Attractors & Universality in 3+1D Relativistic Boltzmann Transport

Vincenzo Nugara

mostly based on:
V. Nugara, S. Plumari, L. Oliva, and V. Greco, *Eur.Phys.J.C 84 (2024) 8, 861*;
V. Nugara, S. Plumari, V. Greco *Eur.Phys.J.C 85 (2025) 3, 311*
V. Nugara, S. Plumari, V. Greco, N. Borghini *in preparation*

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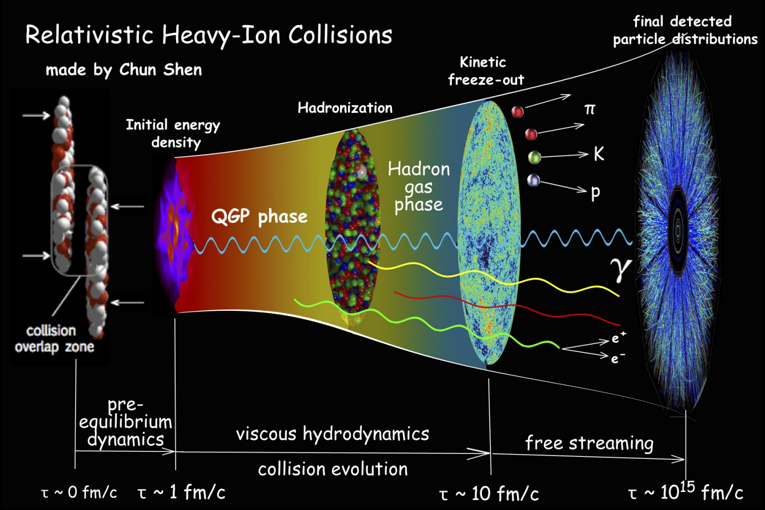
Laboratori Nazionali del Sud

Meeting of the SIM and PRIN 2022SM5YAS projects

Torino, July 2-3

ultra-Relativistic Heavy-Ion Collisions...



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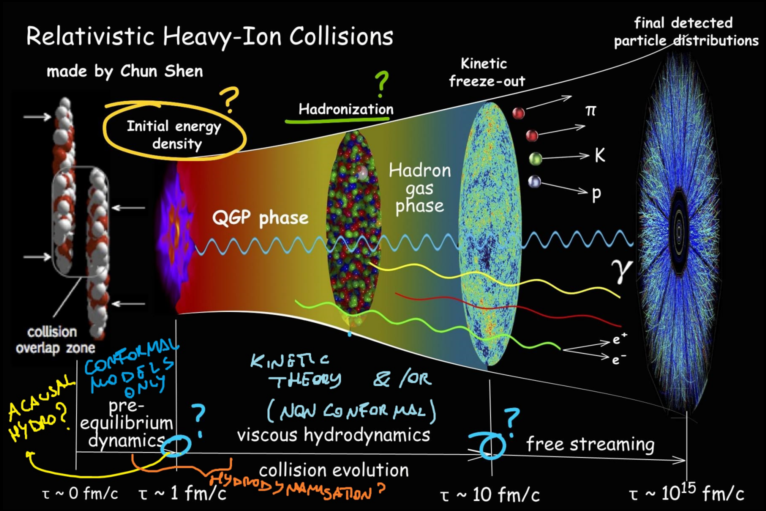
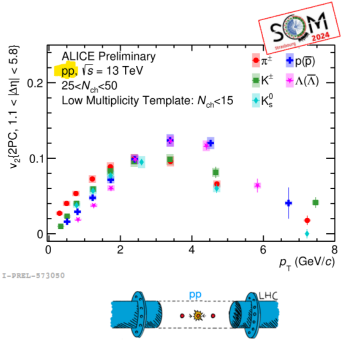
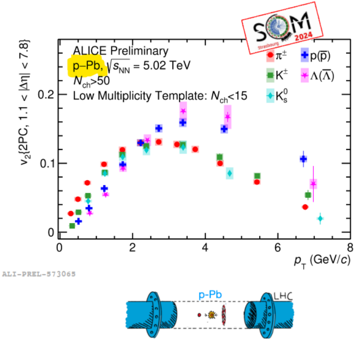
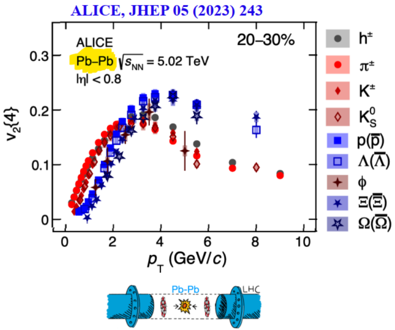


...but not only

Collectivity signatures observed also in small systems (*pp* and *pA*)







(You Zhou, *Collectivity in high energy proton proton collisions*, SQM2024)

Good description by hydrodynamics!

Attractors

What is an attractor?

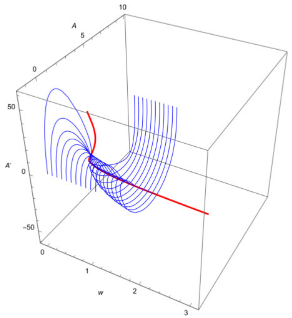
Subset of the phase space to which all trajectories converge

Why do we look for attractors?

- Uncertainties in initial conditions affect final observables? Memory of initial conditions?
- Attractors and hydrodynamisation (small systems)
- Universality as a hint for collective phenomena

Where do we look for attractors?

- Full distribution function $f(x, p)$
- Moments of $f(x, p)$ and anisotropic flows v_n



Jankowski, Spalinski, *Hydrodynamic attractors in ultrarelativistic nuclear collisions*, 2023

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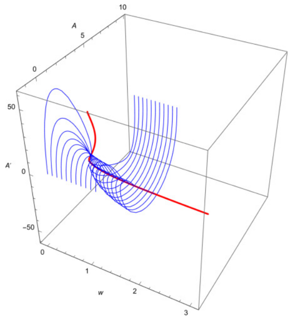

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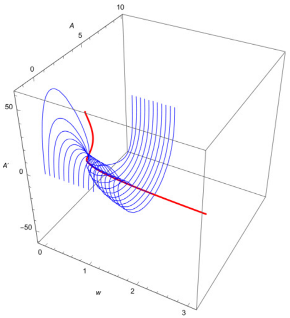

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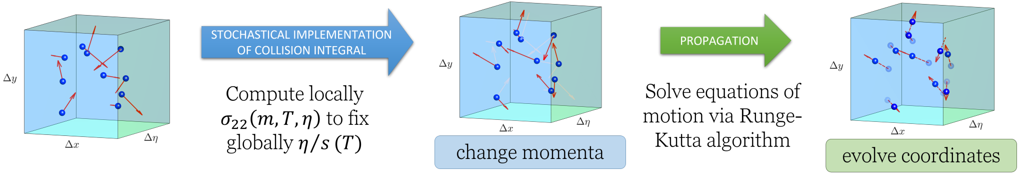
Relativistic Boltzmann Transport (RBT) Code

- Solve Boltzmann Equation: $p^\mu \partial_\mu f(x,p) = C_{2\leftrightarrow 2}[f(x,p)]_p$
- Large number of Test Particles sample the distribution function



Unique tool from $\eta/s \lesssim 1/4\pi$ (hydro limit) to $\eta/s \to +\infty$ (free streaming limit)

Preserving causality by construction: Particles velocity $\leq c$, $\Delta t > \Delta x$

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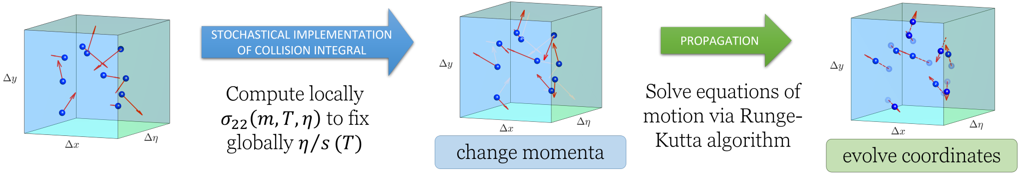

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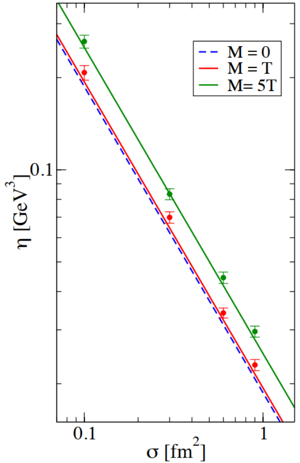
Fixing η/s

- $2 \leftrightarrow 2$ collisions $\Rightarrow$ Particle conservation $\Rightarrow$ Fugacity $\neq 1$.
- Test particles can collide with probability $P_{22} \propto \sigma_{22}(x)$
  Notice: They are not physical collisions, but a numerical method to implement $C_{22}\left[f(x,p)\right]_p$
- Fix σ_{22} (total cross section) locally via the Champan-Enskog formula (Plumari, Puglisi, Scardina, Greco, PRC 86 (2012)):

$$\eta = f(m/T)\frac{T}{\sigma_{22}} \overset{m=0}{=} 1.2\frac{T}{\sigma_{22}} \Rightarrow \sigma_{22}(x) = 1.2\frac{T(x)}{\eta/s\, s(x)}$$

$\eta/s \to 0$: ideal hydro; $\eta/s \to \infty$: free streaming
$(\eta/s)_{QGP} \sim 1/4\pi$: most ideal fluid!



Green-Kubo vs Chapman-Enskog estimations of η.

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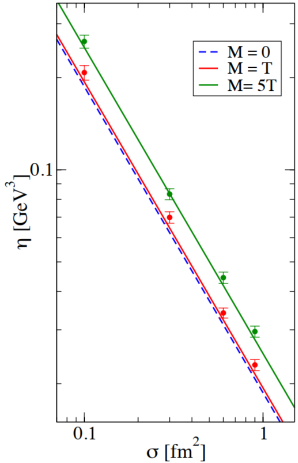

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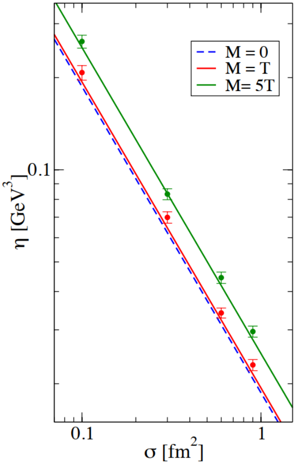

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Code setup for 1D boost-invariant systems (Bjorken flow)

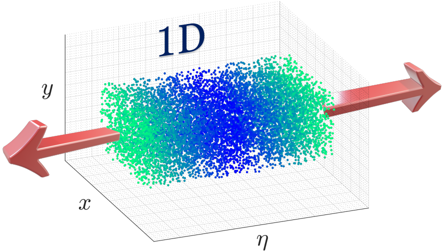
- Conformal system ($m = 0$)
- One-dimension Homogeneous distribution and periodic b.c. in the transverse plane.
- Boost-invariance. No dependence on η_s $dN/d\eta_s = \text{const.}$ in $[-\eta_{s\,\max}, \eta_{s\,\max}]$
- Normalised moments: $\overline{M}^{nm}(x) = \dfrac{\int dP\,(p\cdot u)^n (p\cdot z)^{2m}\, f(x,p)}{\int dP\,(p\cdot u)^n (p\cdot z)^{2m}\, f_{eq}(x,p)}$ (e.g. $\overline{M}^{01} = P_L/P_{eq}$)

Romatschke-Strickland Distribution Function

$$f_0(\mathrm{p}; \gamma_0, \Lambda_0, \xi_0) = \gamma_0 \exp\left(-\frac{1}{\Lambda_0}\sqrt{p_\perp^2 + p_w^2(1+\xi_0)}\right)$$

where $p_\perp^2 = p_x^2 + p_y^2$ and $p_w = (p\cdot z)$

ξ_0 fixes initial P_L/P_T, γ_0 and Λ_0 fix initial ε and n



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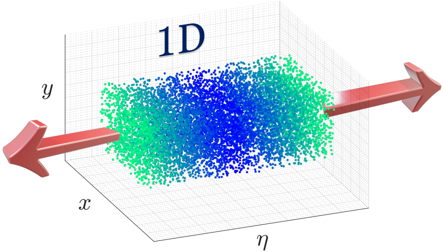
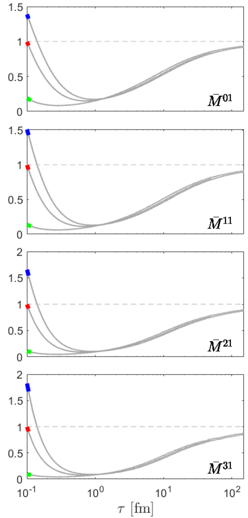
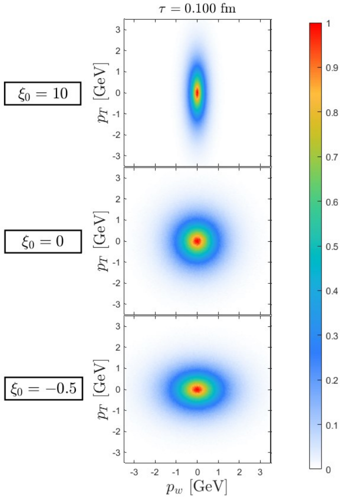


Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

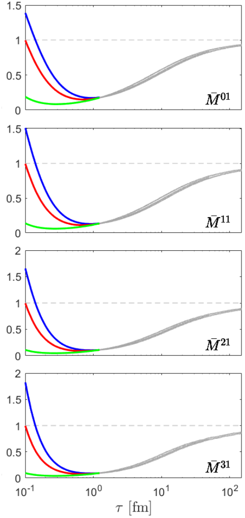
- At $\tau = \tau_0$, three different distributions in momentum space: oblate ($\xi_0 = 10$), spherical ($\xi_0 = 0$) and prolate ($\xi_0 = -0.5$).

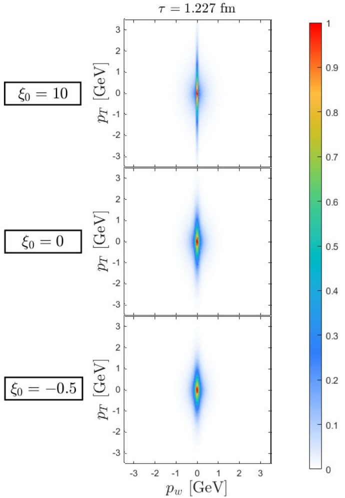




Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

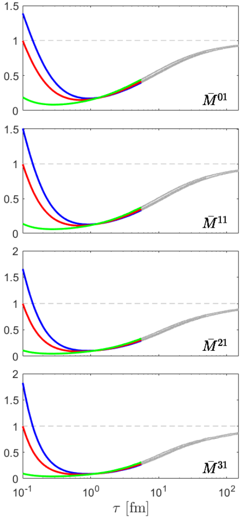
- Already at $\tau \sim 1$ fm, strong initial longitudinal expansion brings the system away from equilibrium
- Distribution functions have similar (but not identical) shape.

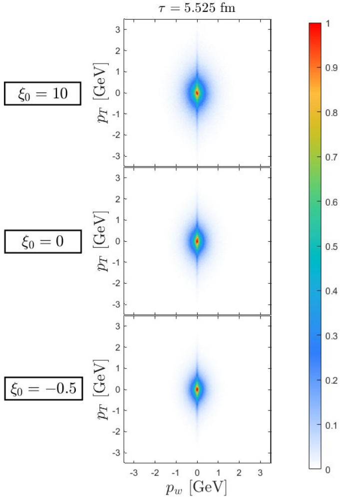




Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

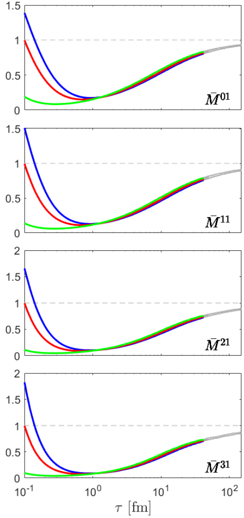
- At $\tau \sim 5$ fm, clear universal behaviour also for the distribution functions.
- Two components: strongly peaked p_w distribution and a more isotropic one
  (Strickland, *JHEP* 12, 128)

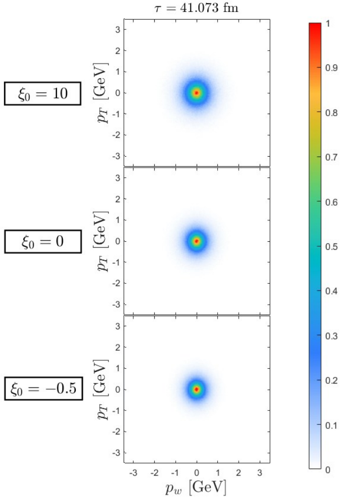




Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

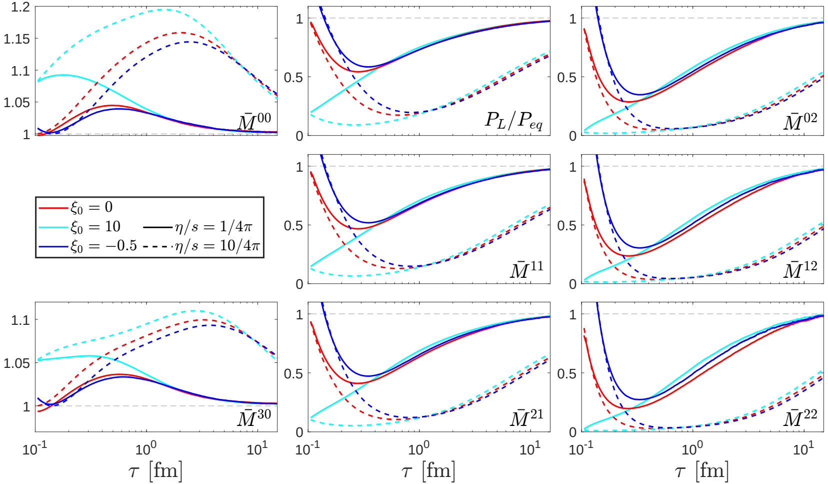
- For large τ the system is almost completely thermalized and isotropized.





Forward Attractor vs τ

Different initial anisotropies $\xi_0 = -0.5,\ 0,\ 10,\ \infty$, for $\eta/s = 1/4\pi$ and $\eta/s = 10/4\pi$.



- $\eta/s = 1/4\pi$: attractor at $\tau \sim 0.5$ fm
- $\eta/s = 10/4\pi$: attractor at $\tau \sim 1.0$ fm
- Not 10 times larger!
- Less collisions to reach the attractor?
- **Different attractors for different η/s?**

Mean free time & Pull-back attractors

Only one relevant time-scale

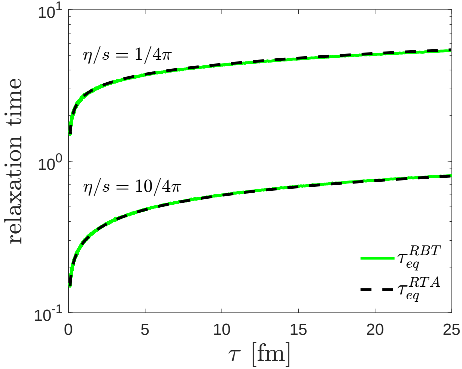
Mean free time

$$\tau_{coll} = \frac{1}{2}\left(\frac{1}{N_{\text{part}}}\frac{\Delta N_{\text{coll}}}{\Delta t}\right)^{-1}$$

Notice: $\tau_{coll} \propto \lambda_{\text{mfp}}$.

$$\tau_{eq}^{RBT} = \frac{3}{2}\tau_{coll} = \tau_{tr} = \tau_{eq}^{\text{RTA}} = \frac{5\eta/s}{T}$$

(Denicol *et al.PRD* 83, 074019)



Same relaxation time as RTA

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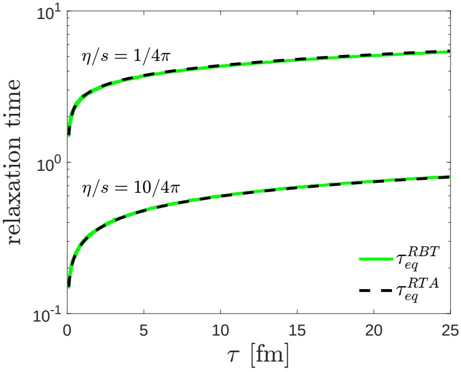

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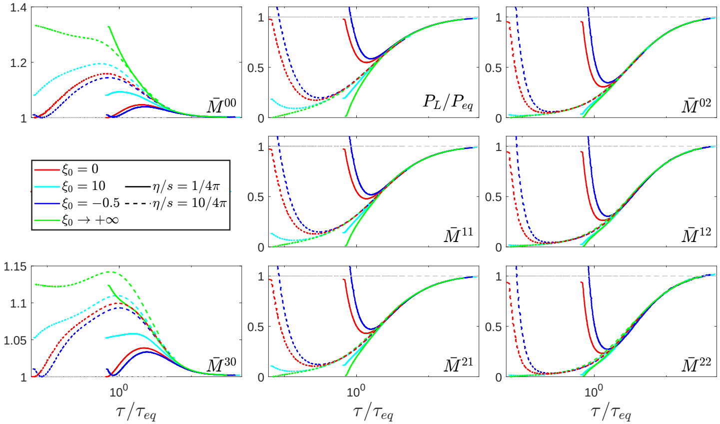
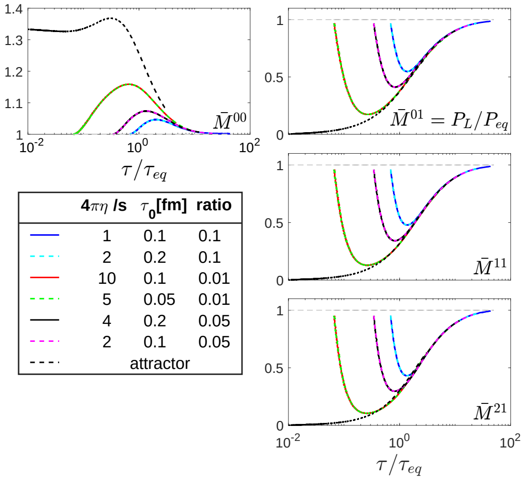
Mean free time & Pull-back attractors

Only one relevant time-scale $\Longrightarrow$ Solution rescaling: Pull-back attractor



- **Unique attractor!**
- $\eta/s = 1/4\pi$: attractor at $\tau \sim 1.5\,\tau_{eq}$
- $\eta/s = 10/4\pi$: attractor at $\tau \sim 0.2\,\tau_{eq}$
- Initial free streaming expansion

Universality in 1D



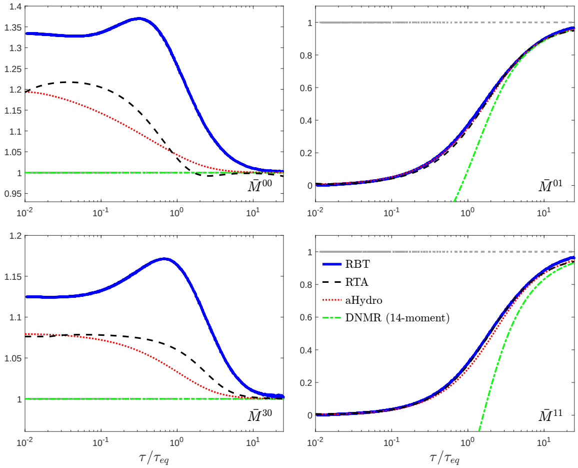
- Fix $(\tau/\tau_{eq})_0 = \tau_0 T_0/(\eta/s) \implies$ same results in terms of scaled time τ/τ_{eq}
- Same in RTA and hydro.

In DNMR $\bar{w} \equiv \frac{\tau T}{5\eta/s}$ and $\varphi = \frac{\pi}{4\varepsilon} + \frac{2}{3}$:

$$\begin{cases} \bar{w}\varphi\varphi' + 4\varphi^2 + (\bar{w} - \frac{34}{7})\varphi - \frac{442}{315} - 2\frac{\bar{w}}{3} = 0 \\ \varphi(\bar{w}_0) = \varphi_0 \end{cases}$$

(Strickland *et al.* *PRD* 97 036020 (2018))

Comparison with different models



Who is ***the*** attractor?
Go to the limit $\xi_0 \to \infty$ $(P_L \to 0)$,
$(\tau/\tau_{eq})_0 = \tau_0 T_0/(\eta/s) \to 0$;
in agreement with RTA and aHydro

(M. Strickland *et al.PRD* 97, 036020 (2018),
P. Romatschke *PRL* 120, 012301 (2018))

- Very good agreement with other models for M^{nm}, $n > 0$, get slightly worse for higher order moments
- Worse agreement for M^{0m}: sensitivity to slowly thermalising particles with $p_z \sim 0$

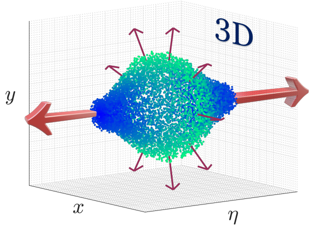
Code setup for 3D systems

- Conformal system ($m = 0$)
- Relax boundary conditions in the transverse plane $\Longrightarrow$ Transverse expansion

Romatschke-Strickland Distribution Function

$$f_0(x, \mathrm{p}) = \gamma_0 \exp\left(-\frac{\sqrt{p_T^2 + p_w^2(1+\xi_0)}}{\Lambda_0}\right) e^{-x_\perp^2/R^2}\theta(2.5 - |\eta_s|)$$

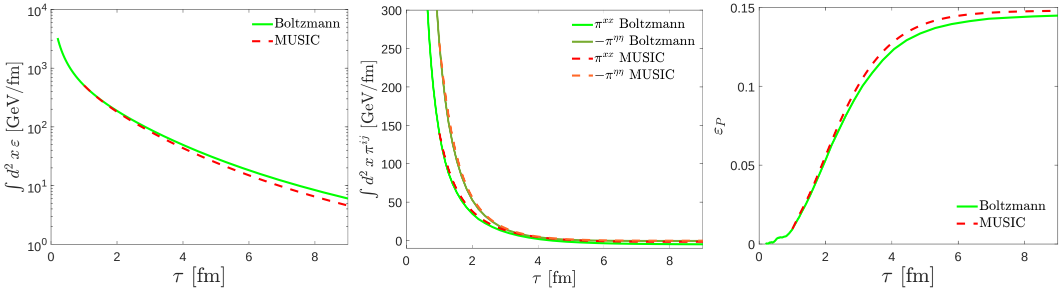
- γ_0 and Λ_0 fix initial ε and n (Landau matching conditions);
- ξ_0 fixes initial P_L/P_T
- Gaussian distribution in the transverse plane
- Uniform distribution in η_s: [-2.5, 2.5]



Comparison with hydro

Very good agreement with 3D conformal hydro (MUSIC) with $\eta/s = 1/(4\pi)$:

- Matching time at 1.0 fm via full $T^{\mu\nu}$
- Conformal EOS, same $\eta/s = 1/4\pi$
- Fugacity: $\Gamma(t) \neq 1$ in Boltzmann $\neq$ $\Gamma(t) = 1$ in hydro.



Transverse expansion

$0 < t < R$

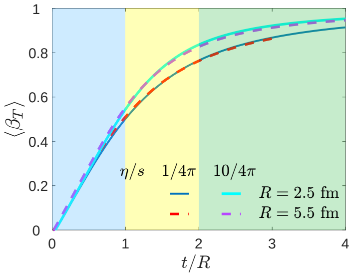
Longitudinal expansion ($\sim$ 1D)

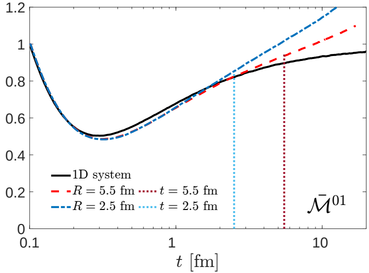
$t > R$

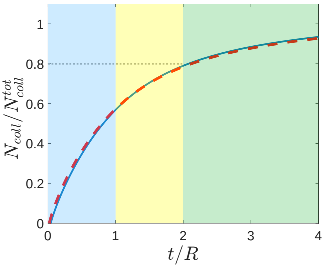
Onset of transverse expansion

$t > 2R$

Quasi free streaming ($\langle\beta_\perp\rangle > 0.8$)







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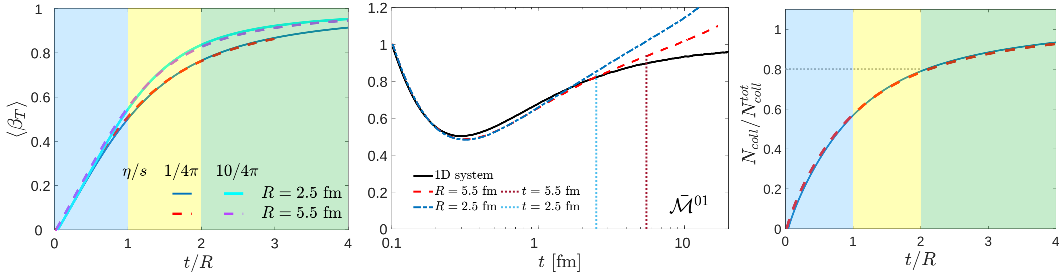

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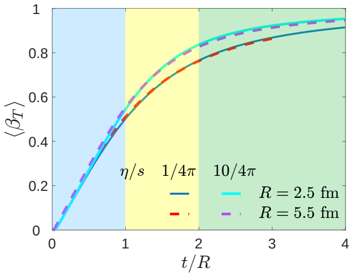
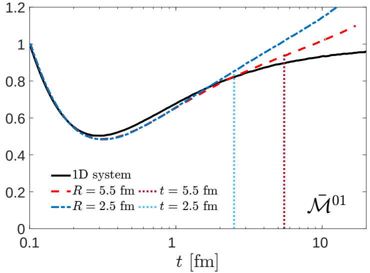
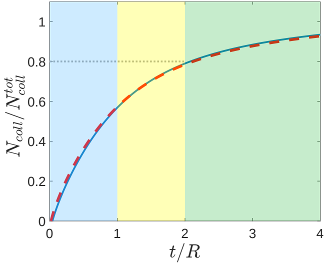


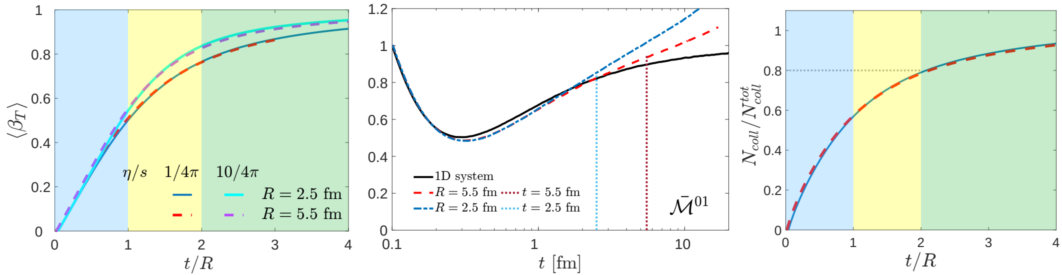




Transverse expansion

New relevant time/length scale

Transverse dimension R



Boltzmann Equation in Relaxation Time Approximation

$$\partial_\tau f + \vec{v}_\perp f - \frac{p_z}{\tau}\partial_{p_z} f = -\frac{v_\mu u^\mu}{\tau_{rel}}(f - f_{eq})$$

Solve the equation for the p-integrated moment...

$$\mathcal{F}(\tau, \vec{x}_\perp; \Omega) \equiv \int \frac{4\pi dp\, p^3}{(2\pi)^3} f(\tau, x_\perp; p_\perp, p_z)$$

...after writing it in a dimensionless fashion:

$$\partial_\tau \mathcal{F} + \vec{v}_\perp \cdot \partial_{\vec{x}_\perp}\mathcal{F} - \frac{1}{\tau} v_z(1 - v_z^2)\partial_{v_z}\mathcal{F} + \frac{4v_z^2}{\tau}\mathcal{F} = -\hat{\gamma}\left[\varepsilon_{\mathcal{F}}^{1/4}(-v \cdot u)\mathcal{F} - \frac{\varepsilon_{\mathcal{F}}^{5/4}}{(-v \cdot u)^3}\right]$$

The only free parameter is the opacity $\hat{\gamma} = \frac{1}{5\eta/s}\left(\frac{R}{\pi a}\frac{dE_\perp^0}{d\eta}\right)^{1/4}$

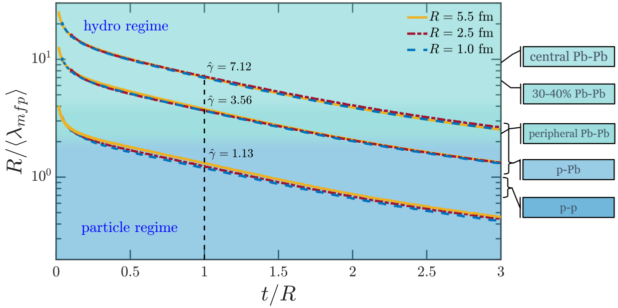
(Kurkela et al., PLB 783, 274 (2018); Ambrus et al. PRD 105, 014031 (2022))

Opacity vs Inverse Knudsen Number R/λ_{mfp}

The Inverse Knudsen Number is the ratio between the two main physical scales:

$$\text{Kn}_R^{-1} = \frac{R}{\lambda_{\text{mfp}}}(t = R) \approx \hat{\gamma}$$

It also defines the regime of rigorous applicability of hydro



Universality classes in R/λ_{mfp}

Link with 1D: $\hat{\gamma} = \frac{1}{5\eta/s}\left(\frac{R}{\pi a}\frac{dE_\perp^0}{d\eta}\right)^{1/4} = \frac{\tau_0 T_0}{5\eta/s}\left(\frac{R}{\tau_0}\right)^{3/4} = (\tau/\tau_{eq})_0\left(\frac{R}{\tau_0}\right)^{3/4}$

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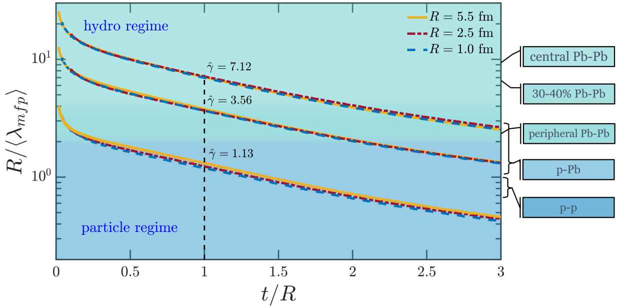

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Forward attractors

3+1D, with azimuthal symmetry at $\eta_s \sim 0 \implies \overline{M}^{nm} = \overline{M}^{nm}(t, x_\perp)$.
Fix $\eta/s = 1/4\pi$. Change ξ_0 (P_L/P_T) and R.

- Same trend of 1D: attractor due to initial longitudinal expansion (identical in 1D and 3D)
- Reached at same t for different R (transverse size doesn't matter)
- Differentiate when transverse expansion starts to play a role

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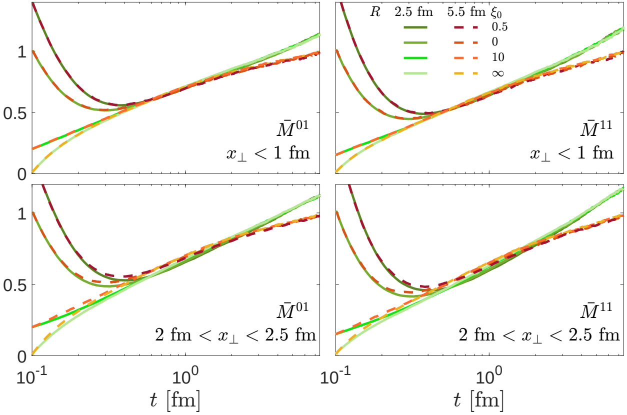

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Pull-back attractors

We do not have a unique time-scale any more.
How do we rescale time? Do we expect pull-back attractors at all?

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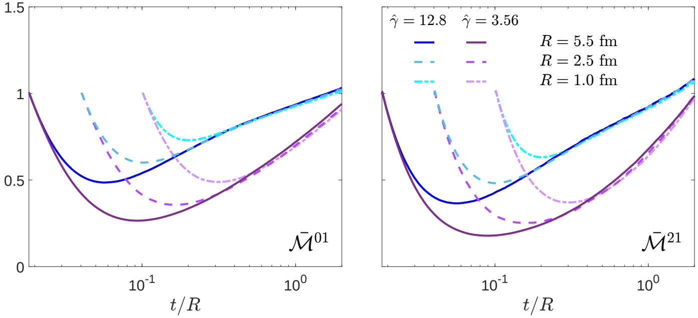

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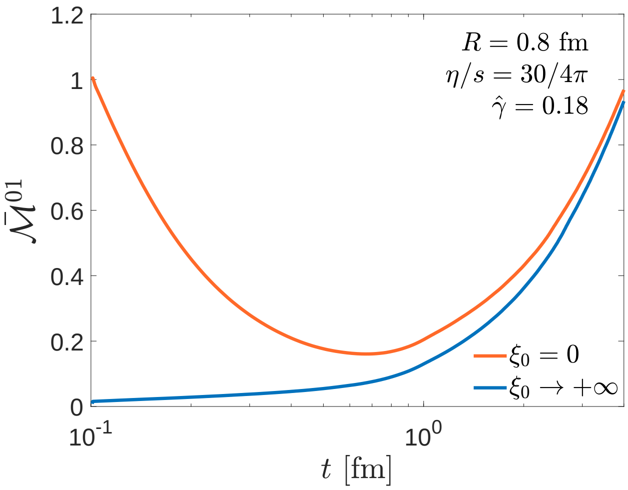
Loss of attractors for extremely small $\hat{\gamma}$



- Attractor do not reached even for $t = 4$ fm $\approx 5R$.
- This case is strongly unphysical! Low estimates for $\hat{\gamma}_{pp} \gtrsim 0.4$

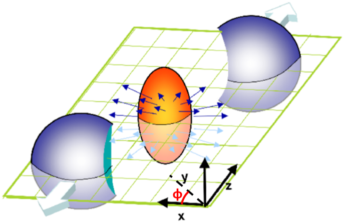
Eccentricities and anisotropic flows

Reproduce eccentricity in coordinate space by shifting (x, y):

$$z = x + iy \to z' = z - \alpha \bar{z}^{n-1}$$

$$\epsilon_n = \frac{\sqrt{\langle x_\perp^n \cos(n\phi)\rangle^2 + \langle x_\perp^n \sin(n\phi)\rangle^2}}{\langle x_\perp^n \rangle} \overset{\alpha \ll 1}{\simeq} n\alpha \frac{\langle x_\perp^{2(n-1)}\rangle}{\langle x_\perp^n \rangle}.$$

(S. Plumari, G. L. Guardo, V. Greco, J.-Y. Ollitrault, Nucl. Phys. A 941, 87 (2015))



Viscosity converts space anisotropies in momentum space. Expand distribution function as:

$$\frac{dN}{d\phi\, p_\perp\, dp_\perp} \propto 1 + 2\sum_{n=1} v_n(p_\perp) \cos[n(\phi_p - \Psi_n(p_\perp))].$$

Anisotropic flows $v_n = \langle \cos(n\phi) \rangle$

How efficiently does this conversion happen? How does it depend on η/s, $\hat{\gamma}$ and R/λ_{mfp}?

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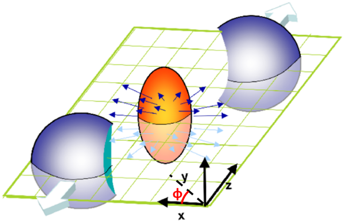

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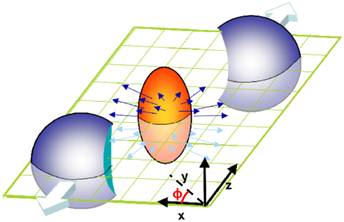

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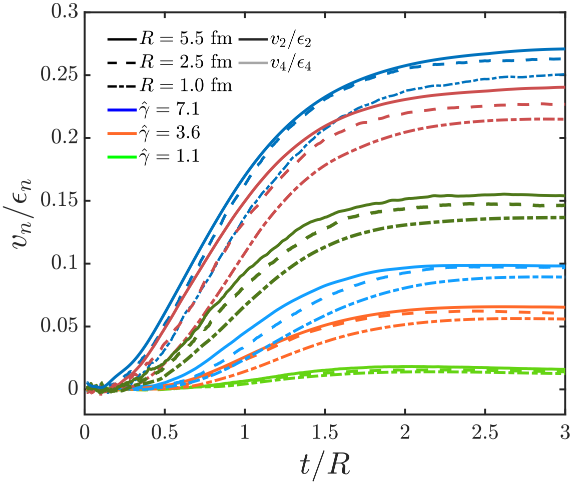
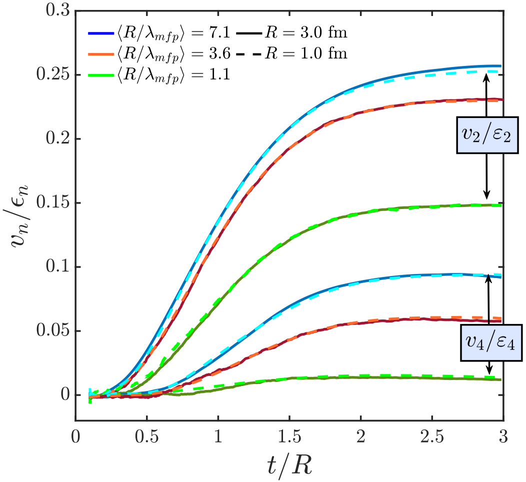
Response functions v_n/ϵ_n: Knudsen number vs opacity



- No dependence on ϵ_n
- Clusters in $\hat{\gamma}$ within 10%. Spreading decreases with increasing $\hat{\gamma}$
- For fixed $\hat{\gamma}$, monotonic ordering in R

Response functions v_n/ϵ_n: Knudsen number vs opacity

Universality w.r.t Knudsen number!



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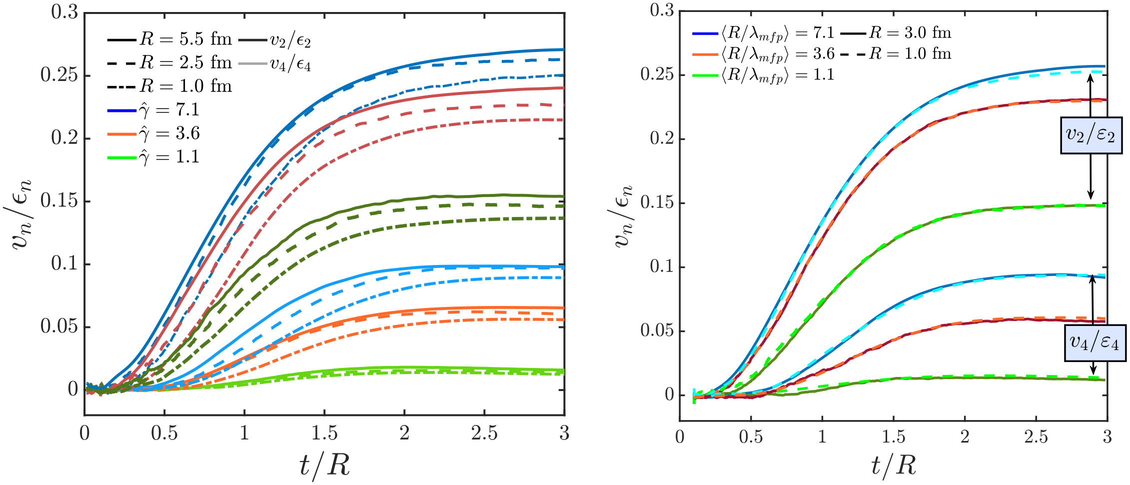
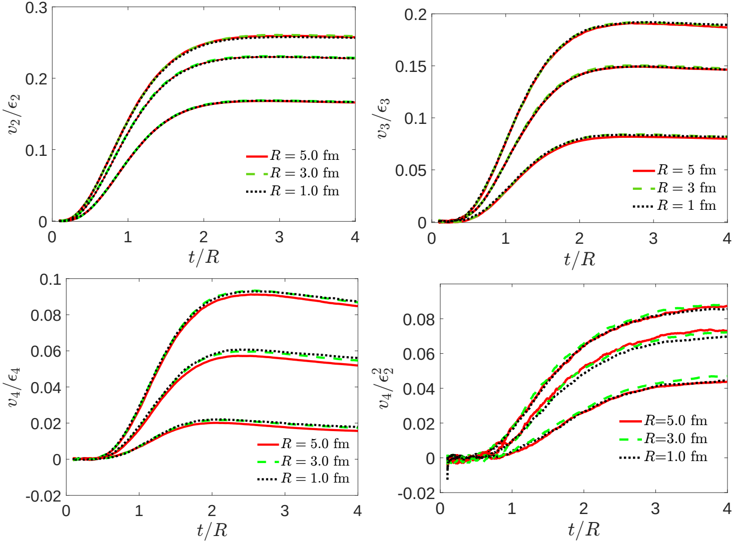


Response functions



- Very good scaling from small ($R = 1$ fm) to large systems ($R = 5$ fm) with $\eta/s = 0.5/4\pi - 10/4\pi$
- Scaling is slightly worse for higher order harmonics
- Universality also for quadratic response functions $v_4 \approx (\varepsilon_2)^2$

(in preparation)

Dissipation of initial v_2

- Initial ($\tau_0 \sim 0.1 - 0.4$ fm) v_n from CGC model prediction
- Mimic initial $v_2 = 0.025$ by $\psi_0 = -0.1 \implies f \propto \exp\left(-\sqrt{p_x^2(1+\psi_0) + p_y^2 + p_z^2}/T\right)$
- How does this initial v_2 impact on the observed $v_2(t = 2R)$?

- $\sim$ Universality in $\hat{\gamma}$ (same colour curves)
- For AA systems really small impact: collisions cancel initial correlation
- For pp strong impact $\gtrsim 15\%$

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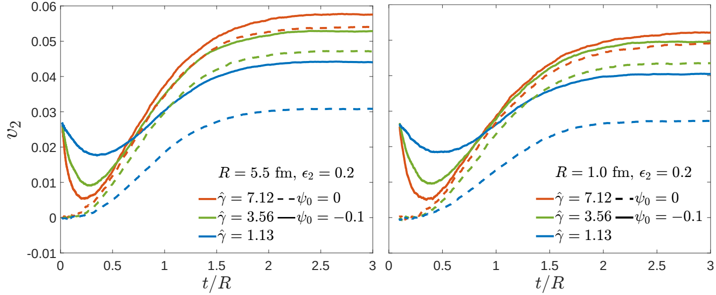

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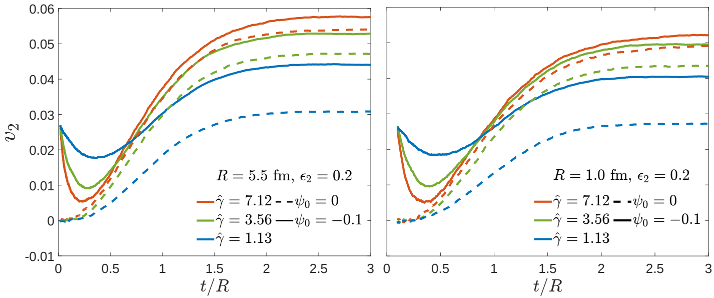

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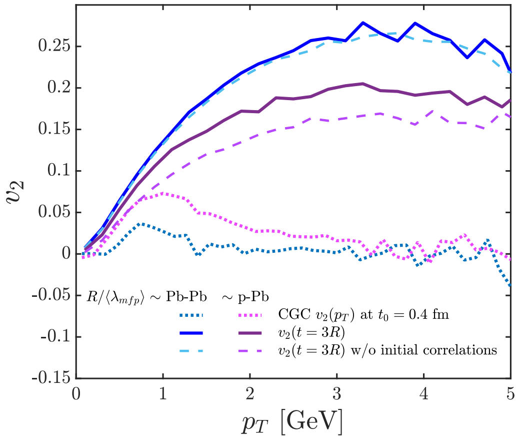
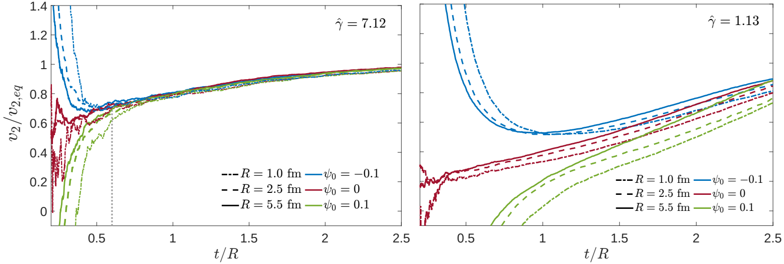
Memory of initial v_2 in pA vs AA



- Minijets + $m = 0.3$ GeV ($\approx$ QPM) + $\eta/s(T)$
- Initial $v_2(p_T)$ from CGC
  (Schenke et al., PLB 747 (2015))
- Initial eccentricity $\epsilon_2 = 0.3$
  (Sun et al., EPJC (2020))
- No memory of initial $v_2(p_T)$ in AA
- Sensitive impact of initial $v_2(p_T)$ in pA

Attractors in $v_2/v_{2,eq}$

Equilibrium v_n: $v_n^{eq} = \dfrac{\int d^2x_\perp \int d^3p \cos(n\phi)\,\Gamma(x_\perp)\exp(-p_\mu \cdot u^\mu(x_\perp)/T(x_\perp))}{\int d^2x_\perp \int d^3p\,\Gamma(x_\perp)\exp(-p_\mu \cdot u^\mu(x_\perp)/T(x_\perp))}$.



- Fix opacity $\hat{\gamma}$, change $R, \eta/s, \psi_0$
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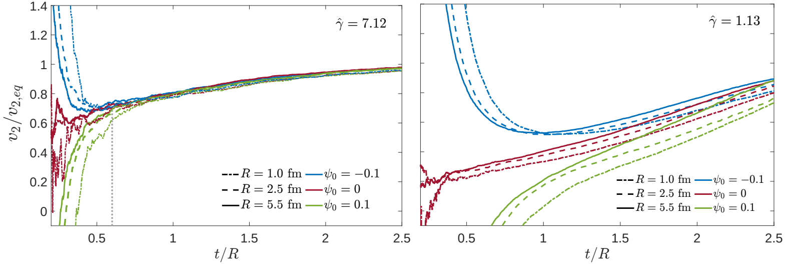

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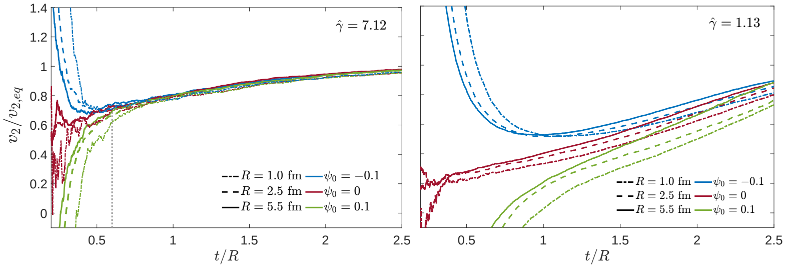

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Summary

1D systems

- Attractors in all the examined cases in the distribution function and its moments
- One relevant time scale (τ_{eq}) driving the evolution

3D systems

✓ Forward and pull-back attractors ($\sim$ 1D), difference w.t.r. 1D for $t > R$

✓ Inverse Knudsen number $R/\lambda_{\rm mfp}$ very good universal parameter

✓ Memory of initial momentum correlations in $\sim pA$ systems, not in $\sim AA$

Outlook

- Non-conformal equation of state implemented
- Initial fluctuations for event-by-event simulation implemented
- Pre-hydrodynamic transport + transport/hydro without discontinuity in bulk pressure Π

Thank you for your attention.

LRF and matching conditions

Define the Landau Local Rest Frame (LRF) via the fluid four-velocity:

$$T^{\mu\nu} u_\nu = \varepsilon u^\mu,$$
$$n = n^\mu u_\mu$$

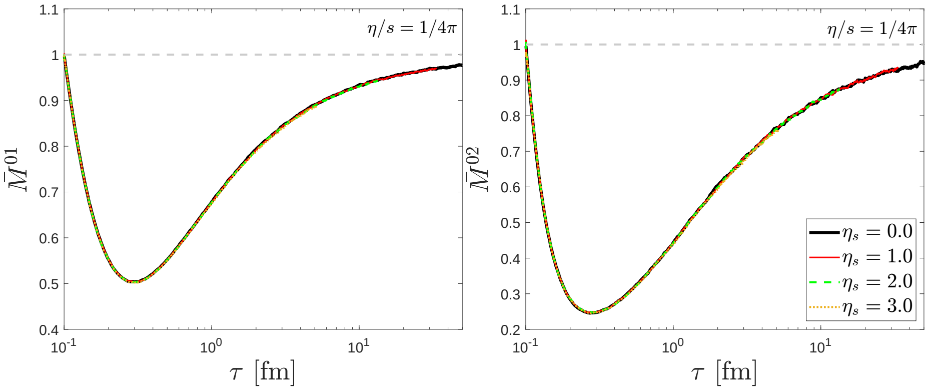
ε and n are the energy and particles density in the LRF.
Fluid is not in equilibrium $\implies$ define locally effective T and Γ via Landau matching conditions:

$$T = \frac{\varepsilon}{3\,n}, \qquad \Gamma = \frac{n}{d\,T^3/\pi^2},$$

d is the # of dofs, fixed $d = 1$.

Testing boost-invariance

Compute normalized moments at different η_s's within an interval $\Delta\eta_s = 0.04$.



No dependence on η! We look for them at midrapidity: $\eta \in [-0.02, 0.02]$

Boltzmann RTA Equation for number-conserving systems

Boltzmann equation in Relaxation Time Approximation (RTA) (Strickland, Tantary, JHEP10(2019) 069)

$$p^\mu \partial_\mu f_p = -\frac{p \cdot u}{\tau_{eq}}(f_{eq} - f_p).$$

Exactly solvable, by fixing number and energy conservation.
Two coupled integral equations for $\Gamma_{eff} \equiv \Gamma$ and $T_{eff} \equiv T$:

$$\Gamma(\tau) T^4(\tau) = D(\tau, \tau_0)\Gamma_0 T_0^4 \frac{\mathcal{H}(\alpha_0 \tau_0/\tau)}{\mathcal{H}(\alpha_0)} + \int_{\tau_0}^{\tau} \frac{d\tau'}{2\tau_{eq}(\tau')} D(\tau, \tau')\Gamma(\tau') T^4(\tau')\mathcal{H}\left(\frac{\tau'}{\tau}\right),$$

$$\Gamma(\tau) T^3(\tau) = \frac{1}{\tau}\left[D(\tau, \tau_0)\Gamma_0 T_0^3 \tau_0 + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{eq}(\tau')} D(\tau, \tau')\Gamma(\tau') T^3(\tau')\tau'\right].$$

Here $\alpha = (1+\xi)^{-1/2}$. System solvable by iteration.

vHydro equations

Second-order dissipative viscous hydrodynamics equations according to DNMR derivation, starting from kinetic theory (G. S. Denicol *et al.*, *PRL*105, 162501 (2010)) :

$$\partial_\tau \varepsilon = -\frac{1}{\tau}(\varepsilon + P - \pi),$$
$$\partial_\tau \pi = -\frac{\pi}{\tau_\pi} + \frac{4}{3}\frac{\eta}{\tau_\pi \tau} - \beta_\pi \frac{\pi}{\tau},$$

where $\tau_\pi = 5(\eta/s)/T$ and $\beta_\pi = 124/63$.
Solved with a Runge-Kutta-4 algorithm.

aHydro for number-conserving systems

Formulation of dissipative anisotropic hydrodynamics with number-conserving kernel (Almaalol, Alqahtani, Strickland, PRC 99, 2019).

System of three coupled ODEs:

$$\partial_\tau \log\gamma + 3\partial_\tau \log\Lambda - \frac{1}{2}\frac{\partial_\tau\xi}{1+\xi} + \frac{1}{\tau} = 0;$$

$$\partial_\tau \log\gamma + 4\partial_\tau \log\Lambda + \frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau\xi = \frac{1}{\tau}\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right];$$

$$\partial_\tau\xi - \frac{2(1+\xi)}{\tau} + \frac{\xi(1+\xi)^2\mathcal{R}^2(\xi)}{\tau_{eq}} = 0.$$

Solved with a Runge-Kutta-4 algorithm.

Computation of moments in other models

- RTA:

$$M^{nm}(\tau) = \frac{(n+2m+1)!}{(2\pi)^2}\Big[D(\tau,\tau_0)\alpha_0^{n+2m-2}T_0^{n+2m+2}\Gamma_0\frac{\mathcal{H}^{nm}(\alpha\tau_0/\tau)}{[\mathcal{H}^{20}(\alpha_0)/2]^{n+2m-1}}+$$
$$+\int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{eq}(\tau')}D(\tau',\tau')\Gamma(\tau')T^{n+2m+2}(\tau')\mathcal{H}^{nm}\left(\frac{\tau'}{\tau}\right)\Big];$$

- DNMR:

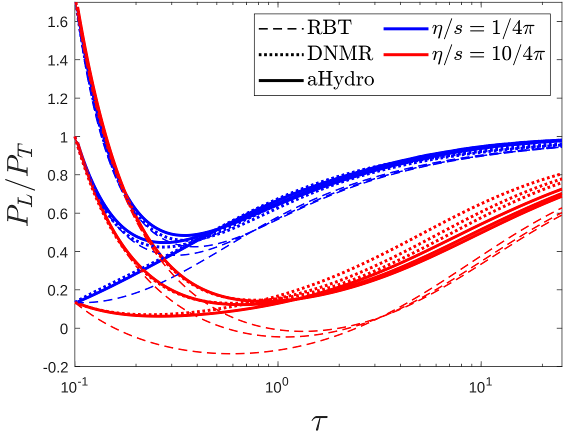
$$\overline{M}^{nm}_{\mathrm{DNMR}} = 1 - \frac{3m(n+2m+2)(n+2m+3)}{4(2m+3)}\frac{\pi}{\varepsilon};$$

- aHydro:

$$\overline{M}^{nm}_{\mathrm{aHydro}}(\tau) = (2m+1)(2\alpha)^{n+2m-2}\frac{\mathcal{H}^{nm}(\alpha)}{[\mathcal{H}^{20}(\alpha)]^{n+2m-1}};$$

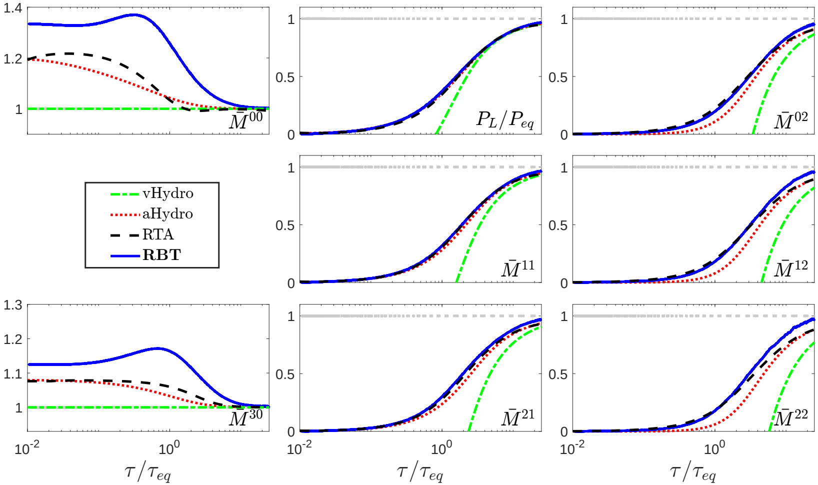
Pressure anisotropy in different frameworks

For $\eta/s = 1/4\pi$ and $\eta/s = 10/4\pi$, compute P_L/P_T from three different initial anisotropies: $\xi_0 = -0.5,\ 0,\ 10.$



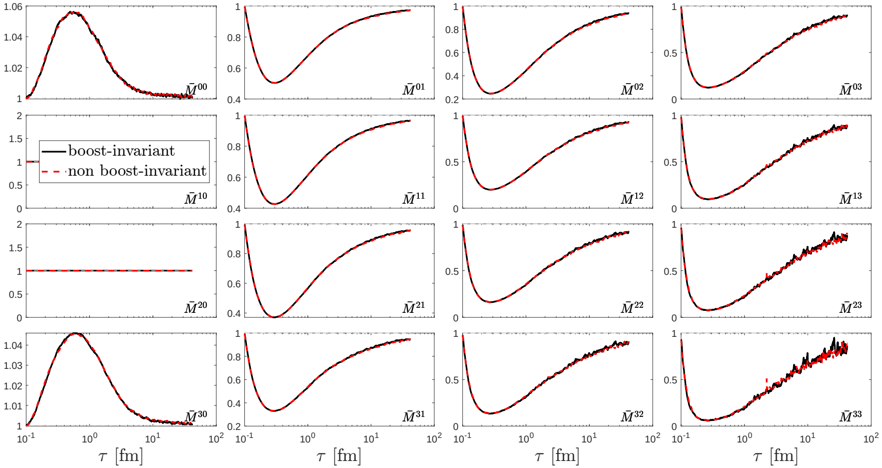
- RTA (not showed) really similar to aHydro
- aHydro attractor reached ∼ time than RBT
- vHydro attractor reached at later time, especially for larger η/s

Attractors in different models



- $\overline{M}^{nm}$, $m > 0$: very good agreement
- Higher order moments $\rightarrow$ stronger departure between models
- RBT thermalizes earlier
- No agreement for $\overline{M}^{n0}$

Midrapidity

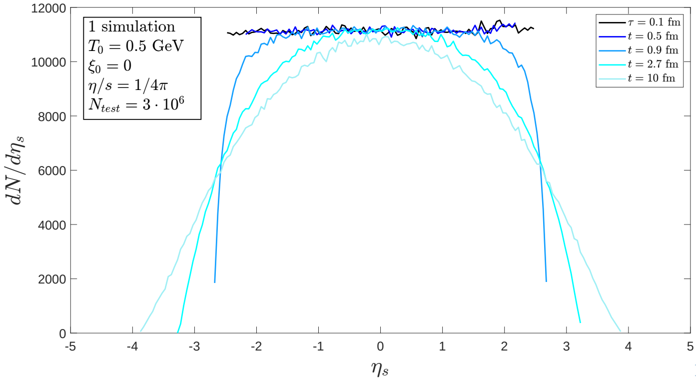


At midrapidity no difference w.r.t. the boost invariant case.

Finite distribution in η

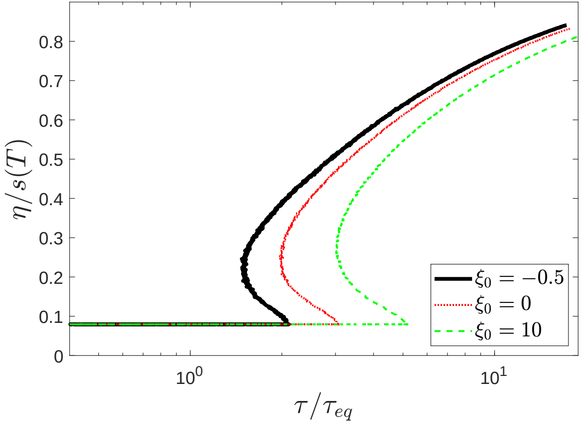
Breaking boost-invariance: $$\frac{dN}{d\eta_s}(\eta_s;\tau_0) = \begin{cases} \text{const.} & |\eta_s| < 2.5 \\ 0 & \text{elsewhere} \end{cases}$$

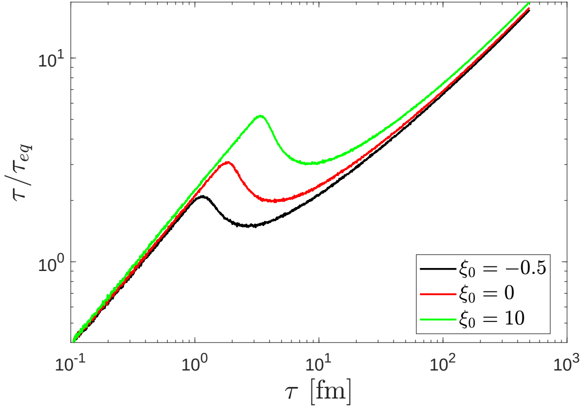
- Tails of the distribution function at $|\eta_s| > 1$
- Discontinuity in initial distribution $\rightarrow$ non-analyticity points in moments' evolution



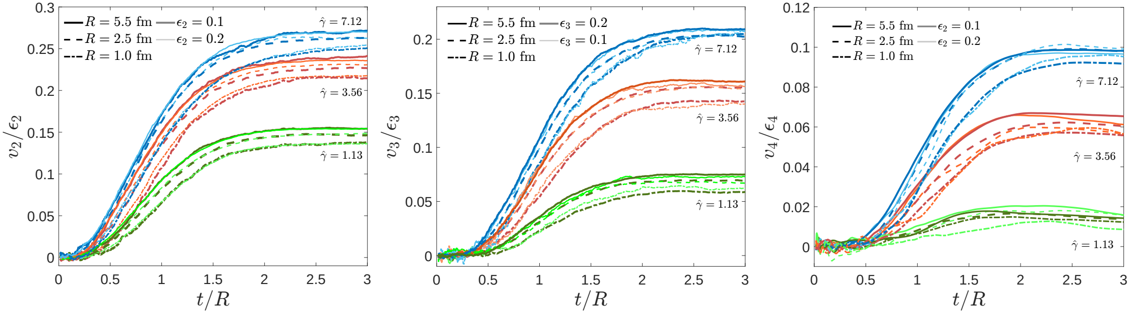
Non-monotonic τ/τ_{eq} for Case 1

Loops when τ/τ_{eq} is no more a monotonic function: $\tau_{eq} \propto \eta/s(T)/T$ grows faster than τ.



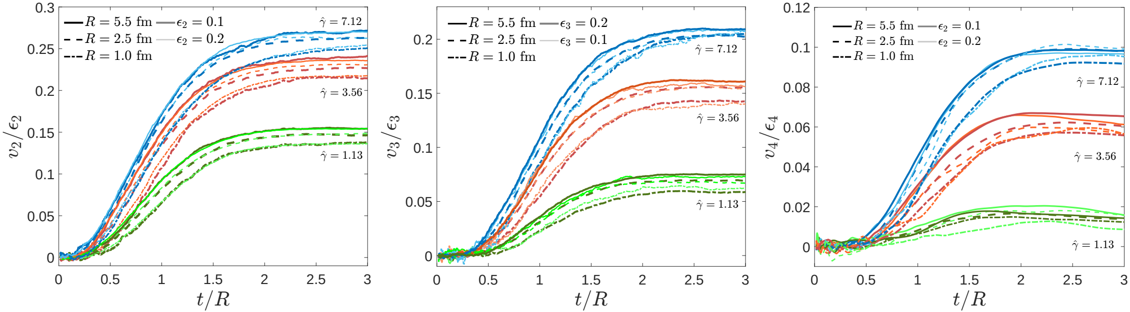


Response functions v_n/ϵ_n at fixed opacity $\hat{\gamma}$



- No dependence on ϵ_n
- Clusters in $\hat{\gamma}$ within 10%. Spreading decreases with increasing $\hat{\gamma}$
- For fixed $\hat{\gamma}$, monotonic ordering in R

Response functions v_n/ϵ_n at fixed opacity $\hat{\gamma}$



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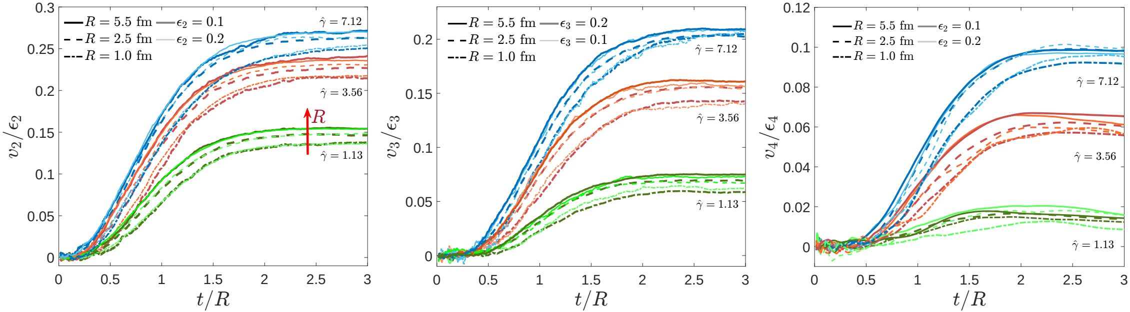
Response functions v_n/ϵ_n at fixed opacity $\hat{\gamma}$



- No dependence on ϵ_n
- Clusters in $\hat{\gamma}$ within 10%. Spreading decreases with increasing $\hat{\gamma}$
- For fixed $\hat{\gamma}$, monotonic ordering in R