Hydrodynamics with Anomalies and Effective Field Theory

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Outline

- Intro: anomaly inflow
- Intro: 1+1d bosonization of fermions
- Bosonization and hydrodynamics in 1+1d (perfect fluid)
- 3+1d hydrodynamics with anomalies from inflow

Hydrodynamics: Introduction

- Effective low-energy description of fluid phases, low-energy expansion
- Like effective field theory, written in terms of currents as bosonic "fields"

→ "non-particle" field theory (Jackiw, Nair, Pi, Polychronakos, 2004)

- Local equilibrium, local thermodynamics, based on symmetries only
- A theory of everything (….many of them)
- Old but poorly understood (non-linear)
- Many potential applications to strongly-interacting (fermionic) fluids
- Relation with gravity
- Semiclassical, quantization is postponed; one-point functions and responses

→ Try to bridge with effective field theory, using recent advances

Topological phases of matter & hydrodynamics

- Topological phases of matter provide new kinds of effective theories

  - BULK: gapped, but non-trivial global effects (Aharonov-Bohm phases)
    described by topological gauge theories (Chern-Simons theory)

  - BOUNDARY: massless excitations, described by field theory & anomalies

  - topological theories are hydrodynamic! (currents, dual gauge fields)

→ Find here the relation with hydrodynamics

  - perfect fluid: T=0, dynamics from pressure & density $P = P(\rho)$

  - Lagrangian formulation of Euler equations

→ 1+1d: hydro equal to standard bosonic QFT for fermions

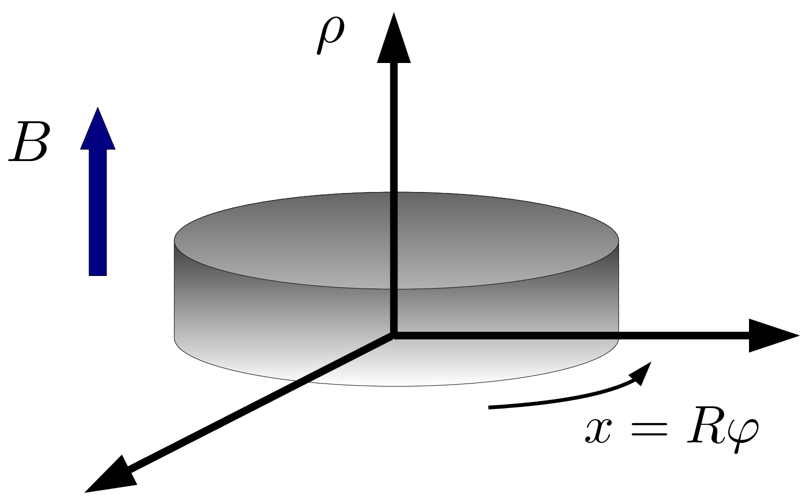
→ 3+1d: implement all anomalies in hydro, nice universal/geometrical picture

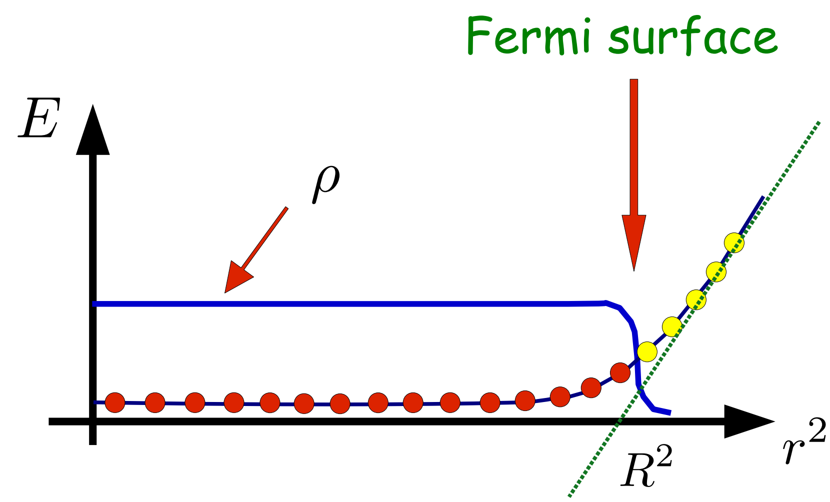
→ 3+1d: hydro provides a “cheap” bosonization

→ 3+1d: topological theories suggest further generalized hydros (ongoing)

Topological phases: Quantum Hall Effect

Landau levels: first level is filled, bulk gap, massless fermion at the edge

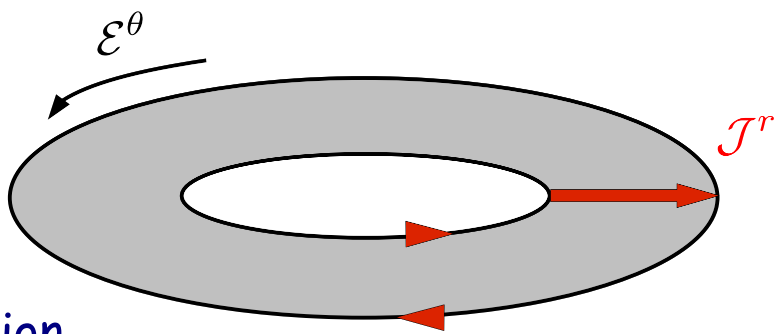




edge ~ Fermi surface: linearize energy $\varepsilon(k) \sim (k - k_F)$, $\quad (J = rk = n, \quad r \sim R)$

→ massless chiral fermion on (1+1) dimensional boundary

Anomaly inflow



- tangential electric field $\mathcal{E}^\theta$
- Hall current $\mathcal{J}^r = -\frac{1}{2\pi}\mathcal{E}^\theta = -\frac{1}{2\pi}\varepsilon^{r\alpha\beta}\partial_\alpha A_\beta$
- Hall current = chiral anomaly of 1+1d boundary fermion

$$\oint d\theta\ \mathcal{J}^r = \frac{dQ_{edge}}{dt} = \oint d\theta\ \partial_\alpha J^\alpha \qquad \partial_\alpha J^\alpha = -\frac{1}{2\pi}\varepsilon^{\alpha\beta}\partial_\alpha A_\beta, \quad \alpha,\beta = 0,1 = \hat{t},\hat{\theta}$$

J^α 1+1d current
$\mathcal{J}^\mu$ 2+1d current

- Hall current described by Chern-Simons topological theory in 2+1d

$$S_{CS}[A] = -\frac{1}{4\pi}\int d^3x\ AdA, \qquad \mathcal{J}^\mu = \frac{\delta S_{CS}}{\delta A_\mu}, \qquad A = A_\mu dx^\mu, \quad \mu = 0,1,2$$

➡ current is conserved in the bulk+boundary system (gauge invariance)

➡ inflow: quantum anomaly compensated by classical current in extra dimension

➡ effective theory is Chern-Simons, hallmark of topological phases of matter

1+1d bosonization and anomalies

- many derivations of bosonization, non-perturbative, endless number of applications

$$S = \frac{1}{2}\int d^2x\ (\partial_\mu\theta)^2\,, \qquad \theta(t,x)$$

- U(1) symmetry $\theta \to \theta + \text{const.}$ but two conserved currents

$$J^\mu = \partial_\mu\theta, \qquad \partial_\mu J^\mu = 0 \qquad \text{Noether current}$$

$$\tilde{J}^\mu = \varepsilon^{\mu\nu}\partial_\nu\theta, \qquad \partial_\mu \tilde{J}^\mu = 0 \qquad \text{topological (axial) current}$$

- like Dirac fermion: $J^\mu = \bar{\psi}\gamma^\mu\psi, \qquad \tilde{J}^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$
- couple to corresponding backgrounds $A_\mu, \tilde{A}_\mu$

$$S = \int d^2x \frac{1}{2}\left(\partial_\mu\theta - A_\mu\right)^2 + \tilde{A}_\mu\varepsilon^{\mu\nu}\left(\partial_\nu\theta - A_\nu\right)$$

- gauge-invariant currents ("covariant currents") are anomalous

$$J^\mu = \partial_\mu\theta - A_\mu, \qquad \partial_\mu J^\mu = -\varepsilon^{\mu\nu}\partial_\mu\tilde{A}_\nu, \qquad (e/\pi \to 1)$$

$$\tilde{J}^\mu = \varepsilon^{\mu\nu}\left(\partial_\nu\theta - A_\nu\right), \qquad \partial_\mu\tilde{J}^\mu = -\varepsilon^{\mu\nu}\partial_\mu A_\nu$$

→ Anomalies reproduced at classical level in the bosonic effective theory

Checking the anomaly inflow

- Topological insulators in 2+1d: BF theory, `hydrodynamic' gauge fields $p_\mu, \tilde{q}_\mu$ expressing conserved bulk currents, e.g. $\mathcal{J}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu \tilde{q}_\rho, \quad \partial_\mu \mathcal{J}^\mu = 0,$

$$S_{BF}[p, \tilde{q}, A, \tilde{A}] = \int_{\mathcal{M}_3} p d\tilde{q} + \tilde{A} dp + A d\tilde{q}, \qquad \mathcal{J} = * d\tilde{q}, \quad \tilde{\mathcal{J}} = * dp$$

- equations of motion

$$dp + dA = 0 \quad \to \quad p = d\theta - A$$

$$d\tilde{q} + d\tilde{A} = 0 \quad \to \quad \tilde{q} = d\psi - \tilde{A}$$

$$S_{BF}[p, \tilde{q}, A, \tilde{A}] \underset{eom}{\to} S_{resp}[A, \tilde{A}] = -\int_{\mathcal{M}_3} \tilde{A} dA$$

- check anomaly inflow

$$\tilde{\mathcal{J}}^{\hat{r}} \underset{eom}{=} \varepsilon^{\hat{r}\alpha\beta}\partial_\alpha \left(\partial_\beta \theta - A_\beta\right) = \partial_\alpha \tilde{J}^\alpha = -\varepsilon^{\alpha\beta}\partial_\alpha A_\beta, \qquad \alpha, \beta = 0, 1$$

$$\mathcal{J}^{\hat{r}} \underset{eom}{=} \varepsilon^{\hat{r}\alpha\beta}\partial_\alpha (\partial_\beta \psi - \tilde{A}_\beta) = \partial_\alpha J^\alpha = -\varepsilon^{\alpha\beta}\partial_\alpha \tilde{A}_\beta$$

➡ hydrodynamic fields $p, \tilde{q}$ express the edge currents, once reduced to 1+1d θ, ψ

$$\tilde{J}^\alpha = \varepsilon^{\alpha\beta}\left(\partial_\beta \theta - A_\beta\right)$$

$$J^\alpha = \varepsilon^{\alpha\beta}(\partial_\beta \psi - \tilde{A}_\beta)$$

Checking the anomaly inflow

- obtained by inflow
  $$\tilde{J}^\alpha = \varepsilon^{\alpha\beta}(\partial_\beta\theta - A_\beta)$$
  $$J^\alpha = \varepsilon^{\alpha\beta}(\partial_\beta\psi - \tilde{A}_\beta)$$
  gauge invariant
- compare with bosonic theory
  $$\tilde{J}^\alpha = \varepsilon^{\alpha\beta}(\partial_\beta\theta - A_\beta)$$
  $$J^\alpha = \partial_\alpha\theta - A_\alpha$$
- bulk topological theory + inflow indicates `compensating' scalar fields θ, ψ ensuring gauge invariance of edge currents $J^\alpha, \tilde{J}^\alpha$ (θ, ψ also appearing in WZW action):
  - θ earlier scalar field
  - ψ is the dual field: $\partial^\mu\theta - A^\mu = \varepsilon^{\mu\nu}(\partial_\nu\psi - \tilde{A}_\nu)$
- Summing up so far

→ bosonic theory can reproduce the 1+1d chiral anomalies

→ anomaly inflow from 2+1d topological theory (QHE, topological insulator, reservoir,...) derive the currents and the `hydrodynamic' fields θ, ψ to use in 1+1d

Lagrangian formulation of perfect fluid

- long history (Lichnerowicz; Carter; Arnold; Marsden;……; Abanov, Wiegmann '22)
- no viscosity, no heat flow, yet physical; also T=0, s= const.
- R & NR fluids described by action of fluid momentum p_α

$$S = -\int \hat{J}^\alpha p_\alpha + \varepsilon(\hat{J}) \quad \to \quad S = \int P(p_\alpha), \qquad dP = \rho\, d\mu, \quad \mu = \mu(p_\alpha)$$

P	pressure
μ	chemical potential
ρ	fluid density NR
$p_\alpha = \mu u_\alpha, \quad u^2 = -1$	

- Euler hydrodynamics is a constrained system
- Diffeo variations: $\delta S[p] = 0$ for $\delta_\epsilon p_\nu = \epsilon^\alpha \partial_\alpha p_\nu + p_\alpha \partial_\nu \epsilon^\alpha, \quad \delta_\epsilon p = \mathcal{L}_\epsilon p$
- Carter-Lichnerowicz equations of motion:

$$\hat{J}^\nu (\partial_\nu p_\mu - \partial_\mu p_\nu) = 0,$$ particle current $$\hat{J}_\nu = -\frac{\delta S}{\delta p_\nu}, \quad p_\nu = -\frac{\delta S}{\delta \hat{J}^\nu}$$

- solution in 1+1d: $\partial_\nu p_\mu - \partial_\mu p_\nu = 0 \quad \to \quad p_\mu = \partial_\mu \theta$ $\quad \hat{J}$-independent = constraint

→ completely equivalent to earlier bosonic theory for $P(p_\alpha) = \frac{1}{2} p_\alpha^2 = \frac{1}{2}(\partial_\alpha \theta - A_\alpha)^2$ which is actually 1+1d hydrodynamics!

→ add anomalies as before

Anomalies in 3+1d perfect fluids

$$S[p] = \int d^4x \; P(p)$$

(Abanov, Wiegmann '22)

- CL equations of motion: $i_{\tilde{J}} dp = 0 \quad \rightarrow \quad i_{\tilde{J}}(dpdp) = 0 \quad \rightarrow \quad dpdp = 0$

→ again a constraint: $\tilde{J} = *pdp, \qquad \tilde{J}^\mu = \varepsilon^{\mu\nu\rho\sigma} p_\nu \partial_\rho p_\sigma, \qquad \partial_\mu \tilde{J}^\mu = 0$

- $\tilde{J}$ helicity current

$$\tilde{Q} = \int d^3x \; \tilde{J}^0 \sim \int \vec{v} \cdot \vec{\omega}, \qquad \vec{\omega} = \nabla \times \vec{v}$$

- idea: identify it as the axial current!

(Son, Surowka '09; ....)

$$S[\pi, A, \tilde{A}] = \int d^4x P(\pi - A) + \tilde{A}(\pi - A)d(\pi + A), \qquad p = \pi - A \quad \text{gauge inv.}$$

- obtain (some of) the anomalies

$$\partial_\mu \tilde{J}^\mu = *d[(\pi - A)d(\pi + A)] = - * dAdA \qquad \partial_\mu \tilde{J}^\mu = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \qquad (e/2\pi = 1)$$

$$\partial_\mu J^\mu = -2 * dAd\tilde{A}, \qquad \partial_\mu J^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} \tilde{F}_{\rho\sigma},$$

(equation of motion now $d\pi d\pi = 0$)

→ hydrodynamics can describe interacting 3+1d fermionic fluids with anomalies

something is missing...

3+1d hydrodynamics from inflow

- 4+1d topological theory leading to 3+1d anomalies (Dirac fermion $\alpha = \frac{1}{3}$)

$$S^{(II)} = \int_{\mathcal{M}_5} \tilde{\pi} d\pi d\pi + \alpha\, \tilde{\pi} d\tilde{\pi} d\tilde{\pi} - \tilde{A} dA dA + \alpha \tilde{A} d\tilde{A} d\tilde{A}, \quad \underset{eom}{\rightarrow} \quad S^{(II)}_{resp} = -\int_{\mathcal{M}_5} \tilde{A} dA dA + \alpha \tilde{A} d\tilde{A} d\tilde{A}$$

- obtain 3+1d hydrodynamics by inflow (and using gauge invariance)

$$S = S^{(II)} + \int_{\mathcal{M}_4} P(\pi - A, \tilde{\pi} - \tilde{A}) + \tilde{A}(\pi - A) d(\pi + A) + \alpha \tilde{A}(\tilde{\pi} - \tilde{A}) d(\tilde{\pi} - \tilde{A})$$

→ two fluid variables: $p_\beta = \pi_\beta - A_\beta, \qquad \tilde{p}_\beta = \tilde{\pi}_\beta - \tilde{A}_\beta$ too many

→ reduction to one-fluid theory (minimal choice) $\tilde{\pi} = d\psi, \qquad P = P(\pi - A)$

$$p = \pi - A, \qquad \tilde{p} = d\psi - \tilde{A}, \qquad \text{(gauge inv.)}$$

$$S[\pi, \psi, A, \tilde{A}] = \int_{\mathcal{M}_4} P(\pi - A) + \tilde{A}(\pi - A) d(\pi + A) + \psi\left(d\pi d\pi + \frac{1}{3} d\tilde{A} d\tilde{A}\right) - \int_{\mathcal{M}_5} \tilde{A} dA dA + \frac{1}{3} \tilde{A} d\tilde{A} d\tilde{A}$$

→ we recover and extend earlier hydro

3+1d Hydrodynamics from inflow

$$S[\pi,\psi,A,\tilde{A}] = \int_{\mathcal{M}_4} P(\pi-A) + \tilde{A}(\pi-A)d(\pi+A) + \psi(d\pi d\pi + \frac{1}{3}d\tilde{A}d\tilde{A}) - \int_{\mathcal{M}_5} \tilde{A}dAdA + \frac{1}{3}\tilde{A}d\tilde{A}d\tilde{A}$$

$$p = (\pi - A), \quad \tilde{q} = (d\psi - \tilde{A}) \quad \text{gauge inv.}$$

- Results

- ψ is Lagrange multiplier enforcing CL equation: can do ordinary variations of S
- ψ pseudoscalar WZW field was missing, now all anomalies are reproduced
- can add dynamics to ψ ($\tilde{A}_0 - \dot{\psi}$ ~axial chemical potential), add chiral breaking
- hydro provides an effective 3+1d bosonic theory for interacting fermions
- "geometric" description of anomalies, independent of specific dynamics $P(p_\alpha)$
- can also describe mixed axial-gravitational anomaly
- straightforward derivation from Dirac fermions path integral (to appear)

Mixed axial-gravitational anomaly

$$D_\mu \tilde{J}^\mu = - * dAdA - 3\alpha * d\tilde{A}d\tilde{A} - \beta * \mathrm{Tr}(R^2)$$

$$R_{ab} = \frac{1}{2} R_{\mu\nu,ab} dx^\mu dx^\nu, \quad \beta = 1/24$$

- add term to 4+1d topological action (from anomaly literature) and use inflow

$$\Delta S = \beta \int_{\mathcal{M}_5} (d\psi - \tilde{A}) \mathrm{Tr}(R^2)$$

- no extra hydro fields needed; reproduce above anomaly plus additional force

$$D_\nu T^\nu_\mu = F_{\mu\nu} J^\nu + \tilde{F}_{\mu\nu} \tilde{J}^\mu + \mathrm{Tr}(R_{\mu\nu} \Sigma^\nu),$$

$\Sigma^{\mu,ab}$ spin current

- spin current is related to axial current (axial bkgd equivalent to torsion bkgd)

  - free Dirac identity $\Sigma^{\mu,ab} = \frac{1}{4} \bar{\Psi} \{\gamma^\mu, \sigma^{ab}\} \Psi = \frac{1}{2} \varepsilon^{\mu\nu,ab} \tilde{J}_\nu$

  - ψ corresponds to spinor rotation (local-Lorentz transf.)

Conclusions

- Topological phases of matter: massive topological bulk and massless boundary

→ new effective field theories, new view on anomalies

- anomaly inflow helps writing Euler hydrodynamics/bosonic effective field theory

→ bosonization in 1+1d recovered; in 3+1d hydro is a "cheap" bosonization

→ clearly show that anomalies parameterize universal, geometric effects/responses

Generalizations

- 3+1d hydrodynamics with 2 fluids → purely chiral fluid (Weyl fermions)
- 4+1d topological theories with 2-, 3-form fields suggest further hydrodynamic theories involving generalized symmetries
- 2+1d hydrodynamics? (global anomaly)
- add temperature and entropy; extend to many species & non-Abelian symmetries
- Applications to topological phases, heavy-ion collisions, cosmology,….

  (chiral magnetic effect, Kharzeev '11)