Large density QCD: (no) critical point from lattice simulations?

Paolo Parotto, Università di Torino e INFN Torino

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Happy birthday, Wanda!

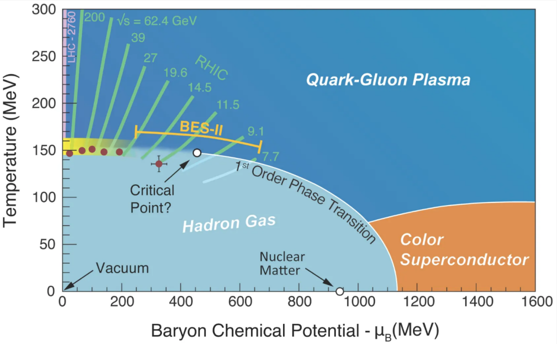
with
Wuppertal-Budapest collaboration: A. Adam, S. Borsányi, Z. Fodor, J. Guenther, P. Kumar, A. Pásztor, L. Pirelli, C. Ratti, C. H. Wong and V. Vovchenko (U. Houston)

The phase diagram of QCD

What do we know about QCD thermodynamics at finite T, μ_B?

From a combination of approaches (experiment, models, first principle calculations, ...), *we are pretty sure* of some things, and *suspect* others.

- Hadron phase at low $T \,\&\, \mu$, QGP at high $T \,||\, \mu$
- Crossover at zero density at $T \simeq 160\,\mathrm{MeV}$
- **Heavy-ion collisions** probe high T, varying density with energy scans
- Ordinary nuclear matter at $T \simeq 0$ and $\mu_B \simeq 922\,\mathrm{MeV}$
- **Critical point?** Exotic phases?



Confinement and chiral symmetry breaking

The QCD transition is characterized by the spontaneous breaking of two approximate symmetries

- **Chiral symmetry**: exact for $m_q \to 0$. Order parameter is **chiral condensate**

  $$\langle \bar{\psi}\psi \rangle = \frac{T}{V}\frac{\partial \log \mathcal{Z}}{\partial m_{ud}}$$

  Symmetric phase $\langle \bar{\psi}\psi \rangle = 0$ at high-T, and SSB $\langle \bar{\psi}\psi \rangle \neq 0$ at low-T

- $\mathbb{Z}(3)$ **center symmetry** (responsible for confinement): exact for $m_q \to \infty$. Order parameter is **Polyakov loop**

  $$P(\vec{x}) = \prod_{x_4=0}^{N_\tau - 1} U_4(\vec{x}, x_4) \sim e^{-F/T}$$

  Symmetric phase $P = 0$ $(F \to \infty)$ at low-T and SSB $P = 0$ $(F \to 0)$ at high-T

Lattice formulation of QCD

In a nutshell, lattice QCD amounts to calculating path integrals like

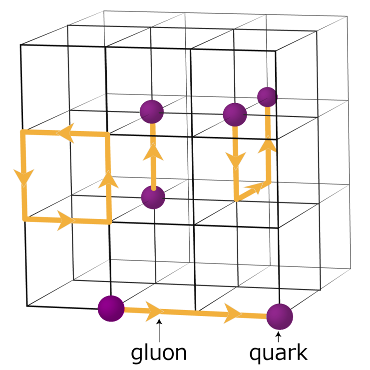
$$\mathcal{Z}[A,\bar{\psi},\psi] = \int \mathcal{D}A^a_\mu(x)\,\mathcal{D}\bar{\psi}(x)\,\mathcal{D}\psi(x)\,e^{-\int d^4x\,\mathcal{L}_E[A,\bar{\psi},\psi]}$$

by defining the theory on a discretized 3+1d lattice with $N_s^3 \times N_\tau$ sites. This allows us to reduce the (otherwise infinite) dimensionality of the problem.

- The quark fields $\bar{\psi},\psi$ are defined on the lattice sites, the gauge fields A_μ are defined on the lattice links as $U_\mu = \exp[iaA_\mu]$
- Now, one can calculate a *finite* number of integrals to evaluate expressions of the like:

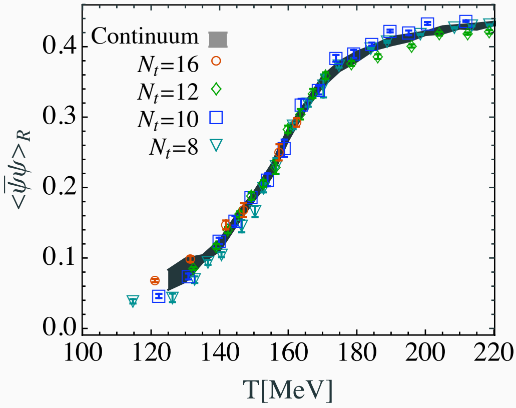
$$Z[U,\bar{\psi},\psi] = \int \mathcal{D}U\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,e^{-S_G[U,\bar{\psi},\psi]-S_F[U,\bar{\psi},\psi]}$$

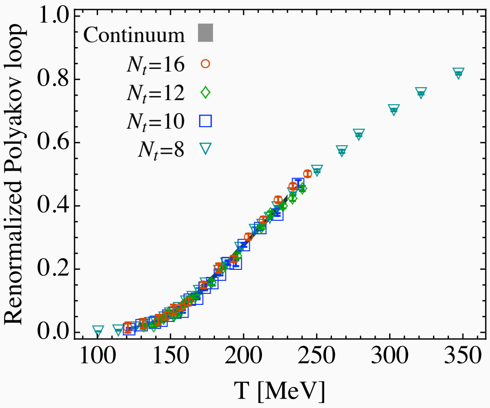
where S_G and S_F are the gauge (gluonic) and fermionic actions



QCD transition at zero density

Lattice results show a smooth crossover





Borsányi *et al.*, JHEP 1009:073 (2010)

The sign/complex action problem

Euclidean path integrals are calculated with MC methods using importance sampling and the Boltzmann weights $\det M[U]\, e^{-S_G[U]}$

$$Z(V,T,\mu) = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{-S_F(U,\psi,\bar{\psi})-S_G(U)}$$
$$= \int \mathcal{D}U\; \det M(U) e^{-S_G(U)}$$

When a chemical potential is introduced, a problem appears:

$$[\det M(\mu)]^* = \det M(-\mu^*)$$

in general the determinant is complex and cannot serve as a statistical weight.

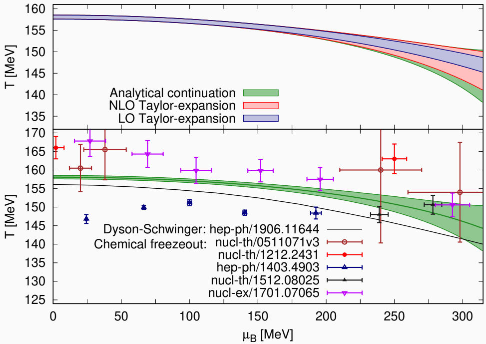
However, that is **not the case if**:

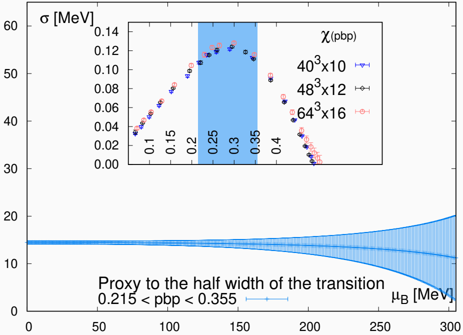
- there is particle-antiparticle-symmetry ($\mu = 0$):
- the chemical potential is *purely imaginary* ($\mu^2 < 0$)

$$[\det M(\mu)]^* = \det M(-\mu^*) = \det M(\mu) \in \mathbb{R}$$

QCD transition at finite density

We resort to extrapolations, but (chiral) transition is well established up to $T \simeq 300$ MeV





and shows no hint of strenghtening

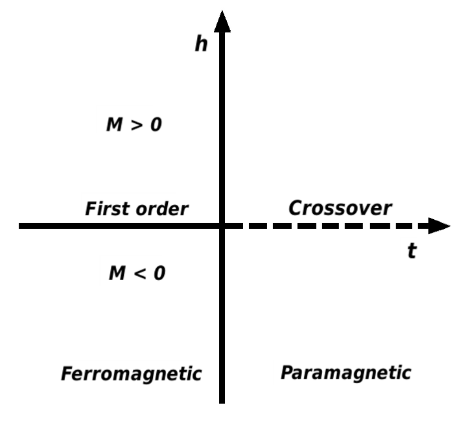
Borsányi, PP, *at el.*, PRL 125 (2020) 052001

The critical point of QCD: universality

The critical point of QCD is in the same universality class as the 3D Ising model

Pisarski, Wilczek, PRD 29 (1984) 338

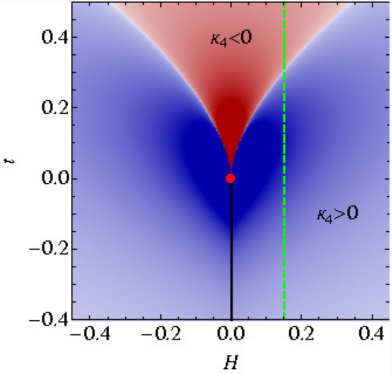
- α : specific heat at $h = 0$ behaves as $C \sim |t|^{\alpha}$;
- β : spontaneous magnetization (i.e. in the limit $h \to 0^{+}$) scales as $M \sim (-t)^{\beta}$;
- γ : zero-field susceptibility $\chi \equiv (\partial M/\partial H)_{H=0} \sim |t|^{-\gamma}$;
- δ : along the h axis, i.e. for $T = T_C$, the magnetization follows $M \sim \mathrm{sign}(h)\,|h|^{1/\delta}$;
- with $\beta \simeq 0.326$, $\alpha \simeq 0.11$ and $2 = \alpha + \beta(1+\delta)$

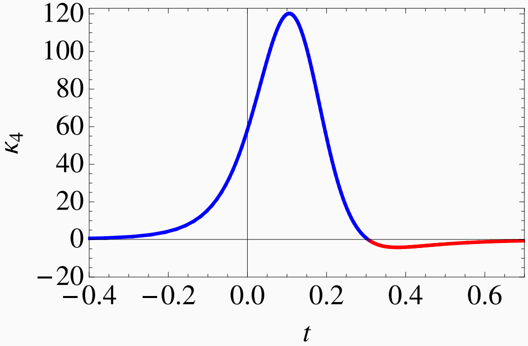


In QCD, the critical exponents of QCD will be the same, *if* there is a critical point!

Signals of critical behavior

Baryon fluctuations diverge at the critical point with increasing powers of the correlation length → higher order net-proton fluctuations are most promising



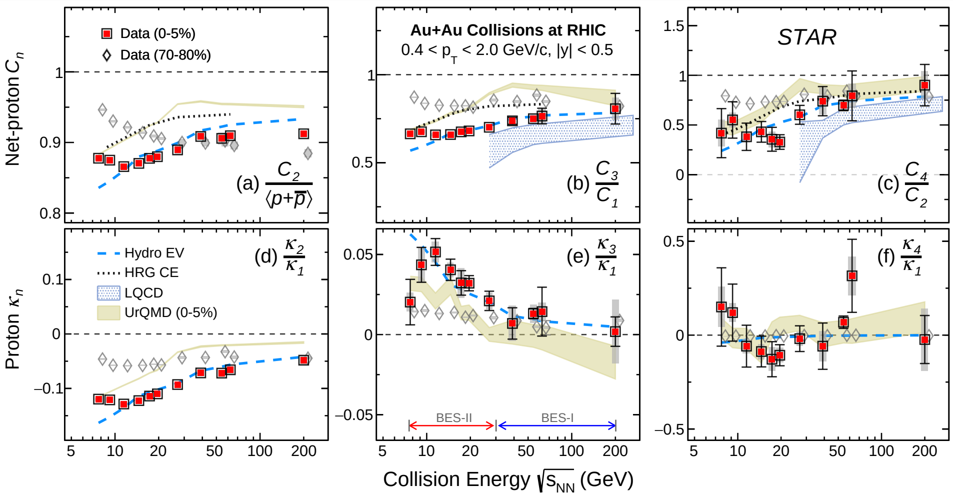


Expected non monotonic behaviour as a function of μ_B in net-baryon kurtosis.

Stephanov, PRL 107 (2011) 052301

Signals of critical behavior?

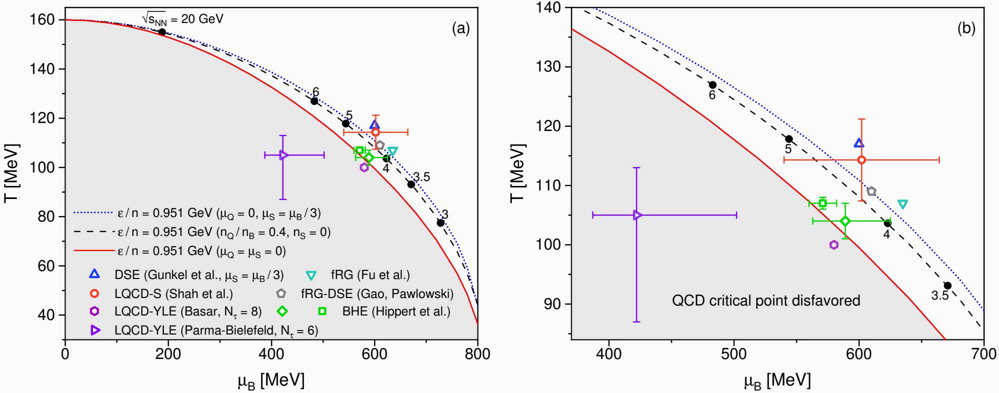
Recent data: not clear if and where here is non-monotonicity



STAR Collaboration, 2504.00817

The QCD critical point

From the theory side, many different models predicting a critical point



Vovchenko, PRC 111 (2025) 054903

→ and recent estimates seem to “converge”

Large density QCD from the lattice

I. Large lattices in the continuum limit

Exclusion region for the critical point from lattice simulations!

Borsànyi, PP *et al.*, 2502.10267

II. Monster statistics on a $16^3 \times 8$ lattice

Yang-Lee edge singularities

Adam, PP *et al.*, 2507.xxxxx

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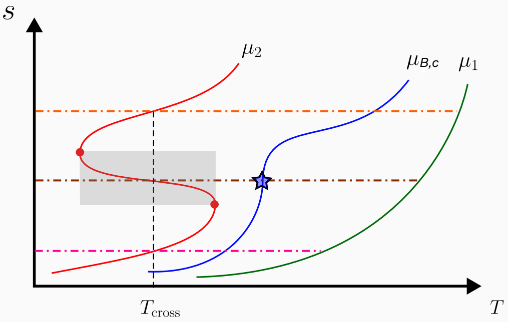
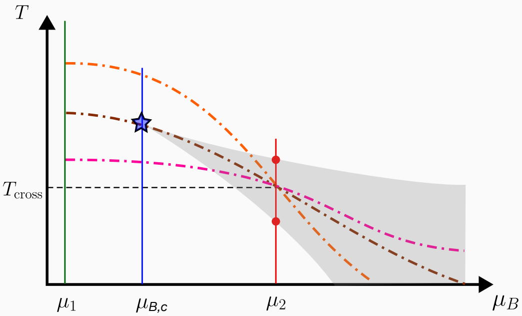
Critical point location from entropy contours

Recently proposed to look at contours of constant entropy, and search for where they meet

Shah *et al.*, 2410.16206



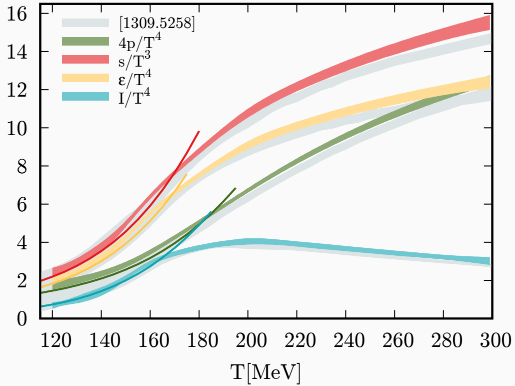


Through Taylor expansion, predicted a critical point at $T \sim 100$ MeV, $\mu_B \sim 600$ MeV

Critical point location from entropy contours

Full quantitative analysis with same method, but:

- analytical continuation from imaginary μ_B
- new equation of state at $\mu_B = 0$ with 2x better precision

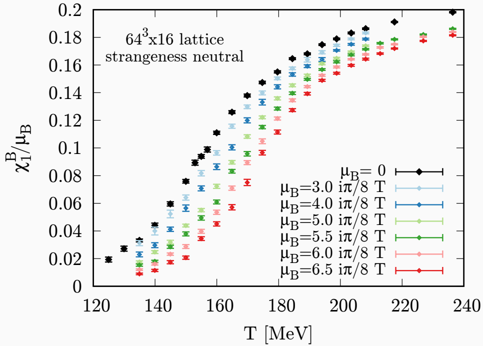


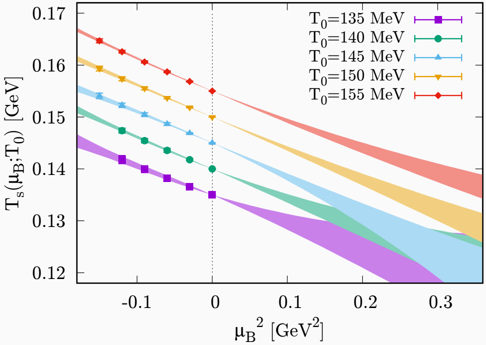
and **different goal**: try to exclude the CP somewhere in the phase diagram

Our entropy contours

We generalized the approach and employed new data. Starting from baryon density at imaginary μ_B, obtain the entropy entropy at imaginary μ_B (**total** 16x systematics here)

$$s(T,\mu_B) = s(T,\mu_B=0) + \int_0^{\mu_B} d\mu'_B \frac{\partial n_B(T,\mu'_B)}{\partial T}$$





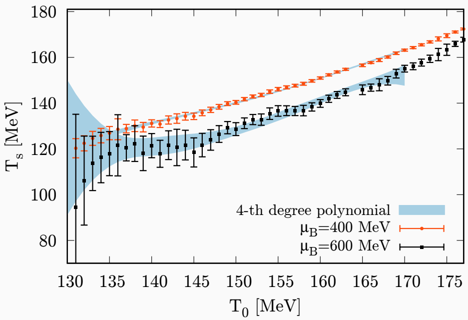
The points (right) are where the entropy has the same exact value (color-by-color)

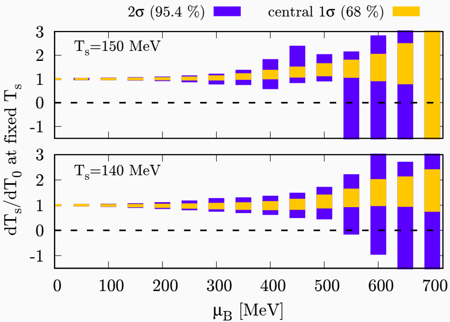
Our entropy contours

Now we extrapolate $T_s(\mu_B, T_0)$, i.e. the temperature at which the entropy has the value it has at $T_0, \mu_B = 0$. We use a rational function (checked linear and parabolic too):

$$T_s(\mu_B^2, T_0) = \frac{T_0 + a\mu_B^2}{1 + b\mu_B^2}$$

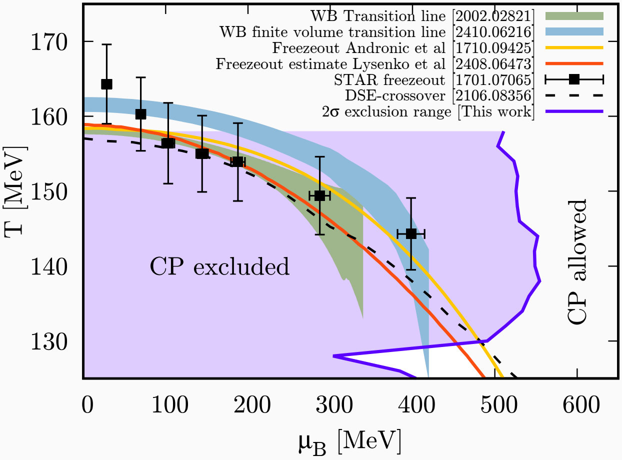
If there is a critical point, T_s is non-monotonic above $\mu_B > \mu_{BC}$, so we look for the smallest values of the derivative at 1σ and 2σ levels (**total** 24x systematics per μ_B value)





The exclusion region

This gives us indication of where the derivative is compatible with 0 at the 1σ and 2σ levels, temperature by temperature $\rightarrow$ 2σ **exclusion region**



Not too strict a constraint, but still the first rigorous exclusion from lattice QCD.

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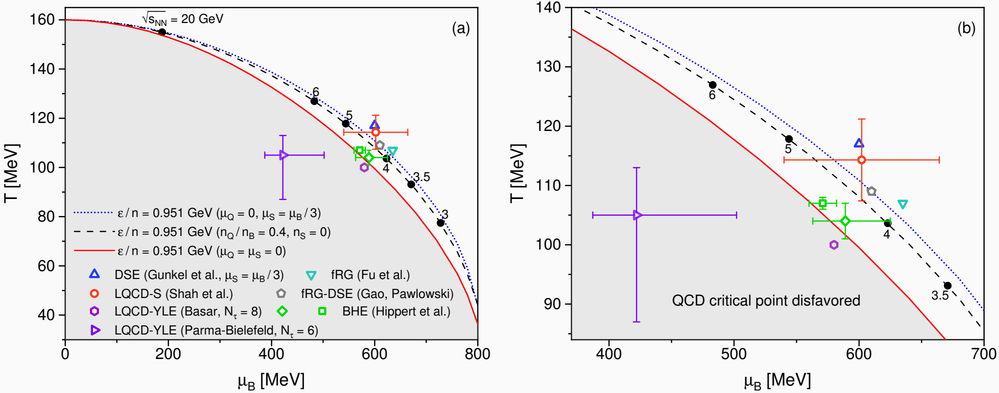

Vovchenko, PRC 111 (2025) 054903

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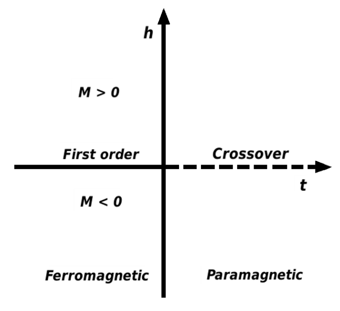
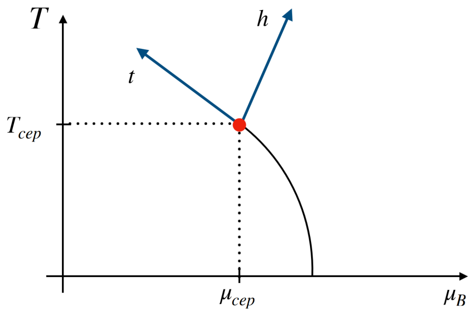
Critical point and universality

In the Ising model, scaling fields are the reduced temperature $t = \frac{T-T_c}{T_c}$ and the magnetic field h. They can be mapped onto QCD coordinates as:

$$t = A_t \Delta T + B_t \Delta \mu_B$$
$$h = A_h \Delta T + B_h \Delta \mu_B$$

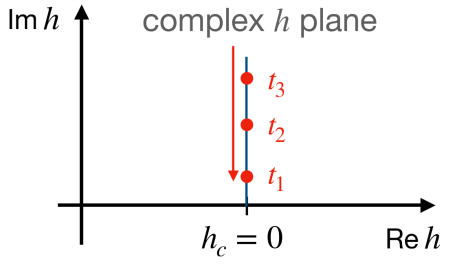
with $\Delta T = T - T_c$, $\Delta \mu_B = \mu_B - \mu_{BC}$.





Critical point: Lee-Yang edge singularities

- The partition function of a thermodynamic system has in general complex zeroes called Lee- Yang zeroes
- When the critical point is approached, these zeroes approach the real (μ_B) axis
- Zeroes of $\mathcal{Z}$ are singularities of the free energy $f \sim \log \mathcal{Z}$, and they accumulate at the so-called Yang-Lee edge (LYE) singularities



C. Schmidt, Lattice 24

- In the vicinity of a critical point, the **scaling variable** $z = t/h^{\beta\delta}$ is the only "coordinate", and the LYE are universally located at:

$$z_c = |z_c| \exp\left(\frac{i\pi}{2\beta\delta}\right)$$

- In QCD this translates to expected scaling forms for the real and imaginary parts:

$$\mathrm{Re}\Delta\mu_B = \mu_{BC} + c_1\Delta T \; (+c_2\Delta T^2) \qquad \text{and} \qquad \mathrm{Im}\Delta\mu_B = c_3\Delta T^{\beta\delta}$$

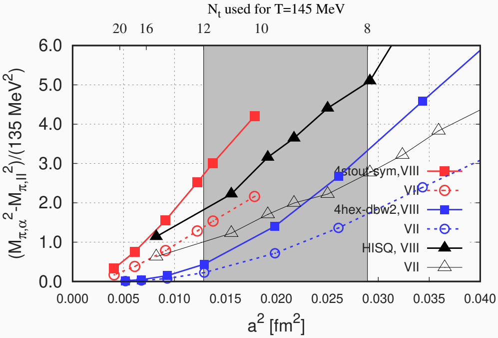
Critical point: Lee-Yang edge singularities

So, the idea is:

- **determine the complex locations** of the YLE
- **use the expected scaling** to find where $\mathrm{Im}\Delta\mu_B = 0$, i.e. where the critical point is

Our setup:

- Single $16^3 \times 8$ lattice, huge statistics ($\mathcal{O}(10^6)$ at each T (!))
- $N_f = 2+1$ flavours of physical quark masses
- Our recent 4HEX action has smaller discretization effects (i.e., $N_\tau = 8$ is not that small)
- Range $T = 110 - 300$ MeV

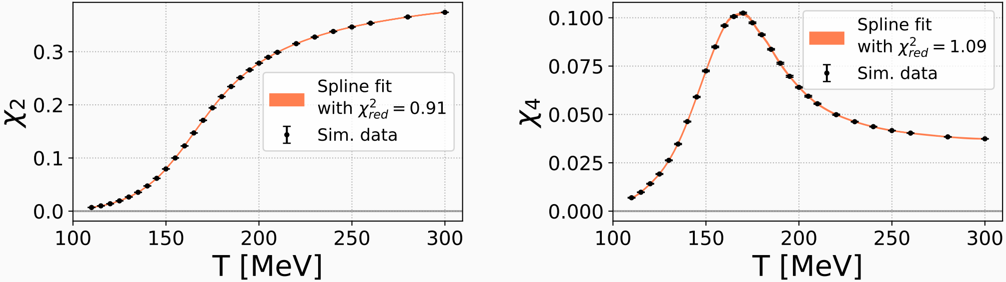


Finally, we will pay particular attention to several sources of systematics

Input: fluctuation observables

The fluctuations are the expansion coefficients of the pressure:

$$\chi_n^B(T) = \frac{\partial^n (p/T^4)}{\partial(\mu_B/T)^n}\bigg|_{\mu_B=0}$$



The huge statistics is reflected in tiny errors, crazy precision

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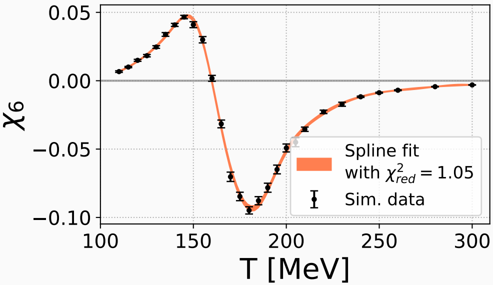

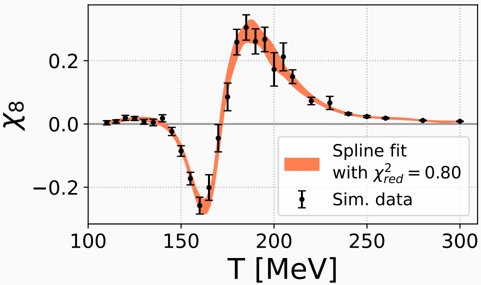

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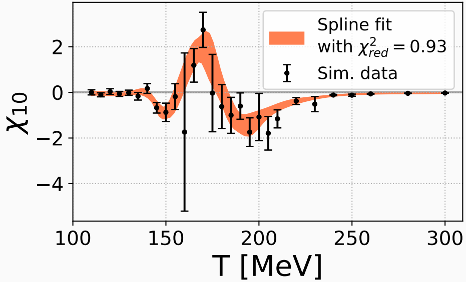


The huge statistics is reflected in tiny errors, crazy precision

Our strategy

Model the pressure with a functional form allowing for singularities, possibly retaining the known symmetries (charge conjugation $\mu_B \leftrightarrow -\mu_B$ and Roberge-Weiss periodicity

- For each T, we use a 2/2 Padé in $\cosh(\mu_B) - 1$,:

$$F(\mu_B) = \frac{a(\cosh(\mu_B) - 1)}{1 + c(\cosh(\mu_B) - 1) + d(\cosh(\mu_B) - 1)^2}$$

  Systematics # 1: repeat the procedure for related quanties (χ_1^B, χ_2^B), also singular!

- Universality fixes the approach to the critical point:

$$\mathrm{Im}(\mu_{LY})^{1/\beta\delta} = c_3(T - T_c)$$

  Systematics # 2: but other asymptotically equivalent ansätze are allowed:

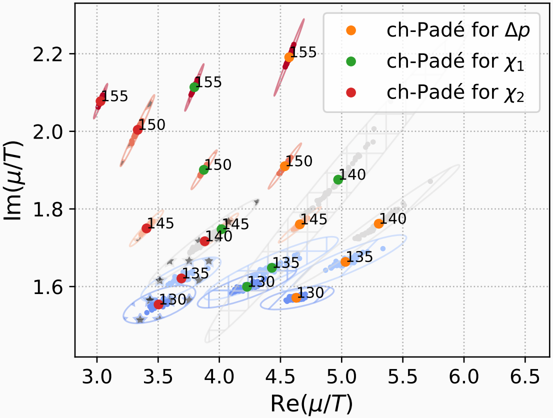
$$\mathrm{Im}(\mu_{LY}/T)^{1/\beta\delta} \sim \mathrm{Im}(\mu_{LY}^2)^{1/\beta\delta} \sim \mathrm{Im}((\mu_{LY}/T)^2)^{1/\beta\delta}$$

- The range where the ansätze hold is not known a priori (non universal)

  Systematics # 3: vary the fit range in T

Systematics #1, different observables

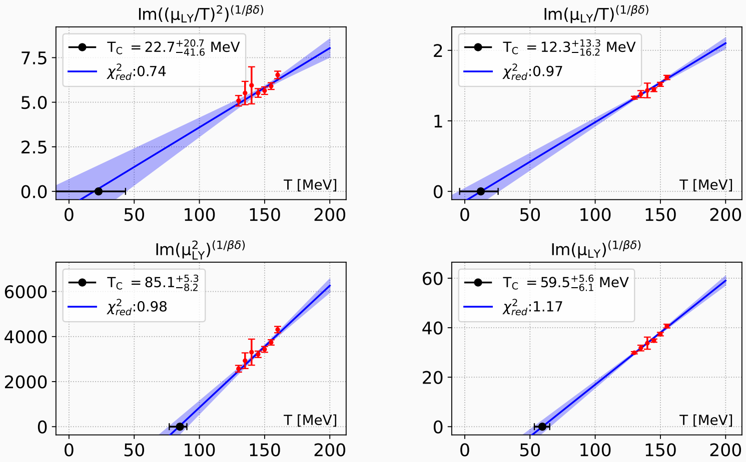
Estimates for the LYE singularities from p (χ_0), χ_1 and χ_2



NOTE: here the spreads reflect the statistical errors only

Systematics #2, different ansätze for T dependence

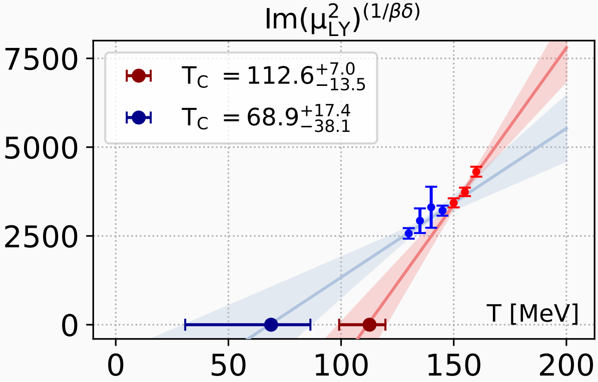
Different ansätze all give good fits



but rather different results!

Systematics #3, fit range

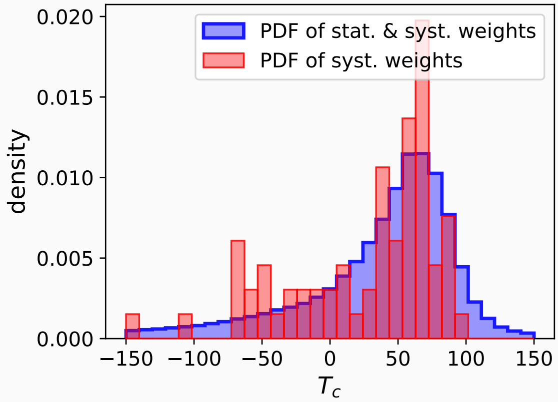
Since we can't know *a priori* where the scaling ansatz is valid, fits with different T ranges must be treated equally



The dependence on the fit range is very large!

Systematics: wrap up

Putting together the $3 \times 4 \times 10 = 120$ analyses:



When considering the systematic errors, it is extremely hard to make any predictions on the location of the CP.

Summary

Is there a critical point in the QCD phase diagram? Can lattice say something?

YES! (it can say something)

i. **Critical point exclusion range**: first time, though admittedly not very stringent yet, but method is systematically improvable

ii. **Yang-Lee edge singularities** offer an intriguing chance, but the numerics tell us the predictive power is (still) small

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Happy birthday, Wanda!

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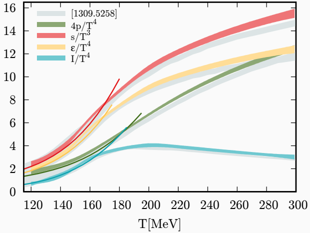
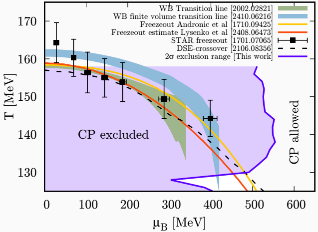
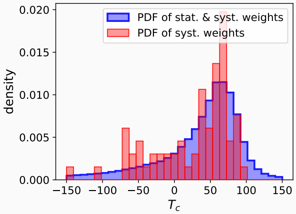


[1309.5258]
4p/T^4
s/T^3
ε/T^4
I/T^4
T[MeV]

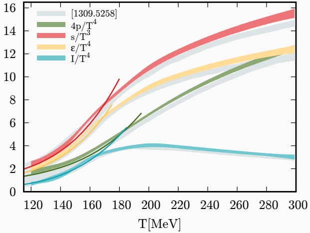


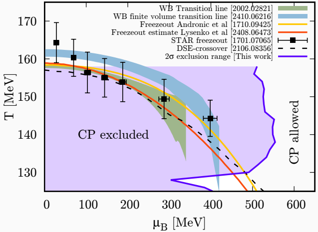
WB Transition line [2002.02821]
WB finite volume transition line [2410.06216]
Freezeout Andronic et al [1710.09425]
Freezeout estimate Lysenko et al [2408.06473]
STAR freezeout [1701.07065]
DSE-crossover [2106.08356]
2σ exclusion range [This work]
CP excluded
CP allowed
T [MeV]
μB [MeV]

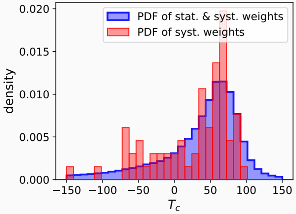


PDF of stat. & syst. weights
PDF of syst. weights
density
Tc

Summary







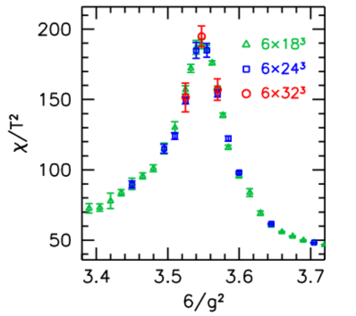
THANK YOU!

BACKUP

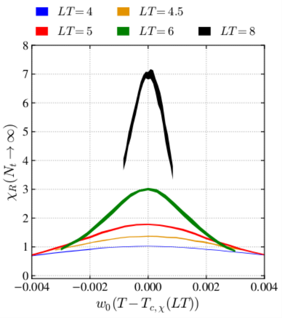
The QCD transition: crossover vs. first order

On the lattice we study the volume scaling of certain quantities to determine the order of the transition

Left: physical masses



Right: infinite masses (pure gauge)

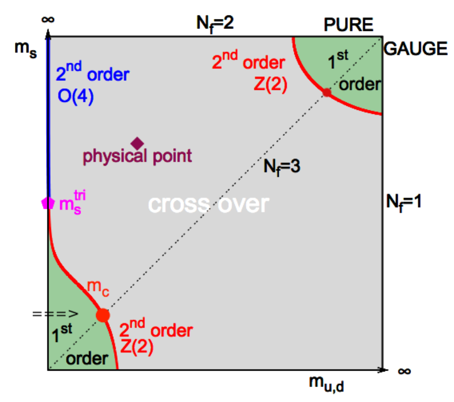


- For a crossover (left), **the peak height is independent of the volume**
- For a first order transition, **it scales linearly with the volume**

Aoki *et al.* Nature 443 (2006), Borsányi, PP *et al.*, PRD 105 (2022)

The QCD transition: Columbia plot

As a function of the light (u,d) and strange quark masses, the order of the transition changes

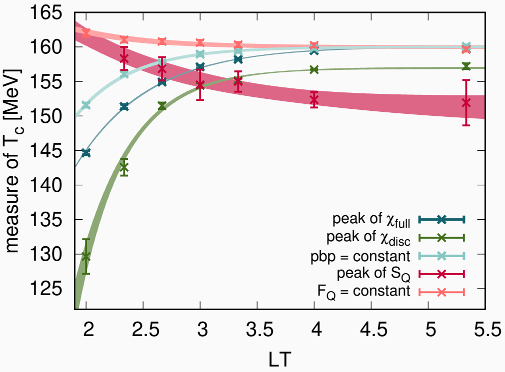


- At the physical point $m_s/m_{ud} \simeq 27$, the transition is a smooth crossover!
- In the heavy-quark limit (pure gauge), the transition is first order

Measures of T_c vs V

Combining the estimates of T_c from different observables and volumes we can draw some conclusions:

- Chiral transition T_c estimates have larger V-dependence and decrease with the volume
- Deconfinement T_c estimates have milder V-dependence and increase with the volume
- The spread is ~ 10 MeV for $LT > 2.5$
- Clear ordering $T_c^{\chi} < T_c^{S_Q}$ appears above $LT = 3$

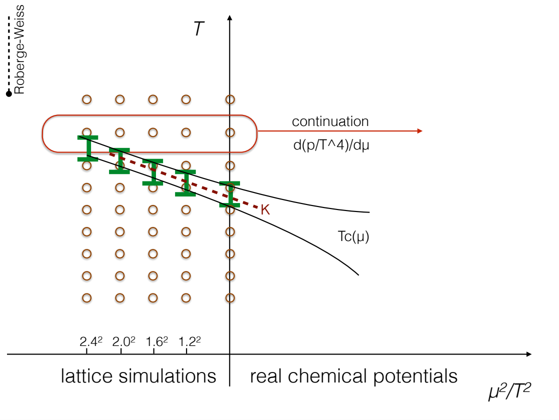


This suggests that studies of T_c can be performed on lattice with smaller volumes based on deconfinement-related observables

Simulations at imaginary chemical potential

- While for real chemical potential ($\mu^2 > 0$) $\det M(U)$ is complex, for **imaginary** chemical potential ($\mu^2 < 0$) $\det M(U)$ is real
- We perform simulations at imaginary chemical potentials:

$$\hat{\mu}_B = i\frac{j\pi}{8} \quad j = 0, 1, 2, \ldots$$



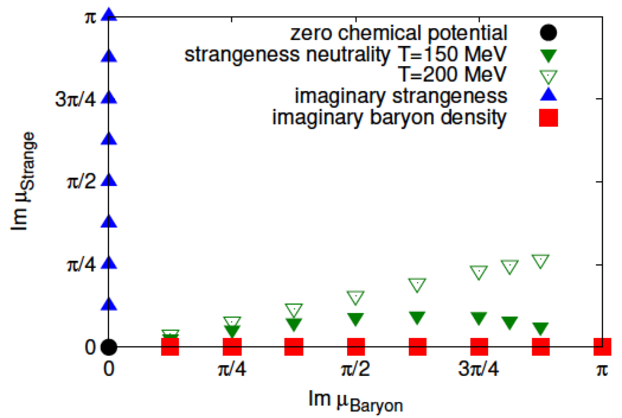
We then analytically continue to $\mu^2 > 0$ by means of suitable extrapolation schemes

Simulations at imaginary chemical potential

Strangeness neutrality (or not)

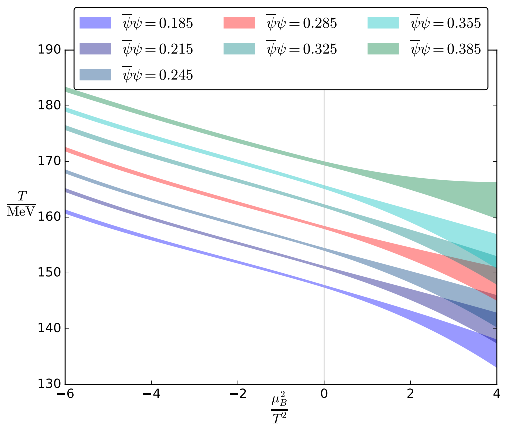
Set the chemical potentials for heavy-ion collisions scenario, or simpler setup:

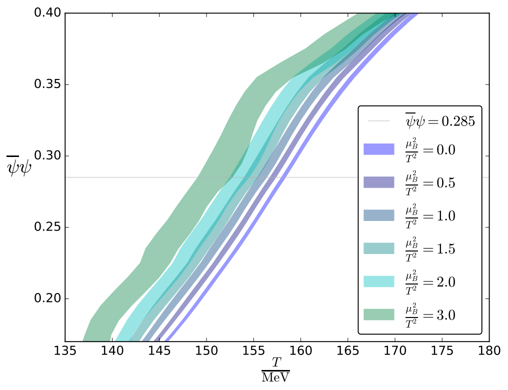
$$\langle n_S \rangle = 0 \qquad \langle n_Q \rangle = 0.4 \langle n_B \rangle \qquad \text{or} \qquad \mu_Q = \mu_S = 0$$



The width of the transition at finite chemical potential

We can extrapolate our results for $\langle\bar{\psi}\psi\rangle$ along contours of constant $\langle\bar{\psi}\psi\rangle$ (left) or constant μ_B/T (right)





The extrapolated $\langle\bar{\psi}\psi\rangle$ at finite μ_B is quite precise for $\mu_B < 3T$