{\psi}} = \frac{1}{s}\int d^4z\delta(z^+)\langle x|\frac{i\not{p}}{p^+(p^2+i\epsilon)}|z\rangle \times\gamma_\perp^\mu\Big(Q(z_\perp)\theta(x^+)\theta(-y^+)-\tilde{Q}(z_\perp)\theta(-x^+)\theta(y^+)\Big)\gamma_\mu^\perp\langle z|\frac{i\not{p}}{p^2+i\epsilon}|y\rangle+O(\lambda^{-2})$$

$$Q_{ij}^{\alpha\beta}(x_\perp)\equiv g^2\frac{s}{2}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\Big([\infty n_1,z^+]_x t^a\psi^\alpha(z^+,x_\perp)[z^+,z'^+]_x^{ab}\bar{\psi}^\beta(z'^+,x_\perp)t^b[z'^+,-\infty n_1]_x\Big)_{ij}$$

$$\tilde{Q}_{ij}^{\alpha\beta}(x_\perp)\equiv g^2\frac{s}{2}\int_{-\infty}^{+\infty}dz^+\int_{z^+}^{+\infty}dz'^+\Big([-\infty n_1,z^+]_x t^a\psi^\alpha(z^+,x_\perp)[z^+,z'^+]_x^{ab}\bar{\psi}^\beta(z'^+,x_\perp)t^b[z'^+,\infty n_1]_x\Big)_{ij}$$