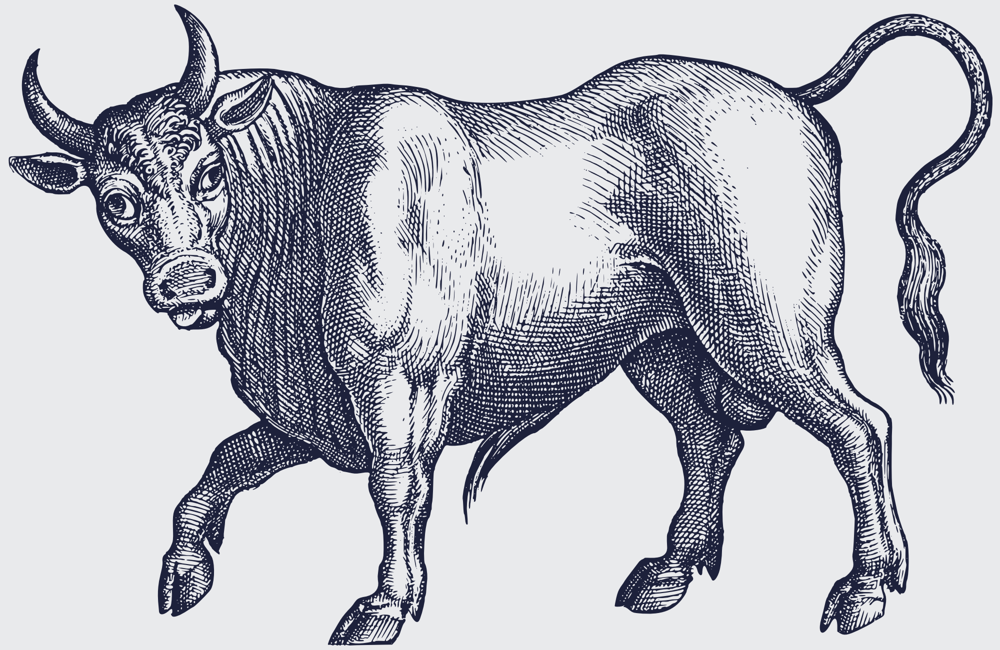
12/05/2025

Giornata tematica sulla

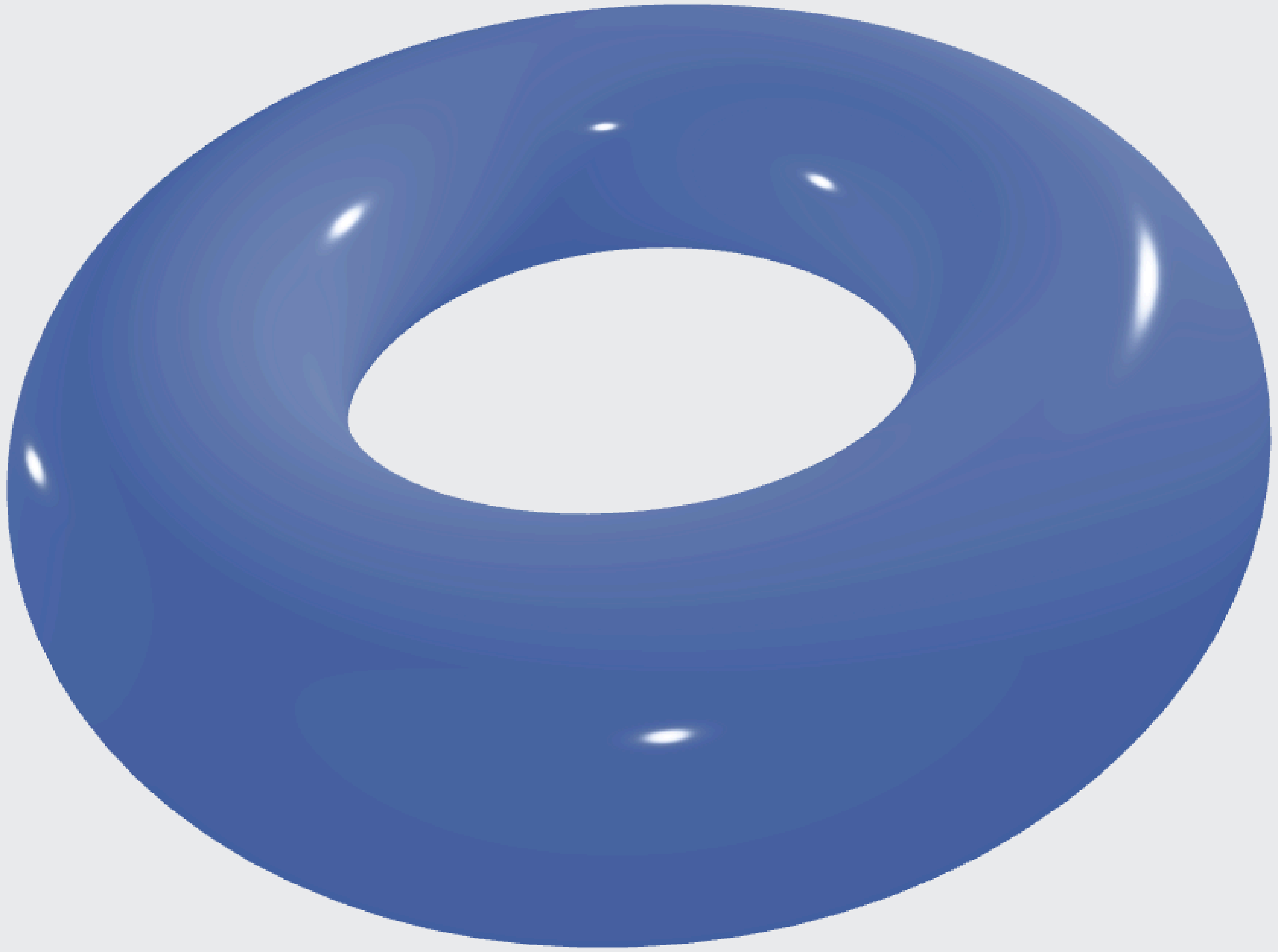
Fisica del Neutrino

IL TORO NELLA FISICA DEL NEUTRINO



Gabriele Battimelli

The Torus:



Describing the torus: the modulus τ

$$\tau \in C, \quad Im(\tau) > 0$$

Changing description:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \tau \to \frac{a\tau + b}{c\tau + d}$$

a, b, c, d *integers* $\qquad ad - bc = 1$

Modular Forms

A modular form is a holomorphic function

$$f(\tau)$$

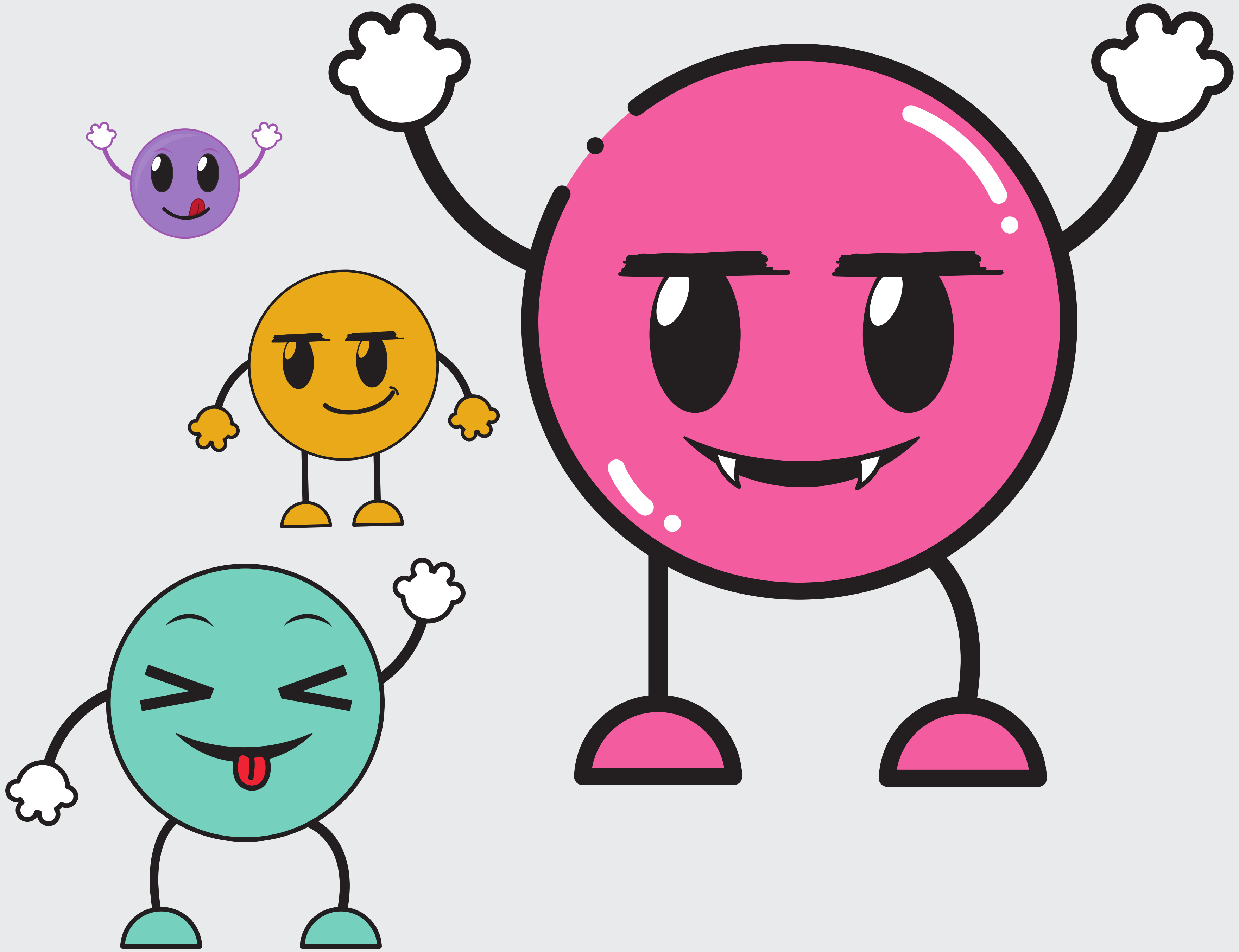
which transforms as

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

k *is called weight*

Explaining flavour

Different particle families, great mass differences



Mass matrices

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \xrightarrow{\text{Diagonalization}} \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}$$

Masses are eigenvalues

Can we give structure to these matrices?

Matrices derived from ***Lagrangian terms***

- **Building blocks, fields, transform as** $\Phi^{(i)} \rightarrow \sim (c\tau + d)^{-k_i} \Phi^{(i)}$

- **Modular forms as** $Y(\gamma\tau) \sim (c\tau + d)^{k_Y} Y(\tau)$

A typical term looks like $\sim \alpha Y(\tau) \Phi^{(1)} \ldots \Phi^{(n)}$

Invariance under symmetry transformation if

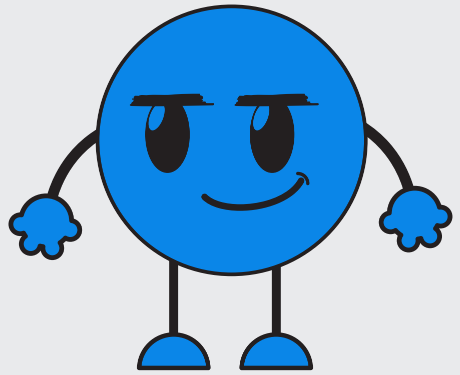
$$k_Y = k_1 + \ldots + k_n$$

Example: modular S3 for quarks

$$M_u = v_u \begin{pmatrix} a_u\left(Y_2^{(1)}\right)_1 & a_u\left(Y_2^{(1)}\right)_2 & 0 \\ b_u\left(Y_2^{(3)}\right)_2 & -b_u\left(Y_2^{(3)}\right)_1 & B_u Y_1^{(4)} \\ c_u\left(Y_2^{(3)}\right)_2 & -c_u\left(Y_2^{(3)}\right)_1 & C_u Y_1^{(4)} \end{pmatrix}$$

$$M_d = v_d \begin{pmatrix} a_d\left(Y_2^{(1)}\right)_1 & a_d\left(Y_2^{(1)}\right)_2 & 0 \\ b_d\left(Y_2^{(2)}\right)_2 & -b_d\left(Y_2^{(2)}\right)_1 & B_d Y_1^{(3)} \\ c_d\left(Y_2^{(2)}\right)_1 & c_d\left(Y_2^{(2)}\right)_2 & C_d Y_{1'}^{(3)} \end{pmatrix}$$

- “0” entries
- Same coefficients in different entries
- Various modular forms: hierarchical → hierarchy in matrix structure

Thank you! 

References and backup slides

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The modular group(s)

We are interested in transformations of the complex variable τ, the modulus, restricted to

$$\mathscr{H} = \{\tau \in \mathbb{C} \,|\, \operatorname{Im}(\tau) > 0\}$$

After introducing $SL(2,Z) = \Gamma$, the special linear group of 2×2 matrices with integer entries and determinant $ad - bc = 1$ also called *homogeneous modular group*,

let's consider the groups:

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,Z), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod N \right\} \qquad (\Gamma(1) = \Gamma)$$

Action on τ through linear fractional transformations: $\tau \to \gamma\tau = \dfrac{a\tau + b}{c\tau + d}$, $\quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

γ and $-\gamma$: same linear fractional transformation →

Groups of linear fractional transformation are isomorphic to groups $\overline{\Gamma}(N)$

$\overline{\Gamma} = \overline{\Gamma}(1) = \Gamma/\{\pm 1\}$: *inhomogeneous modular group* or simply *modular group*

(Also $\overline{\Gamma}(2) = \Gamma(2)/\{\pm 1\}$, while for $N > 2 \to \overline{\Gamma}(N) = \Gamma(N)$ as $-1 \notin \Gamma(N)$)

The *finite modular groups* are: $\Gamma_N = \overline{\Gamma}/\overline{\Gamma}(N)$

For $2 \leq N \leq 5$, Γ_N are isomorphic to permutation groups, widely used in model building:

$\Gamma_2 \cong S_3$, $\Gamma_3 \cong A_4$, $\Gamma_4 \cong S_4$, $\Gamma_5 \cong A_5$

Modular forms

A modular form is a holomorphic function $f(\tau)$ which transforms under $\Gamma(N)$ as

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

k (integer) is the weight, N is the level

- $k < 0$: no modular forms
- $k = 0$: constants only
- $k > 0$: they form a linear space $\mathscr{M}_k(\Gamma(N))$ of finite dimension $d_k(\Gamma(N))$

For example, if k even:

- $d_k(\Gamma(2)) = \dfrac{k}{2} + 1,$
- $d_k(\Gamma(3)) = k + 1,$
- $d_k\Gamma(4) = 2k + 1 ...$

Few independent modular forms of low weight

Crucial point: modular forms of level N and weight k transform under Γ_N

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

ρ representation of Γ_N; it is possible to choose a basis in $\mathcal{M}_k(\Gamma(N))$ such that it is unitary

Yukawa couplings will be modular forms and fields will transform under Γ_N, too

τ gets a vev $\rightarrow$ modular symmetry breaks

Supersymmetry

Standard Model extension: bosonic partners for fermions, fermionic partners for bosons

Building blocks are superfields Φ^i : they simultaneously describe a field and its partner

Yukawa interactions emerge from the superpotential

$$W(\Phi^i) = a_j \Phi^j + \frac{1}{2} m_{jk} \Phi^j \Phi^k + \frac{1}{3} g_{jkl} \Phi^j \Phi^k \Phi^l + \dots$$

The superpotential is a holomorphic function of superfields (required to preserve supersymmetry)

$\rightarrow$ Modular forms can appear inside it (holomorphic functions of τ)

Modular symmetry as a flavour symmetry

Given a particular Γ_N, Φ^i transforms: $\Phi^i \to (c\tau + d)^{-k_i} \rho(\gamma)_{ij} \Phi^j$, ρ mixes generations

Matter superfields are not modular forms → there are no restrictions on k_i

A term in the superpotential: $\alpha Y(\tau) \Phi^{(1)} \Phi^{(2)} \dots \Phi^{(n)}$, $Y(\tau)$ modular form in representation ρ_Y

Invariance if $\begin{cases} k_Y = k_1 + k_2 + \dots + k_n \\ \rho_Y \times \rho_1 \times \rho_2 \times \dots \times \rho_n \text{ contains a singlet} \end{cases}$

In minimal models: no other fields are needed, symmetry breaking only depends on the vev of τ

Looking at Yukawa interactions which will generate masses and mixing:

Few modular forms → few terms → small number of free parameters