Precision physics on the lattice

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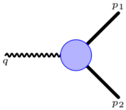
a_μ: definition and SM theory prediction - I



$$= ie\bar{u}(p_2)\mathcal{V}^{\mu}_{\ell\ell\gamma}(P,q)u(p_1)\ , \quad q = p_2 - p_1\ , \quad P = p1 + p_2\ , \quad q \cdot P = 0$$

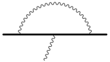
the 1PI vertex $\mathcal{V}^{\mu}_{\ell\ell\gamma}$ in the SM, due to equation of motions and symmetries, reads

$$\mathcal{V}^{\mu}_{\ell\ell\gamma}(\text{P,q}) = \gamma^\mu F_E(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_\ell}F_M(q^2) + \left(\gamma^\mu - \frac{2m_\ell q^\mu}{q^2}\right)\gamma_5 F_A(q^2) + \frac{\sigma^{\mu\nu}\gamma_5 q_\nu}{2m_\ell}F_D(q^2)$$

at $q^2 = 0 \ \Rightarrow \ F_E(0) = 1\ , \qquad F_A(0) = 0\ , \qquad F_M(0) = a_\ell = \dfrac{g_\ell - 2}{2}\ , \qquad F_D(0) = -2m_\ell\, d_\ell$

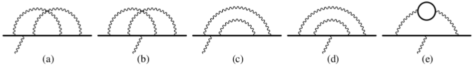
with a_ℓ the anomalous (quantum effect) magnetic moment of the lepton $\ell = e, \mu, \tau$

At 1-loop order: $a^{(1)}_{e,\mu,\tau} = \dfrac{\alpha}{\pi}\dfrac{1}{2}$ (Schwinger, 1947)



In SM: $a_\mu = a_\mu^{lept} + a_\mu^{EW} + a_\mu^{had}$ with $a_\mu^{lept} \sim 0.00116\ldots$ known in QED up to 5 loops

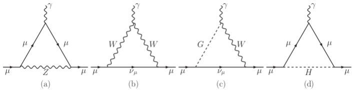
e.g. 2-loop QED diagrams:



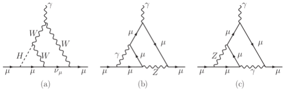
a_μ: definition and SM theory prediction - II

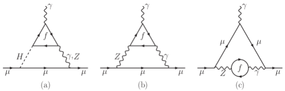
a_μ^{EW} = sum of loop diagrams with at least one W, Z or H-field boson; at 1 loop:

$$a_\mu^{EW\,(1)} = \frac{G_F m_\mu^2}{\sqrt{2} 8\pi^2}\left[\frac{5}{3} + \frac{1}{3}(1 - 4 s_W^2)^2\right] \simeq 15.4 \cdot 10^{-10}$$

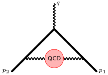


at 2 loops e.g.

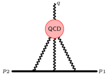




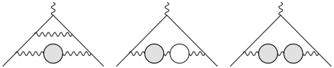
a_μ^{had} = sum of diagrams with at least one quark loop: HVP = O(α^2), HLBL = O(α^3)



HVP, LO $\sim$ $700 \cdot 10^{-10}$ $\Leftarrow$ PT kernel & $\Pi_{\mu\nu}(q) \equiv \int d^4x\, e^{iqx} \langle 0|T\{J_\mu^{em,h}(x) J_\nu^{em,h}(0)\}|0\rangle$



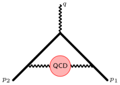
HLBL $\sim$ $11 \cdot 10^{-10}$ $\Leftarrow$ $\Pi_{\mu\nu\rho\sigma}(q_{1,2,3}) \equiv \int \Pi_i d^4x_i e^{iq_i x_i} \langle 0|T\{J_\mu^{em,h}(x_1) J_\nu^{em,h}(x_2) J_\rho^{em,h}(x_3) J_\sigma^{em,h}(0)\}|0\rangle$



HVP, NLO $\sim$ - $9 \cdot 10^{-10}$ $\Leftarrow$ $PT kernels \& 1 - loop\ VP's$

$a_\mu^{HVP,LO}$ from data-driven approach or Lattice QCD+QED

Uncertainty in the SM theory prediction of a_μ is dominated by the error on $a_\mu^{HVP,LO}$

- $a_\mu^{HVP,LO} =$ 

$= \frac{\alpha^2}{\pi^2} 4\pi^2 \int_0^\infty dQ^2\, K_E(Q^2)\, \underline{[\Pi^E(Q^2) - \Pi^E(0)]}$

with the kernel $K_E(Q^2)$ analitically known from the (real) $a_\mu^{(1)}$. Dispersion relation $\Rightarrow$

- $a_\mu^{HVP,LO} = \frac{\alpha^2}{3\pi^2} \underline{\int_0^\infty \frac{ds}{s} R_{had}(\sqrt{s})} \int_0^\infty dQ^2\, \underline{\frac{Q^2}{s+Q^2}} K_E(Q^2) \equiv \frac{\alpha^2}{3\pi^2} \int_0^\infty \frac{ds}{s} R_{had}(\sqrt{s})\, K(s)$

with $K(s)$ also known (see e.g. WP 2006.04822, eq. (2.2)). Writing $\Pi^E(Q^2)$ in terms of $G(t)$ $\Rightarrow$

- $a_\mu^{HVP,LO} = 4\alpha^2 m_\mu \int_0^\infty dt\, G(t)\, t^3 \tilde{K}(t)$ (time-momentum form) (Meyer - Bernecker 2011)

with known $\tilde{K}(t) = \frac{2}{m_\mu t^3} \int_0^\infty \frac{d\omega}{\omega} K_E(\omega^2)\, [\omega^2 t^2 - 4\sin^2(\omega t/2)]$, $\tilde{K}(t) \sim t$ as $t \to 0$, $\tilde{K}(t) \sim t^{-1}$ as $t \to \infty$

$a_\mu^{HVP,LO}$: from SM via Lattice QCD + QED (●) [pioneered by T. Blum, 2002 ; ETMC, K. Jansen et al. 2011] or from R_{had} data via (●)

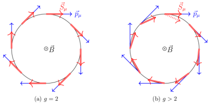
Experiments on $g_\mu - 2$ and on $e^+e^- \to$ hadrons

- Recent experiments on $g_\mu - 2$ (good internal consistency)

CERN I,II,III (1961-'76); BNL (from 1997 to 2006) E-821: PRD 73 (2006) 072003 ; Fermilab E989: 2104.03281, 2308.06230 (PRL papers).

$$\vec{\omega}_a = \frac{-e}{mc}\left[a_\mu \vec{B} - a_\mu \frac{\gamma}{\gamma+1}(\vec{\beta}\cdot\vec{B})\vec{\beta} - (a_\mu - \frac{1}{\gamma^2-1})\vec{\beta}\times\vec{E}\right]$$

$\mu^+ \to e^+\nu_e\bar{\nu}_\mu$: direction of $\vec{S}_{\mu^+}$ and $\vec{p}_{e^+}$ correlated



Planned: J-PARC muon g-2/EDM (Ibaraki, Japan). Planned MuonE (CERN): μ–e scattering, $q^2 \lesssim -0.01\,GeV^2 < 0$

- Recent experiments on $e^+e^- \to$ hadrons $\Rightarrow$ R_{had} (increasing internal tensions?)

BABAR: 1205.2228 [hep-ex] (PRD) ... ; KLOE: 1711.03085 [hep-ex] (JHEP) ... ; BESIII: 1507.08188, 2009.05011 [hep-ex] (PLB) ... ;
SND: 1809.07631 (PRD) ... ; CMD-2 hep-ex/0409030 (PLB) ... ; CMD-3: 2302.08834 (PRD), 2309.12910 (PRL)

Experiments measure $\sigma(e^+e^- \to \text{ hadrons})$ with cuts and systematic errors: the bare

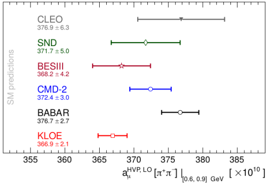
inclusive $\sigma^0(\mathrm{e}^+e^- \to \text{hadrons }) \equiv \frac{4\pi\alpha^2}{3s} R_{had}(\sqrt{s})$ is reconstructed via MC's with

approximate radiative corrections [σ^0 $\leftrightarrow$ σ with QED-FSR but no γ VP, QED ISR, virtual InFin, interference] (WP2020, sect. 2.2)

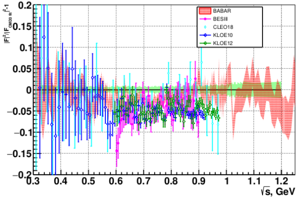
and accuracy depending on experiment, method (direct scan, ISR), $\sqrt{s}$, channel ($\pi 0\gamma$, 2π, 3π, 4π, K^+K^-, ... J/ψ, ...)

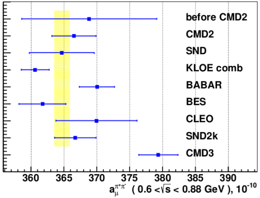
$a_\mu^{HVP,LO}$ from data-driven approach: experimental uncertainties & tensions

In the key $\pi^+\pi^-$ channel, giving $\sim 70\%$ of a_μ^{LO-HVP}: till end 2022 only a moderate tension (KLOE/BABAR)



but ... CMD-3 2023-papers $\Rightarrow$ big puzzle in $e^+e^- \to$ had. experiments; do SM and exp. agree ($\sim 5 \pm 6 \times 10^{-10}$) on $a_\mu^{HVP,LO}$?





First suggested by lattice results: on $a_\mu^{HVP,LO}$ in 2020 by BMW (Nature); on interm. window term a_μ^W in 2020-22 (BMW, Mainz, ETMC) !

White Paper 2025 muon $g-2$: SM prediction and role of lattice field theory

- a very precise experiment [Muon g-2 Coll. - Phys.Rev.Lett. 131 (2023)]: $a_\mu^{\rm exp} = 116592059(22) \times 10^{-11}$
  abs. experimental error $\sim 2.2 \times 10^{-10}$, expected to be further reduced by update in (early) June 2025

- very accurate theory predictions [WP2020] for QED (dominating) and EW (sub-sub-leading) contributions:
  $a_\mu^{QED} = 116584718.9(1) \times 10^{-11}$ and $a_\mu^{EW} = 153.6(1.0) \times 10^{-11}$ (similar in WP2025)

- largest uncertainty from the HVP hadronic (main sub-leading) contribution:
  - dispersive way: $a_\mu^{HVP,LO+N(N)LO} = 684.2(4.0) \times 10^{-10}$, from $a_\mu^{HVP,LO} = 693.1(4.0) \times 10^{-10}$ [WP2020]
  - lattice way: $a_\mu^{HVP,LO} = 711.6(18.4) \times 10^{-10}$ [WP2020] $\longrightarrow$ $a_\mu^{HVP,LO}$ similar & 0.7% error [WP2025]

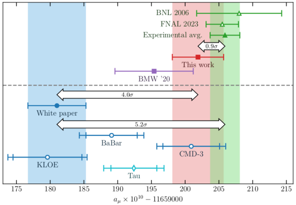
- smaller uncertainty from the (sub-sub-leading) HLbL contribution: (see also talk by L. Cappiello)
  $a_\mu^{HLbL-NLO} = 9.2(1.8) \times 10^{-10}$ [WP2020] $\longrightarrow$ similar ($\sim 20\%$ larger, halved error) [WP2025]

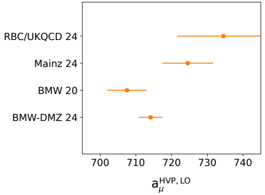
SM value of [WP2025] from QED + EW (leptons) + lattice (quarks) or phenomenology/models for HLbL
yet no dispersive prediction for HVP due to large CMD3-BaBar-Kloe tension in $e^+e^- \to$ hadron exp. data

Recent lattice results for $a_\mu^{HVP,LO}$

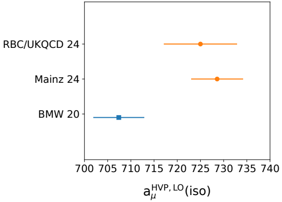
After BMW20 paper, in 2024 a few more published results on $a_\mu^{HVP,LO}$ in QCD+QED (at 1st order in $\alpha_{\rm em}$ and $(m_d - m_u)\Lambda_{\rm QCD}^{-1}$):

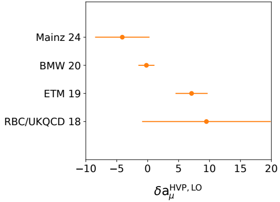
BMW-DHMZ 24: low energy tail ($\lesssim 5\%$ of total) from e^+e^- data; RBC/UKQCD 24: a -0.5% shift for missing SIB effects in WP2025





$$a_\mu^{HVP,LO} = a_\mu^{HVP,LO}(\mathrm{iso}) + \delta a_\mu^{HVP,LO}$$ **= total isospin symmetric QCD (BMW20 $\leftrightarrow$ WP25 scheme) contribution + IB correction**

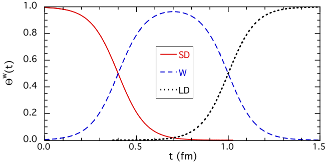


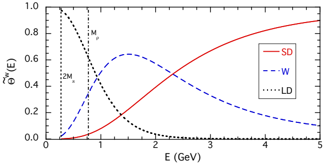


Window approach to $a_\mu^{HVP,LO}$

RBC/UKQCD'18: $a_\mu^{\rm HVP} \equiv a_\mu^{\rm SD} + a_\mu^{\rm W} + a_\mu^{\rm LD}$, **with smoothed** $\Theta^{\rm SD,W,LD}(t)$

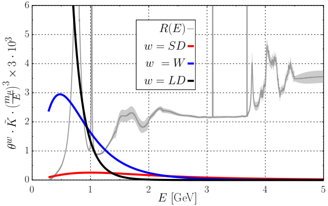
$$a_\mu^w = 2\alpha_{em}^2 \int_0^\infty dt \left[t^2 K(m_\mu t)\right] \underline{\Theta^w(t)}\, G(t) \qquad w = \{SD, W, LD\} ,$$





probe the R_{had} ratio of $e^+e^- \to$ hadrons in different regions of c.o.m. energy E

$$a_\mu^w = \frac{2\alpha_{em}^2 m_\mu^{-1}}{9\pi^2} \int_{E_{thr}}^\infty dE \left\{ \frac{m_\mu^3}{E^3} \widetilde{K}\left(\frac{E}{m_\mu}\right) \underline{\widetilde{\Theta}^w(E)} \right\} R^{had}(E)$$



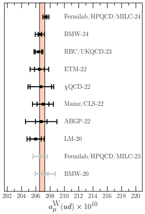
note E-profile of the different kernels $\{\ldots\}$

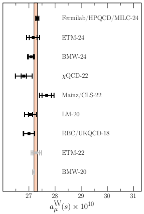
a_μ^W**: noise, finite-L and cutoff effects moderate;** a_μ^{SD}**: larger O(a)**

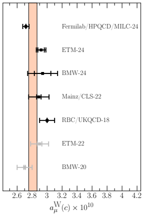
$\Rightarrow$ a test of SM (lattice QCD+QED) versus experimental data (independent of a_μ !)

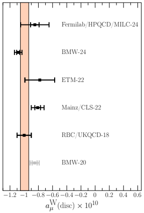
Intermediate window a_μ^W: partial contributions & high precision

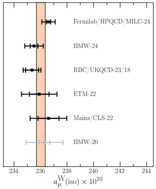
$$a_\mu^W = a_\mu^W(\mathrm{iso}) + \delta a_\mu^W = a_\mu^W(ud) + a_\mu^W(s) + a_\mu^W(c) + a_\mu^W(\mathrm{disc}) + \delta a_\mu^W$$

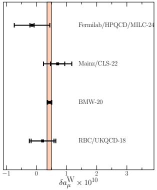










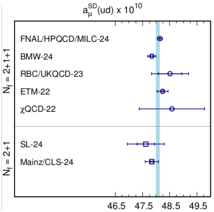


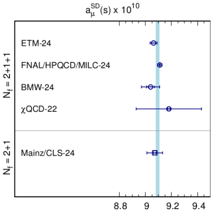
$$a_\mu^W(\mathrm{iso}) = 235.97(38) \times 10^{-10}\,, \quad \delta a_\mu^W = 0.42(07) \times 10^{-10}\,, \quad \Rightarrow \quad a_\mu^W = 236.39(39) \times 10^{-10}$$

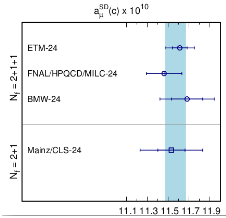
accuracy better than 0.2%, measured [$> 5\sigma$ tension with pre-CMD3 data]/ measurable in e^+e^- to hadrons exp.s

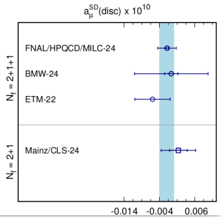
Short distance window and total HVP: a_μ^{SD} and a_μ^{HVP}

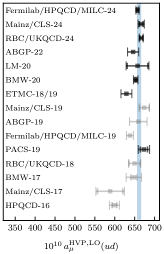
Analogous splitting in partial contributions and small $\delta a_\mu^{SD} = 0.04(04) \times 10^{-10}$, $\delta a_\mu^{HVP} = 0.6(2.3) \times 10^{-10}$

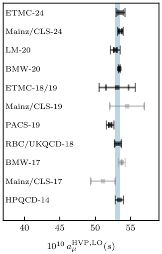


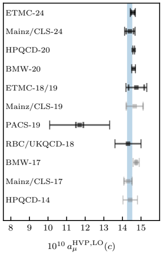


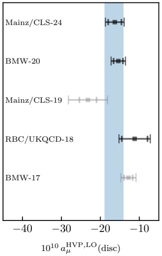












only 3 to 4 results for LD (ud; disc) and HVP (ud; disc) – good overall consistency within errors !

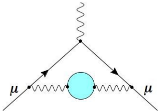
White Paper 2025 muon g − 2 (on arXiv for May 22nd) & Outlook

More work on $a_\mu^{LD}(ud)$ and on the IB correction δa_μ still needed (ongoing) to match experimental $g_\mu - 2$ error

Precise lattice results, e.g. for a_μ^{W}, are useful benchmarks for experiments on e^+e^- to hadrons

ETMC plans to give results on $a_\mu^{LD}(ud)$ within Fall 2025 and on δa_μ within Spring 2026

Methods for QCD+QED beneficial (sea IB effects necessary) to other studies, e.g. hadronic inclusive decays of τ



Thanks to colleagues of ETMC and the Muon $g - 2$ Theory Initiative for most enjoyable collaboration

Thanks to A.X. El-Khadra for advise/guidance about this presentation (few days prior to posting of WP2025)

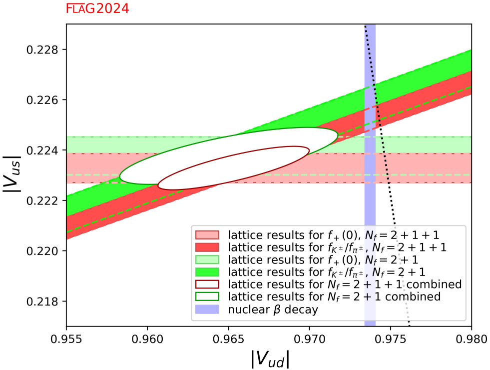
Plots by G.Gagliardi, D. Giusti, S. Gottlieb, S. Kuberski, S. Lahert, M.K. Marinkovic, A. Portelli, J.T. Tsang

THANK YOU for attention!

Inclusive hadronic τ lepton decay rate give access to the CKM matrix elements V_{ud} and V_{us}

- $|V_{ud}|$
  - superallowed nuclear β-transitions $\longrightarrow$ systematic nuclear effects difficult to quantify
  - $\tau \to X_d \nu$ inclusive decay

- $|V_{us}|$
  - $\Gamma\left(K_0 \to \pi^- \ell \nu\right)$ semileptonic decay
  - $\frac{\Gamma\left(K^\pm \to \ell^\pm \nu_\ell[\gamma]\right)}{\Gamma\left(\pi^\pm \to \ell^\pm \nu_\ell[\gamma]\right)}$ leptonic decays
  - $\tau \to K \nu$ exclusive decay
  - $\tau \to X_s \nu$ inclusive decay



FLAG24 - `https://arxiv.org/abs/2411.04268`

experimentally can be measured with extreme precision

$$|V_{us}|f_+(0) = 0.21654(41)$$

$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.27599(41)$$

f_+ represent the form factors relevant for the semileptonic decay $K_0 \to \pi^- \ell \nu$

$f_\pi^\pm$ and $f_K^\pm$ are the QCD axial matrix elements between the pion and kaon states and the vacuum

QED effects have been subtracted from experimental data by using χPT or direct lattice calculation

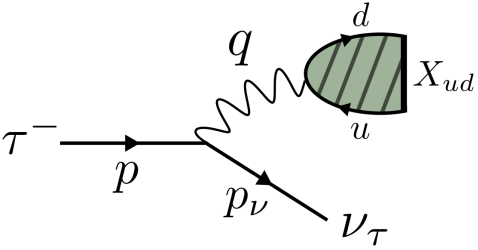
V. Cirigliano *et. al* - `https://doi.org/10.1016/j.physletb.2011.04.038`

J.L. Rosner *et al.* - `https://arxiv.org/abs/1509.02220`

M. Di Carlo *et al.* - `https://doi.org/10.1103/PhysRevD.100.034514`

A. Evangelista *et al.* - https://doi.org/10.1103/PhysRevD.108.074513

C. Alexandrou *et al.* - https://doi.org/10.1103/PhysRevLett.132.261901



$$\left|\mathcal{A}\left(\tau \to X_{fg}\,\nu_\tau\right)\right|^2 = \frac{G_F^2\left|V_{fg}\right|^2}{2}\,\mathcal{L}^{\mu\nu}(p, p_\nu)\,\rho^{\mu\nu}(q)$$

$$\rho^{\mu\nu}(q) = \sum_X \langle 0|\; J^\mu_{fg}(0)\,|X\rangle\,\langle X|\; J^\mu_{fg}(0)^\dagger\,|0\rangle$$

$$R^{(\tau)}_{fg} \equiv \frac{\Gamma\left(\tau \to X_{fg}\,\nu_\tau\right)}{\Gamma\left(\tau \to e\bar{\nu}_e\nu_\tau\right)} = 6\pi\, S_{EW}\left|V_{fg}\right|^2 \sum_{I=L,T}\int_0^1 \mathrm{d}s\; K_I(s)\,\rho_I(s)\,, \qquad s = \frac{q^2}{m_\tau^2}$$

$$C_{\mathrm{L}}(t) = \int \mathrm{d}^3\boldsymbol{x} \left\langle 0 \middle| J^0_{fg}(\boldsymbol{x}, t) J^0_{fg}(\mathbf{0}, 0)^\dagger \middle| 0 \right\rangle \qquad C_{\mathrm{T}}(t) = \frac{1}{3} \sum_i \int \mathrm{d}^3\boldsymbol{x} \left\langle 0 \middle| J^i_{fg}(\boldsymbol{x}, t) J^i_{fg}(\mathbf{0}, 0)^\dagger \middle| 0 \right\rangle$$

hadronic spectral densities $\updownarrow$ **lattice vector-vector correlators**

$$C_I(t) = \int_0^\infty \frac{\mathrm{d}E}{2\pi}\, e^{-Et}\, E^2 \rho_I(E^2)$$

smearing the spectral densities through the phase space

M.T. Hansen *et al.* - https://doi.org/10.1103/PhysRevD.96.094513

P. Gambino *et al.* - https://doi.org/10.1103/PhysRevLett.125.032001

$$R_{fg}^{(\tau)} \propto |V_{fg}| \sum_{l=L,T} \int_0^\infty \mathrm{d}E \; K_l^\sigma\left(\frac{E^2}{m_\tau^2}\right) E^2 \rho_l(E^2)$$

$$K_l^\sigma\left(\frac{E^2}{m_\tau^2}\right) = K_l\left(\frac{E^2}{m_\tau^2}\right) \Theta_\sigma\left(1 - \frac{E^2}{m_\tau^2}\right)$$

$$\downarrow$$

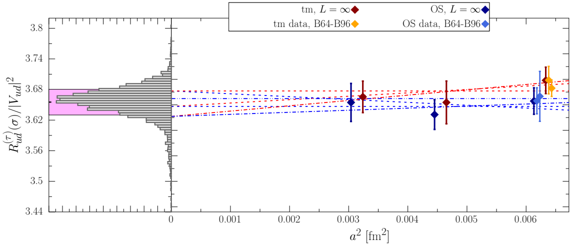
HLT method for extracting spectral densities

M. Hansen *et. al* - https://doi.org/10.1103/PhysRevD.99.094508

$\bar{u}d$ channel

A. Evangelista *et al.* -

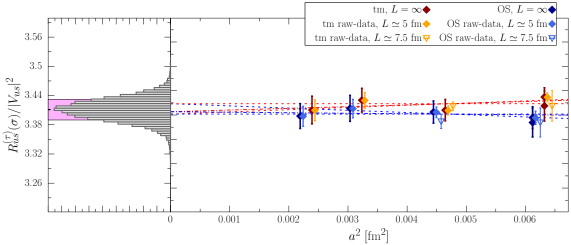
https://doi.org/10.1103/PhysRevD.108.074513



$\bar{u}s$ channel

C. Alexandrou *et al.* -

https://doi.org/10.1103/PhysRevLett.132.261901

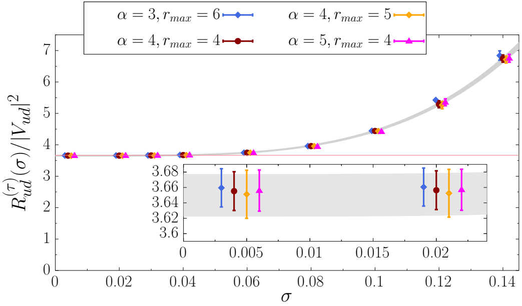


results do not show large lattice artifacts

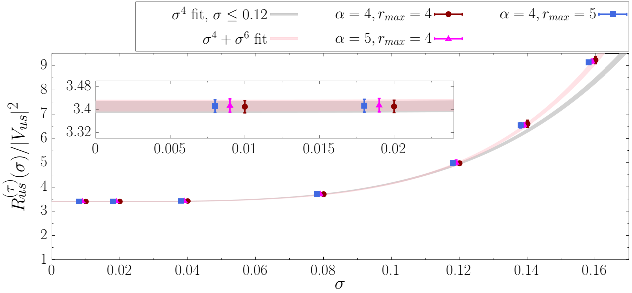
↓

very well controlled continuum extrapolations

$\bar{u}d$ channel

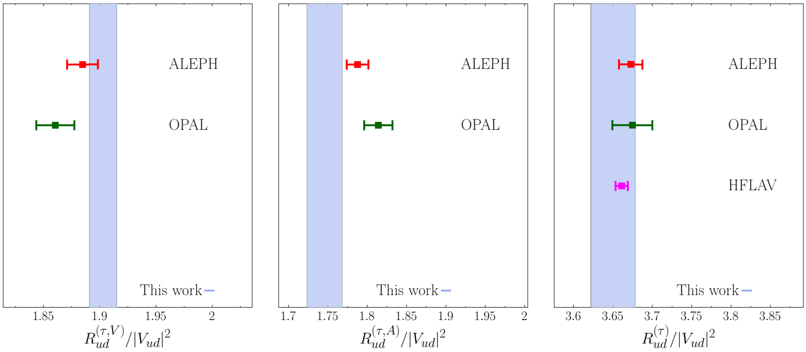


$\bar{u}s$ channel



$$R^{(\tau)}_{fg}(\sigma) - R^{(\tau)}_{fg} = \mathcal{O}\left(\sigma^4\right)$$

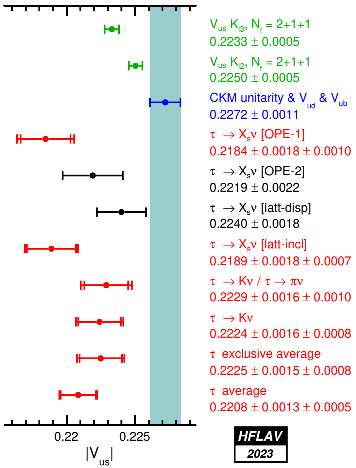
$\sigma \to 0$ extrapolation taken in full confidence



$$|V_{ud}|_{(\tau-\mathrm{incl})} = 0.9752(39) \qquad |V_{ud}|_{(\mathrm{super}-\beta)} = 0.97373(31)$$

$$\mathcal{O}(0.4\%) \quad \longleftrightarrow \quad \mathcal{O}(0.03\%)$$

missing QED effects | **systematic effects really under control?**



HFLAV - https://arxiv.org/abs/2411.18639

C. Alexandrou *et al.* - https://doi.org/10.1103/PhysRevLett.132.261901

- our result agrees well within errors with the ones from OPE
- confirms the tension between inclusive and exclusive decays
- confirms the tension with the CKM unitarity determination

see also talk by L. Vittorio

long-distance QED effects are neglected in our computation

$\mathcal{O}(5\%)$ QED effects to reconcile with $K_{\ell 2}$ determination

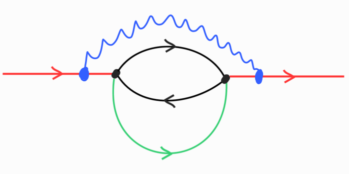
$\downarrow$

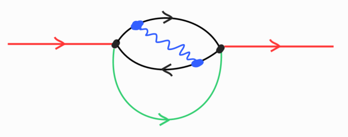
are experiments missing something?

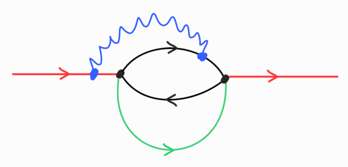
QED effects are essential at this level of accuracy

↓

we are working on computing these effects







↓ kernel modification

↓ a_μ-like corrections

↓ computationally challenging

Thanks for the attention!