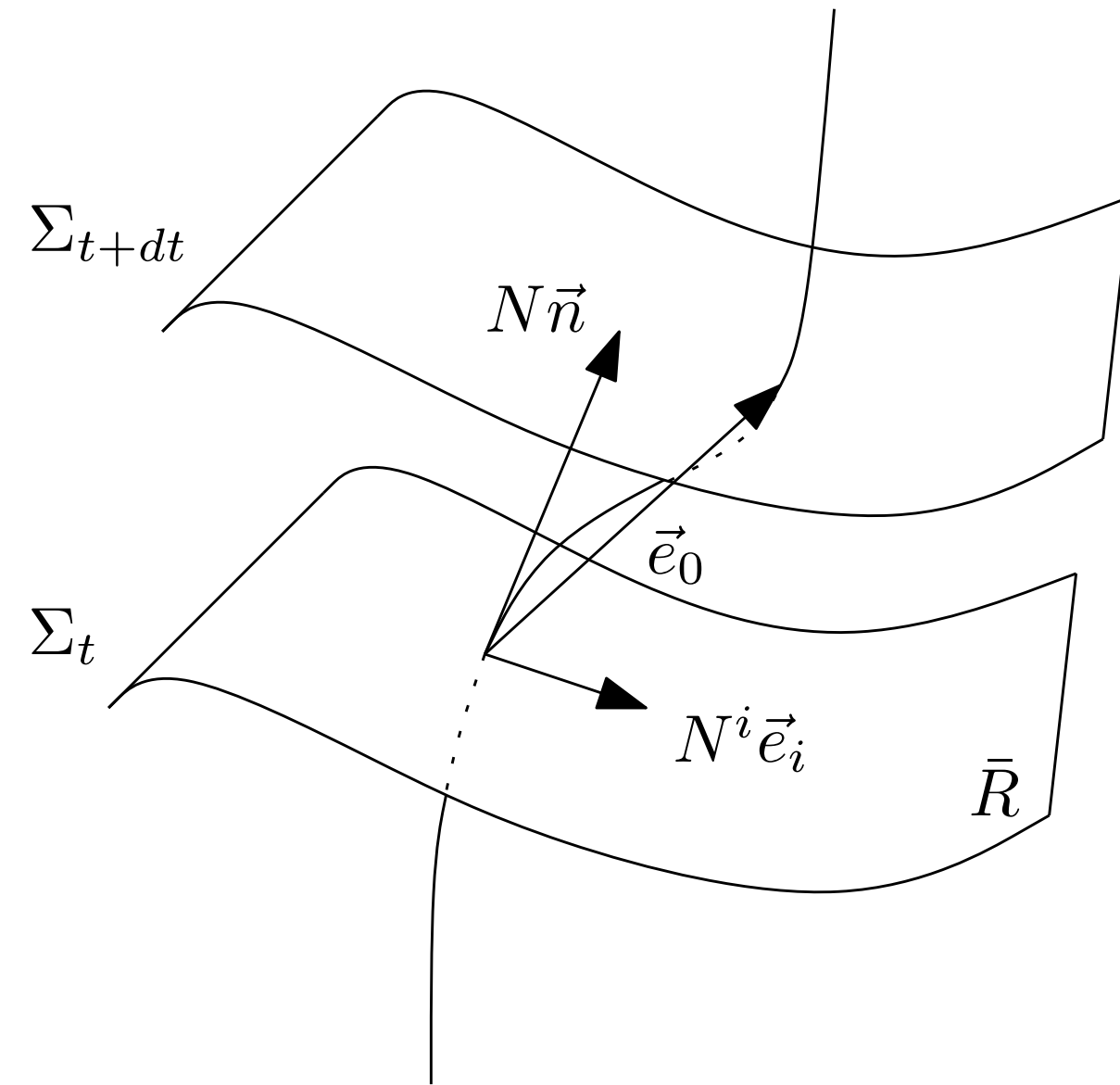


A stochastic gradient flow approach to nuclear and atomic physics, and the measurement problem

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Stochastic quantization, symmetry breaking and gravity



$$\frac{\partial}{\partial s} \phi_A(x^\mu, s) = -\frac{\delta S[\phi]}{\delta \phi_A} + \eta_A(x^\mu, s)$$

Additive (white) noise associated

$$\langle \eta_A(x, s) \rangle = 0$$

$$\langle \eta_A(x, s) \eta_B(x', s) \rangle = \alpha_\eta \delta_{AB} \delta(x - x') \delta(s - s')$$

Symmetries breakdown out-of-equilibrium, effective RG-noise and relaxation toward equilibrium

$$\frac{\partial}{\partial s} g_{\mu\nu} = -2 [R_{\mu\nu} - R_{\mu\nu}^T] + \eta_{\mu\nu}$$

$$= -2 \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{8\pi G}{c^4} T_{\mu\nu} \right] - g_{\mu\nu} (R - T) + \eta_{\mu\nu}$$

$$R_{\mu\nu}^T = \frac{8\pi G}{c^4} \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right]$$

Symmetries in GR

- **Generators of gauge symmetries** found from the **Hamiltonian** [1]

$$\mathcal{L}_{\text{ADM}} = N\sqrt{h} [K^2 - K^{ij}K_{ij} - \bar{R}]$$

$$\mathcal{H}_{\text{ADM}} = \Pi\dot{N} + \Pi_i\dot{N}^i + \Pi^{ij}\dot{h}_{ij} - \mathcal{L}_{\text{ADM}},$$

$$\boxed{\Pi = \frac{\partial \mathcal{L}_{\text{ADM}}}{\partial \dot{N}} = 0, \quad \Pi_i = \frac{\partial \mathcal{L}_{\text{ADM}}}{\partial \dot{N}^i} = 0,}$$

$$[\Pi, \mathcal{H}_{\text{ADM}}]_{\text{PB}} = 0,$$

$$[\Pi_i, \mathcal{H}_{\text{ADM}}]_{\text{PB}} = 0,$$

$$\mathcal{H} = G_{ijkl}\Pi^{ij}\Pi^{kl} - \sqrt{h}\bar{R} = 0,$$

Super-Hamiltonian

$$\mathcal{H}_i = 2h_{ij}\nabla_k^{(3)}\Pi^{kj} = 0,$$

Super-Momentum

Secondary constraints: eq. of motion

$$R^{0i} - \frac{1}{2}g^{0i}R = \frac{N\mathcal{H}^i + N^i\mathcal{H}}{2N^2\sqrt{h}} = 0,$$

$$R^{00} - \frac{1}{2}g^{00}R = -\frac{\mathcal{H}}{2N^2\sqrt{h}} = 0,$$

Symmetries in GR

- **Generators of gauge symmetries** found from the **Hamiltonian** [1]

$$\mathcal{L}_{\text{ADM}} = N\sqrt{h} [K^2 - K^{ij}K_{ij} - \bar{R}]$$

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$$\boxed{\Pi = \frac{\partial \mathcal{L}_{\text{ADM}}}{\partial \dot{N}} = 0, \quad \Pi_i = \frac{\partial \mathcal{L}_{\text{ADM}}}{\partial \dot{N}^i} = 0,}$$

**Generators of gauge transformation:
time, space diffeomorphisms**

$$[\Pi, \mathcal{H}_{\text{ADM}}]_{\text{PB}} = 0,$$

$$[\Pi_i, \mathcal{H}_{\text{ADM}}]_{\text{PB}} = 0,$$

$$\xi [\mathcal{H}, h_{ij}]_{\text{PB}} = \delta_t h_{ij} = \mathcal{L}_{\xi \mathbf{n}} h_{ij}$$

$$\mathcal{H} = G_{ijkl}\Pi^{ij}\Pi^{kl} - \sqrt{h}\bar{R} = 0,$$

Super-Hamiltonian

$$\mathcal{H}_i = 2h_{ij}\nabla_k^{(3)}\Pi^{kj} = 0,$$

Super-Momentum

$$\xi^k [\mathcal{H}_k, h_{ij}]_{\text{PB}} = \delta_{\xi} h_{ij} = \mathcal{L}_{\xi} h_{ij}$$

$$\mathcal{H}_{\text{ADM}} = \Pi\dot{N} + \Pi_i\dot{N}^i + N\mathcal{H} + N^i\mathcal{H}_i = 0,$$

The Hamiltonian vanishes

Symmetries in GR

Path-integral Quantization [1]:

Path integral

$$= \int [Dx^\mu] \prod_{t,\alpha} \delta(\chi_\alpha) \prod_t (\text{sdet} [\chi_\alpha, \chi_\beta])^{1/2} \exp iS[x^\mu(t)]. \quad (16.5)$$

Delta on the constraints



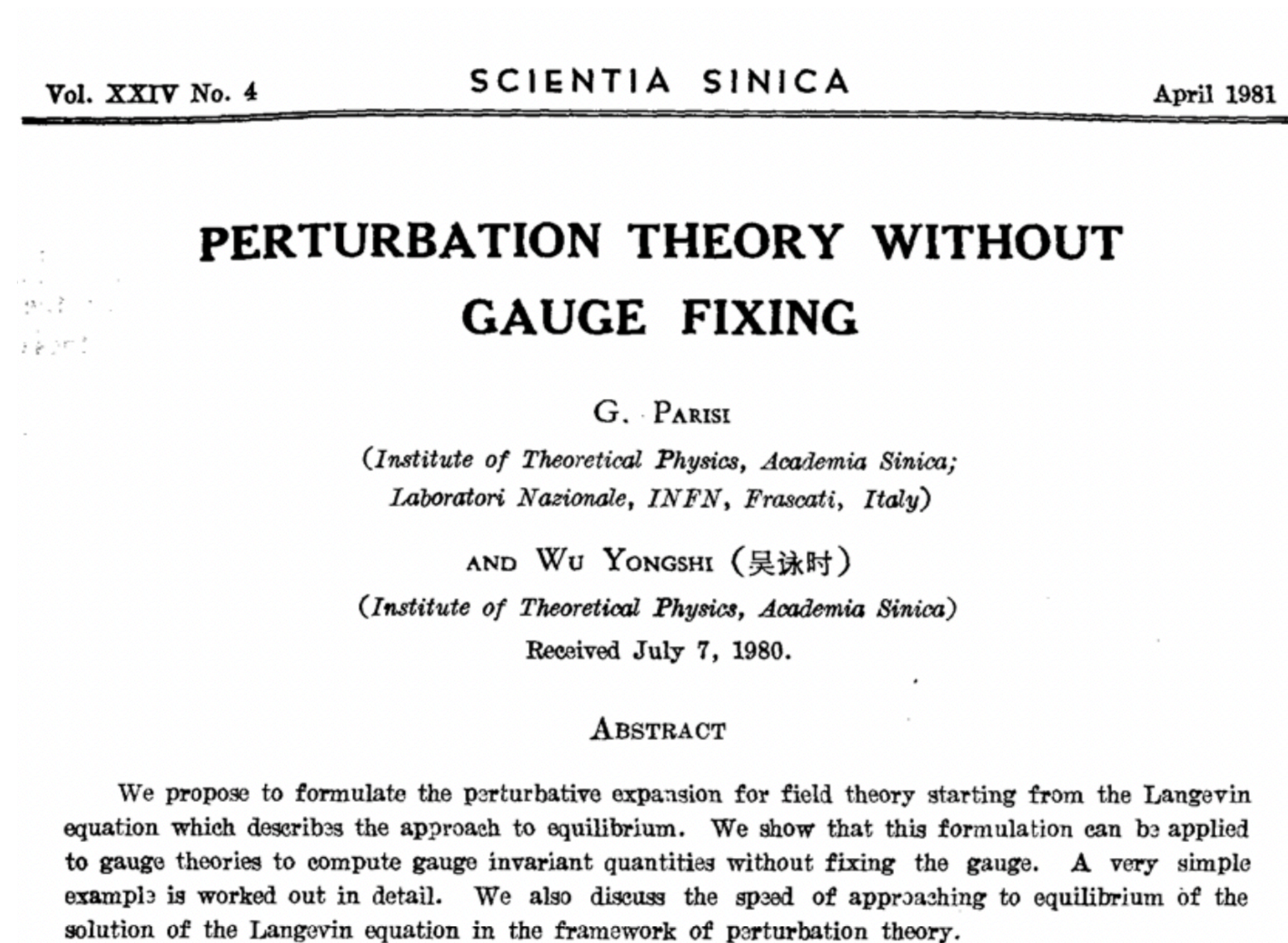
We are asking the path integral not to fluctuate around the constraints
In GR constraints generate **external** symmetries

Symmetries are imposed in the quantization of GR rather than being **allowed to emerge**

Quantization Methods - Continued

1. Canonical Quantization - Dynamic Perspective
2. Path-Integral Quantization - Ensemble Average Perspective
3. Stochastic Quantization

Stochastic Dynamic Relaxation of the Ensemble Average



Quantization Methods - Continued

1. Canonical Quantization - Dynamic Perspective
2. Path-Integral Quantization - Ensemble Average Perspective
3. Stochastic Quantization

A. Introduce a “stochastic time” variable s and noise - Langevin dynamics

$$\frac{\partial}{\partial s} \phi_A (x^\mu, s) = -\frac{\delta S [\phi]}{\delta \phi_A} + \eta_A (x^\mu, s),$$

$$\langle \eta_A (x, s) \eta_B (x', s') \rangle = 2\alpha \delta_{AB} \delta (x - x') \delta (s - s') \quad \langle \eta_A \rangle = 0$$

$$p (\eta) \sim [D\eta] \exp -\frac{1}{4\alpha} \int d^4x ds [\eta (x, s)]^2$$

The noise is **additive** and **Gaussian**: higher-order (even) correlations are functions of $\langle \eta_A \eta_B \rangle$

Quantization Methods - Continued

1. Canonical Quantization - Dynamic Perspective
2. Path-Integral Quantization - Ensemble Average Perspective
3. Stochastic Quantization

- A. Introduce a “stochastic time” variable s and noise - Langevin dynamics
- B. Expectation values: (i) with respect to the noise, (ii) with respect to $P(\phi, s)$

Average over histories

$$\langle \phi_{A_1, \eta}(x_1, s) \cdots \phi_{A_N, \eta}(x_N, s) \rangle = \frac{\int [D\eta] \exp \left[-\frac{1}{4} \int d^4x d\sigma \eta^2(x, \sigma) \right] \phi_{A_1, \eta}(x_1, s) \cdots \phi_{A_N, \eta}(x_N, s)}{\int [D\eta] \exp \left[-\frac{1}{4} \int d^4x d\sigma \eta^2(x, \sigma) \right]}$$

Average over dynamic measure

$$\langle \phi_{A_1, \eta}(x_1, s) \cdots \phi_{A_N, \eta}(x_N, s) \rangle = \int [D\phi_B] P(\phi, s) \phi_{A_1}(x_1) \cdots \phi_{A_N}(x_N)$$

The Fokker-Planck Eq., associated to the Langevin, dictates the dynamics of $P(\phi, s)$

Quantization Methods - Continued

1. Canonical Quantization - Dynamic Perspective
2. Path-Integral Quantization - Ensemble Average Perspective
3. Stochastic Quantization

- A. Introduce a “stochastic time” variable S and noise - Langevin dynamics
- B. Expectation values: (i) with respect to the noise, (ii) with respect to $P(\phi, s)$
- C. The associated Fokker-Planck ensures the correct “equilibrium” limit of $P(\phi, s)$

$$\frac{\partial F}{\partial s} = - \frac{\partial F}{\partial \phi_A} \frac{\delta S[\phi]}{\delta \phi_A} + \frac{1}{2} \frac{\partial^2 F}{\partial \phi_A^2} + \frac{\partial F}{\partial \phi_A} \eta_A \quad \text{Stochastic Calculus}$$

$$\frac{\partial P(\phi, s)}{\partial s} = \frac{\partial}{\partial \phi_A} \left[P(\phi, s) \frac{\delta S[\phi]}{\delta \phi_A} \right] + \frac{1}{2} \frac{\partial^2 P(\phi, s)}{\partial \phi_A^2} \quad \text{Fokker-Planck}$$

$$\frac{\partial P(\phi, s)}{\partial s} = 0 \quad \longrightarrow \quad \lim_{s \rightarrow \infty} P(\phi, s) = P(\phi) \sim \exp[-S]$$

The Euclidean Path-Integral measure is recovered at equilibrium

$$\lim_{s \rightarrow +\infty} \langle \phi_A(x, s) \phi_B(x', s) \rangle_{p(s)} = \langle \phi_A(x) \phi_B(x') \rangle_E$$

Quantization Methods - Continued

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2. Path-Integral Quantization - Ensemble Average Perspective

3. Stochastic Quantization

- A. Introduce a “stochastic time” variable s and noise - Langevin dynamics
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- C. The associated Fokker-Planck ensures the correct “equilibrium” limit of $P(\phi, s)$
- D. Gauge theories: the **propagator splits** in a **gauge-invariant** part (**finite**) and a **gauge-dependent** part (**divergent**)

$$\langle \phi_A(x, s) \phi_B(x', s) \rangle_{p(s)} = \underbrace{\langle \phi_A(x, s) \phi_B(x', s) \rangle_{\text{gauge-inv}}}_{\text{Convergent to gauge-invariant propagator}} + \underbrace{\langle \phi_A(x, s) \phi_B(x', s) \rangle_{\text{not-gauge-inv}}}_{\text{Divergent in eq. limit and gauge-dependent}},$$

No need to use Faddeev-Popov approach when applied to gauge theories

Stochastic Quantization of General Relativity à la Ricci Flow

- Let's consider the presence of matter $S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}_M,$

**We propose the Langevin equation [1]
with scalar multiplicative noise**

$$\frac{\partial}{\partial s} g_{\mu\nu}(x, s) = \imath \mathcal{G}_{\alpha\beta\mu\nu}(\lambda = -1) \frac{\delta S}{\delta g_{\alpha\beta}} + g_{\mu\nu}(x, s) e^{\imath \frac{\gamma}{2}} \sqrt{2\alpha} \tilde{\eta}(x, s)$$

$$\langle \tilde{\eta}(x, s) \tilde{\eta}(x', s') \rangle = \delta(s - s') \delta^{(4)}(x - x')$$

- Starting point: seminal Rumpf work [2]
with additive tensorial noise

$$\frac{\partial}{\partial s} g_{\mu\nu}(x, s) = \imath \mathcal{G}_{\alpha\beta\mu\nu}(\lambda) \frac{\delta S}{\delta g_{\alpha\beta}} + \eta_{\mu\nu}(x, s)$$

$$\langle \eta_{\alpha\beta}(x, s) \eta_{\mu\nu}(x', s') \rangle = \mathcal{G}_{\alpha\beta\mu\nu}(\lambda) \delta(s - s') \delta^{(4)}(x - x')$$

- DeWitt Supermetric: special choice [2] $\lambda = -1$
- Strong link to Horava-Lifshitz gravity

$$\mathcal{G}^{\alpha\beta\mu\nu}(\lambda) = \frac{\sqrt{-g}}{2\kappa} [g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - \lambda g^{\alpha\beta} g^{\mu\nu}]$$

$$\mathcal{G}_{\alpha\beta\mu\nu}(\lambda) = \frac{2\kappa}{\sqrt{-g}} \left[g_{\alpha\mu} g_{\beta\nu} + g_{\alpha\nu} g_{\beta\mu} - \frac{\lambda}{2\lambda + 1} g_{\alpha\beta} g_{\mu\nu} \right]$$

[1] A. M. Lulli, A. Marciano, X. Shan, arXiv:2112.01490 (2021)

[2] H. Rumpf, Phys. Rev.D, 33, 4, 1986

Stochastic Quantization of General Relativity à la Ricci Flow

- Let's consider the presence of matter $S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}_M,$

**We propose the Langevin equation [1]
with scalar multiplicative noise**

$$\frac{\partial g_{\mu\nu}}{\partial s} = -2\imath \left[R_{\mu\nu} - \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \right] + g_{\mu\nu} e^{\imath \frac{\gamma}{2}} \sqrt{2\alpha} \tilde{\eta}$$
$$\langle \tilde{\eta}(x, s) \tilde{\eta}(x', s') \rangle = \delta(s - s') \delta^{(4)}(x - x')$$

- Zero-noise limit: looks like a Ricci-flow with a target fixed point
- Noise amplitude to be determined at the saddle-point of the equilibrium

[1] A. M. Lulli, A. Marciano, X. Shan, arXiv:2112.01490 (2021)

[2] H. Rumpf, Phys. Rev.D, 33, 4, 1986

Emergent Cosmological Constant

$$\frac{\partial g_{\mu\nu}}{\partial s} = -2\imath \left[R_{\mu\nu} - \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \right] + g_{\mu\nu} e^{\imath \frac{\gamma}{2}} \sqrt{2\alpha} \tilde{\eta},$$

- We need to “interpret” the Langevin equation: Itô calculus [1]
- Different variables coupled to the same noise
- Need to compute Itô rules for this case: for a generic F

$$\frac{\partial F}{\partial s} = -2\imath \frac{\partial F}{\partial g_{\mu\nu}} \left[R_{\mu\nu} - R_{\mu\nu}^T \right] + \alpha e^{\imath \gamma} g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2 F}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} + e^{\imath \frac{\gamma}{2}} \sqrt{2\alpha} \frac{\partial F}{\partial g_{\mu\nu}} g_{\mu\nu} \tilde{\eta}$$

- The associated Fokker-Planck reads

$$\partial_s p = - \frac{\partial}{\partial g_{\mu\nu}} \left(-2\imath \left[R_{\mu\nu} - R_{\mu\nu}^T \right] p \right) + \alpha e^{\imath \gamma} \frac{\partial^2}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} (g_{\mu\nu} g_{\alpha\beta} p)$$

Emergent Cosmological Constant

- We can look for the steady state solution of the Fokker-Planck

$$\partial_s p = -\frac{\partial}{\partial g_{\mu\nu}} \left(-2\imath [R_{\mu\nu} - R_{\mu\nu}^T] p \right) + \alpha e^{\imath\gamma} \frac{\partial^2}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} (g_{\mu\nu} g_{\alpha\beta} p)$$

- After setting $\partial_s p = 0$ one obtains

$$p(g_{\mu\nu}) \simeq \mathcal{C} \exp \left\{ \frac{1}{D\alpha e^{\imath\gamma}} \int^{g^{\mu\nu}} \mathcal{D}\bar{g}^{\mu\nu} \left[-2\imath (R_{\mu\nu} - R_{\mu\nu}^T) - \alpha e^{\imath\gamma} (1 + n_g) \bar{g}_{\mu\nu} \right] \right\}$$

Number of independent
metric components



- The saddle-point $\frac{\delta p}{\delta g^{\mu\nu}} = 0$ condition reads $R_{\mu\nu} - R_{\mu\nu}^T - \frac{1}{2}\alpha (1 + n_g) e^{\imath(\gamma + \frac{\pi}{2})} g_{\mu\nu} = 0$

- After setting $\frac{1}{2}\alpha (1 + n_g) = \Lambda$, $\alpha = \frac{2\Lambda}{1 + n_g}$,

**The cosmological constant
appears as a consequence of the noise at the saddle point at equilibrium (!)**

$$R_{\mu\nu} - \Lambda e^{\imath(\gamma - \frac{\pi}{2})} g_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

Stochastic Quantization à la Ricci-Flow in ADM

Let us go from the metric tensor to the ADM variables

We need to follow the Itô transformation rule

$$\frac{\partial F}{\partial s} = -2\imath \frac{\partial F}{\partial g_{\mu\nu}} [R_{\mu\nu} - R_{\mu\nu}^T] + \alpha e^{\imath\gamma} \left(g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2 F}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} \right) + e^{\imath\frac{\gamma}{2}} \sqrt{2\alpha} \frac{\partial F}{\partial g_{\mu\nu}} g_{\mu\nu} \tilde{\eta}$$

The final set of equations reads

$$\frac{\partial N}{\partial s} = -\frac{N}{2} \left[\imath \left(\frac{\mathcal{H}}{\sqrt{h}} + R \right) + \frac{\Lambda e^{-\imath\gamma}}{1 + n_g} + e^{\imath\frac{\gamma}{2}} \sqrt{\frac{4\Lambda}{1 + n_g}} \tilde{\eta} \right]$$

$$\boxed{\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}}$$

$$\frac{\partial h_{ij}}{\partial s} = \frac{1}{N} \mathcal{L}_{Nn} [\mathcal{H}, h_{ij}] + [\mathcal{H}, [\mathcal{H}, h_{ij}]] + \frac{h_{ij} \mathcal{H}}{2\sqrt{h}} - h_{ij} e^{\imath\frac{\gamma}{2}} \sqrt{\frac{4\Lambda}{1 + n_g}} \tilde{\eta}$$

Projective connection & out-of-equilibrium torsional DOFs

$$\bar{\Gamma}_{\alpha\beta}^{\gamma} = \Gamma_{\alpha\beta}^{\gamma} + \mathcal{C}_{\alpha\beta}^{\gamma}$$

$$\mathcal{C}_{\alpha\beta}^{\gamma} = \lambda_1 \delta_{\alpha}^{\gamma} u_{\beta} + \lambda_2 u_{\alpha} \delta_{\beta}^{\gamma} + \lambda_3 w_{\alpha\beta} u^{\gamma} + \lambda_4 u_{\alpha} u_{\beta} u^{\gamma}$$

$$\sqrt{-g} \bar{R} = \sqrt{-g} R + \sqrt{-g} g^{\beta\delta} \left(\mathcal{C}_{\beta\delta}^{\mu} \mathcal{C}_{\mu\alpha}^{\alpha} - \mathcal{C}_{\beta\alpha}^{\mu} \mathcal{C}_{\mu\delta}^{\alpha} \right)$$

Cosmological term induced in the action

$$\begin{aligned} & \sqrt{-g} g^{\beta\delta} \left(\mathcal{C}_{\beta\delta}^{\mu} \mathcal{C}_{\mu\alpha}^{\alpha} - \mathcal{C}_{\beta\alpha}^{\mu} \mathcal{C}_{\mu\delta}^{\alpha} \right) \\ &= \sqrt{-g} \left[(\lambda_2^2 + \lambda_3^2) (D - 1) u_{\mu} u^{\mu} \right] \end{aligned}$$

Out-of-equilibrium torsional DOFs and topology changes

E_0 trivial $SU(3)$ bundle and E_1 non-trivial $SU(3)$ bundle

$$a : E_0 \rightarrow E_1$$

Map between trivial and non-trivial bundles induces topology changes

$$c_2(E_1) = \frac{1}{8\pi^2} \int_M \text{tr}(F \wedge F), \quad c_2(E_0) = 0$$

Singularities of the maps produce non-trivial instantons $F = \pm * F$

map a induces singular gauge transformation g_a

$$\oint_{\gamma=\partial S} A = \int_S dA = \int_S d(g_a^{-1} dg_a) = \int_S F$$

Stokes theorem and area-law for the holonomy

Breakdown of the conformal symmetry and the scalaron

$$h_{\mu\nu} = h_{\mu\nu}^{\perp} + \partial_{\mu} a_{\nu}^{\perp} + \partial_{\nu} a_{\mu}^{\perp} \\ + \left(\partial_{\mu} \partial_{\nu} - \frac{1}{4} \eta_{\mu\nu} \square \right) a + \frac{1}{4} \eta_{\mu\nu} \varphi \\ \Phi = \varphi - \bar{\square} a$$

Einstein-Hilbert expanded on dS or AdS backgrounds

$$\mathcal{S}_{\text{EH}}^{(2)} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-\bar{g}} \left[\frac{1}{4} h_{\perp}^{\mu\nu} \left(\bar{\square} - \frac{\bar{R}}{6} \right) h_{\mu\nu}^{\perp} - \frac{3}{32} \Phi \left(\bar{\square} + \frac{\bar{R}}{3} \right) \Phi \right]$$

Residual gauge transformation: conformal Killing vector and disappearance of the ghost

$$h_{\mu\nu}^{\perp} \rightarrow h_{\mu\nu}^{\perp} + \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu}, \quad \Phi \rightarrow \Phi + 2\nabla^{\mu} k_{\mu}$$

Hamiltonian analysis of the Ricci flow

$$\eta_{\mu\nu} = \eta g_{\mu\nu}$$

Multiplicative choice of the noise source entails additivity in the Hamiltonian ADM picture

$$\frac{\partial N}{\partial s} = -\frac{N}{2} \left[\imath \left(\frac{\mathcal{H}}{\sqrt{h}} + R \right) + \frac{\Lambda e^{-\imath\gamma}}{1+n_g} + e^{\imath\frac{\gamma}{2}} \sqrt{\frac{4\Lambda}{1+n_g}} \tilde{\eta} \right]$$

$$\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}$$

$$\frac{\partial h_{ij}}{\partial s} = \frac{1}{N} \mathcal{L}_{Nn} [\mathcal{H}, h_{ij}] + [\mathcal{H}, [\mathcal{H}, h_{ij}]] + \frac{h_{ij} \mathcal{H}}{2\sqrt{h}} - h_{ij} e^{\imath\frac{\gamma}{2}} \sqrt{\frac{4\Lambda}{1+n_g}} \tilde{\eta}$$

Shift-vector and Navier-Stokes

- Let's consider the shift-vector equation

$$\frac{\partial N^k}{\partial s} = \frac{i N \mathcal{H}^k}{\sqrt{h}}$$

- **The noise term cancels out**

- The fixed point solution automatically implements super-Momentum constraint

$$\mathcal{H}_i = 2h_{ij} \nabla_k^{(3)} \Pi^{kj} = 0,$$

- In the non-relativistic limit the super-momentum reduces to Navier-Stokes [1]

$$\Pi^{ij} = \sqrt{h}(K h^{ij} - K^{ij}) = \frac{1}{2} \sqrt{h} T^{ij} \longrightarrow \boxed{\text{Brown-York stress tensor}}$$

Its divergence yields Navier-Stokes $r_c^{3/2} \partial^k T_{ki} = \partial_t v_i - \zeta \partial^2 v_i + \partial_i P + v^k \partial_k v_i = 0$

Incompressibility $r_c^{3/2} \partial_k T^{kt} = \partial_k v^k = 0$

Shift-vector and Navier-Stokes

- It reduces to a forced incompressible Navier-Stokes equation (for Euclidean signature)

$$\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}$$



$$\partial_t v^i - \zeta \partial^2 v^i + \partial^i P + v^k \partial_k v^i = -\frac{\imath}{N} \frac{\partial N^i}{\partial s}$$

- **The noise term cancels out**
- The fields can enter a turbulent regime
- Turbulent fluctuations are **intermittent**
- Possibility to investigate the multi-fractal hypothesis [1] in GR

The only equation that loses noise responds with intermittency!

$$r_c^{3/2} \partial_k T^{kt} = \partial_k v^k = 0$$

Possible role of turbulence in cosmological first-order phase transitions (!)

Out-of-equilibrium dynamics of Black Holes

- Let us consider a “static” spherical symmetric metric $ds^2 = -e^{\nu(r)} dt^2 + e^{\mu(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$
- After performing Itô transformation and taking the **quasi-equilibrium limit** $\mu = -\nu$

$$\frac{\partial \nu}{\partial s} = \frac{d\nu}{dg_{00}} \frac{\partial g_{00}}{\partial s} + e^{\nu} g_{00}^2 \frac{d^2 \nu}{dg_{00}^2},$$

$$\frac{\partial \nu}{\partial s} = -\imath \left[\frac{d^2 \nu}{dr^2} + \frac{2}{r} \frac{d\nu}{dr} + \left(\frac{d\nu}{dr} \right)^2 \right] e^{\nu} - \frac{1}{2} e^{\imath \frac{\gamma}{2}} \sqrt{2\alpha} \left(\tilde{\eta} + e^{\imath \frac{\gamma}{2}} \sqrt{2\alpha} \right)$$

- Let's normalise the equation to 2α

$$\nu = \nu_0(r) + \bar{\nu}(r, s) \quad \frac{\bar{\nu}}{\sqrt{2\alpha}} = \bar{h} \quad s\sqrt{2\alpha} = t \quad \bar{\eta} = -\frac{1}{2} \frac{e^{\imath \frac{\gamma}{2}}}{\sqrt{2\alpha}} \tilde{\eta} \quad h = \bar{h} + \frac{1}{2} e^{\imath \frac{\gamma}{2}} t$$

$$\frac{1}{2\alpha} \frac{\partial \bar{\nu}}{\partial s} = \frac{\partial \bar{h}}{\partial t} = \imath \left[\frac{e^{\nu_0}}{\sqrt{2\alpha}} \left(\frac{d^2 \bar{h}}{dr^2} + \frac{2}{r} \frac{d\bar{h}}{dr} \right) + e^{\nu_0} \left(\frac{d\bar{h}}{dr} \right)^2 \right] + \bar{\eta} - \frac{1}{2} e^{\imath \frac{\gamma}{2}}$$

$$\frac{\partial h}{\partial t} = \imath \left[\nu_{\text{KPZ}}(r, \alpha) \nabla^2 h(r, s) + \lambda_{\text{KPZ}}(r) \left(\frac{dh}{dr} \right)^2 \right] + \bar{\eta} \quad \textbf{Kardar-Parisi-Zhang Eq. [1]}$$

for a spherical surface

$$\nu_{\text{KPZ}}(r, \alpha) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \quad \lambda_{\text{KPZ}}(r) = e^{\nu_0(r)}$$

Black Holes and Kardar-Parisi-Zhang

$$\frac{\partial h}{\partial t} = \nu \left[\nu_{\text{KPZ}}(r, \alpha) \nabla^2 h(r, s) + \lambda_{\text{KPZ}}(r) \left(\frac{dh}{dr} \right)^2 \right] + \bar{\eta}$$

Kardar-Parisi-Zhang Eq. [1]
for a spherical surface

$$\nu_{\text{KPZ}}(r, \alpha) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \quad \lambda_{\text{KPZ}}(r) = e^{\nu_0(r)}$$

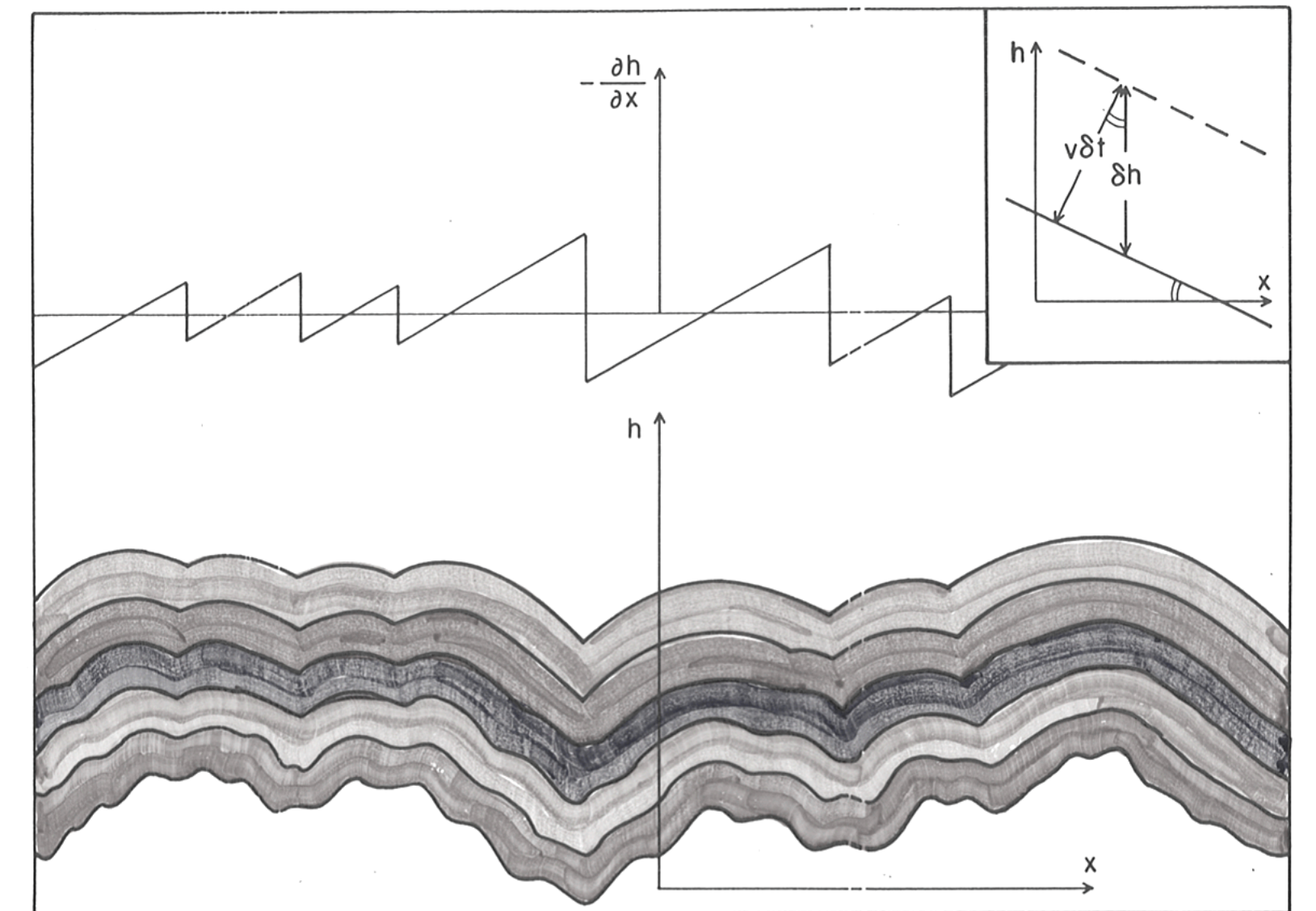
- KPZ defines its own **dynamic universality class** $z = 3/2$
- It models **surface growth** by deposition: **surface height** h
- It is related to Burgers' equation via a change of variable

$$u = -\lambda \frac{\partial h}{\partial x} \longrightarrow \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

- Both models exhibit **intermittency** for small dissipation
- The small dissipation limit is obtained near the horizon

$$\lim_{r \rightarrow r_s} \nu_{\text{KPZ}}(r, \alpha) = \lim_{r \rightarrow r_s} \lambda_{\text{KPZ}}(r) \propto \lim_{r \rightarrow r_s} g_{00} = \lim_{r \rightarrow r_s} \left(1 - \frac{r_s}{r} \right) = 0$$

[1] M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)



Stochastic Ricci RG flow of Λ

$$ds^2 = -N^2 dt^2 + a^2(t) \left[\frac{(dr)^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

FLRW background

$$S = 6 \int d^4x N a^3 R + \int d^4x N a^3 (D-1) \lambda_2^2 \epsilon(\lambda) \quad R = 6 \left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}^2}{a^2} \right) + \frac{k}{a^2} \right)$$

$$\begin{aligned} \frac{\partial a}{\partial s} &= -\frac{2\iota}{N^2} \left(a\dot{H} + 3aH^2 + \epsilon N^2 \lambda_2^2 \right) + a\eta, \\ \frac{\partial N}{\partial s} &= -2\iota \left(\frac{3}{2N} (\dot{H} + H^2) + \frac{1}{16} N (\Lambda_0 + 8\lambda_2^2) \right) \\ &\quad - N\eta, \\ \frac{\partial \lambda_2}{\partial s} &= \iota (-2\epsilon - \imath\eta) \lambda_2, \end{aligned}$$

Ricci flow equations

Hubble tension: a macroscopic QG effect?

$$\langle \lambda_2^k(s) \rangle = \exp \left[\left(i(-2\varepsilon) + \frac{\Lambda_0}{2} \right) s \right] \langle \lambda_2^k(0) \rangle$$

Thermal time oriented as the proper time implies mild increase of Λ

[1] M. Lulli, A. Marciano, X. Shan, arXiv:2112.01490 (2021)

[2] M. Lulli, A. Marciano, L. Visinelli, in preparation

Cosmological measurements $67.4 \pm 1.4 \quad (\text{km/s})/\text{Mpc}$

Astronomical measurements $74.03 \pm 1.42 \quad (\text{km/s})/\text{Mpc}$

Diosi-Penrose model from Stochastic Gradient Flow

Diosi-Penrose quantum collapse master equation as a
the non-relativistic limit of the stochastic geometry (Ricci) flow

Running of the lapse function in a stochastic thermal time

Strategy: describe a discrete system of masses undergoing
gravitational interaction

First principle discussion and RG flow induced by a stochastic gradient (Ricci) flow

[M. Lulli, A. Marciano & X. Shan, arXiv:2112.01490v2](#)

Derivation of the Diosi Penrose quantum collapse master equation from the
stochastic geometry (Ricci) flow and discussion of the parameter space

[M. Lulli, A. Marciano & K. Piscicchia, arXiv:2307.10136](#)

Non-relativistic limit of the Stochastic Gradient Flow

Consider a discrete system of gravitationally interacting bodies

Ideal gas approximation for the energy-momentum tensor

Use the non-relativistic semi-classical limit of

$$\frac{\partial N}{\partial s} = -\frac{N}{2} \left[i \left(\frac{\mathcal{H}}{\sqrt{h}} + R \right) + \frac{\Lambda e^{-i\gamma}}{1 + n_g} + e^{i\frac{\gamma}{2}} \sqrt{\frac{4\Lambda}{1 + n_g}} \tilde{\eta} \right]$$

$$d|\psi_t\rangle = \left[-\frac{i}{\hbar} H dt + \sqrt{\frac{G}{\hbar}} \int d\mathbf{x} (\mathcal{M}(\mathbf{x}) - \langle \mathcal{M}(\mathbf{x}) \rangle_t) dW(\mathbf{x}, t) \right. \\ \left. - \frac{G}{2\hbar} \int d\mathbf{x} \int d\mathbf{y} \frac{(\mathcal{M}(\mathbf{x}) - \langle \mathcal{M}(\mathbf{x}) \rangle_t)(\mathcal{M}(\mathbf{y}) - \langle \mathcal{M}(\mathbf{y}) \rangle_t)}{|\mathbf{x} - \mathbf{y}|} dt \right] |\psi_t\rangle,$$

Assumptions to recover Diosi-Penrose

DP quantum collapse master equation as out-of-equilibrium relaxation
described through the stochastic geometry (Ricci) flow

Similarity: both realizations have multiplicative noise

New task: explain the role of the lapse, recalling that $N=1+V+\dots$

Tools: use the Ito calculus to account for the variation of the lapse

Langevin equation and perfect gas model

Langevin equation for the shift

$$\frac{\partial N}{\partial s} = -2\imath \frac{\partial N}{\partial g_{\mu\nu}} [R_{\mu\nu} - R_{\mu\nu}^T] + \alpha e^{\imath\gamma} g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2 N}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} + e^{\imath\gamma/2} \sqrt{2\alpha} \frac{\partial N}{\partial g_{\mu\nu}} g_{\mu\nu} \tilde{\eta}$$

$$\frac{1}{2}\alpha (1 + n_g) = \Lambda, \quad \alpha = \frac{2\Lambda}{1 + n_g},$$

with matter Ricci target

$$R_{\mu\nu}^T = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

$$T_{\nu}^{\mu} = \text{diag} \{ -\rho, \rho\sigma T, \rho\sigma T, \rho\sigma T \}$$

The role of the Ito calculus

Using the Ito calculus

$$dN = \left\{ -\frac{\imath}{2N} [\nabla^2 N^2 + \kappa N^2 \rho (3\sigma T + 1)] - \frac{\alpha}{4} e^{\imath\gamma} N \right\} ds + e^{\imath\gamma/2} \sqrt{\frac{\alpha}{2}} N dW$$

$$dN = a ds + b dW$$



$$df = \int d^4x m_x \left[\frac{\delta f}{\delta N_x} a_x + \frac{1}{2} \int d^4y n_y b_x b_y k_\varepsilon(x, y) \frac{\delta}{\delta N_x} \frac{\delta}{\delta N_y} f \right] ds + \int d^4x m_x \frac{\delta f}{\delta N_x} b_x dW_x$$

for a functional $f=f[N]$



$$\left\langle \frac{df}{ds} \right\rangle = - \int d^4x dN_x m_x f \frac{\delta}{\delta N_x} \left[(a_x p[N_x]) - \frac{1}{2} \int d^4y d^4z n_y \frac{\delta}{\delta N_y} (b_x b_z p[N_x]) k_\varepsilon(x, y) k_\varepsilon(y, z) \right]$$

Stationarized Fokker-Planck and WKB approximation

Stationary Fokker-Planck

$$a_x p[N_x] - \frac{1}{2} \int d^4 y d^4 z n_y \frac{\delta}{\delta N_y} (b_x b_z p[N_x]) k_\varepsilon(x, y) k_{\varepsilon'}(y, z) = 0$$



$$p[N_x] = B \exp \left\{ \int^{N_x} \prod_y dN_y \frac{4}{e^{i\gamma} \alpha N_y^2} \left[a_y - \frac{e^{i\gamma} \alpha}{2} N_y \right] \right\} = B \exp \left[\frac{i}{\hbar} S_0 \right]$$

GKLS equation and non-cocommutativity in stochastic time

Non-unitary but trace-preserving and positive

$$\begin{aligned} d\Psi = & \left\{ \int d^4x m_x \left[2\imath\kappa\rho_x (2 + 3\sigma T) + \frac{\alpha}{4} e^{\imath\gamma} \right] ds \right\} \Psi \\ & - \left\{ \frac{e^{-\imath\frac{\gamma}{2}}}{\sqrt{2\alpha}} \int d^4x m_x [2\imath\kappa\rho_x (3\sigma T + 2) + 3\alpha e^{\imath\gamma}] dW_x \right\} \Psi \\ & + \left\{ \frac{1}{2} \int d^4x m_x \int d^4y n_y \frac{1}{2} k_\varepsilon(x, y) \frac{1}{e^{\imath\gamma}\alpha} ds \times \right. \\ & \left. [2\imath\kappa\rho_x (2 + 3\sigma T) + 3\alpha e^{\imath\gamma}] [2\imath\kappa\rho_y (2 + 3\sigma T) + 3\alpha e^{\imath\gamma}] \right\} \Psi \end{aligned}$$



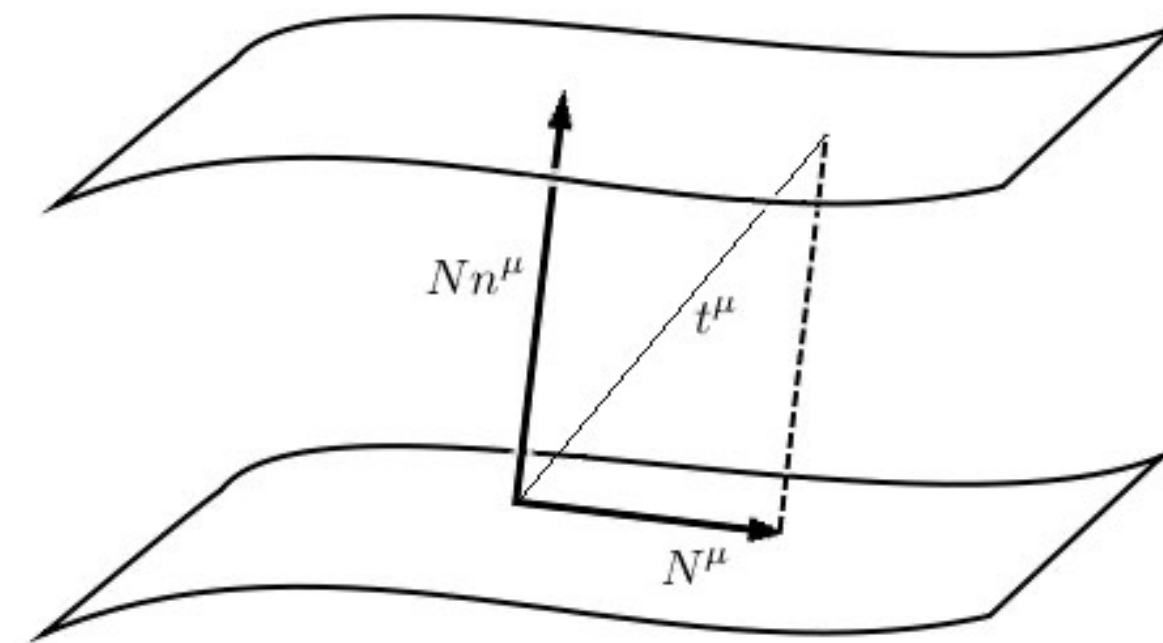
$$\begin{aligned} \frac{d}{ds} \rho_\Psi & \equiv \left\langle \frac{d}{ds} \rho_\Psi \right\rangle_{\tilde{\eta}} = \imath \int d^4x m_x [\mathcal{H}_x, \rho_\Psi] \\ & + \int d^4x m_x \int d^4y n_y \tilde{k}_\varepsilon(x, y) \mathcal{L}_x \rho_\Psi \mathcal{L}_y^\dagger \\ & - \frac{1}{2} \int d^4x m_x \int d^4y n_y \tilde{k}_\varepsilon(x, y) \{ \mathcal{L}_x^\dagger \mathcal{L}_y, \rho_\Psi \} \end{aligned}$$

Generalization of GKLS Equation

Berry phase and stochastic gradient flows

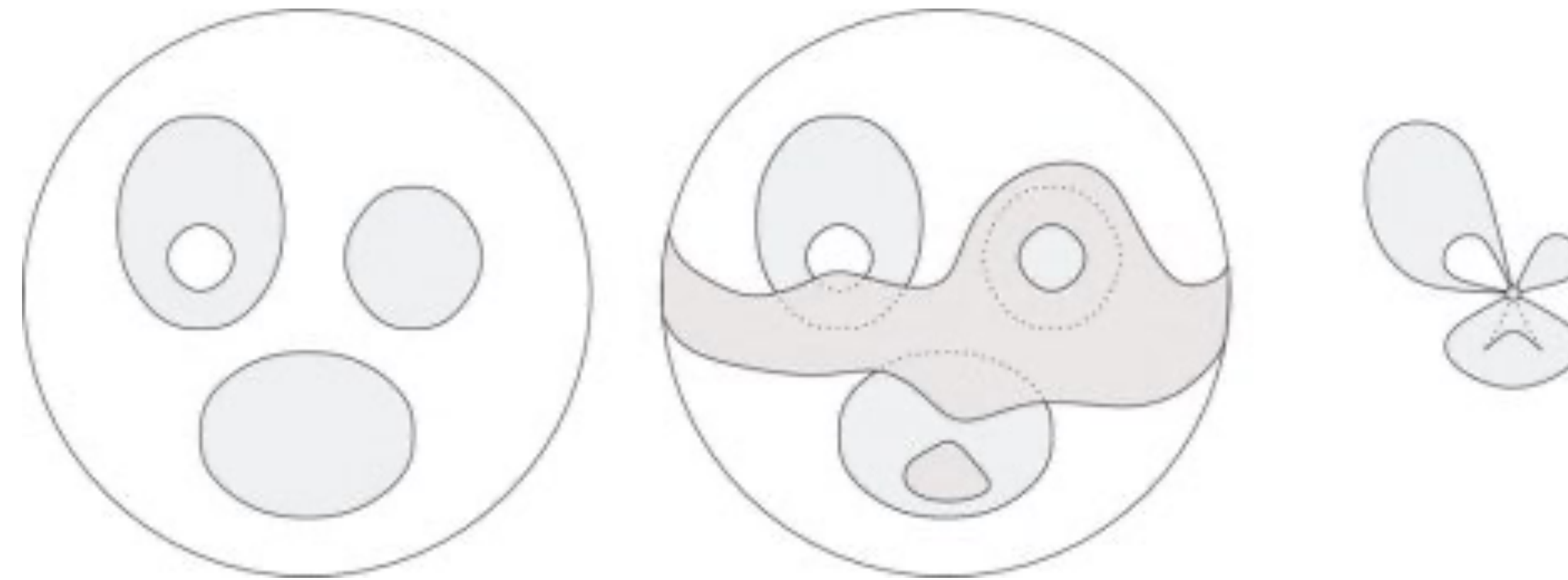
$$\begin{aligned}\frac{\partial}{\partial s} g_{\mu\nu} &= -2 \left[R_{\mu\nu} - R_{\mu\nu}^T \right] \\ &= -2 \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{8\pi G}{c^4} T_{\mu\nu} \right] - g_{\mu\nu} (R - T)\end{aligned}$$
$$R_{\mu\nu}^T = \frac{8\pi G}{c^4} \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right]$$

Changes of topologies through defects are induced by singularities in the stochastic gradient flow



Topological features of ground-states

The gauge-geometry flow allows for topology changes (fluctuations) out of equilibrium



Geometrical interpretation of ground-states

Quark confinement and center symmetry

Elitzur theorem and necessity of breakdown of global symmetry to distinguish phases of a gauge theory

$$G(x) = \exp\left(\frac{i}{2}\Xi^a(x)\sigma^a\right),$$
$$\Xi^a(x) = \epsilon_\mu^a x^\mu - g \frac{1}{\partial^2} (A_\mu \times \epsilon^\mu)^a + \mathcal{O}(g^2)$$

Occurrence of N-ality classes for SU(N)

$$R(zg) = z^k R(g) \quad \text{for } z \text{ element of } \mathbb{Z}_N$$

For holonomies and related (Wilson) loops

$$H_{\gamma(\tau)}(A) \rightarrow G_{\gamma(\tau_s)} H_{\gamma(\tau_0)}(A) G_{\gamma(\tau_t)}^\dagger$$

$$G_{\circ\gamma} = z^* G_{\circ\gamma}$$

Geometric phase and quark confinement

Intuition from notable example of U(1)

$$W_{\circ\gamma} = e^{ie\Phi_B}$$

Non-vanishing magnetic fluxes recovered from singular gauge transformations applied to vanishing fields

$$W_{\circ\gamma} \rightarrow e^{\pm ie\Phi_B} W_{\circ\gamma}$$

Holonomy defined as circuitation of gauge connection along path (Wilson loop when paths are closed)

$$W_\alpha = \mathcal{P} \exp \left[ig \oint dx^\mu A_\mu(x) \right]$$

Action of center symmetry

$$G_{\gamma(\tau_s)} = z G_{\gamma(\tau_t)} \qquad W_{\circ\gamma} \rightarrow (z^*)^l W_{\circ\gamma} \qquad \circ\gamma \text{ space-like loop and } l \text{ linking number}$$

For SU(N) the geometric phase is replaced by a center symmetry group element of $\mathbb{Z}_N$

$$G_{\gamma(\tau_s)} = z G_{\gamma(\tau_t)} . \qquad W_{\circ\gamma} \rightarrow (z^*)^l W_{\circ\gamma}$$

't Hooft loop and smeared phase space

Creation of thin cents vortices at the base points of loops

$$B_\alpha W_{\alpha'} = z W_{\alpha'} B_\alpha \quad l=1 \text{ linked loops}$$

Smeared conjugated variables

Magnetic order

$$\mathcal{W}_\alpha \sim e^{-a A_\alpha} \quad \leftrightarrow \quad \langle B_\alpha \rangle \sim e^{-b P_\alpha} \quad \text{Confinement}$$

Magnetic disorder

$$\mathcal{W}_\alpha \sim e^{-a' P_\alpha} \quad \leftrightarrow \quad \langle B_\alpha \rangle \sim e^{-b' A_\alpha} \quad \text{SSB of center symmetry}$$

Wilson loops

Parallel transport along closed paths of a principal bundle

$$W_\alpha = \mathcal{P} \exp \left[ig \oint dx^\mu A_\mu(x) \right]$$

Particle pairs interacting through strings

$$\mathcal{C}(T_E) = \psi^\dagger(0, T_E) H_{\gamma_R} \psi(R, T_E)$$

Wilson loops on Euclidean rectangular path

$$\langle \mathcal{C}(T_E)^\dagger \mathcal{C}(0) \rangle \sim \mathcal{W}_\alpha(R, T_E)$$

Stochastic gauge-geometry flow

Consider the geometry back-reaction (fluctuation of topologies)

$$\mathcal{S}_{\text{EYM}} = +\frac{1}{2\kappa} \int dx_M^0 d^{D-1}x \sqrt{-g} R[g_{\mu\nu}] - \frac{1}{4g_s^2} \int dx_M^0 d^{D-1}x \sqrt{-g} F_{\mu\nu}^a F^{a\mu\nu}$$

Stochastic flow complemented with noise multiplicative ansatz

$$\begin{aligned} \frac{\partial A_\mu^a(x, \tau)}{\partial \tau} &= \imath \nabla^\nu F_{\nu\mu}^a(x, \tau) + \xi_\mu^a(x, \tau) \\ \frac{\partial g_{\mu\nu}(x, \tau)}{\partial \tau} &= -\imath \frac{\sqrt{-g}}{2\kappa} \left[R_{\mu\nu}(x, \tau) - \frac{1}{2} R(x, \tau) g_{\mu\nu}(x, \tau) - \kappa T_{\mu\nu}(x, \tau) \right] + \xi_{\mu\nu}(x, \tau) \end{aligned}$$

$$\xi_\mu^a(x, \tau) \equiv \xi(x, \tau) A_\mu^a(x, \tau)$$

$$\xi_{\mu\nu}(x, \tau) \equiv \xi'(x, \tau) g_{\mu\nu}(x, \tau)$$

Topology changes

Stochastic noise induce fluctuations of the manifold's (hadronic ground states) topology

$$\bar{\Gamma}_{\beta\gamma}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha} + \delta_{\gamma}^{\alpha} U_{\beta}$$

$$\frac{d}{d\tau} g_{\mu\nu}(\tau) = -2n^{\alpha}(\tau)U_{\alpha}(\tau)g_{\mu\nu}(\tau) = -2\lambda(\tau)g_{\mu\nu}(\tau)$$

Geometric flow and appearance of singularities at the geometry level



Singularities of the gauge transformations and appearance of thin vortices

Dual fluxes are created dynamically in the stochastic time and source the area-law

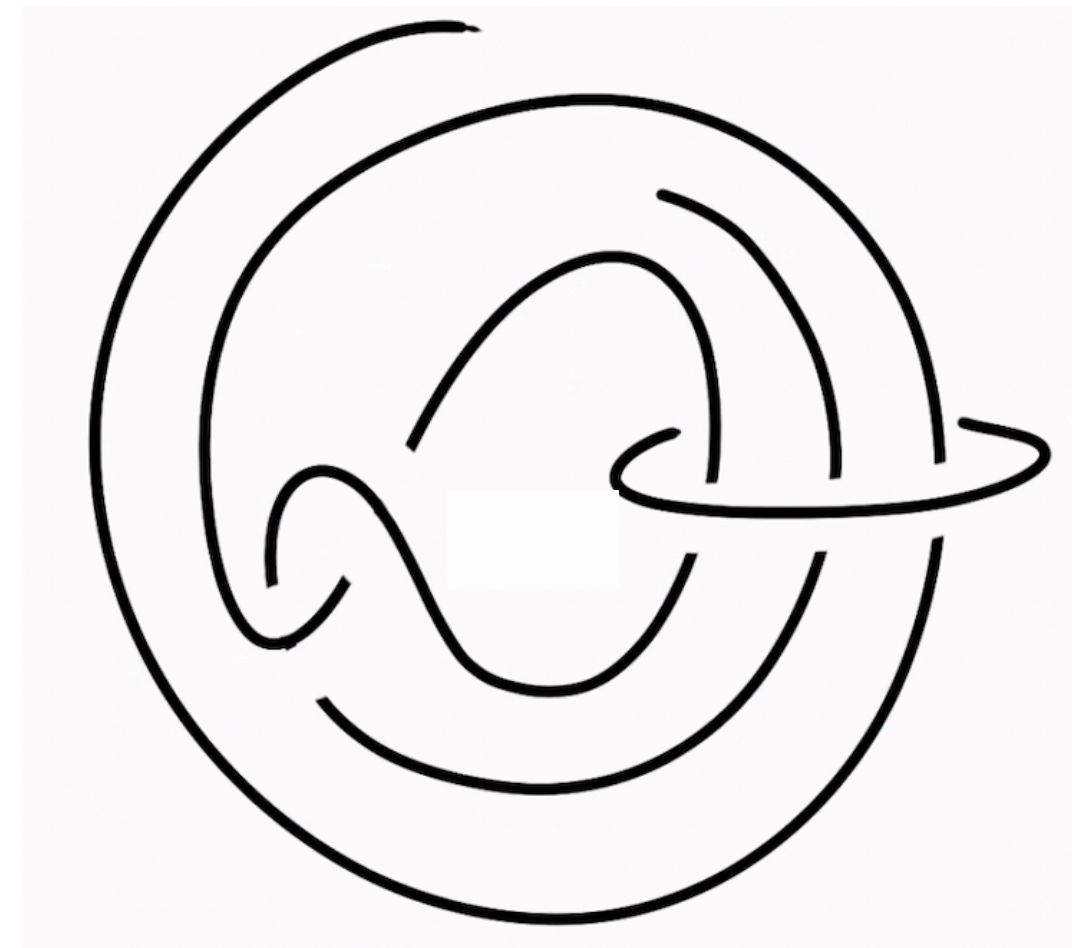
Quark Confinement

$$\langle W_{\gamma_1}(A) W_{\gamma_2}(A) \rangle = \frac{1}{Z} \int \mathcal{D}A \operatorname{Tr} \left[\mathcal{P} e^{i \oint_{\gamma_1} A_\mu dx^\mu} \right] \operatorname{Tr} \left[\mathcal{P} e^{i \oint_{\gamma_2} A_\mu dx^\mu} \right] e^{i S_{\text{YM}}[A]}$$

VEV of the product of two Wilson loops

$$S_{\text{YM}} = \int_M F \wedge *F$$

$$\oint_{\gamma=\partial\Gamma} A = \int_\Gamma F$$



$$\langle W_{\gamma_1}(A) W_{\gamma_2}(A) \rangle \sim e^{i \text{Area}(\Gamma_1) + i \text{Area}(\Gamma_2)} e^{2i \text{lk}(\gamma_1, \gamma_2)}$$

Linking number between Wilson and 't Hooft loops

Dimensional transmutation

Rescaling between leaf of the bindle with changes of topology

$$ds_T^2 = \frac{da_T^2}{a_T^2} = d\left(\frac{t}{L}\right)^2 \longleftrightarrow a_T(t, L) = a_0 \cdot \exp\left(\frac{t}{L}\right)$$

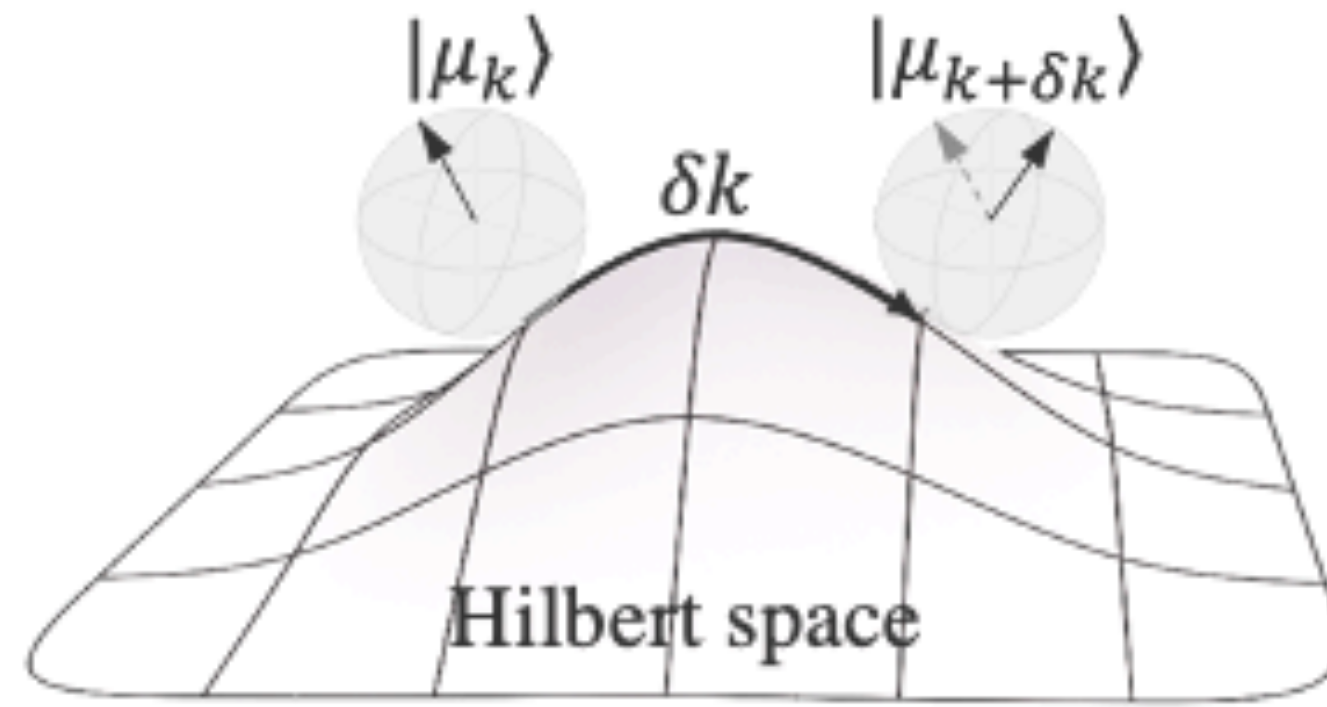
Level fo the Chen-Simons action and rescaling

$$\int_{\Sigma} tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) = 8\pi^2 CS(\Sigma)$$
$$a = L_P \cdot \exp\left(\frac{3}{2 \cdot CS(\Sigma(2, 5, 7))}\right)$$

QCD scale dynamically generated in the stochastic time

$$a_1 = L_P \cdot \exp\left(\frac{140}{3}\right) \approx 7.5 \cdot 10^{-15} m$$

Topology changes in out-of-equilibrium condensed matter



Quantum Geometry:

$$T = \mathbf{g} - \frac{i}{2} \mathbf{\Omega}$$

P. Törmä, Phys. Rev. Lett.
131, 240001 (2023)

Quantum metric $\mathbf{g}$:

The **amplitude distance** between $|\mu_k\rangle$
and $|\mu_{k+\delta k}\rangle$

Berry curvature $\mathbf{\Omega}$:

The **phase distance** between $|\mu_k\rangle$
and $|\mu_{k+\delta k}\rangle$

Quantum metric can induce a lot of interesting phenomena:

Intrinsic nonlinear response;
Fractional Chern insulator states ;
Superconductivity and other phenomena in flat band systems;
Linear displacement current;

.....

Y. Gao, et al. Phys. Rev. Lett. 112, 166601 (2014)
C. Wang, et al. Phys. Rev. Lett. 127, 277201 (2021)
T. Liu, et al. Nat. Sci. Rev. nwae334 (2024)
L. Xiang, et al., Phys. Rev. B 109, 115121 (2024)

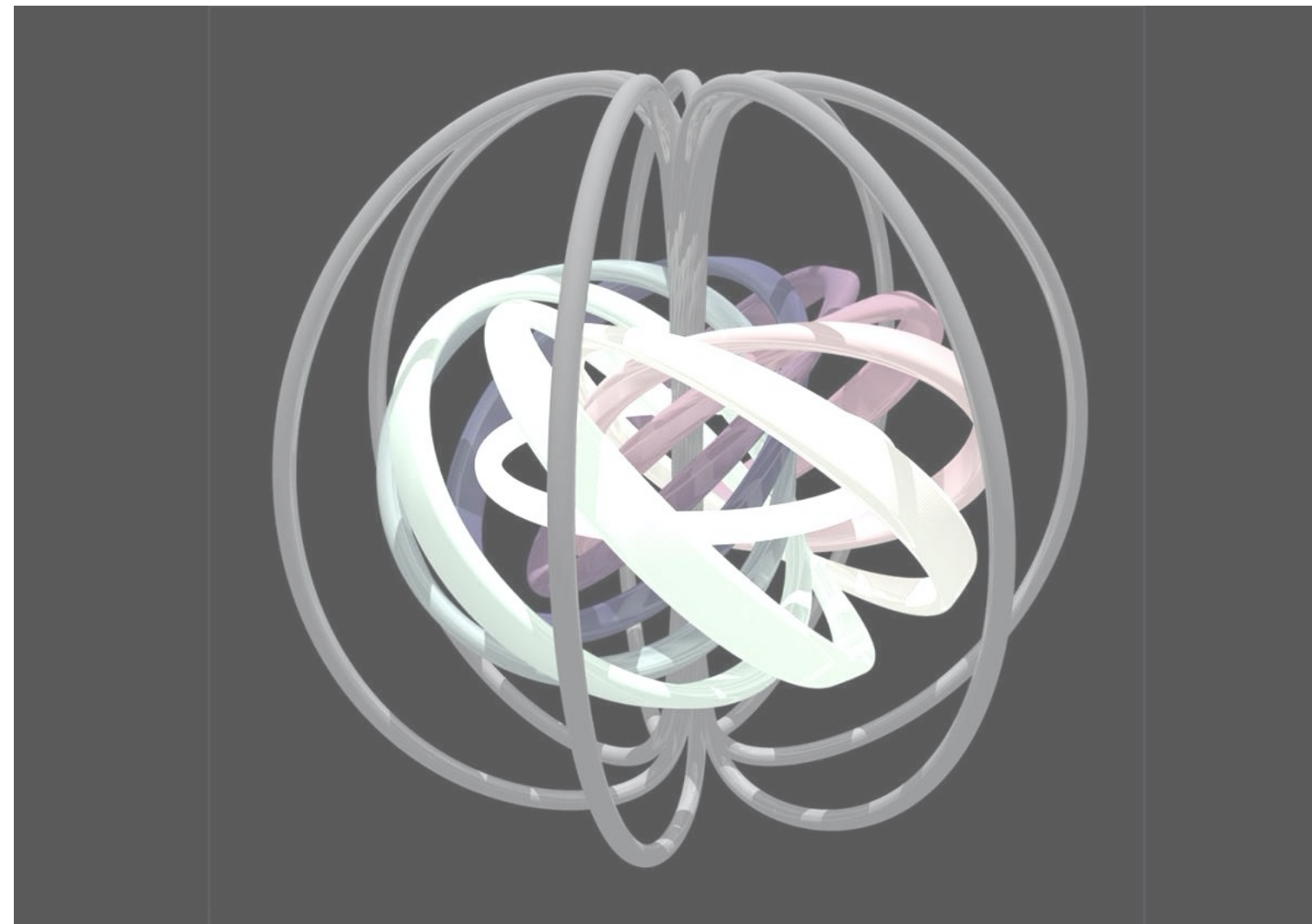
Anomalous Hall effect

Spin Hall effect

Valley Hall effect

BCD nonlinear Hall

Perspectives for Nuclear Physics



Experiments side

Any data set not reproduced by theoretical predictions?

Open questions to solve at the phenomenological level?

Theory side

Can we define topology classes and topology changes?

Any effect that to be addressed in terms of Berry phase?

Outlook and conclusions

Stochastic quantisation and out-of-equilibrium breakdown of symmetries

Geometric RG flow complemented with stochastic (multiplicative) noise

Consequences of the SGF: astrophysics, cosmology, particle physics and foundational aspects

DP model from the Stochastic Ricci flow in non-relativistic limit and a following WKB approximation

Symmetry breaking & topology changes in out-of-equilibrium condensed matter systems

谢谢

Thank you!



Grazie!

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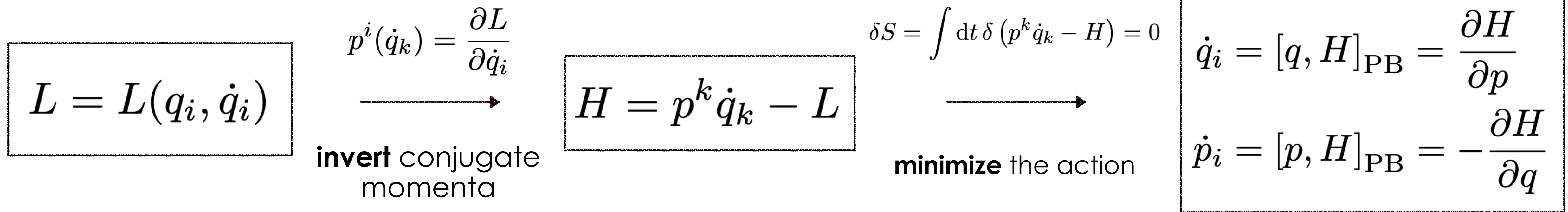
marciano@Inf.infn.it

Quantization Methods

All methods start from a classical description

Canonical Quantization - Dynamic Perspective

Considering a mechanical system...



**Replace PB with
commutators**

$$[A, B]_{\text{PB}} \rightarrow -\frac{i}{\hbar} [\hat{A}, \hat{B}]$$

$$[A, B]_{\text{PB}} = \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p^k} - \frac{\partial B}{\partial q_k} \frac{\partial A}{\partial p^k}$$

Poisson Brackets provide
a “metric” (symplectic)
structure in phase space

“Any dynamical theory must first be put in the Hamiltonian form before one can quantize it.”

P.A.M. Dirac, The theory of gravitation in Hamiltonian form, Proc. Roy. Soc. A, **246**, 1246 (1958)

Quantization Methods

All methods start from a classical description

1. Canonical Quantization - Dynamic Perspective

2. Path-Integral Quantization - Ensemble Average Perspective

Considering fields in four dimensions - Connection with Statistical Field Theory

Generating
Functional, or
Partition Function
(Wick rotation)

$$Z [J^A] = \int [D\phi] \exp \frac{i}{\hbar} \int d^4x [\mathcal{L}(\phi_A, \partial_\mu \phi) + J^A \phi_A]$$

$$\langle \phi_{A_1}(x_1) \cdots \phi_{A_N}(x_N) \rangle = \frac{(-i)^N}{Z[J^A]} \frac{\delta^N Z[J^A]}{\delta J^{A_1}(x_1) \cdots \delta J^{A_N}(x_N)} \Big|_{J=0}$$

GF of Connected
Feynman diagrams,
Free Energy (WR)

$$W [J^A] = \log Z [J^A]$$

$$\langle \phi_{A_1}(x_1) \cdots \phi_{A_N}(x_N) \rangle_c = (-i)^N \frac{\delta^N W[J^A]}{\delta J^{A_1}(x_1) \cdots \delta J^{A_N}(x_N)} \Big|_{J=0}$$

Quantization Methods

All methods start from a classical description

1. Canonical Quantization - Dynamic Perspective
2. Path-Integral Quantization - Ensemble Average Perspective

Difficulties in handling Symmetries

Dynamics limited to a hypersurface in phase-space — Constraints Hypersurface $\chi_i = 0$

- 1. Dirac Prescription
define *physical states*
as

$$\chi_i |\psi\rangle = 0$$

- 2. Gribov copies - Faddeev Popov determinant

Path integral

$$= \int [Dx^\mu] \prod_{t,\alpha} \delta(\chi_\alpha) \prod_t (\det [\chi_\alpha, \chi_\beta])^{1/2} \exp iS[x^\mu(t)]. \quad (16.5)$$

Stochastic Ricci flow & quantum collapse VI

Stochastic Ricci-flow equation for a wave functional depending on N

$$\begin{aligned} d\Psi = & \left[\int d^4x m_x \frac{i}{\hbar} \frac{\delta S_0}{\delta N_x} a_x ds + \int d^4x m_x \frac{i}{\hbar} \frac{\delta S_0}{\delta N_x} b_x dW_x \right] \Psi \\ & + \frac{1}{2} \int d^4x m_x \int d^4y n_y b_x b_y k_\varepsilon(x, y) \frac{\delta}{\delta N_x} \left(\Psi \frac{i}{\hbar} \frac{\delta S_0}{\delta N_y} \right) ds \end{aligned}$$

WKB ansatz for the wave functional

$$\Psi = \exp \left[\frac{i}{\hbar} S_0 \right] = \exp \left[\int^{N_x} \prod_y dN_y \frac{4}{e^{i\gamma} \alpha N_y^2} \left[a_y - \frac{e^{i\gamma} \alpha}{2} N_y \right] \right], \quad b_x = e^{i\gamma/2} \sqrt{\frac{\alpha}{2}} N_x$$

$$a_x = -\frac{i}{2} \left[2 \left(N_x (\nabla \log N_x)^2 + \nabla^2 N_x \right) + \kappa N_x \rho_x (3\sigma T + 1) \right] - \frac{\alpha}{4} e^{i\gamma} N_x$$

Stochastic Ricci flow & quantum collapse VII

Non-relativistic weak-gravity limit

$$d|\psi_t\rangle = \left[-\frac{i}{\hbar} H dt + \sqrt{\frac{G}{\hbar}} \int d\mathbf{x} (\mathcal{M}(\mathbf{x}) - \langle \mathcal{M}(\mathbf{x}) \rangle_t) dW(\mathbf{x}, t) - \frac{G}{2\hbar} \int d\mathbf{x} \int d\mathbf{y} \frac{(\mathcal{M}(\mathbf{x}) - \langle \mathcal{M}(\mathbf{x}) \rangle_t)(\mathcal{M}(\mathbf{y}) - \langle \mathcal{M}(\mathbf{y}) \rangle_t)}{|\mathbf{x} - \mathbf{y}|} dt \right] |\psi_t\rangle,$$

$$\Psi \int d^4x m_x \left[2\imath \kappa \rho_x (2 + \sigma T) + \frac{\alpha}{4} e^{\imath\gamma} \right] ds$$

$$\Psi e^{-\imath\gamma/2} \sqrt{\kappa} \sqrt{\frac{\kappa}{2\alpha}} \int d^4x m_x \left[-2\imath \rho_x (3\sigma T + 2) - 3e^{\imath\gamma} \frac{\alpha}{\kappa} \right] dW_x$$

Stochastic Ricci flow & quantum collapse VIII

Non-relativistic weak-gravity limit

$$d|\psi_t\rangle = \left[-\frac{i}{\hbar} H dt + \sqrt{\frac{G}{\hbar}} \int d\mathbf{x} (\mathcal{M}(\mathbf{x}) - \langle \mathcal{M}(\mathbf{x}) \rangle_t) dW(\mathbf{x}, t) - \frac{G}{2\hbar} \int d\mathbf{x} \int d\mathbf{y} \frac{(\mathcal{M}(\mathbf{x}) - \langle \mathcal{M}(\mathbf{x}) \rangle_t)(\mathcal{M}(\mathbf{y}) - \langle \mathcal{M}(\mathbf{y}) \rangle_t)}{|\mathbf{x} - \mathbf{y}|} dt \right] |\psi_t\rangle,$$

$$\begin{aligned} & \frac{1}{2} \int d^4x m_x \int d^4y n_y \frac{1}{2} k_\varepsilon(x, y) \frac{16}{e^{\imath\gamma} \alpha} ds \Psi \\ & \times \left[-\frac{\imath}{2} [\kappa \rho_y + \kappa \rho_y (3\sigma T + 1)] - \frac{3}{4} \alpha e^{\imath\gamma} \right] \\ & \times \left[-\frac{\imath}{2} [\kappa \rho_x + \kappa \rho_x (3\sigma T + 1)] - \frac{3}{4} \alpha e^{\imath\gamma} \right] \end{aligned}$$

Stochastic Quantization - Examples

Abelian gauge field

$$\frac{\partial}{\partial s} A_\mu = \partial^2 A_\mu - \partial_\mu \partial_\nu A_\nu + \eta_\mu$$

Gauge symmetry broken by boundary condition (restored at equilibrium)

$$A_\nu(x, s)|_{s=0} = 0$$

Solution again a Gaussian random variable

$$A_\mu(k, s) = \int_0^s ds' G_{\mu\nu}(k, s - s') \eta_\nu(k, s')$$

Retarded Green function

$$G_{\mu\nu}(k, s - s') = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) e^{-k^2(s-s')} + \frac{k_\mu k_\nu}{k^2}$$

Correlation function

$$D_{\mu\nu}(k, s, s') = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{e^{-k^2|s-s'|} - e^{-k^2|s+s'|}}{k^2} + \frac{k_\mu k_\nu}{k^2} 2\min(s, s')$$

$$D_{\mu\nu}(k, s, s) = \underbrace{\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2}}_{\text{Feynman propagator Landau gauge}} + \underbrace{2s \frac{k_\mu k_\nu}{k^2}}_{\text{longitudinal term secular divergence}}$$

Decompose in transverse A_μ^T and longitudinal part

$$A_\mu(x, s) = A_\mu^T(x, s) + \partial_\mu \alpha(x, s) \quad \partial_\mu A_\mu^T(x, s) = 0$$

Equilibrium Limit $\langle A_\mu^T(k, s) A_\mu^T(-k, s) \rangle = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2}$

Slower evolution: random walk $\langle \alpha(k, s) \alpha(-k, s) \rangle = \frac{2}{k^2} s$

Stochastic Quantization - Examples

Massive scalar field with potential

$$S[\phi] = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} g \phi^3 \right]$$

The stochastic equations

$$\frac{\partial \phi}{\partial s} = \partial^2 \phi - m^2 \phi + g \phi^2 + \eta$$

$$\langle \eta(x, s) \eta(x', s') \rangle = 2 \delta(s - s') \delta(x - x')$$

Non-interacting case $g = 0$

$$\frac{\partial}{\partial s} G(x, s) - (\partial^2 - m^2) G(x, s) = \delta(x) \delta(s)$$

$$G(x, s) = 0, \quad (\text{for } s < 0)$$

$$G(x, s) = \int \frac{d^4 k}{(2\pi)^4} \exp[-s(k^2 + m^2) + i\mathbf{k} \cdot \mathbf{x}] \theta(s)$$

Associated Green function - **No FT on stoch. time**

$$\phi(x, s) = \int_0^s d\tau \int d^4 y G(x - y, s - \tau) \eta(y, \tau)$$

The field is a Gaussian variable

$$\langle \phi(x, s) \phi(x', s') \rangle \equiv D(x - x'; t, t')$$

correlation function

$$= \left\langle \int_0^\infty d\tau d\tau' \int d^4 y d^4 y' G(x - y, s - \tau) G(x' - y', s' - \tau') \eta(y, \tau) \eta(y', \tau') \right\rangle$$

$$= 2 \int_0^\infty d\tau \int d^4 y G(x - y, s - \tau) G(x' - y, s' - \tau)$$

$$\text{Non-Eq. FT} \quad D(k; t, t') = \frac{\exp[-(k^2 + m^2)(t - t')]}{k^2 + m^2} [1 - \exp[-2(k^2 + m^2)t']]$$

$$\text{Equilibrium limit} \quad \lim_{t \rightarrow \infty} D(k; t, t) = \frac{1}{k^2 + m^2}$$

Stochastic Quantization - Examples

Massive scalar field with potential

$$S[\phi] = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} g \phi^3 \right]$$

The stochastic equations

$$\frac{\partial \phi}{\partial s} = \partial^2 \phi - m^2 \phi + g \phi^2 + \eta$$

$$\langle \eta(x, s) \eta(x', s') \rangle = 2 \delta(s - s') \delta(x - x')$$

Non-interacting case $g = 0$

$$\frac{\partial}{\partial s} G(x, s) - (\partial^2 - m^2) G(x, s) = \delta(x) \delta(s)$$

$$G(x, s) = 0, \quad (\text{for } s < 0)$$

$$G(x, s) = \int \frac{d^4 k}{(2\pi)^4} \exp[-s(k^2 + m^2) + i\mathbf{k} \cdot \mathbf{x}] \theta(s)$$

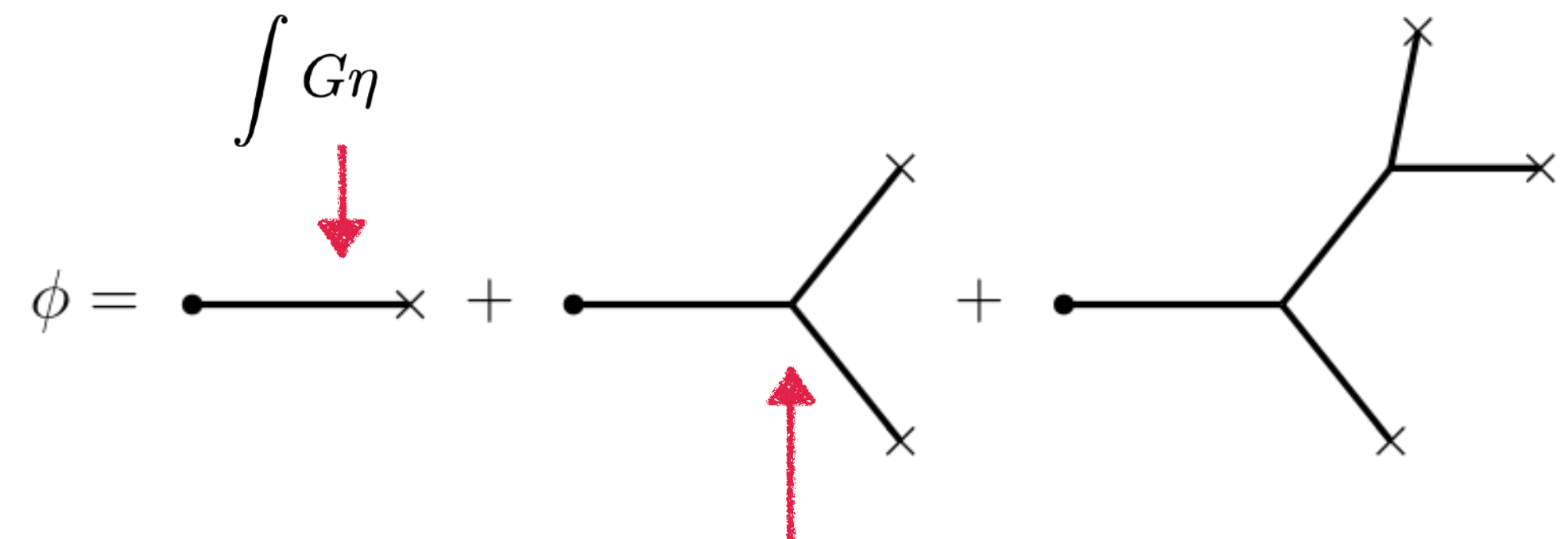
Associated Green function - **No FT on stoch. time**

$$\phi(x, s) = \int_0^s d\tau \int d^4 y G(x - y, s - \tau) \eta(y, \tau)$$

The field is a Gaussian variable

Interacting case $g \neq 0$

$$\phi(x, s) = \int_0^s d\tau \int d^4 y G(x - y, s - \tau) [\eta(y, \tau) + g \phi^2(y, \tau)]$$



$$\int G \left[\eta + g \int^* \int^{**} G^* \eta^* G^{**} \eta^{**} \right]$$

Stochastic time and scale transformation

- Let us look for a link between the proper time and the stochastic time
- Assume stochastic-time dependence in space-time coordinates

$$\frac{d}{ds} = \frac{\partial}{\partial s} + \frac{dx^\mu}{ds} \nabla_\mu$$

- Assume that the total derivatives are proportional to the normal to the hyper surface

$$\ell(s) \frac{dx^\mu}{ds} = n^\mu = -g^{\mu\alpha} N \partial_\alpha t$$

- The normalisation reads

$$g_{\mu\nu} n^\mu n^\nu = g_{\mu\nu} \ell^2(s) \frac{dx^\nu}{ds} \frac{dx^\mu}{ds} = -g_{\mu\nu} \ell(s) \frac{dx^\nu}{ds} g^{\mu\alpha} N \partial_\alpha t \quad \rightarrow \quad \varepsilon(s) \frac{\delta s}{\ell(s)} = -N \delta t = -\delta\tau$$

$$\boxed{\frac{d\tau}{ds} = -\frac{\varepsilon(s)}{\ell(s)}} \quad \rightarrow \quad \boxed{\delta s = \frac{ds}{d\tau} \delta\tau = \ell(s) \sqrt{\frac{1}{c^2 \varepsilon^2(s)} g_{\mu\nu} dx^\mu dx^\nu}}$$

Stochastic time flow - Effective scale transformation
Stochastic dynamics related to RG group transformation

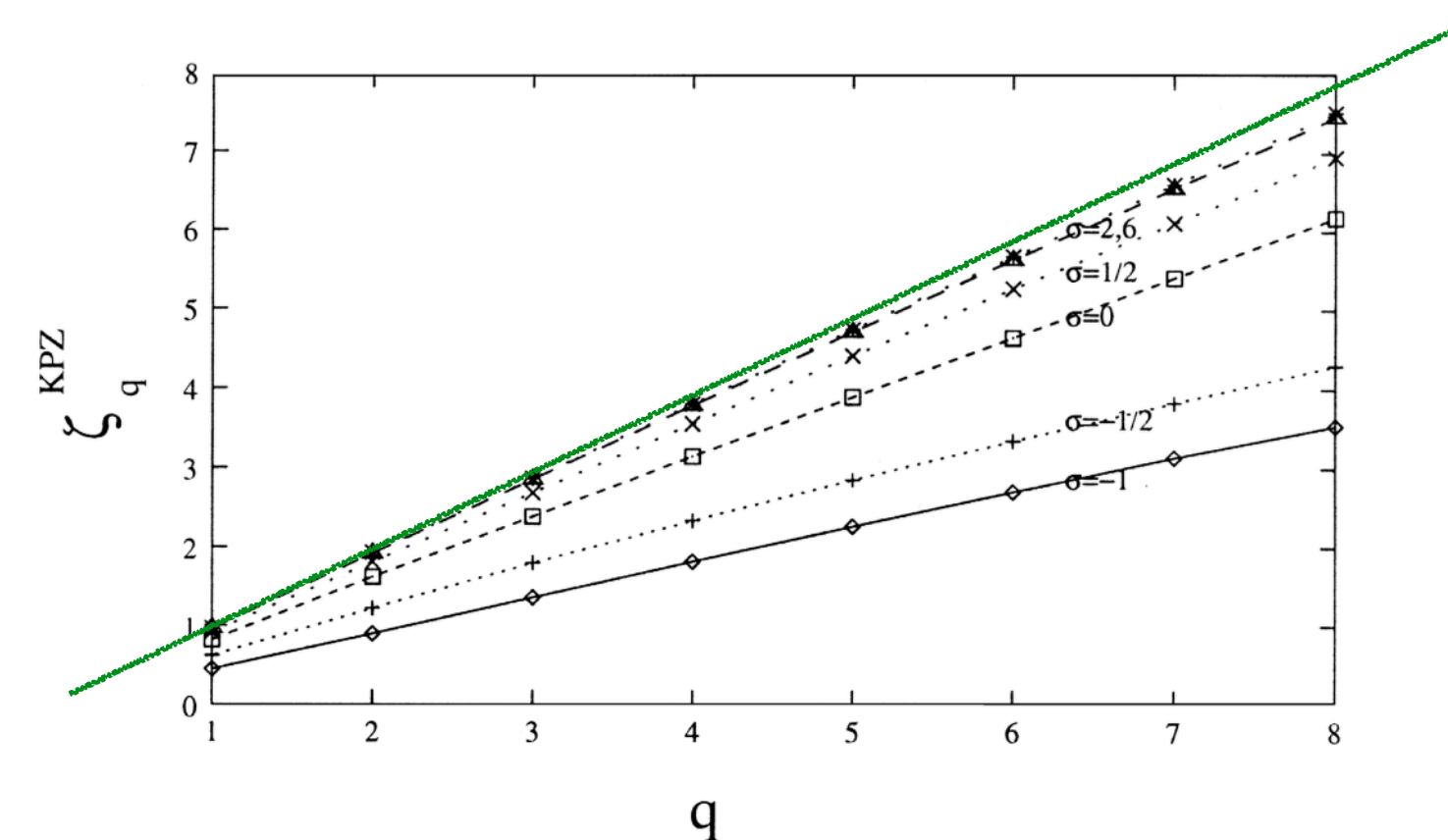
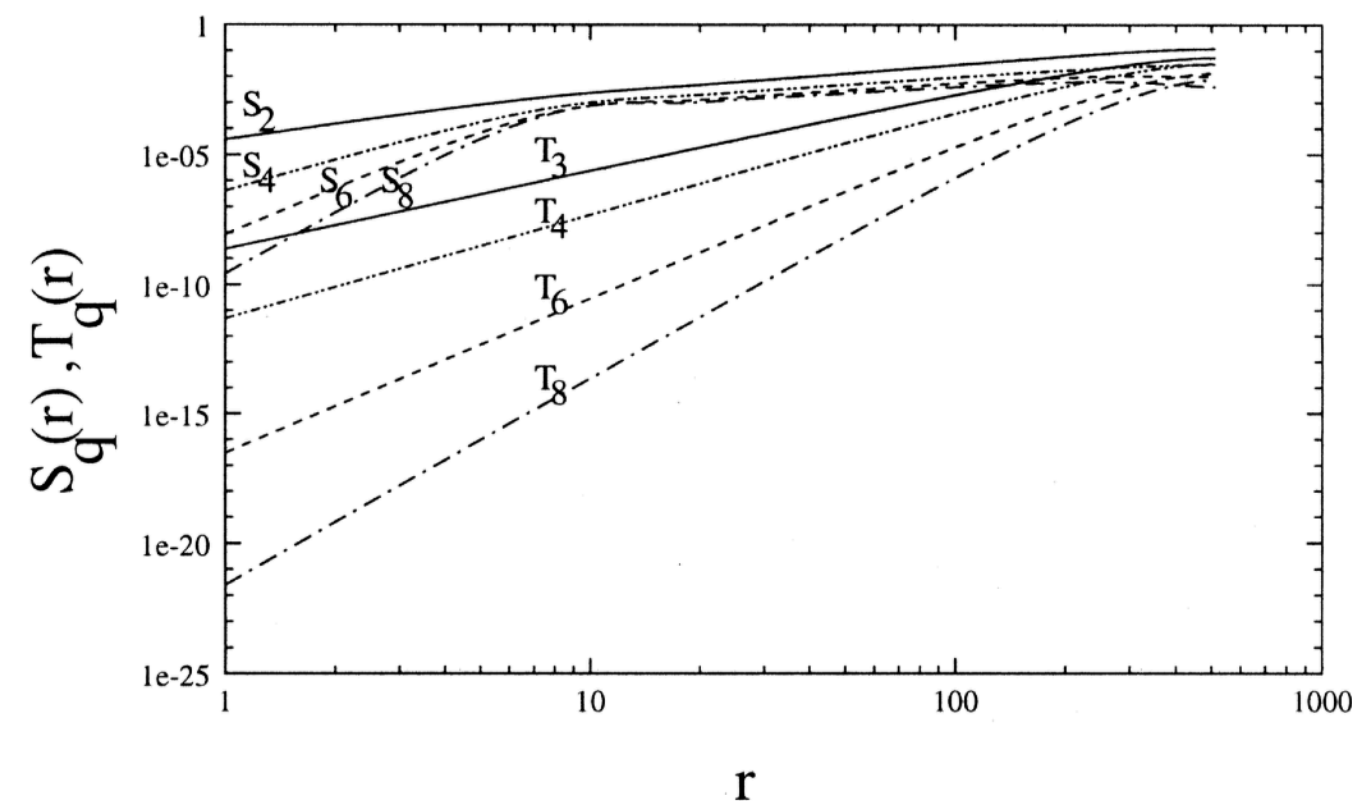
Black Holes and KPZ

$$\frac{\partial h}{\partial t} = \nu \left[\nu_{\text{KPZ}}(r, \alpha) \nabla^2 h(r, s) + \lambda_{\text{KPZ}}(r) \left(\frac{dh}{dr} \right)^2 \right] + \bar{\eta}$$

Kardar-Parisi-Zhang Eq. [1] for a spherical surface

$$\nu_{\text{KPZ}}(r, \alpha) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \quad \lambda_{\text{KPZ}}(r) = e^{\nu_0(r)}$$

- What is exactly **intermittency**?
- The KPZ structure function is defined as $T_q(r) = \langle |\Delta h(r)|^q \rangle$ $\Delta h(r) = h(x+r) - h(x)$
- Display power-law scaling $T_q(r) = \langle |\Delta h(r)|^q \rangle \propto r^{\zeta_q}$
- For **Gaussian statistics** one would expect **linear scaling of the exponents** $\zeta_q = \chi q$
- **Different probability distribution at every scale**



[1: M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)]

[2: M. K. Verma, Phys. A 277 (2000) 359-388]

Stochastic dynamics and the Ricci RG flow

Holography in 4+1D and dynamics in the stochastic time parameter

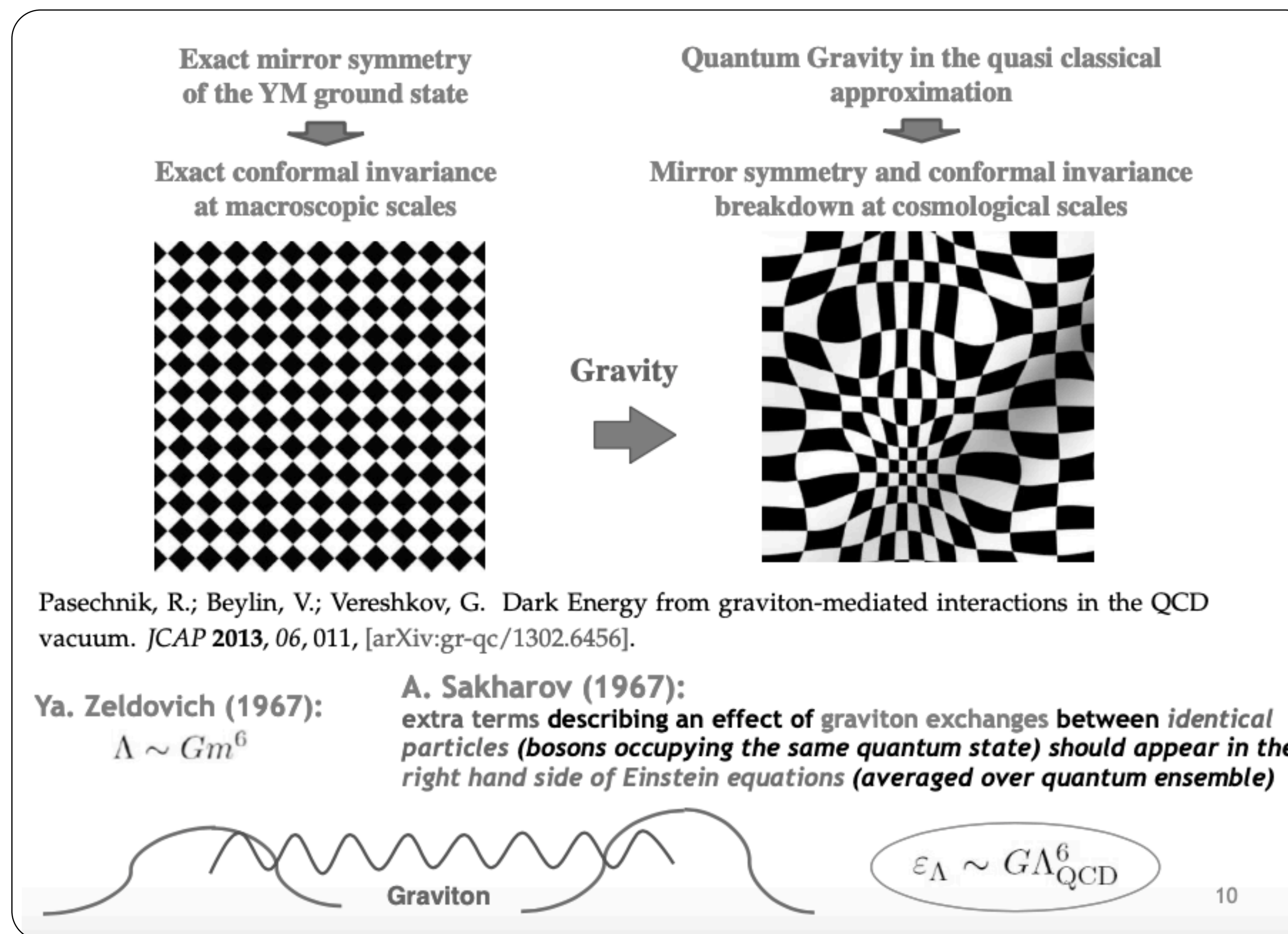
The Ricci flow amounts to a conformal transformation of the 3D-hypersurfaces

The Langevin equation and the probability distributions for manifolds with Lorentzian signature and complex structure

Manifolds with Lorentzian signature enable to fully take into account dynamics of out-of-equilibrium systems and relaxations features

Vortices and turbulences (both for fluids and gauge fields) can be addressed as a by-product of the Ricci flow driven relaxation processes

Gravitational back-reaction to YM



[Credit to R. Pasechnik](#)

Stochastic Quantization: trivial example

Field defined only in one point (reminiscent mini-superspace)

$$\hat{H} = \frac{1}{2}P^2 + U(Q)$$

The potential $V(Q)$ increasing fast at infinity; H has a discrete spectrum

$$\hat{H}\psi_n(Q) = \lambda_n\psi_n(Q), \quad \lambda_{i+1} > \lambda_i$$

Correlation function at equal time

$$\langle Q(s)^k \rangle = \sum_{n=0}^{\infty} c_n e^{-2\lambda_n s} \int dQ Q^k \psi_n(Q) \psi_0(Q)$$

$\exp(-V/2)$ is an eigenvector of H , and its ground-state (function without zeros)

→ at large stochastic time

$$\langle Q(s)^k \rangle = \frac{\int dQ Q^k e^{-V(Q)}}{\int dQ e^{-V(Q)}} + \mathcal{O}[\exp(-2\lambda_1 s)]$$

Stochastic Quantization: less trivial example

Field defined not restricted in one point

$$H = \int d^D x \left\{ \frac{1}{2} \pi(x)^2 + U(\phi(x)) \right\}, \quad [\pi(x), \phi(y)] = -i\delta(x - y)$$

The solution of the Schroedinger functional with $\lambda_0 = 0$

$$\left\{ -\frac{1}{2} \frac{\delta^2}{\delta \phi(x)^2} + U(\phi(x)) \right\} \psi_0(\phi) = \lambda_0 \psi_0(\phi) \quad \longrightarrow \quad \psi_0(\phi) = e^{-\frac{1}{2} \int d^D x V(\phi(x))}$$

A more general Langevin equation can be considered

$$\frac{\partial}{\partial s} \phi(x, s) = - \int d^D y M(x, y) \frac{\delta V}{\delta \phi(y)} + \eta(x, s)$$

still equivalent to the previous one if the matrix M is positive and if

$$\langle \eta(x, s) \eta(y, s') \rangle = 2M(x, y) \delta(s - s')$$

Stochastic Quantization: diagrammatics I

For simplicity consider the case

$$V(\phi) = \int d^D x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} g \phi^3 \right\}, \quad M(x, y) = \delta(x - y)$$

With associated Langevin equation and gaussian noise

$$\frac{\partial}{\partial s} \phi(x, s) = \partial^2 \phi - m^2 \phi + g \phi^2 + \eta(x, s), \quad \langle \eta(x, s) \eta(x', s') \rangle = 2 \delta(x - x') \delta(s - s')$$

Solution for $g=0$ is gaussian stochastic

$$\phi(x, s) = \int_0^s d\sigma \int d^D y G(x - y, s - \sigma) \eta(y, \sigma)$$

$G(x, s)$ retarded Green function

$$\begin{aligned} \frac{\partial}{\partial s} G(x, s) &= (\partial^2 - m^2) G(x, s) + \delta(x) \delta(s) \\ G(x, s) &= 0 \quad \text{for } t < 0 \end{aligned}$$

→

$$G(x, s) = \int \frac{d^D k}{(2\pi)^D} e^{-s(k^2 + m^2) + i k \cdot x} \theta(t)$$

Stochastic Quantization: diagrammatics II

Correlation function

$$\langle \phi(x, s) \phi(x', s') \rangle \equiv D(x - x'; s, s') = 2 \int_0^\infty d\sigma \int d^D y G(x - y, s - \sigma) G(x' - y, s' - \sigma)$$

In the momentum space, for $s' < s$

$$D(x - x'; s, s') = \frac{e^{-(k^2 + m^2)(s - s')}}{k^2 + m^2} \left(1 - e^{-2(k^2 + m^2)s'} \right)$$

For s' that tends to infinity, second term negligible

At equal time we recover the propagator at equilibrium

$$\frac{1}{k^2 + m^2}$$

Stochastic Quantization: diagrammatics III

$$g \neq 0 \quad \text{case} \quad \phi(x, s) = \int_0^s d\sigma \int d^D y G(x - y, s - \sigma) [\eta(y, \sigma) + g\phi^2(y, \sigma)]$$

Denote Green function as a line, noise as a cross and the field as a point

Assign a factor g to each three-line vertex

Integrate over stochastic times and coordinates of all crosses and vertices

Iterate expression for the stochastic self-interacting field

$$\phi = \bullet \longrightarrow \times + \bullet \longrightarrow \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \end{array} + \bullet \longrightarrow \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \quad \times \end{array} + \dots$$

Crosses must coincide for the mean over the noise not to vanish

Stochastic Quantization: diagrammatics IV

$$\langle \phi(t_1) \phi(t_2) \rangle = \text{(a)} + \text{(b)} + \text{(c)} + \text{(d)} + \dots$$

$$b = g^2 \int \frac{d^D k_1}{(2\pi)^D} \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 G(k; t_1 - \tau_1) G(k; t_2 - \tau_2) \\ \times D(k_1; \tau_1, \tau_2) D(k - k_1; \tau_1, \tau_2),$$

$$c + d = g^2 \int \frac{d^D k_1}{(2\pi)^D} \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 \{ D(k - k_1; \tau_1, \tau_2) \\ \times [D(k; t_1, \tau_1) G(k_1; \tau_2 - \tau_1) G(k; t_2 - \tau_2) \\ + D(k; t_2, \tau_2) G(k_1; \tau_1 - \tau_2) G(k; t_1 - \tau_1)] \\ + \text{terms obtained by } k_1 \longleftrightarrow k - k_1 \}.$$

Parisi & Wu

Correct equilibrium limit recovered for $s=s'$ that tends to infinity

$$b = g^2 \frac{d^D k_1}{(2\pi)^D} \frac{1}{(k^2 + m^2)(k_1^2 + m^2)(k_2^2 + m^2)(k^2 + k_1^2 + k_2^2 + 3m^2)},$$

$$c + d = g^2 \int \frac{d^D k_1}{(2\pi)^D} \left(\frac{1}{k_1^2 + m^2} + \frac{1}{k_2^2 + m^2} \right) \frac{1}{(k^2 + m^2)^2 (k^2 + k_1^2 + k_2^2 + 3m^2)},$$

$$b + c + d = g^2 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{(k_1^2 + m^2)(k_2^2 + m^2)(k^2 + m^2)^2}.$$

Stochastic Quantization: simple example I

Potential for n-dimensional vector

$$V(q) = -\frac{1}{2}\mu^2 q^2 + \frac{1}{4}g (q^2)^2, \quad q^2 = \sum_{i=1}^n (q^i)^2$$

Compute perturbatively in g, around the minimum

$$\langle q^2 \rangle \propto \int d[q] q^2 e^{-V(q)}$$

O(n) symmetry $\longrightarrow$ choose the minimum

$$q_0 \equiv \left(\sqrt{\frac{\mu^2}{g}}, 0, \dots, 0 \right)$$

Find the potential in the new variable, shifted around the vacuum

$$q = \left(\sqrt{\frac{\mu^2}{g}} + q_L, \mathbf{q}_T \right)$$

$$V(q) = \mu^2 q_L^2 + \sqrt{\mu^2 g} q_L (q_L^2 + \mathbf{q}_T^2) + \frac{1}{4}g (q_L^2 + \mathbf{q}_T^2)^2 - \frac{1}{4g} \mu^4$$

Stochastic Quantization: simple example II

Usually, change to the radial variable $r = \sqrt{q^2}$ and the angular coordinate on S_{n-1}

$$\langle q^2 \rangle = \frac{\int dr r^{n-1} r^2 e^{-(\frac{g}{4}r^4 - \frac{\mu^2}{2}r^2)}}{\int dr r^{n-1} e^{-(\frac{g}{4}r^4 - \frac{\mu^2}{2}r^2)}}$$

The Langevin equation can be used to by-pass non-linear transformation

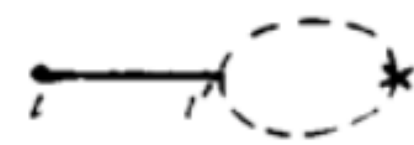
We start from $\langle q^2 \rangle = \frac{\mu^2}{g} + A(n-1) + B$ and write the Langevin equation in power of g

$$\begin{aligned} \dot{\mathbf{q}}_T &= \eta_T + \mathcal{O}(g^{\frac{1}{2}}) \\ q_L &= -2\mu^2 q_L - \sqrt{\mu^2 g} (3q_L^2 + \mathbf{q}_T) + \mathcal{O}(g) + \eta_L \end{aligned} \quad \longrightarrow \quad \langle q_T^i(s) q_T^j(s') \rangle = 2\delta_{ij} \min(s, s')$$

Use the out-equilibrium correlation function to compute $q_L @ \mathcal{O}(g^{\frac{1}{2}})$

Stochastic Quantization: simple example III

For the term proportional to $(n-1)$ the only non-vanishing diagram is



$$q_L(s) = \mu g^{\frac{1}{2}} \int_0^s ds' \langle \mathbf{q}_T^2(s') \rangle e^{-2\mu^2(s-s')} = -(n-1) \sqrt{\frac{g}{\mu^2}} \left(s - \frac{1}{2\mu^2} \right)$$

with dashed lines transverse propagator and neglecting vanishing terms for s infinity

Computing terms proportional to $(n-1)$

$$\begin{aligned} \langle q^2 \rangle &= \frac{\mu^2}{g} + 2 \sqrt{\frac{\mu^2}{g}} \langle q_L \rangle + \langle q_L^2 \rangle + \langle \mathbf{q}_T^2 \rangle \\ &= -2(n-1) \left(s - \frac{1}{2\mu^2} \right) + 2(n-1)s = \frac{n-1}{\mu^2} \end{aligned}$$

Terms proportional to s cancel each other:
finite contribution with correct result at the saddle-point

Stochastic Quantization: gauge theories I

Euclidean Hamiltonian in D-dimensions

$$V = \int d^D x \left\{ D_\mu \phi^\dagger D_\mu \phi + \frac{1}{2} \text{Tr} F_{\mu\nu}^2 \right\}, \quad D_\mu \phi = (\partial_\mu - ie A_\mu) \phi, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie [A_\mu, A_\nu],$$
$$A = A_\mu^a \tau_a, \quad F_{\mu\nu} = F_{\mu\nu}^a \tau_a, \quad \text{Tr}(\tau_a \tau_b) = \frac{1}{2} \delta_{ab}$$

with associated Langevin equations

$$\frac{\partial}{\partial s} \phi = D^2 \phi + \eta_\phi, \quad \frac{\partial}{\partial s} \phi^\dagger = D^2 \phi^\dagger + \eta_\phi^\dagger, \quad \frac{\partial}{\partial s} A_\mu = D_\nu F_{\nu\mu} + J_\mu + \eta_\mu$$

complemented with interaction currents and gaussian noises

$$J_\mu = J_\mu^a \tau_a, \quad J_\mu = J_\mu^a = ie \phi^\dagger \tau_a \partial_\mu \phi + e^2 \phi^\dagger \{ \tau_a, A_\mu \} \phi,$$
$$\langle \eta_\phi(x, s) \eta_\phi^\dagger(x', s') \rangle = 2 \delta(x - x') \delta(s - s'),$$
$$\langle \eta_\mu(x, s) \eta_\nu(x', s') \rangle = 2 \delta_{\mu\nu} \delta(x - x') \delta(s - s') (\delta_{ab} \tau_a \tau_b)$$

Stochastic Quantization: gauge theories II

Abelian case

$$\frac{\partial}{\partial s} A_\mu = \partial^2 A_\mu - \partial_\mu \partial_\nu A_\nu + \eta_\mu$$

with boundary condition that break the gauge symmetry (restored at equilibrium)

$$A_\nu(x, s)|_{s=0} = 0$$

Solution again a gaussian random variable

$$A_\mu(k, s) = \int_0^s ds' G_{\mu\nu}(k, s - s') \eta_\nu(k, s')$$

with retarded Green function defined only for $s > s'$

$$G_{\mu\nu}(k, s - s') = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) e^{-k^2(s-s')} + \frac{k_\mu k_\nu}{k^2}$$

Correlation function

$$\longrightarrow D_{\mu\nu}(k, s, s') = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{e^{-k^2|s-s'|} - e^{-k^2|s+s'|}}{k^2} + \frac{k_\mu k_\nu}{k^2} 2\min(s, s')$$

Stochastic Quantization: gauge theories III

At large equal stochastic times

$$D_{\mu\nu}(k, s, s) = \underbrace{\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{1}{k^2}}_{\text{Feynman propagator in Landau gauge}} + \underbrace{2s \frac{k_\mu k_\nu}{k^2}}_{\text{Divergent in } s \text{ longitudinal term}}$$

Decompose the connection in a gauge invariant transverse part and longitudinal one

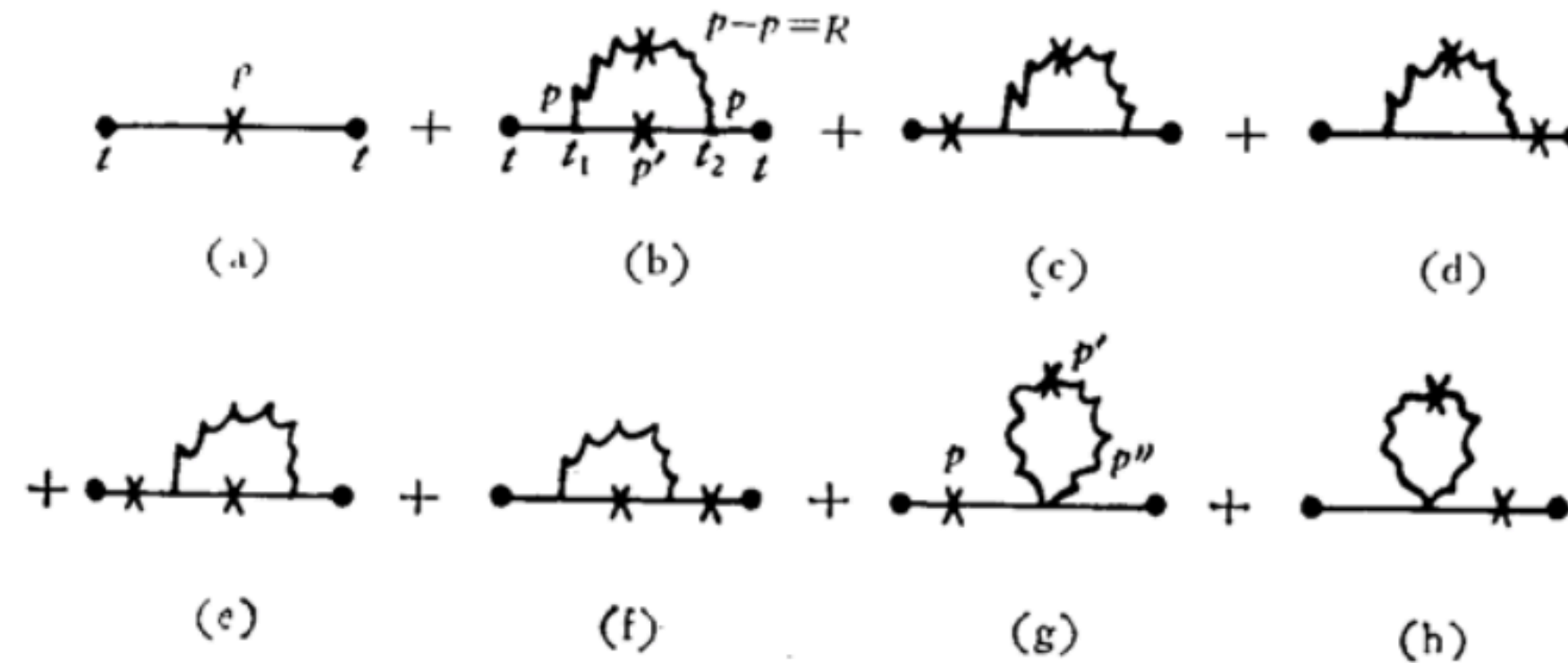
$$A_\mu(x, s) = A_\mu^T(x, s) + \partial_\mu \alpha(x, s) \quad \text{with} \quad \partial_\mu A_\mu^T(x, s) = 0$$

Faster evolution for gauge invariant quantities, random walk for not gauge-invariant ones

$$\langle A_\mu^T(k, s) A_\nu^T(-k, s) \rangle = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{1}{k^2}, \quad \langle \alpha(k, s) \alpha(-k, s) \rangle = \frac{2}{k^2} s$$

Stochastic Quantization: gauge theories IV

Scalar propagator at equal stochastic times



Same diagrammatic rules as before pause usual QED ones for $A_\mu \phi^\dagger \phi$, $A_\mu A_\nu \phi^\dagger \phi$

$$\begin{aligned}
 a &= 2 \int_0^t dt' \exp(-2p^2 t') = \frac{1}{p^2}, & \text{contribution to transverse part} \\
 b &= e^2 \int \frac{d^D p'}{(2\pi)^D} \frac{1}{p^2 p'^2 k^2 (p^2 + p'^2 + k^2)} \left[(p + p')^2 - \frac{(p^2 - p'^2)^2}{k^2} \right] \\
 &\quad + e^2 \int \frac{d^D p'}{(2\pi)^D} \frac{1}{p^2 p'^2 (p^2 + p'^2)} \left[2t - \frac{2}{p^2 + p'^2} - \frac{1}{p^2} \right] \frac{(p^2 - p'^2)^2}{k^2}, & \text{contribution to longitudinal part} \\
 c + e = d + f &= \frac{e^2}{2} \int \frac{d^D p'}{(2\pi)^D} \frac{1}{p^2} \left(\frac{1}{p'^2} + \frac{1}{k^2} \right) \frac{1}{p^2 + p'^2 + k^2} \left[(p + p')^2 - \frac{(p^2 - p'^2)^2}{k^2} \right] \\
 &\quad + \frac{e^2}{2} \int \frac{d^D p'}{(2\pi)^D} \frac{1}{p^4 (p^2 + p'^2)} \left[\frac{1}{p'^2} + 2t - \frac{2}{p^2 + p'^2} - \frac{1}{p^2} \right] \frac{(p^2 - p'^2)^2}{k^2}, \\
 g + h &= -3e^2 \int \frac{d^D p'}{(2\pi)^D} \frac{1}{p^4 p'^2} - e^2 \int \frac{d^D p'}{(2\pi)^D} \frac{1}{p^4} \left[2t - \frac{1}{p^2} \right].
 \end{aligned}$$

Parisi & Wu

Contributions at equal and large stochastic times clarified at $k=p-p'$

Stochastic Quantization: gauge theories V

Summing all the contribution of the transverse part, we recover result in Landau gauge

Contribution of the longitudinal part at large stochastic times equals variation of the equilibrium propagator induced by the adding to the connection propagator the gauge term $s \frac{k_\mu k_\nu}{k^2}$

At large stochastic time the charged field propagator tends to zero, for $s \gg 1/e^2 \omega(x-y)$

$$\langle \phi^\dagger(x, s) \phi(y, s) \rangle \sim_{s \rightarrow \infty} \langle \phi^\dagger(x) \phi(y) \rangle_{\text{free}} e^{-e^2 s \omega(x-y)}, \quad \omega(x) \simeq \frac{1}{|x|^{D-2}}$$

Using standard theorem for the variation of the of Green functions under gauge transformations

Consider now the gauge-invariant quantity $\phi^\dagger(x, s) \phi(x, s)$

Contribution proportional to s , corresponding to a gauge-transformation, is vanishing

Other non-vanishing contribution give the propagator in the Landau gauge

At the lowest order quantities are gauge-invariant at equilibrium, while not gauge-invariant quantities converge dynamically (in the stochastic time) to zero

At the higher order, need for dimensional or lattice regularisation of diagrams and M

Stochastic Quantization: gauge theories VI

Non-abelian case

As in the abelian, we can regularize the matrix M and a) either send stochastic time to infinity before removing the (dimensional or lattice) regulator; b) viceversa

One can show that gauge-invariant quantities are approached uniformly in g and in s , so that the Taylor expansion in g and the limit for s to infinity can be exchanged

$$\partial_\mu A_\mu^T(x, s) = 0, \quad \imath e A_\mu^T(x, s) = e^{-\imath e \alpha(x, s)} (\partial_\mu - \imath e A_\mu) e^{\imath e \alpha(x, s)}, \quad \phi^T(x, s) = e^{-\imath e \alpha(x, s)} \phi(x, s)$$

Langevin equations for longitudinal component and the others decoupled

$$\frac{\partial}{\partial s} A_\mu^T(x, s) = F_1(A_\mu^T, \phi^T, \eta^T), \quad \frac{\partial}{\partial s} \phi^T(x, s) = F_2(A_\mu^T, \phi^T, \eta^T)$$

Approach toward equilibrium controlled by smallest positive eigenvalues of H

No term linear in s appears in the expectation value of gauge-invariant quantities

Correct results found, with contribution of Faddeev-Popov ghost

Fokker-Planck and cosmological constant

Langevin equation with complex multiplicative noise

$$\frac{\partial g_{\mu\nu}}{\partial s} = \imath \mathcal{G}_{\alpha\beta\mu\nu} \frac{\delta S}{\delta g_{\alpha\beta}} + g_{\mu\nu} \eta \longrightarrow \eta = \sigma_{\tilde{\eta}} \tilde{\eta}$$

Related Fokker-Planck within the Ito differential calculus

$$\frac{\partial p}{\partial s} = -\frac{\delta}{\delta g_{\mu\nu}} \left[\mathcal{G}_{\alpha\beta\mu\nu} \frac{\delta S}{\delta g_{\alpha\beta}} p \right] + \frac{\delta^2}{\delta g_{\mu\nu} \delta g_{\rho\sigma}} [g_{\mu\nu}^2 p]$$

$$p \simeq \frac{D}{g_{\mu\nu}^2} \exp \left[2 \int^{g_{\mu\nu}} \mathcal{D} g_{\alpha\beta} \frac{\mathcal{G}_{\rho\sigma\alpha\beta} \frac{\delta S}{\delta g_{\rho\sigma}}}{\Lambda_0 g_{\alpha\beta}^2} \right] \longrightarrow \mathcal{G}_{\rho\sigma\mu\nu} \frac{\delta S}{\delta g_{\rho\sigma}} - \imath \Lambda_0 g_{\mu\nu} = 0$$

$$\sigma_{\tilde{\eta}} = \sqrt{\Lambda_0} \rightarrow e^{-\imath \frac{\pi}{4}} \sqrt{\Lambda_0}$$

Developing a geometric intuition on the RG flow

$$S = \alpha' \int_{\mathcal{M}} d^2\sigma \sqrt{-h} h^{ab}(\sigma) g_{ij}(X) \frac{\partial X^i}{\partial \sigma^a} \frac{\partial X^j}{\partial \sigma^b}$$

The case of the non-linear sigma models

$$\frac{\partial g_{ij}}{\partial \lambda} = -\alpha' R_{ij} - \frac{\alpha'^2}{2} R_{iklm} R_j{}^{klm} + \dots$$

$$\frac{\partial}{\partial \lambda} g_{\mu\nu} = -2R_{\mu\nu}$$

Hamilton

Black Holes and KPZ

$$\frac{\partial h}{\partial t} = i \left[\nu_{\text{KPZ}}(r, \alpha) \nabla^2 h(r, s) + \lambda_{\text{KPZ}}(r) \left(\frac{dh}{dr} \right)^2 \right] + \bar{\eta} \quad \text{Kardar-Parisi-Zhang Eq. [1]}$$

for a spherical surface

$$\nu_{\text{KPZ}}(r, \alpha) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \quad \lambda_{\text{KPZ}}(r) = e^{\nu_0(r)}$$

- Gravitational waves good probe for near-horizon physics: **no screening**
- Black holes mergers: possibly use GW right before merger and final part of “ringdown” [2]
- Proper-time intermittency: include **BH spin**

$$ds^2 \simeq -e^{\nu_0(r)} \left(1 + \sqrt{2\alpha} h \right) dt^2 + e^{\mu_0(r)} \left(1 - \sqrt{2\alpha} h \right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

- **Low dissipation:** far-from-horizon (linear) but need no large $\sqrt{2\alpha}$
- Possible accumulated effects during gravitational waves propagations [3]

[1] M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)

[2] V. Cardoso, E. Franzin, P. Pani, PRL, 116, 171101 (2016)

[3] M. Lulli, A. Marcianò and N. Yunes, work in progress

Characterisation of confinement

Asymptotic behaviour in the Euclidean time T in the operator formalism

$$\langle \mathcal{C}(T_E)^\dagger \mathcal{C}(0) \rangle \propto \sum_n |c_n|^2 e^{-\Delta E_n T_E} \sim e^{-\Delta E_{\min} T_E}$$

Dominated by the static potential, energy difference between two static charges

$$\mathcal{W}_\alpha = \prod_{i=1,n} \langle H_{\gamma_i} \rangle = \exp[-\sigma_\gamma A_\alpha]$$

Uncorrelated plaquettes

$$\sigma_\gamma = -\frac{\ln \langle H_\gamma \rangle}{A'}$$

Magnetic disorder

Magnetically disorder states characterised by linear growth of the interaction potential

$$V(R) = \sigma R + 2V_0$$