Planar three-loop four-point master integrals with two off-shell legs

Dhimiter Canko

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In collaboration with Mattia Pozzoli

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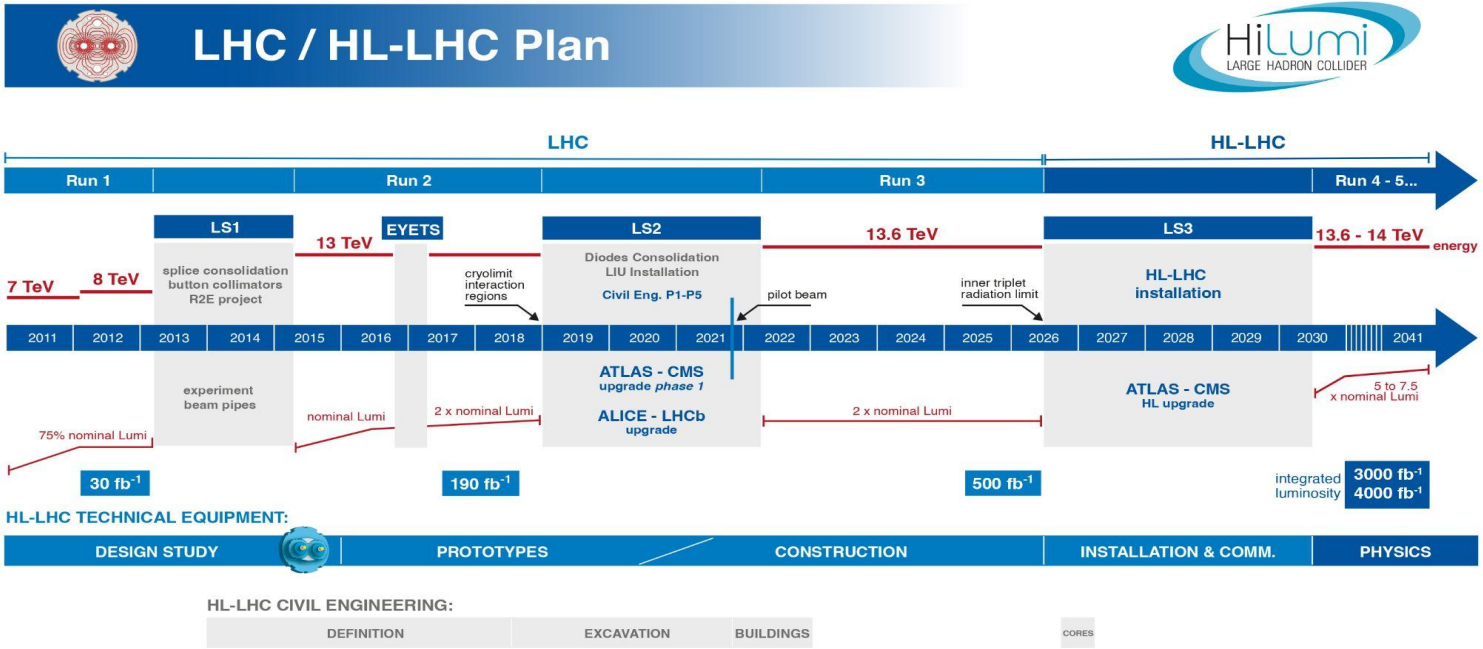
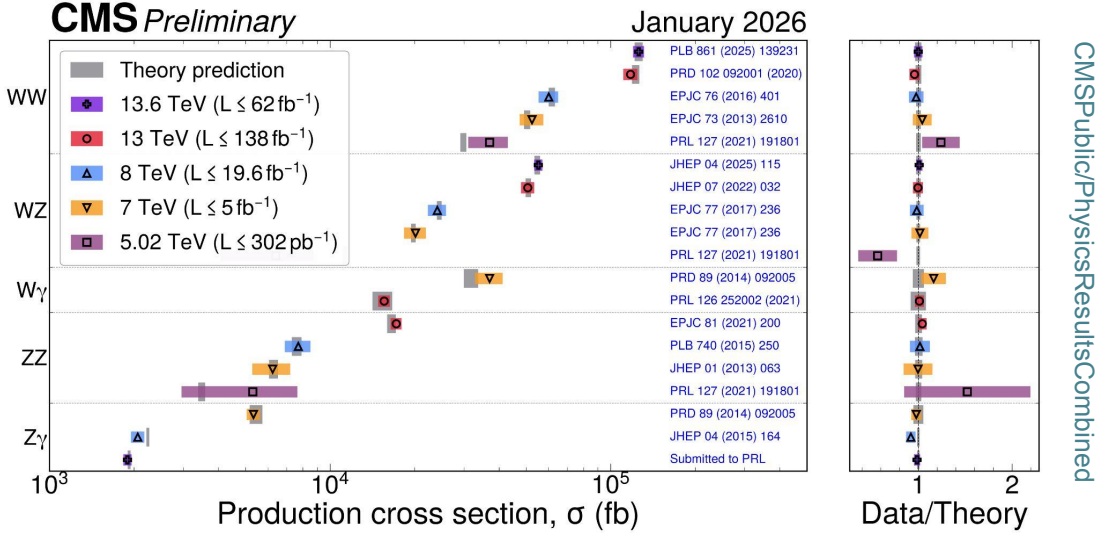
Collider phenomenology



- Observables with **percent accuracy** → NNLO and N3LO predictions demanded

Why diboson production @ N3LO QCD

$$pp \to V_1 V_2 \to (l_1 \bar{l}_1')(l_2 \bar{l}_2') \quad \text{with} \quad V_{1/2} = W^{\pm}, Z, \gamma^*$$



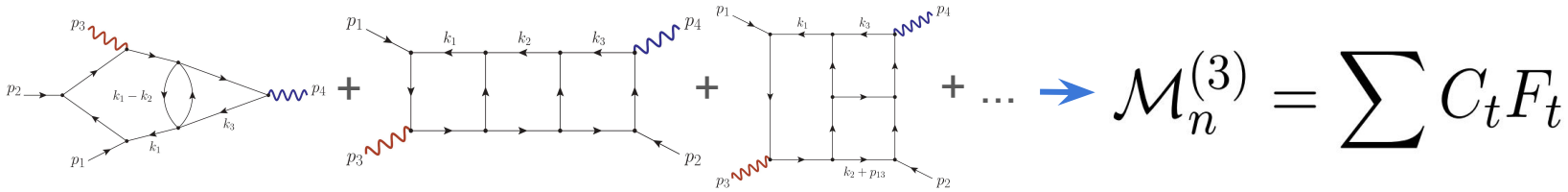
- Abundant production with **clean signatures** → detailed tests of QCD dynamics
- Powerful **probes of EW sector** (sensitive to trilinear gauge couplings)
- Major **backgrounds to Higgs-boson production** measurements

From cross section to Feynman integrals

- QCD amplitudes expanded in a_s: **3-loop contributions to cross section @ N3LO**

$$d\hat{\sigma}^{N3LO} \sim \int d\Phi^{(n-2)} \left(2\,\mathrm{Re}\left[\mathcal{M}_n^{(3)} (\mathcal{M}_n^{(0)})^* \right] \right) + \dots$$

- Amplitude is a sum of Feynman diagrams → Feynman integrals (FIs)



$$+ \quad + \quad + \dots \rightarrow \mathcal{M}_n^{(3)} = \sum C_t F_t$$

where

$$F_t = \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \mathrm{d}^d k_3 \prod_{i \in t} \frac{1}{D_i^{a_i}} \quad \text{and} \quad D_i = (k_i + p_i)^2 - m_i^2$$

From Feynman integrals to master integrals

- FIs satisfy integration-by-parts (IBPs) [Chetyrkin & Tkachov, ’81; Laporta ’04]

$$\int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \mathrm{d}^d k_3 \frac{\partial}{\partial k_l^\mu} \prod_{i \in t} \frac{1}{D_i^{a_i}} = 0 \quad \rightarrow \quad \{\mathcal{J}_i\}$$ Master integrals

- Amplitude written as a sum of Master integrals (MIs) and rational coefficients

$$\mathcal{M}_n^{(3)} = \sum \tilde{C}_i \mathcal{J}_i$$

- **Bottlenecks** for VV-production 3-loop amplitudes: (1) **proliferation of MIs**
  (2) **enormous increase of IBP system**

Kinematics and leading color approximation

$$g/q(p_1) + g/\bar{q}(p_2) \;\to\; V(-p_3)\, V'(-p_4)$$

- **Four-particle** scattering + **two** different external **masses**: **4 independent invariants**

$$s_{12} = (p_1 + p_2)^2, \quad s_{23} = (p_2 + p_3)^2, \quad m_3^2 = p_3^2, \quad m_4^2 = p_4^2$$

This talk: **Computation of MIs contributing in the leading color approximation** (LC)

- **LC@3Loop**: From the color decomposition of amplitudes keep terms proportional $\{N_c^3, N_c^2 n_f, N_c n_f^2, n_f^3\}$

$$M_n^{(3)} = N_c^3 A_{\{3,0\}}^{(3)} + N_c^2 n_f A_{\{2,1\}}^{(3)} + N_c n_f^2 A_{\{1,2\}}^{(3)} + n_f^3 A_{\{0,3\}}^{(3)} + \dots$$

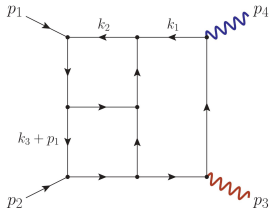
- **LC** approximation → only **planar Feynman integrals** (graphs)!

Computation of MIs: integral families

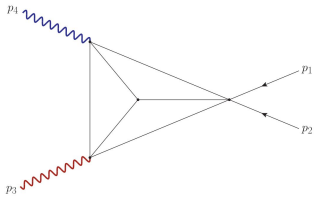
- Group FIs into **integral families** according to loop and kinematic structure

$$G^{(F_x)}_{\vec{\nu}} = \int \prod_{l=1}^{3} \frac{\mathrm{d}^d\, k_l\, e^{\epsilon\gamma_E}}{\mathrm{i}\pi^{\frac{d}{2}}} \prod_{j=1}^{15} \frac{1}{D^{\nu_j}_{(F_x),j}}$$

- Some ν_j are always negative: **irreducible scalar products** (ISP) → 5 in our case
- The set of ν_j that is positive defines the **sector**



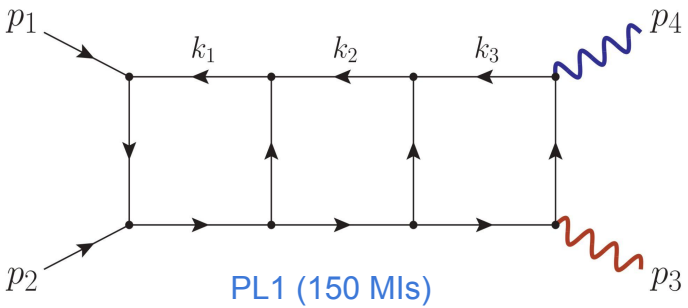
→ top sector



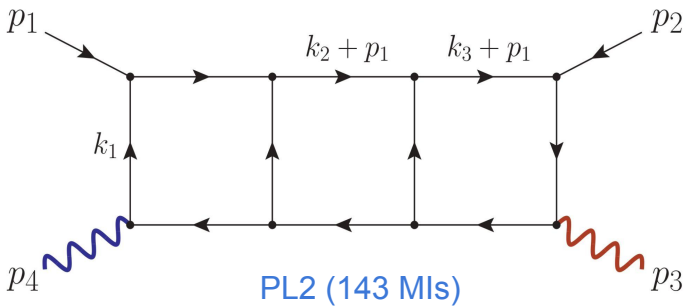
→ lower sector

All planar families: {RL1, RL2, PL1, PL2, PL3, PT1, PT2, PT3, PT4}

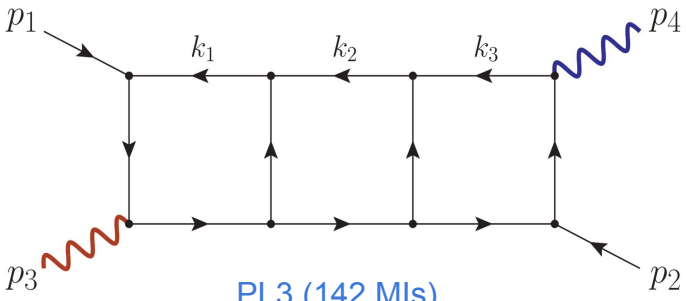
Ladderbox families



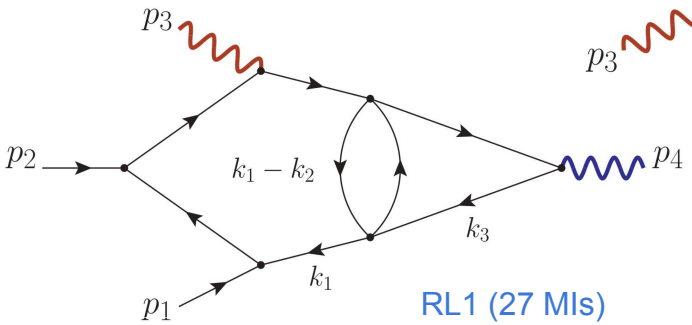
PL1 (150 MIs)



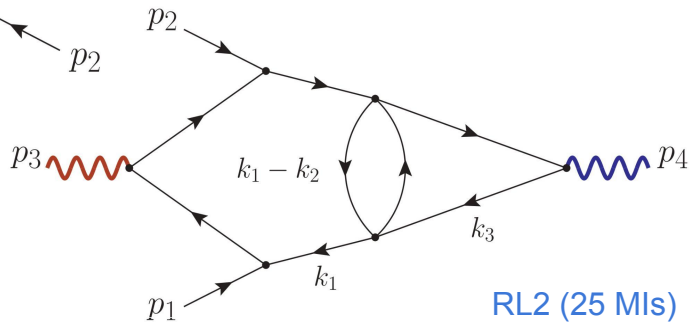
PL2 (143 MIs)



PL3 (142 MIs)

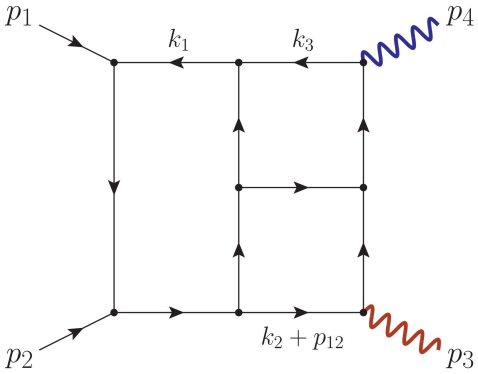


RL1 (27 MIs)

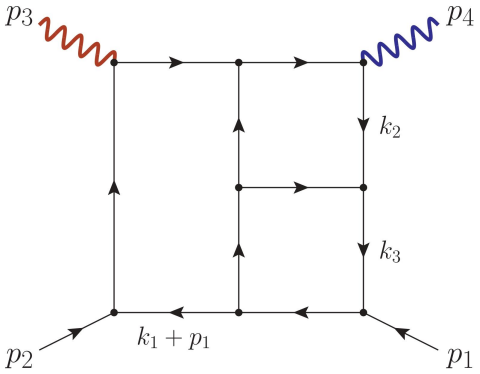


RL2 (25 MIs)

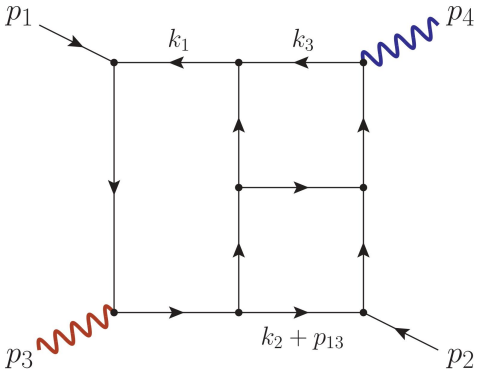
Tenniscourt families



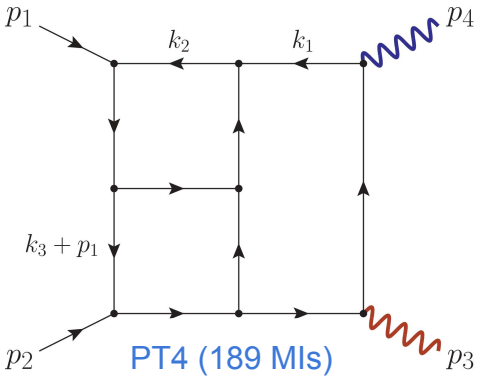
PT1 (344 MIs)



PT2 (252 MIs)



PT3 (240 MIs)



PT4 (189 MIs)

Computation of MIs: Differential Equations

- MIs fulfill **differential equations** (DEs) [Kotikov, '91; Bern et al, '94; Gehrmann et al, '00]

$$\partial_\xi \vec{G}(\vec{s}) = B_\xi(\vec{s};\epsilon)\cdot\vec{G}(\vec{s}) \longrightarrow$$

MIs solved up to desired order on ε

- Basis not unique: Good choice makes DEs easier → **canonical dlog- DEs** [Henn, '14]

$$\mathrm{d}\vec{I}(\vec{s}) = \varepsilon\,\mathrm{d}\tilde{A}(\vec{s})\cdot\vec{I}(\vec{s}) \quad \text{with} \quad \mathrm{d}\tilde{A}(\vec{s}) = \sum a_i\,\mathrm{d}\log w_i(\vec{s})$$

a_i: constant matrices; w_i: letters (alphabet)

- This form allows for an iterative solution in ε (well-understood techniques to handle the solution)

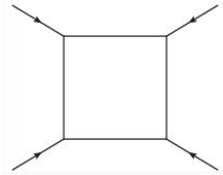
$$\vec{I}(\vec{s};\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k\,\vec{I}^{(k)}(\vec{s}) \quad \text{with} \quad \vec{I}^{(k)}(\vec{s}) = \vec{I}^{(k)}(\vec{s}_0) + \int_{\vec{s}_0}^{\vec{s}} \mathrm{d}\tilde{A}(\vec{s}')\cdot\vec{I}^{(k-1)}(\vec{s}')$$

Basis construction: leading singularities

- How to construct canonical bases? → No general algorithm exists → Conjecture [Arkani-Hamed et al, '12]:

> Integrands with at most simple poles and constant **leading singularities (LS)**

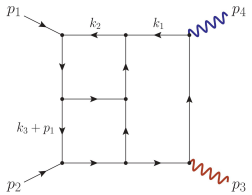
- Public packages: **DlogBasis** [Henn et al, '20]



$$\frac{d^4k}{k^2(k-p_1)^2(k-p_{12})^2(k+p_4)^2} \to \frac{1}{p_{12}^2 p_{23}^2} d\log(\ldots)\wedge d\log(\ldots)\wedge d\log(\ldots)\wedge d\log(\ldots)$$

LS

- Maximal-cut (MC) integrand in the **Baikov representation** [Baikov, 96; Frellesvig & Papadopoulos '17]



$$\frac{1}{D_2^2 D_7(D_7-s_{12})\sqrt{Q(D_2)}} \xrightarrow{\times D_2^2} \frac{d\log(\ldots)\wedge d\log(\ldots)}{s_{12}\sqrt{s_{12}^2+(m_3^2-m_4^2)^2-2s_{12}(m_3^2-m_4^2)}}$$

Basis construction: Magnus Exponential

- Choose (guess) a MI-basis that renders the **MC DEs linear on** ε [Argeri et al, '14; Gehrmann et al, '14]

$$\partial_\xi \vec{G}^{\mathrm{MC}} = (H_{0,\xi} + \varepsilon H_{1,\xi})\vec{G}^{\mathrm{MC}}$$

- **Rescale MIs** $\vec{I}^{\mathrm{MC}} = \tilde{T}^{\mathrm{MC}}\vec{G}^{\mathrm{MC}}$ with a matrix that satisfies $\partial_\xi \tilde{T}^{\mathrm{MC}} = -\tilde{T}^{\mathrm{MC}} H_{0,\xi}$

$$\partial_\xi \vec{I}^{\mathrm{MC}} = \varepsilon A_\xi^{\mathrm{MC}} \vec{I}^{\mathrm{MC}} \quad \text{with} \quad A_\xi^{\mathrm{MC}} = \tilde{T}^{\mathrm{MC}} H_{1,\xi} (\tilde{T}^{\mathrm{MC}})^{-1}$$

- **Relax cut conditions** → subsector entries with linear dependency on ε may appear

$$\partial_\xi \vec{I} = \varepsilon A_\xi^{\mathrm{MC}} \vec{I}^{\mathrm{MC}} + (h_{0,\xi} + \varepsilon h_{1,\xi})\vec{I}^{\mathrm{LS}}$$

- Obtain canonical by rotating with $\vec{I} = \vec{I}^{\mathrm{MC}} + \tilde{T}^{\mathrm{LS}}\vec{I}^{\mathrm{LS}}$, where $\partial_\xi \tilde{T}^{\mathrm{LS}} = -h_{0,\xi}$.

DEs for the families under study

- Computation of derivatives & Generation of IBP system: **FFIntRed** [Peraro]
- IBPs solved over finite fields, reconstructing the canonical DEs: **FiniteFlow** [Peraro, 2019]

$$\mathrm{d}\vec{I}(\vec{s}) = \varepsilon\, \mathrm{d}\tilde{A}(\vec{s}) \cdot \vec{I}(\vec{s}) \quad \text{with} \quad \mathrm{d}\tilde{A}(\vec{s}) = \sum a_i \, \mathrm{d}\log w_i(\vec{s})$$

- **Alphabet gets larger & more complicated** compared to planar two-loop one: 16 → 37 letters!
  Planar MIs with 1 off-shell: same alphabet at 2 & 3 loops [DC & Syrrakos, 2022]
- **4 new square roots** are introduced:

$$r_1 = \sqrt{s_{12}^2 + (m_4^2 - m_3^2)^2 - 2s_{12}(m_4^2 + m_3^2)}$$ ← Root known from 2-loop

$$r_2 = \sqrt{(m_3^2 m_4^2 + s_{13}(s_{12} + s_{13}))^2 - 4m_3^2 m_4^2 s_{13}^2} \qquad r_3 = \sqrt{s_{23}((m_3^2 - m_4^2)^2 s_{23} - 4m_3^2 m_4^2 s_{12})}$$

$$r_4 = \sqrt{(m_4^2 - s_{23})((m_4^2 - s_{12})^2(m_4^2 - s_{23}) - 4m_3^2 m_4^2 s_{12})} \qquad r_5 = \sqrt{(m_3^2 - s_{23})((m_3^2 - s_{12})^2(m_3^2 - s_{23}) - 4m_3^2 m_4^2 s_{12})}$$

- We didn't find a parametrization that rationalises all square roots simultaneously.

Evaluation of DEs: series expansions

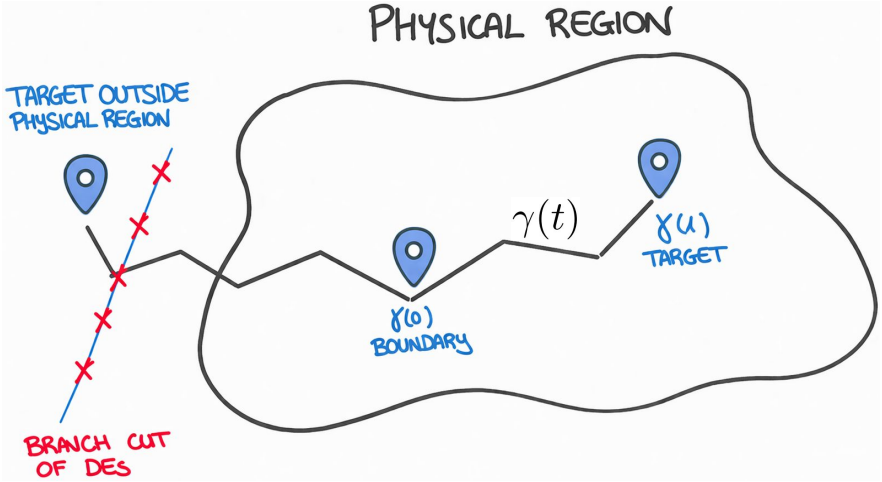
- Semi-numerical method: **generalised series expansions** [Pozzorini & Remiddi, 2006; Moriello, 2020]

Divide $\gamma(t)$ into segments according to singularity structure

↓

Expand and solve the DEs in each segment

$$\vec{I}(t) \sim \sum c_w (t-\tau)^{w_1} \log(t-\tau)^{w_2}$$



$\gamma(t)$ in the physical region

↓

No physical singularities crossed

↓

No analytic continuation needed

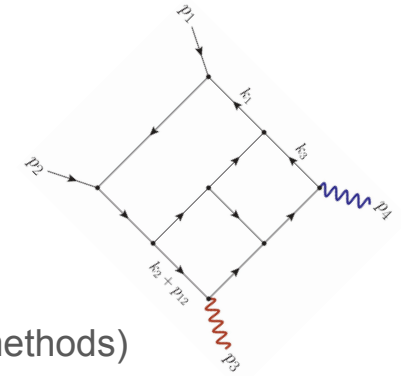
- Employ **DiffExp** [Hidding '20] and **AMFlow**'s [Liu & Ma, '22] DE solvers.
- Boundary values computed with 60 digits precision using AMFlow.
- Cross-check against AMFlow for 3 physical points: perfect agreement!

MIs per family evaluated within minutes!

Conclusions

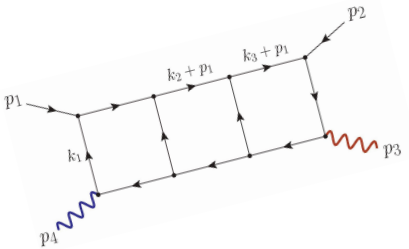
Computed all the MIs relevant for diboson production at N3LO QCD in the LC approximation!

- Constructed canonical DEs for each family
- Appearance of new square roots & letters
- Evaluated DEs using DiffExp & AMFlow solver



Future directions

- Explore faster evaluations (special function basis + other numerical methods)
- Computation of non-planar families
- Calculation of diboson production amplitudes



Thank you very much!