Chiral anomaly: from vacuum to Columbia plot

Francesco Giacosa
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Based (mostly) on:
Michał Zakrzewski, Shahriyar Jafarzade, Robert Pisarski
2502.15617
Győző Kovács, Péter Kovács, Fabian Rennecke, Robert Pisarski
2410.08185 , PRD111 (2025) 1, 1

Celebrating Wanda's birthday:
a career devoted to the richness of nuclear many-body physics
3-4/7/2025, Turin/online

Motivation/outline

Chiral (or axial) anomaly: a classical symmetry of QCD broken by quantum fluctuations

Chiral anomaly important for η and η'.

What about other mesons?

What about its role in the medium?

Summary

QCD Lagrangian: symmetries and anomalies

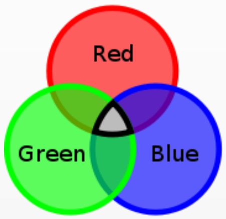
Born	Giuseppe Lodovico Lagrangia 25 January 1736 Turin
Died	10 April 1813 (aged 77) Paris

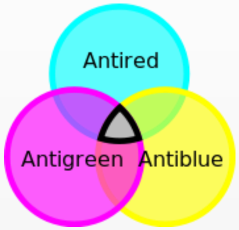
Francesco Giacosa

The QCD Lagrangian

based on local SU(3) color symmetry

Quark: u,d,s and c,b,t R,G,B



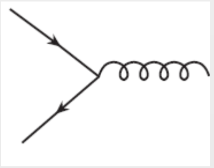


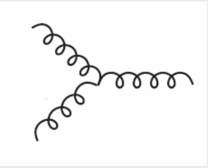
$$q_i = \begin{pmatrix} q_i^R \\ q_i^G \\ q_i^B \end{pmatrix} ; \; i = u,d,s,...$$

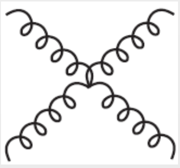
$$\mathcal{L}_{QCD} = \sum_{i=1}^{N_f} \overline{q}_i (i\gamma^\mu D_\mu - m_i) q_i - \frac{1}{4} G_{\mu\nu}^a G^{a,\mu\nu}$$

8 type of gluons ($R\bar{G}$, $B\bar{G}$,…)

$$A_\mu^a \; ; \; a = 1,...,8$$







Btw: **where are glueballs?**

Three pillars of QCD (besides color)

- Trace or dilatation anomaly

- From flavor to chiral symmetry and its spont. breaking

- Chiral or axial anomaly

Trace anomaly: the emergence of a dimension

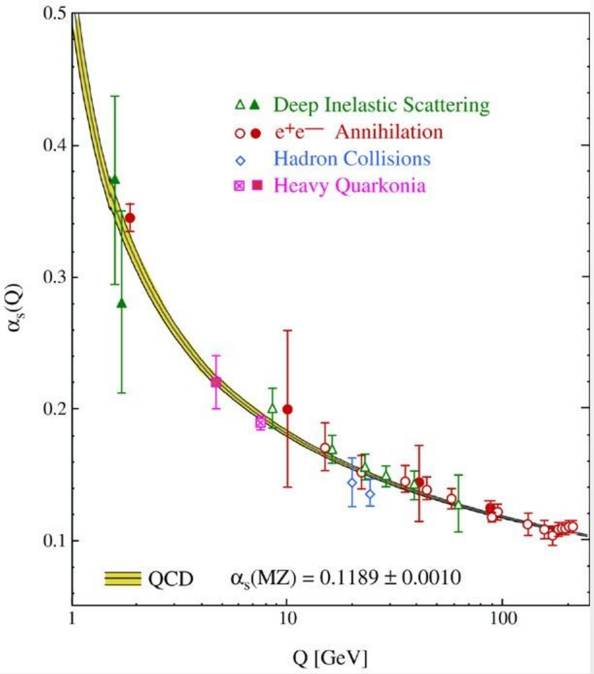
Chiral limit: $m_{.} = 0$

$x^\mu \to x'^\mu = \lambda^{-1} x^\mu$ is a classical symmetry broken by quantum fluctuations (trace anomaly)

$\Lambda_{YM} \approx 250 \text{ MeV}$

Dimensional transmutation

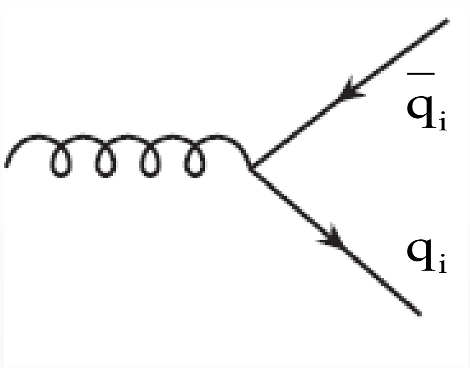
$$\alpha_S(\mu = Q) = \frac{g^2(Q)}{4\pi}$$



Gluon condensate: $\left\langle G^a_{\mu\nu} G^{a,\mu\nu} \right\rangle \neq 0$

Effective gluon mass:
$m_{gluon} = 0 \to m^*_{gluon} \approx 500 - 800\,\text{MeV}$

Flavor symmetry



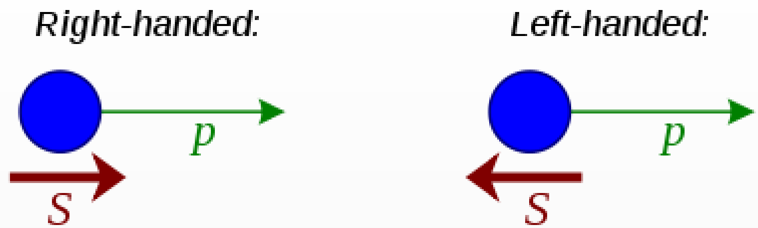
Gluon-quark-antiquark vertex.

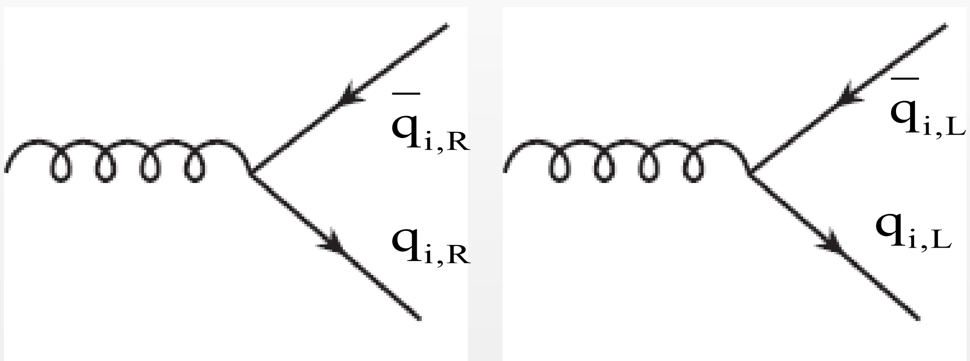
It is democratic! The gluon couples to each flavor with the same strength

$$q_i \rightarrow U_{ij} q_j$$

$$U \in U(3)_V \rightarrow U^+ U = 1$$

Chiral symmetry





$$q_i = q_{i,R} + q_{i,L}$$

$$q_{i,R} = \frac{1}{2}(1+\gamma^5)q_i$$

$$q_{i,L} = \frac{1}{2}(1-\gamma^5)q_i$$

$$q_i = q_{i,R} + q_{i,L} \rightarrow U^R_{ij} q_{j,R} + U^L_{ij} q_{j,L}$$

$$U(3)_R \times U(3)_L = U(1)_{R+L} \times U(1)_{R-L} \times SU(3)_R \times SU(3)_L$$

baryon number **anomaly U(1)A** **SSB into SU(3)V**

In the chiral limit (m_i=0) chiral symmetry is exact

Spontaneous breaking of chiral symmetry

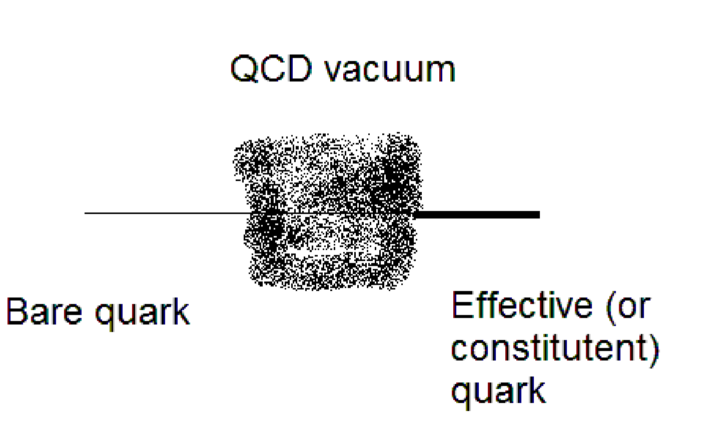
$$U(3)_R \times U(3)_L = U(1)_{R+L} \times U(1)_{R-L} \times SU(3)_R \times SU(3)_L$$

$$\mathrm{SSB:}\ \mathrm{SU(3)_R \times SU(3)_L \to SU(3)_{V=R+L}}$$

Chiral symmetry → Flavor symmetry

$$\langle \bar{q}_i q_i \rangle = \langle \bar{q}_{i,R} q_{i,L} + \bar{q}_{i,L} q_{i,R} \rangle \neq 0$$

$$\mathrm{m \approx m_u \approx m_d \approx 5\ MeV \to m^* \approx 300\ MeV}$$



$$\mathrm{m_{\rho-meson} \approx 2m^*}$$

$$\mathrm{m_{proton} \approx 3m^*}$$

Chiral transformations and axial anomaly

$$SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}} \times U(1)_{\mathrm{A}}$$

$$q_{\mathrm{L,R}} \longrightarrow \mathrm{e}^{\mp \mathrm{i}\alpha/2}\, U_{\mathrm{L,R}}\, q_{\mathrm{L,R}}$$

$U(1)_A$ **$SU(3)_R$x$SU(3)_L$** Chiral

Chiral or axial anomaly: the axial divergence does **not** vanish!

$$\partial^{\mu}(\bar{q}^{i}\gamma_{\mu}\gamma_{5}q^{i}) = \frac{3g^{2}}{16\pi^{2}}\,\varepsilon^{\mu\nu\rho\sigma}\mathrm{tr}(G_{\mu\nu}G_{\rho\sigma})$$

Symmetries of QCD and breakings

$SU(3)_{color}$: exact. Confinement: you never see color, but only white states.

Dilatation invariance: holds only at a classical level and in the chiral limit. Broken by quantum fluctuations (**scale anomaly**) and by quark masses.

$SU(3)_R \times SU(3)_L$: holds in the chiral limit but is broken by nonzero quark masses. Moreover, it is **spontaneously** broken to $U(3)_{V=R+L}$

$U(1)_{A=R-L}$: holds at a classical level, but is also broken by quantum fluctuations (**chiral anomaly**)

Hadrons

The QCD Lagrangian contains ‘colored’ quarks and gluons. However, no ‚colored‘ state has been seen.

Confinement: physical states are white and are called hadrons.

Hadrons can be:

Mesons: bosonic hadrons

Baryons: fermionic hadrons

A meson is **not necessarily** a quark-antiquark state.
A quark-antiquark state is a conventional meson.

Extended Linear Sigma Model: eLSM

eLSM: Chiral model with conv. mesons + glueballs, hybrids…

(since 2008 up to now, for spectroscopy and medium properties)

Recent review paper in:

Ordinary and exotic mesons in the extended Linear Sigma Model

F.G., S. Jafarzade, P. Kovacs

Prog. Part. Nucl.Phys. 143 (2025) 104176

arXiv: 2407.18348 [hep-ph].

eLSM Lagrangian: 2407.18348
(pseudo)scalar, (axial-)vector, ….

$$\mathcal{L} = \mathcal{L}_{\rm dil} + \mathcal{L}_{\Phi} + \mathcal{L}_{U(1)_A} + \mathcal{L}_{LR} + \mathcal{L}_{\Phi LR} \ ,$$

with

$$\mathcal{L}_{\rm dil} = \frac{1}{2}(\partial_\mu G)^2 - \frac{1}{4}\frac{m_G^2}{\Lambda_G^2}\left(G^4 \ln\frac{G^2}{\Lambda_G^2} - \frac{G^4}{4}\right) \ ,$$

$$\mathcal{L}_{\Phi} = \mathrm{Tr}[(D_\mu\Phi)^\dagger(D_\mu\Phi)] - m_0^2\left(\frac{G}{G_0}\right)^2 \mathrm{Tr}(\Phi^\dagger\Phi) - \lambda_1[\mathrm{Tr}(\Phi^\dagger\Phi)]^2 - \lambda_2\,\mathrm{Tr}(\Phi^\dagger\Phi)^2 + \mathrm{Tr}[H(\Phi+\Phi^\dagger)] \ ,$$

$$\mathcal{L}_{U(1)_A} = c_2(\det\Phi - \det\Phi^\dagger)^2 \ ,$$

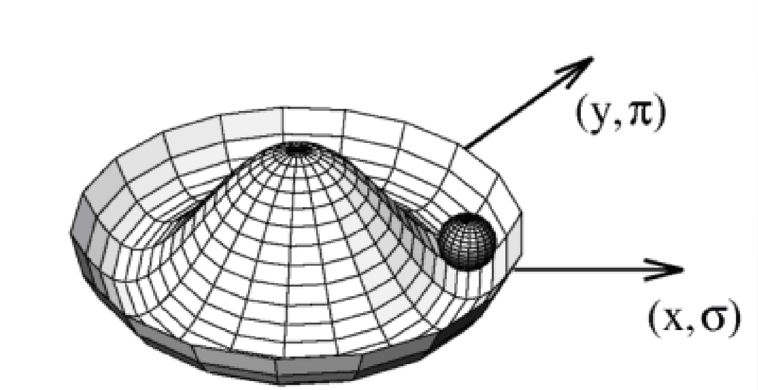
$$\mathcal{L}_{LR} = -\frac{1}{4}\mathrm{Tr}(L_{\mu\nu}^2 + R_{\mu\nu}^2) + \mathrm{Tr}\left[\left(\left(\frac{G}{G_0}\right)^2 + \Delta\right)\frac{m_1^2}{2}(L_\mu^2 + R_\mu^2)\right] + i\frac{g_2}{2}(\mathrm{Tr}\{L_{\mu\nu}[L^\mu, L^\nu]\} + \mathrm{Tr}\{R_{\mu\nu}[R^\mu, R^\nu]\}$$

$$+ g_3[\mathrm{Tr}(L_\mu L_\nu L^\mu L^\nu) + \mathrm{Tr}(R_\mu R_\nu R^\mu R^\nu)] + g_4[\mathrm{Tr}\,(L_\mu L^\mu L_\nu L^\nu) + \mathrm{Tr}\,(R_\mu R^\mu R_\nu R^\nu)]$$

$$+ g_5\,\mathrm{Tr}\,(L_\mu L^\mu)\,\mathrm{Tr}\,(R_\nu R^\nu) + g_6[\mathrm{Tr}(L_\mu L^\mu)\ \mathrm{Tr}(L_\nu L^\nu) + \mathrm{Tr}(R_\mu R^\mu)\ \mathrm{Tr}(R_\nu R^\nu)] \ ,$$

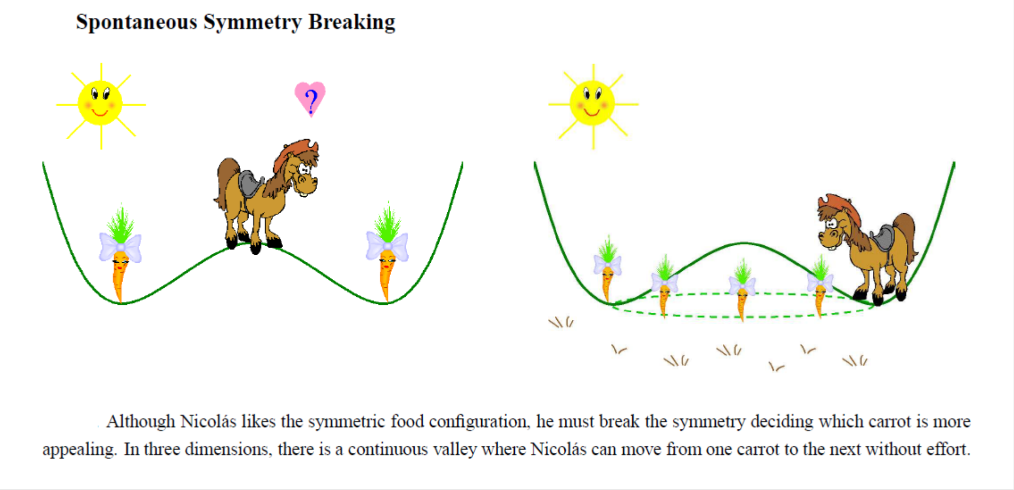
$$\mathcal{L}_{\Phi LR} = \frac{h_1}{2}\mathrm{Tr}(\Phi^\dagger\Phi)\,\mathrm{Tr}(L_\mu^2 + R_\mu^2) + h_2\,\mathrm{Tr}[|L_\mu\Phi|^2 + |\Phi R_\mu|^2] + 2h_3\,\mathrm{Tr}(L_\mu\Phi R^\mu\Phi^\dagger) \ ,$$

SSB and the donkey of Buridan: hadronic approaches



$$\sigma_N \to \sigma_N + \phi$$

Jean Buridan (in Latin, *Johannes Buridanus*) (ca. 1300 – after 1358)



(Pseudo)scalar mesons: heterochiral scalars

Pseudoscalar mesons: {π, K, η(547), η'(958)}
Scalar mesons: {a0(1450), K0*(1430),f0(1370),f0(1500)}

$$\Phi = S + iP = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\sigma_N + a_0^0}{\sqrt{2}} & a_0^+ & K_0^{*+} \\ a_0^- & \frac{\sigma_N - a_0^0}{\sqrt{2}} & K_0^{*0} \\ K_0^{*-} & \bar{K}_0^{*0} & \sigma_S \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\eta_N + \pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta_N - \pi^0}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & \eta_S \end{pmatrix}$$

f0(1710) mostly glueball
See 1408.4921

$$q_{\mathrm{L,R}} \longrightarrow e^{\mp i\alpha/2} U_{\mathrm{L,R}}\, q_{\mathrm{L,R}}$$

⇒

$$\Phi \longrightarrow e^{-2i\alpha} U_{\mathrm{L}} \Phi U_{\mathrm{R}}^{\dagger}$$

$U(1)_A$ **$SU(3)_R \times SU(3)_L$** Chiral

Chirally invariant terms

We call the transformation of the matrix Φ **heterochiral**!
We thus have heterochiral scalars or heteroscalars.

$$\mathrm{tr}(\Phi^\dagger\Phi),\ \mathrm{tr}(\Phi^\dagger\Phi)^2$$

clearly invariant; typical terms for a chiral model.

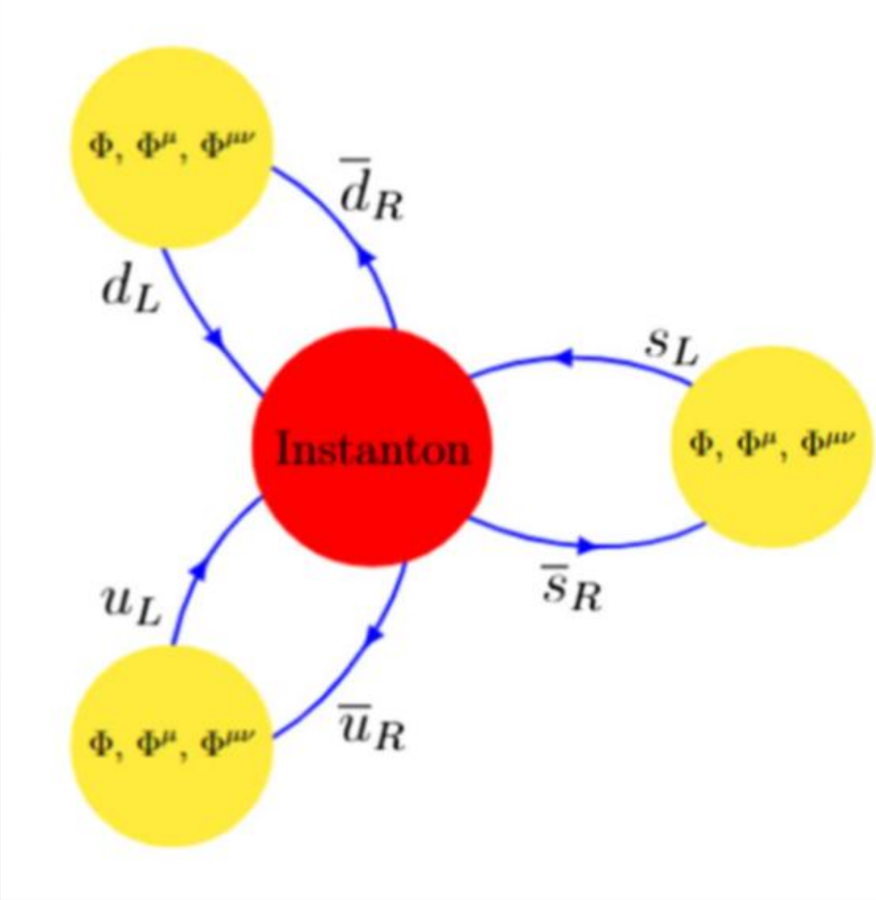
$$\det(\Phi)$$

interesting, since it breaks only $U(1)_A$ axial anomaly

$$\det(\Phi) = \frac{1}{6}\varepsilon^{ijk}\varepsilon^{i'j'k'}\Phi^{ii'}\Phi^{jj'}\Phi^{kk'} \to e^{-3i\alpha}\det(\Phi)$$

Easiest chiral anomalous term

$$\mathcal{L}_{\text{qu}} = -\xi_1 \left(\det \Phi + \det \Phi^\dagger \right)$$



The constant csi1 can be calculated as an overage over instanton density.

See G. ,t Hooft,
How Instantons Solve the U(1) Problem
Phys.Rept. 142 (1986) 357-387

For extension:
Phys.Rev.D 109 (2024) 7, L071502
2309.00086

2309.00086

Axial anomaly: (at least) 2 terms

$$\mathcal{L}_{\text{qu}} = -\xi_1 (\det\Phi + \det\Phi^\dagger) - \xi_2[(\det\Phi)^2 + (\det\Phi^\dagger)^2]$$

Vacuum phenomenology is possible with both.

Probably, both terms are nonzero in Nature.

They arise from Q=1 and Q=2 instantons, respectively.

Important (and peculiar) effects at nonzero temperature.

The chiral anomaly in mesons

There are 8 but not 9 Goldstone bosons:
3 pions, 4 kaons, and one η(547) meson.

The η'(958) meson has a mass of almost 1 GeV.

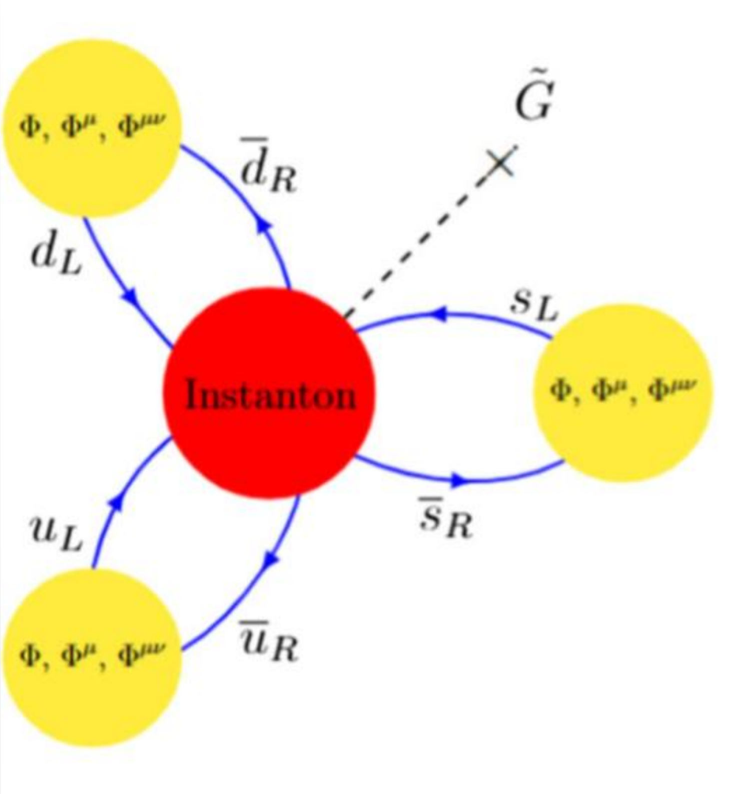
$$m_{\eta'}^2 \sim 1/N_c$$

E. Witten, Current Algebra Theorems for the U(1) Goldstone Boson, Nucl. Phys. B 156 (1979), 269-283

G. 't Hooft, Computation of the quantum effects due to a four-dimensional pseudoparticle, Phys. Rev. D 14, 3432 (1976).

Going further: pseudoscalar glueball

$$\mathcal{L}_{c_g} = -ic_g\tilde{G}_0(\det\Phi - \det\Phi^\dagger).$$



2309.00086 and 1208.6474

PHYSICAL REVIEW LETTERS **132,** 181901 (2024)

Editors' Suggestion

Determination of Spin-Parity Quantum Numbers of $X(2370)$ as 0^{-+} from $J/\psi \to \gamma K_S^0 K_S^0 \eta'$

M. Ablikim *et al.**
(BESIII Collaboration)

Is this meson the pseudoscalar glueball?

Strong coupling to η' is promising!

Decay with of about 200 MeV also ok.

How to include the chiral anomaly for mesons besides ground0state (pseudo)scalars?
A novel mathematical object? Extension of det

Extend the determinan

$$\det[\Phi] = \frac{1}{N!}\varepsilon^{i_1 i_2 \ldots i_N}\varepsilon^{j_1 j_2 \ldots j_N}\Phi^{i_1 j_1}\Phi^{i_2 j_2}\ldots\Phi^{i_N j_N}$$

to the following new object:

$$\varepsilon[\Phi_1, \Phi_2, \ldots, \Phi_N] = \frac{1}{N!}\varepsilon^{i_1 i_2 \ldots i_N}\varepsilon^{j_1 j_2 \ldots j_N}\Phi_1^{i_1 j_1}\Phi_2^{i_2 j_2}\ldots\Phi_N^{i_N j_N}$$

$$\varepsilon[\Phi_1, \Phi_2, \ldots, \Phi_i, \ldots, \Phi_j, \ldots\Phi_N] = \varepsilon[\Phi_1, \Phi_2, \ldots, \Phi_j, \ldots, \Phi_i, \ldots\Phi_N]$$

$$\varepsilon[\Phi_1, \Phi_1, \ldots\Phi_1] = \det\Phi_1$$

Polydeterminant

Emergence of the polydeterminant in QCD

Francesco Giacosa [1,2], Michał Zakrzewski [3], Shahriyar Jafarzade [4,5,6] and Robert D. Pisarski [7]

[1] *Institute of Physics, Jan Kochanowski University, ulica Uniwersytecka 7, P-25-406 Kielce, Poland*

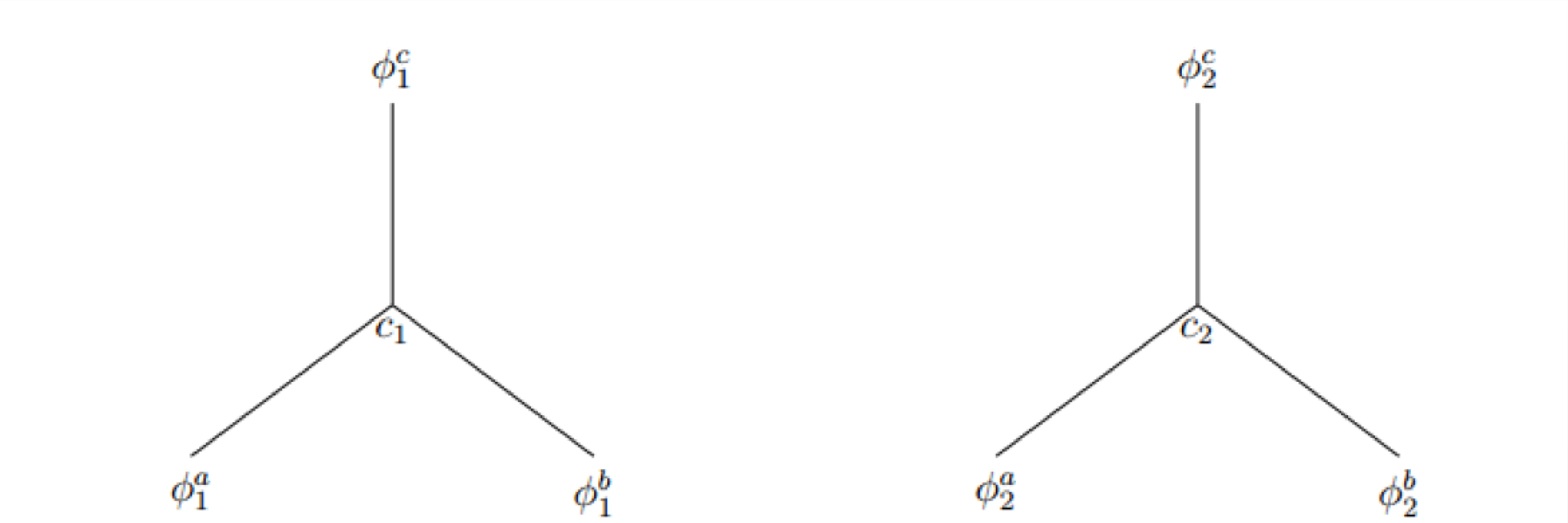
Definition in GPJ in arxiv: 2309.00086, see also review 2407.18348 .
(Implicit def by GKP 1709.07454)

Detailed description in 2504.02113

In mathematics: known as 'mixed discriminant'
Alexandrov, 1939, Panov 1987, Bapak 1989,…

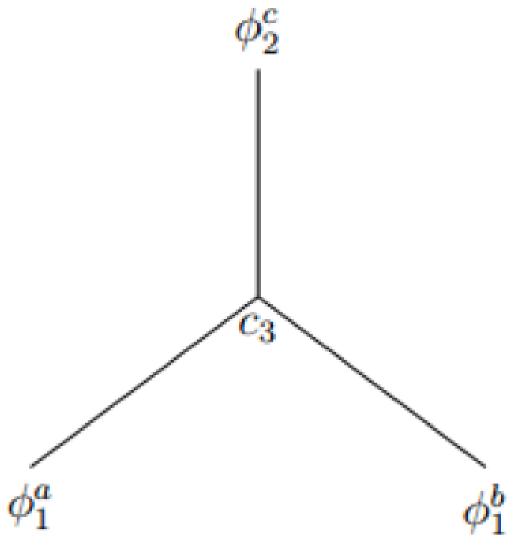
Consider two (pseudo)scalar multiplet Φ_1 and Φ_2.

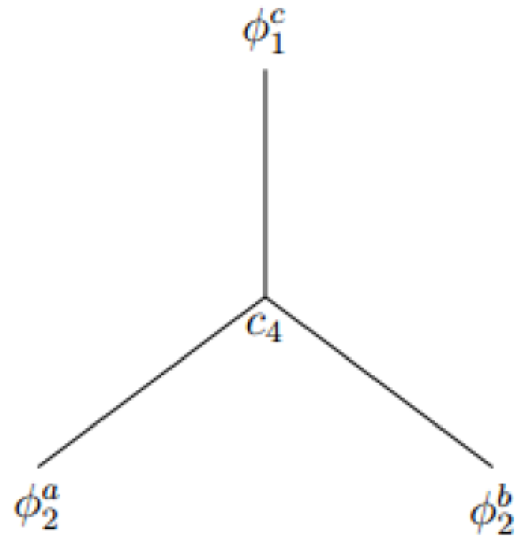
the standard anomalous terms are:



$\det\Phi_1$ $\det\Phi_2$

New interactions via polydeterminant:

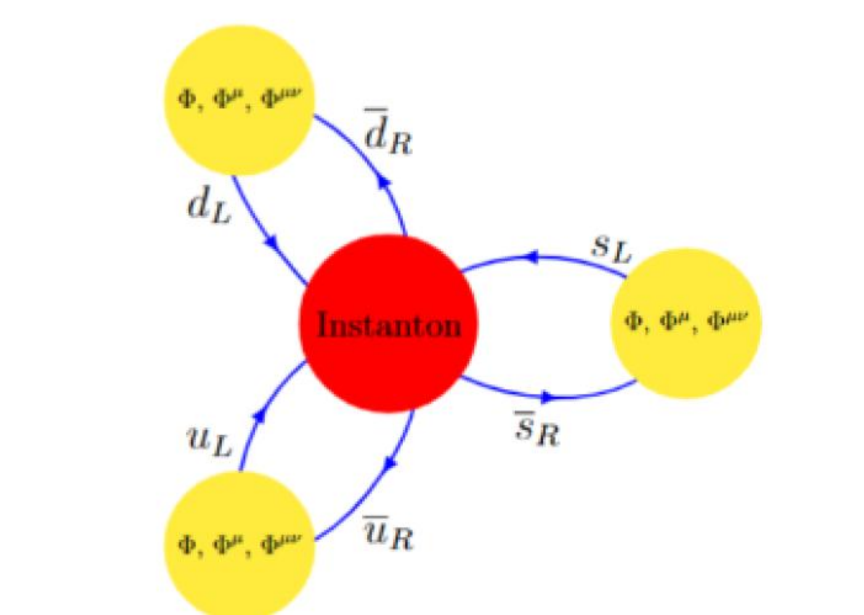




$$\varepsilon(\Phi_1, \Phi_1, \Phi_2) \qquad \varepsilon(\Phi_1, \Phi_2, \Phi_2)$$

Example with mesons with spin: pseudovector heterochiral multiplet

$$\mathcal{L}_{\text{eff}}^{J=1} = a_1\Big(\epsilon[\Phi\,,\Phi_\mu\,,\Phi^\mu] + \text{c.c.}\Big)$$



2309.00086

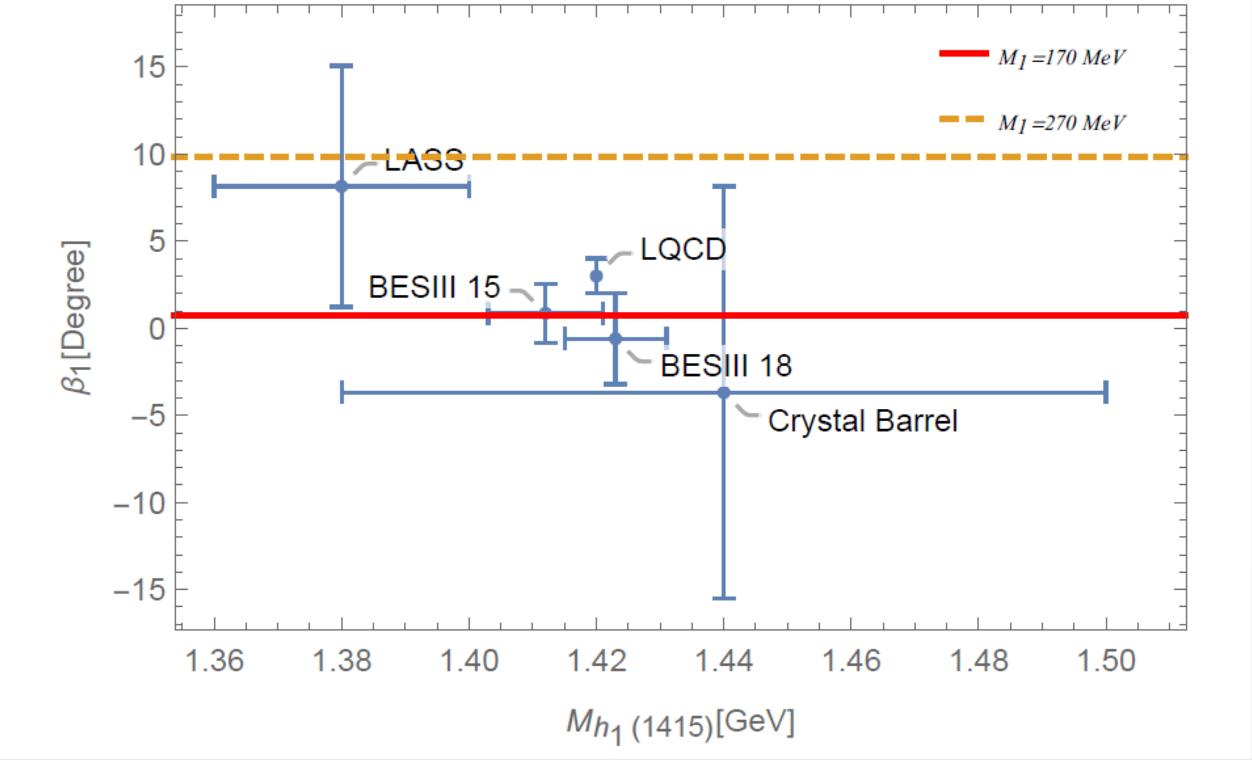
Recall:

$$\mathcal{L}_{\text{qu}} = -\xi_1\left(\det\Phi + \det\Phi^\dagger\right)$$

Isoscalar mixing angle for h1 mesons

$$\begin{pmatrix} h_1(1170) \\ h_1(1380) \end{pmatrix} = \begin{pmatrix} \cos\beta_{AV} & \sin\beta_{AV} \\ -\sin\beta_{AV} & \cos\beta_{AV} \end{pmatrix} \begin{pmatrix} h_{1,N} = \sqrt{1/2}(\bar{u}u + \bar{d}d) \\ h_{1,S} = \bar{s}s \end{pmatrix}$$

Angle between 0-10 degrees: between red and yellow lines

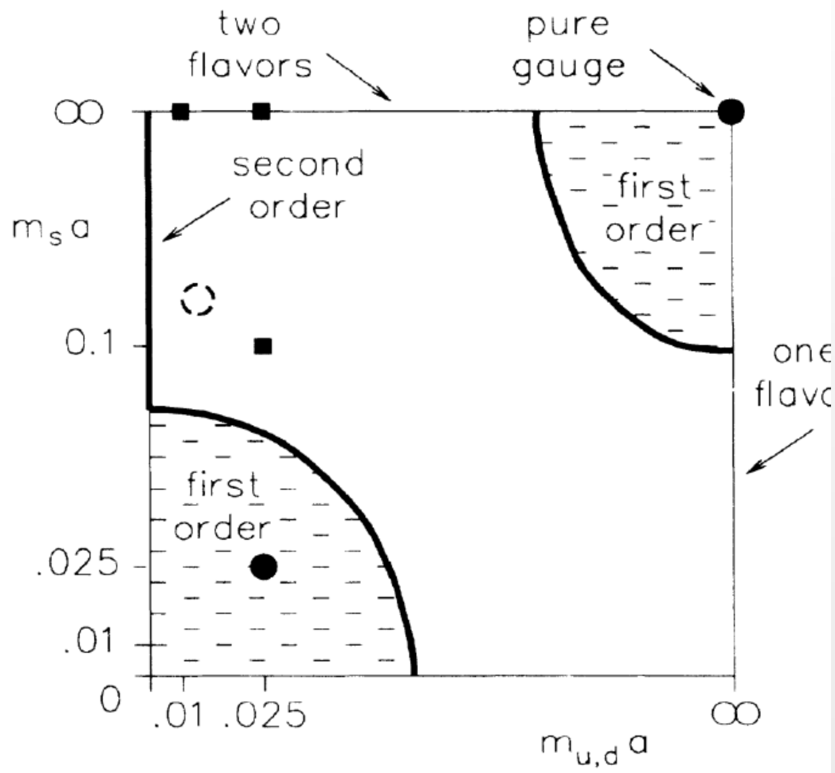


Columbia plot/1

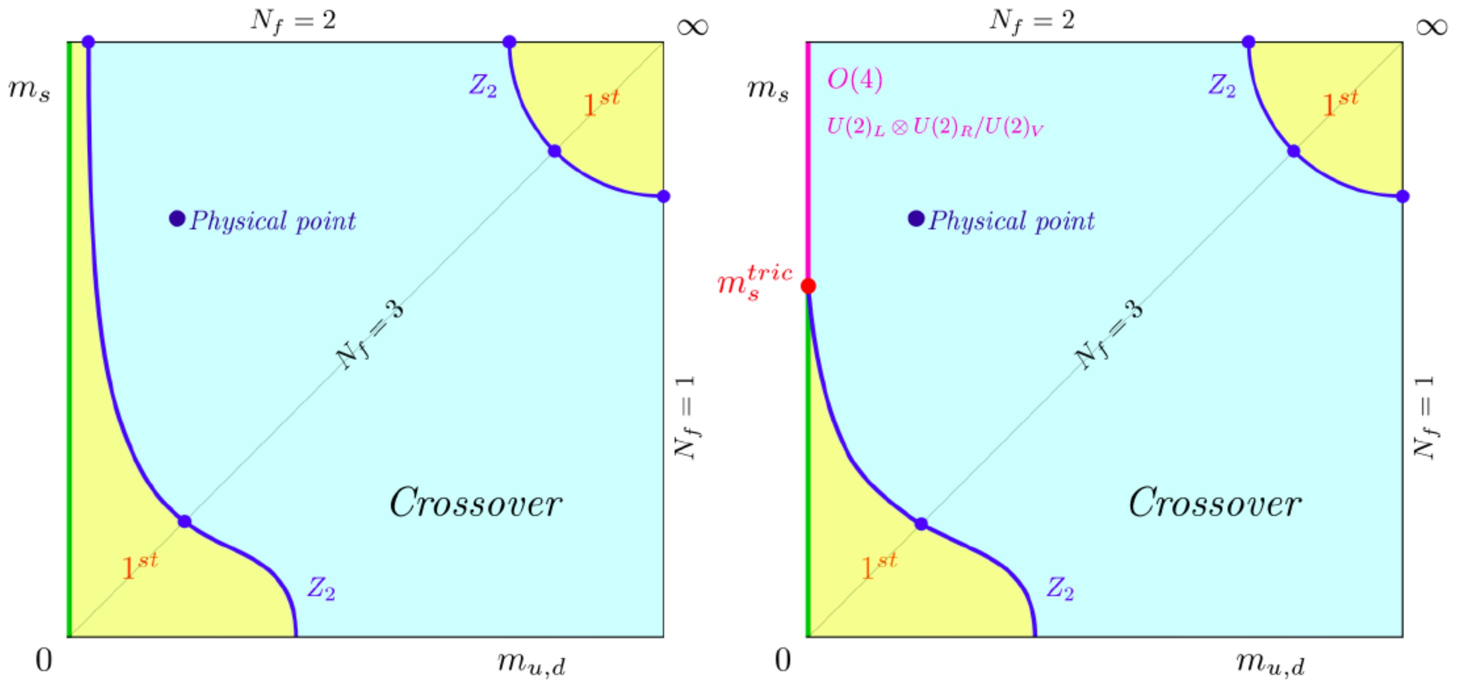
The chiral symmetry is explicitly broken by $m_u \approx m_d > 0$ and $m_s > 0$

Columbia plot:
the order of the phase transition in the limits $m_{u,d}$ and m_s

Phys Rev Lett 65 (1990) 2491



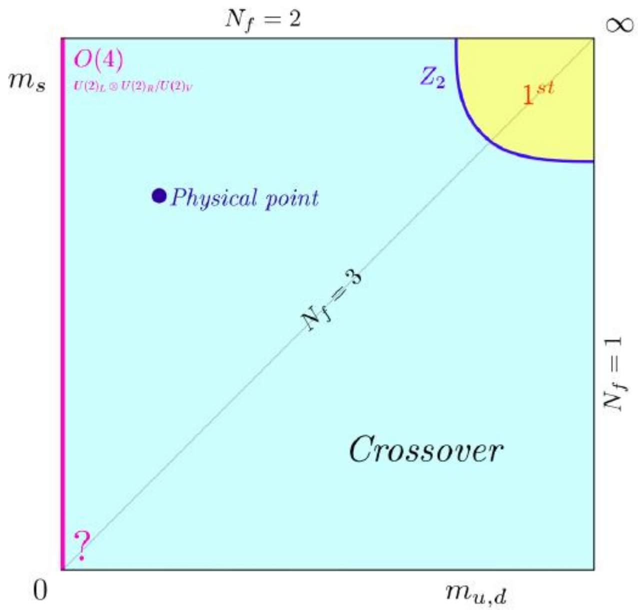
Columbia plot/2: usual expectations



Symmetry 13 (2021) 11, 2079

Slide von G. Kovacs, seminar at UJK, march 2025

Columbia plot/3: recent lattice result



Symmetry 13 (2021) 11, 2079

Lattice calculation to find the tricritical point.

Starting from a coarse lattice
a finite m_s^{tric} cannot be reached
in the continuum limit

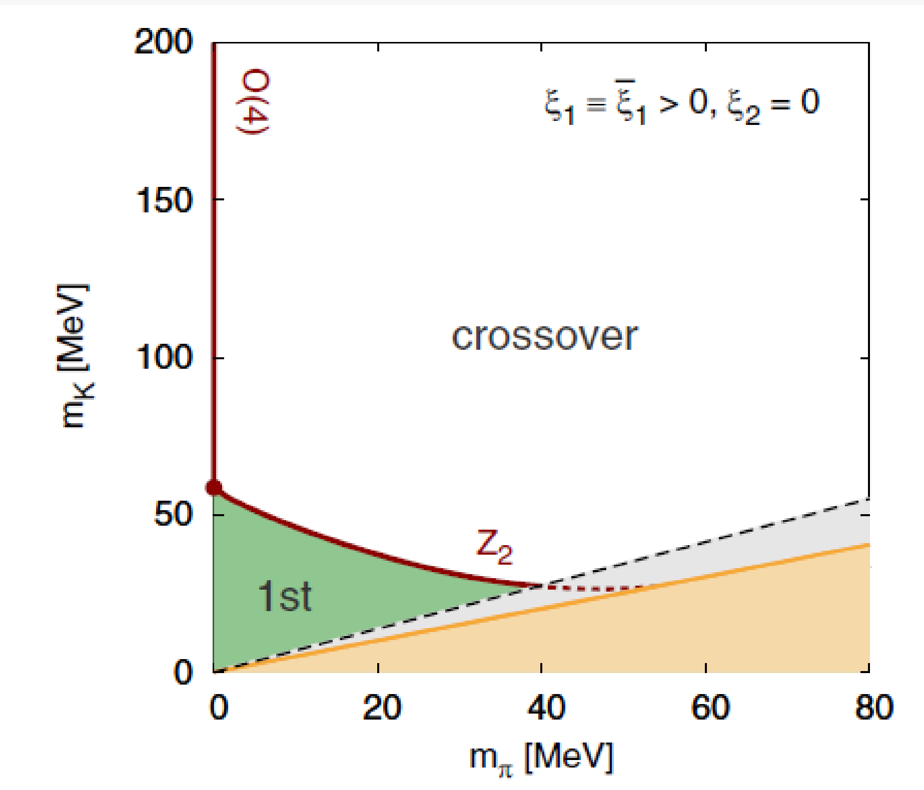
Slide von G. Kovacs, seminar at UJK, march 2025

What about the eLSM?

PHYSICAL REVIEW D **111**, 016014 (2025)

Anomalous $U(1)_A$ couplings and the Columbia plot

Francesco Giacosa,[1,2,*] Győző Kovács,[3,4,†] Péter Kovács,[3,‡] Robert D. Pisarski,[5,§] and Fabian Rennecke[6,7,‖]
[1]Institute of Physics, Jan Kochanowski University, ulica Uniwersytecka 7, P-25-406 Kielce, Poland

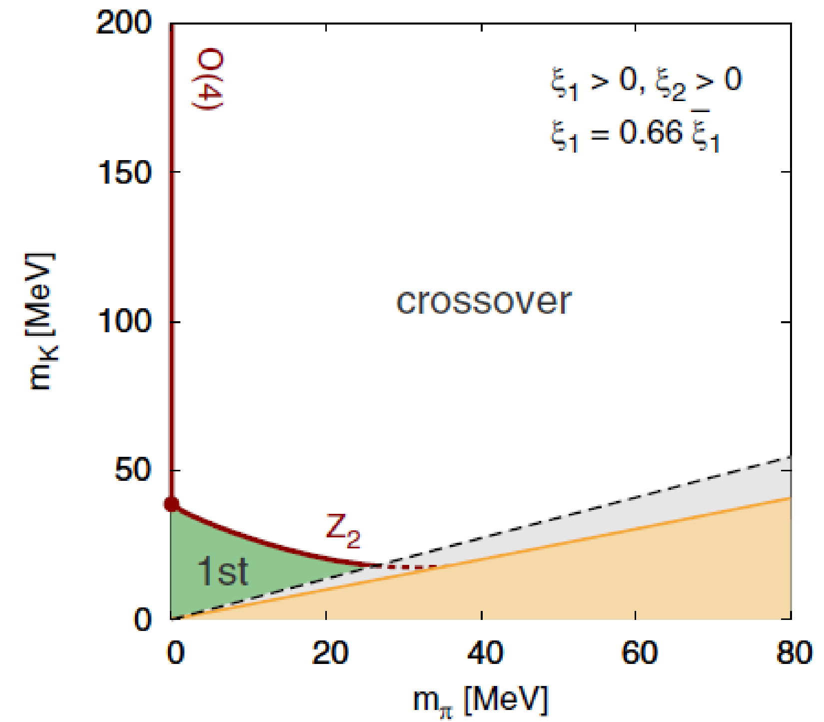


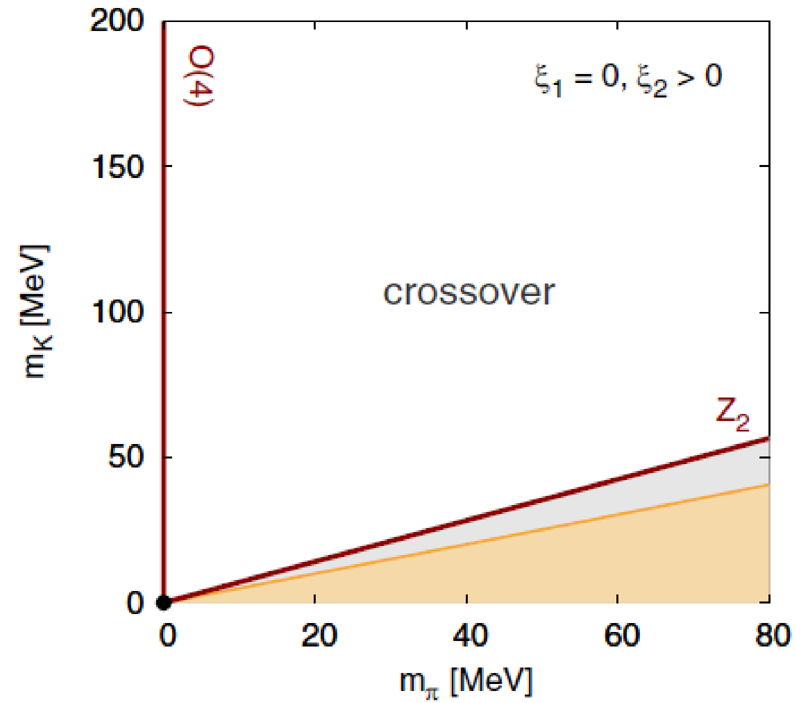
$$\mathcal{L}_{\rm qu} = -\xi_1 \left(\det\Phi + \det\Phi^\dagger\right)$$

$$f_\pi m_\pi^2 = Z_\pi h_N,$$
$$f_K m_K^2 = Z_K\left(\frac{1}{2}h_N + \frac{1}{\sqrt{2}}h_S\right)$$

$h_N \approx (m_u+m_d)/2$, $h_S \approx m_s$

$$\mathcal{L}_{\rm qu} = -\xi_1 \left(\det\Phi + \det\Phi^\dagger\right) - \xi_2\left[(\det\Phi)^2 + (\det\Phi^\dagger)^2\right]$$

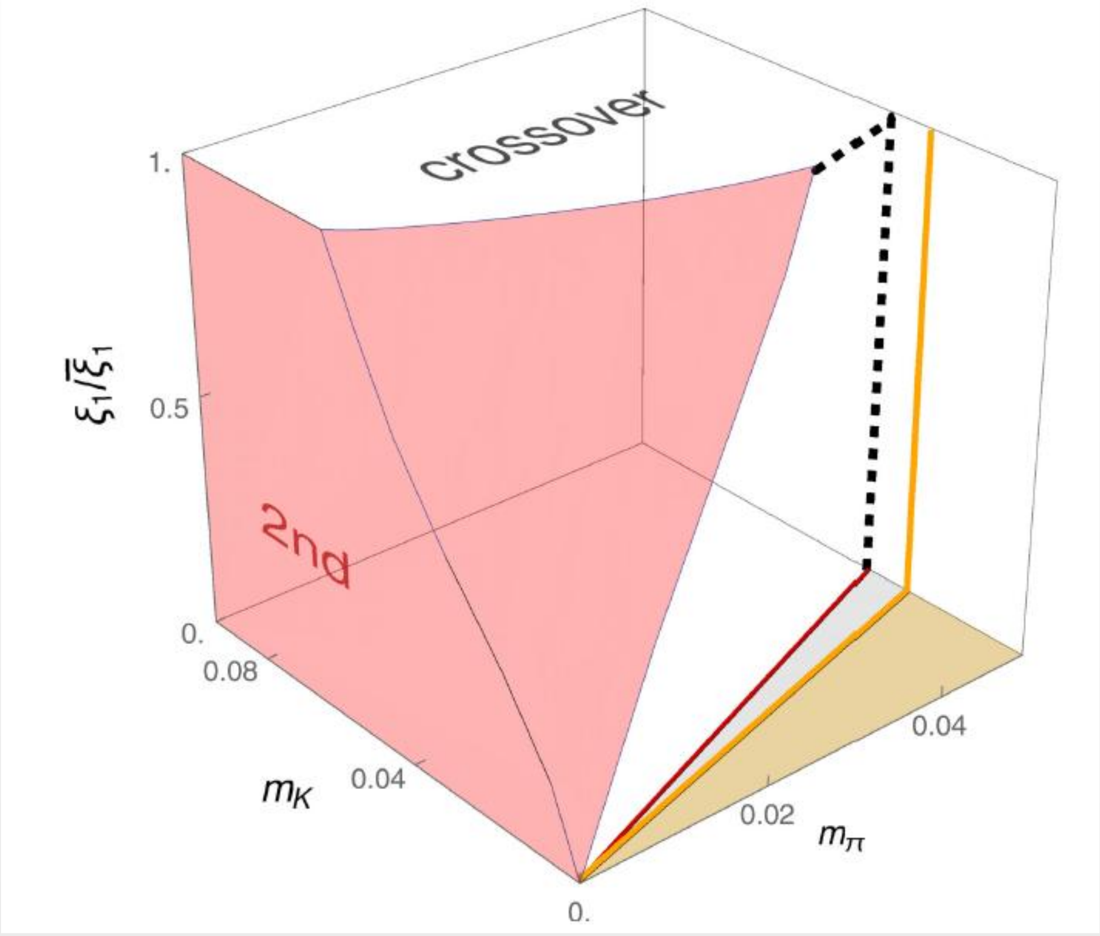




- In the $N_f = 3$ chiral limit the transition is of first order only for $\xi_1 > 0$.
- Moreover, the relative size of ξ_1 determines the size of the first-order region in the bottom left corner of the Columbia pot.

- There might be a CP violating phase in the nonphysical region of the Columbia plot where either m_u or m_s (h_N or h_S) is negative.

3D representation



Conclusions

The axial anomaly is very interesting,
both in vacuum and in the medium.

Connection to mathematics,
influence on vacuum's phenomenology (including glueballs)
and on properties at nonzero T (e.g. Columbia plot)

Thanks

Chiral partners

$n^{2S+1}L_J$	J^{PC}	I=1 $u\bar{d}$, $d\bar{u}$ $\frac{d\bar{d}-u\bar{u}}{\sqrt{2}}$	I=1/2 $u\bar{s}$, $d\bar{s}$ $s\bar{d}$, $s\bar{u}$	I=0 $\approx \frac{u\bar{u}+d\bar{d}}{\sqrt{2}}$	I=0 $\approx s\bar{s}$	Meson names	Chiral Partners
1^1S_0	0^{-+}	π	K	$\eta(547)$	$\eta'(958)$	Pseudoscalar	$J=0$
1^3P_0	0^{++}	$a_0(1450)$	$K_0^\star(1430)$	$f_0(1370)$	$f_0(1500)/f_0(1710)$	Scalar	
1^3S_1	1^{--}	$\rho(770)$	$K^\star(892)$	$\omega(782)$	$\phi(1020)$	Vector	$J=1$
1^3P_1	1^{++}	$a_1(1260)$	K_{1A}	$f_1(1285)$	$f_1'(1420)$	Axial-vector	
1^1P_1	1^{+-}	$b_1(1235)$	K_{1B}	$h_1(1170)$	$h_1(1415)$	Pseudovector	$J=1^\star$
1^3D_1	1^{--}	$\rho(1700)$	$K^\star(1680)$	$\omega(1650)$	$\phi(???)$	Excited-vector	
1^3P_2	2^{++}	$a_2(1320)$	$K_2^\star(1430)$	$f_2(1270)$	$f_2'(1525)$	Tensor	$J=2$
1^3D_2	2^{--}	$\rho_2(???)$	$K_2(1820)$	$\omega_2(???)$	$\phi_2(???)$	Axial-tensor	
1^1D_2	2^{-+}	$\pi_2(1670)$	$K_2(1770)$	$\eta_2(1645)$	$\eta_2(1870)$	Pseudotensor	
1^3D_3	3^{--}	$\rho_3(1690)$	$K_3^\star(1780)$	$\omega_3(1670)$	$\phi_3(1850)$	$J=3$ - Tensor	

TABLE I. Chiral multiplets, their currents, and transformations up to $J = 3$. [* and/or $f_0(1500)$; **a mix of.] The first two columns correspond to the assignment suggested in the Quark Model review of the PDG [8], to which we refer for further details and references (see also the discussion in the text).

J^{PC}, $^{2S+1}L_J$	$I = 1(\bar{u}d, \bar{d}u, \frac{\bar{d}d-\bar{u}u}{\sqrt{2}})$; $I = 1(-\bar{u}s, \bar{s}u, \bar{d}s, \bar{s}d)$; $I = 0(\frac{\bar{u}u+\bar{d}d}{\sqrt{2}}, \bar{s}s)^{**}$	Microscopic currents	Chiral multiplet	Transformation under $SU(3)_L \times SU(3)_R \times \times U(1)_A$
0^{-+}, $^1S_0$	π; K; $\eta, \eta'(958)$	$P^{ij} = \frac{1}{2}\bar{q}^j i\gamma^5 q^i$	$\Phi = S + iP$ ($\Phi^{ij} = \bar{q}^j_R q^i_L$)	$\Phi \to e^{-2i\alpha} U_L \Phi U_R^\dagger$
0^{++}, $^3P_0$	$a_0(1450)$; $K_0^*(1430)$; $f_0(1370), f_0(1710)^*$	$S^{ij} = \frac{1}{2}\bar{q}^j q^i$		
1^{--}, $^1S_1$	$\rho(770)$; $K^*(892)$; $\omega(782), \phi(1020)$	$V^{ij}_\mu = \frac{1}{2}\bar{q}^j \gamma_\mu q^i$	$L_\mu = V_\mu + A_\mu$ ($L^{ij}_\mu = \bar{q}^j_L \gamma_\mu q^i_L$)	$L_\mu \to U_L L_\mu U_L^\dagger$
1^{++}, $^3P_1$	$a_1(1260)$; $K_{1,A}$; $f_1(1285), f_1(1420)$	$A^{ij}_\mu = \frac{1}{2}\bar{q}^j \gamma^5 \gamma_\mu q^i$	$R_\mu = V_\mu - A_\mu$ ($R^{ij}_\mu = \bar{q}^j_R \gamma_\mu q^i_R$)	$R_\mu \to U_R R_\mu U_R^\dagger$
1^{+-}, $^1P_1$	$b_1(1235)$; $K_{1,B}$; $h_1(1170), h_1(1380)$	$P^{ij}_\mu = -\frac{1}{2}\bar{q}^j \gamma^5 \overleftrightarrow{D}_\mu q^i$	$\Phi_\mu = S_\mu + iP_\mu$ ($\Phi^{ij}_\mu = \bar{q}^j_R i\overleftrightarrow{D}_\mu q^i_L$)	$\Phi_\mu \to e^{-2i\alpha} U_L \Phi_\mu U_R^\dagger$
1^{--}, $^3D_1$	$\rho(1700)$; $K^*(1680)$; $\omega(1650), \phi(?)$	$S^{ij}_\mu = \frac{1}{2}\bar{q}^j i\overleftrightarrow{D}_\mu q^i$		
2^{++}, $^3P_2$	$a_2(1320)$; $K_2^*(1430)$; $f_2(1270), f_2'(1525)$	$V^{ij}_{\mu\nu} = \frac{1}{2}\bar{q}^j(\gamma_\mu i\overleftrightarrow{D}_\nu + \cdots)q^i$	$L_{\mu\nu} = V_{\mu\nu} + A_{\mu\nu}$ ($L^{ij}_{\mu\nu} = \bar{q}^j_L(\gamma_\mu i\overleftrightarrow{D}_\nu + \cdots)q^i_L$)	$L_{\mu\nu} \to U_L L_{\mu\nu} U_L^\dagger$
2^{--}, $^3D_2$	$\rho_2(?)$; $K_2(1820)$; $\omega_2(?), \phi_2(?)$	$A^{ij}_{\mu\nu} = \frac{1}{2}\bar{q}^j(\gamma^5\gamma_\mu i\overleftrightarrow{D}_\nu + \cdots)q^i$	$R_{\mu\nu} = V_{\mu\nu} - A_{\mu\nu}$ ($R^{ij}_{\mu\nu} = \bar{q}^j_R(\gamma_\mu \overleftrightarrow{D}_\nu + \cdots)q^i_R$)	$R_{\mu\nu} \to U_R R_{\mu\nu} U_R^\dagger$
2^{-+}, $^1D_2$	$\pi_2(1670)$; $K_2(1770)$; $\eta_2(1645), \eta_2(1870)$	$P^{ij}_{\mu\nu} = -\frac{1}{2}\bar{q}^j(i\gamma^5 \overleftrightarrow{D}_\mu \overleftrightarrow{D}_\nu + \cdots)q^i$	$\Phi_{\mu\nu} = S_{\mu\nu} + iP_{\mu\nu}$ ($\Phi^{ij}_{\mu\nu} = \bar{q}^j_R(\overleftrightarrow{D}_\mu \overleftrightarrow{D}_\nu + \cdots)q^i_L$)	$\Phi_{\mu\nu} \to e^{-2i\alpha} U_L \Phi_{\mu\nu} U_R^\dagger$
2^{++}, $^3F_2$	$a_2(?)$; $K_2^*(?)$; $f_2(?), f_2'(?)$	$S^{ij}_{\mu\nu} = -\frac{1}{2}\bar{q}^j(\overleftrightarrow{D}_\mu \overleftrightarrow{D}_\nu + \cdots)q^i$		
3^{--}, $^3D_3$	$\rho_3(1690)$; $K_3^*(1780)$; $\omega_3(1670), \phi_3(1850)$	⋮	⋮	⋮

Uniwersytet Jana Kochanowskiego w Kielcach

Heterochiral

Homochiral

Heterochiral

Homochiral

Table from:

F.G., R. Pisarski, A. Koenigstein Phys.Rev.D 97 (2018) 9, 091901 e-Print: 1709.07454

(Pseudo)scalar mesons: heterochiral scalars

$$q_{\mathrm{L,R}} \longrightarrow e^{\mp i\alpha/2} U_{\mathrm{L,R}}\, q_{\mathrm{L,R}}$$

J^{PC}, $^{2S+1}L_J$	$\begin{cases} I=1 & (\bar{u}d, \bar{d}u, \frac{\bar{d}d-\bar{u}u}{\sqrt{2}}) \\ I=1 & (-\bar{u}s, \bar{s}u, \bar{d}s, \bar{s}d) \\ I=0 & (\frac{\bar{u}u+\bar{d}d}{\sqrt{2}}, \bar{s}s)^{\star\star} \end{cases}$	microscopic currents	chiral multiplet	transformation under $SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}} \times$ $\times U(1)_{\mathrm{A}}$
0^{-+}, $^1S_0$	$\begin{cases} \pi \\ K \\ \eta, \eta'(958) \end{cases}$	$P^{ij} = \frac{1}{2}\bar{q}^j i\gamma^5 q^i$	$\Phi = S + iP$ $(\Phi^{ij} = \bar{q}^j_{\mathrm{R}} q^i_{\mathrm{L}})$	$\Phi \longrightarrow e^{-2i\alpha} U_{\mathrm{L}} \Phi U_{\mathrm{R}}^\dagger$
0^{++}, $^3P_0$	$\begin{cases} a_0(1450) \\ K_0^*(1430) \\ f_0(1370), f_0(1710)^\star \end{cases}$	$S^{ij} = \frac{1}{2}\bar{q}^j q^i$		

$$\Phi \longrightarrow e^{-2i\alpha} U_{\mathrm{L}} \Phi U_{\mathrm{R}}^\dagger$$

We call the transformation of the matrix Φ **heterochiral**!
We thus have heterochiral scalars.

$\mathrm{tr}(\Phi^\dagger\Phi)$, $\mathrm{tr}(\Phi^\dagger\Phi)^2$ are clearly invariant; typical terms for a chiral model.

$\det(\Phi)$ is interesting, since it breaks only U(1)A axial anomaly $\det\Phi \to e^{-i6\alpha}\det\Phi$

(Axial-)vector mesons: homochiral vectors

J^{PC}, $^{2S+1}L_J$	$\begin{cases} I=1 & (\bar{u}d, \bar{d}u, \frac{\bar{d}d-\bar{u}u}{\sqrt{2}}) \\ I=1 & (-\bar{u}s, \bar{s}u, \bar{d}s, \bar{s}d) \\ I=0 & (\frac{\bar{u}u+\bar{d}d}{\sqrt{2}}, \bar{s}s)^{\star\star} \end{cases}$	microscopic currents	chiral multiplet	transformation under $SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}} \times$ $\times U(1)_{\mathrm{A}}$
1^{--}, $^1S_1$	$\begin{cases} \rho(770) \\ K^*(892) \\ \omega(782), \phi(1020) \end{cases}$	$V_\mu^{ij} = \frac{1}{2}\bar{q}^j\gamma_\mu q^i$	$L_\mu = V_\mu + A_\mu$ $(L_\mu^{ij} = \bar{q}_{\mathrm{L}}^j\gamma_\mu q_{\mathrm{L}}^i)$	$L_\mu \longrightarrow U_{\mathrm{L}} L_\mu U_{\mathrm{L}}^\dagger$
1^{++}, $^3P_1$	$\begin{cases} a_1(1260) \\ K_{1,A} \\ f_1(1285), f_1(1420) \end{cases}$	$A_\mu^{ij} = \frac{1}{2}\bar{q}^j\gamma^5\gamma_\mu q^i$	$R_\mu = V_\mu - A_\mu$ $(R_\mu^{ij} = \bar{q}_{\mathrm{R}}^j\gamma_\mu q_{\mathrm{R}}^i)$	$R_\mu \longrightarrow U_{\mathrm{R}} R_\mu U_{\mathrm{R}}^\dagger$

$$L_\mu \longrightarrow U_{\mathrm{L}} L_\mu U_{\mathrm{L}}^\dagger$$

$$R_\mu \longrightarrow U_{\mathrm{R}} R_\mu U_{\mathrm{R}}^\dagger$$

We have here a **homochiral** multiplet.
We call these states as homochiral vectors.

Ground-state tensors (and their chiral partners): Homochiral tensors

J^{PC}, $^{2S+1}L_J$	$I=1$ $(\bar{u}d, \bar{d}u, \frac{\bar{d}d-\bar{u}u}{\sqrt{2}})$ $I=1$ $(-\bar{u}s, \bar{s}u, \bar{d}s, \bar{s}d)$ $I=0$ $(\frac{\bar{u}u+\bar{d}d}{\sqrt{2}}, \bar{s}s)^{\star\star}$	microscopic currents	chiral multiplet	transformation under $SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}} \times$ $\times U(1)_{\mathrm{A}}$
2^{++}, $^3P_2$	$a_2(1320)$ $K_2^*(1430)$ $f_2(1270), f_2'(1525)$	$V^{ij}_{\mu\nu} = \frac{1}{2}\bar{q}^j(\gamma_\mu \mathrm{i}\overleftrightarrow{D}_\nu + \ldots)q^i$	$L_{\mu\nu} = V_{\mu\nu} + A_{\mu\nu}$ $(L^{ij}_{\mu\nu} = \bar{q}^j_{\mathrm{L}}(\gamma_\mu \mathrm{i}\overleftrightarrow{D}_\nu + \ldots)q^i_{\mathrm{L}})$	$L_{\mu\nu} \longrightarrow U_{\mathrm{L}} L_{\mu\nu} U^\dagger_{\mathrm{L}}$
2^{--}, $^3D_2$	$\rho_2(?)$ $K_2(1820)$ $\omega_2(?), \phi_2(?)$	$A^{ij}_{\mu\nu} = \frac{1}{2}\bar{q}^j(\gamma^5\gamma_\mu \mathrm{i}\overleftrightarrow{D}_\nu + \ldots)q^i$	$R_{\mu\nu} = V_{\mu\nu} - A_{\mu\nu}$ $(R^{ij}_{\mu\nu} = \bar{q}^j_{\mathrm{R}}(\gamma_\mu \mathrm{i}\overleftrightarrow{D}_\nu + \ldots)q^i_{\mathrm{R}})$	$R_{\mu\nu} \longrightarrow U_{\mathrm{R}} R_{\mu\nu} U^\dagger_{\mathrm{R}}$

$$L_{\mu\nu} \longrightarrow U_{\mathrm{L}} L_{\mu\nu} U^\dagger_{\mathrm{L}}$$

$$R_{\mu\nu} \longrightarrow U_{\mathrm{R}} R_{\mu\nu} U^\dagger_{\mathrm{R}}$$

Thus, we have **homochiral** tensors. We do not expect large mixing.

Pseudovectors and orbitally excited vectors: Heterochiral vectors

J^{PC}, ${}^{2S+1}L_J$	$\begin{cases} I=1 & (\bar{u}d, \bar{d}u, \frac{\bar{d}d-\bar{u}u}{\sqrt{2}}) \\ I=1 & (-\bar{u}s, \bar{s}u, \bar{d}s, \bar{s}d) \\ I=0 & (\frac{\bar{u}u+\bar{d}d}{\sqrt{2}}, \bar{s}s)^{\star\star} \end{cases}$	microscopic currents	chiral multiplet	transformation under $SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}} \times$ $\times U(1)_{\mathrm{A}}$
1^{+-}, ${}^1P_1$	$\begin{cases} b_1(1235) \\ K_{1,B} \\ h_1(1170), h_1(1380) \end{cases}$	$P^{ij}_\mu = -\frac{1}{2}\bar{q}^j\gamma^5\overleftrightarrow{D}_\mu q^i$	$\Phi_\mu = S_\mu + \mathrm{i}P_\mu$ $(\Phi^{ij}_\mu = \bar{q}^j_{\mathrm{R}}\mathrm{i}\overleftrightarrow{D}_\mu q^i_{\mathrm{L}})$	$\Phi_\mu \longrightarrow \mathrm{e}^{-2\mathrm{i}\alpha}U_{\mathrm{L}}\Phi_\mu U^\dagger_{\mathrm{R}}$
1^{--}, ${}^3D_1$	$\begin{cases} \rho(1700) \\ K^*(1680) \\ \omega(1650), \phi(?) \end{cases}$	$S^{ij}_\mu = \frac{1}{2}\bar{q}^j\mathrm{i}\overleftrightarrow{D}_\mu q^i$		

$$\Phi_\mu \longrightarrow \mathrm{e}^{-\mathrm{i}\alpha}\, U_{\mathrm{L}}\, \Phi_\mu\, U^\dagger_{\mathrm{R}}$$

The pseudovector mesons and the excited vector mesons form a **heterochiral** multiplet. We thus call them heterochiral vectors.
The chiral transformation is just as the (pseudo)scalar mesons (which is also hetero).
Hence, an anomalous Lagrangian is possible for heterochiral vectors.

Excited vector mesons: phi(1930) predicted to be the missing state, see
M. Piotrowska, C. Reisinger and FG.,
``Strong and radiative decays of excited vector mesons and predictions for a new phi(1930)$ resonance,'' arXiv:1708.02593 [hep-ph], to appear in PRD.

Ground-state tensors (and their chiral partners): Homochiral tensors

Tensor mesons: {a2(1320), K_2^*(1430), f2(1270), f2(1535)}
Axial-vector mesons: {ρ_2(???), K_2(1820), ω_2(???), φ2(???)}

Chiral transformations

$$q_{\mathrm{L,R}} \longrightarrow \mathrm{e}^{\mp \mathrm{i}\alpha/2}\, U_{\mathrm{L,R}}\, q_{\mathrm{L,R}}$$

$$L_{\mu\nu} \longrightarrow U_{\mathrm{L}} L_{\mu\nu} U_{\mathrm{L}}^{\dagger}$$

$$R_{\mu\nu} \longrightarrow U_{\mathrm{R}} R_{\mu\nu} U_{\mathrm{R}}^{\dagger}$$

Thus, we have **homochiral** tensors. We do not expect large mixing.

eLSM, additional lag. details

where

$$D^\mu\Phi \equiv \partial^\mu\Phi - ig_1(L^\mu\Phi - \Phi R^\mu) - ieA^\mu[t_3,\Phi] \ ,$$

$$L^{\mu\nu} \equiv \partial^\mu L^\nu - ieA^\mu[t_3,L^\nu] - \{\partial^\nu L^\mu - ieA^\nu[t_3,L^\mu]\} \ ,$$

$$R^{\mu\nu} \equiv \partial^\mu R^\nu - ieA^\mu[t_3,R^\nu] - \{\partial^\nu R^\mu - ieA^\nu[t_3,R^\mu]\} \ ,$$

and

$$H = H_0t_0 + H_8t_8 = \begin{pmatrix} \frac{h_{0N}}{2} & 0 & 0 \\ 0 & \frac{h_{0N}}{2} & 0 \\ 0 & 0 & \frac{h_{0S}}{\sqrt{2}} \end{pmatrix} \ ,$$

$$\Delta = \Delta_0t_0 + \Delta_8t_8 = \begin{pmatrix} \frac{\tilde{\delta}_N}{2} & 0 & 0 \\ 0 & \frac{\tilde{\delta}_N}{2} & 0 \\ 0 & 0 & \frac{\tilde{\delta}_S}{\sqrt{2}} \end{pmatrix} \equiv \begin{pmatrix} \delta_N & 0 & 0 \\ 0 & \delta_N & 0 \\ 0 & 0 & \delta_S \end{pmatrix} .$$

eLSM, masses

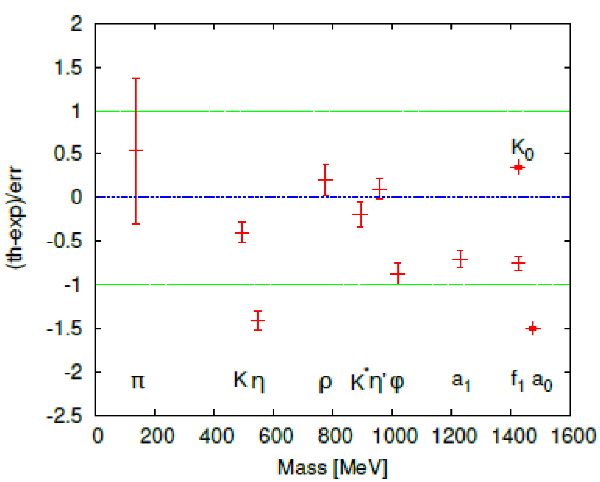
Mass squares	Analytical expressions
m_π^2	$Z_\pi^2\left[m_0^2+\left(\lambda_1+\frac{\lambda_2}{2}\right)\phi_N^2+\lambda_1\phi_S^2\right]\equiv\frac{Z_\pi^2 h_{0N}}{\phi_N}$
m_K^2	$Z_K^2\left[m_0^2+\left(\lambda_1+\frac{\lambda_2}{2}\right)\phi_N^2-\frac{\lambda_2}{\sqrt{2}}\phi_N\phi_S+\left(\lambda_1+\lambda_2\right)\phi_S^2\right]$
$m_{\eta_N}^2$	$Z_\pi^2\left[m_0^2+\left(\lambda_1+\frac{\lambda_2}{2}\right)\phi_N^2+\lambda_1\phi_S^2+c_2\,\phi_N^2\phi_S^2\right]\equiv Z_\pi^2\left(\frac{h_{0N}}{\phi_N}+c_2\,\phi_N^2\phi_S^2\right)$
$m_{\eta_S}^2$	$Z_{\eta_S}^2\left[m_0^2+\lambda_1\phi_N^2+\left(\lambda_1+\lambda_2\right)\phi_S^2+\frac{c_2}{4}\phi_N^4\right]\equiv Z_{\eta_S}^2\left(\frac{h_{0S}}{\phi_S}+\frac{c_2}{4}\phi_N^4\right)$
$m_{\eta_{NS}}^2$	$Z_{\eta_N}Z_{\eta_S}\frac{c_2}{2}\phi_N^3\phi_S$
$m_{a_0}^2$	$m_0^2+\left(\lambda_1+\frac{3}{2}\lambda_2\right)\phi_N^2+\lambda_1\phi_S^2$
$m_{K_0^\star}^2$	$Z_{K_0^\star}^2\left[m_0^2+\left(\lambda_1+\frac{\lambda_2}{2}\right)\phi_N^2+\frac{\lambda_2}{\sqrt{2}}\phi_N\phi_S+\left(\lambda_1+\lambda_2\right)\phi_S^2\right]$
$m_{\sigma_N}^2$	$m_0^2+3\left(\lambda_1+\frac{\lambda_2}{2}\right)\phi_N^2+\lambda_1\phi_S^2$
$m_{\sigma_S}^2$	$m_0^2+\lambda_1\phi_N^2+3\left(\lambda_1+\lambda_2\right)\phi_S^2$
$m_{\sigma_{NS}}^2$	$2\lambda_1\phi_N\phi_S$

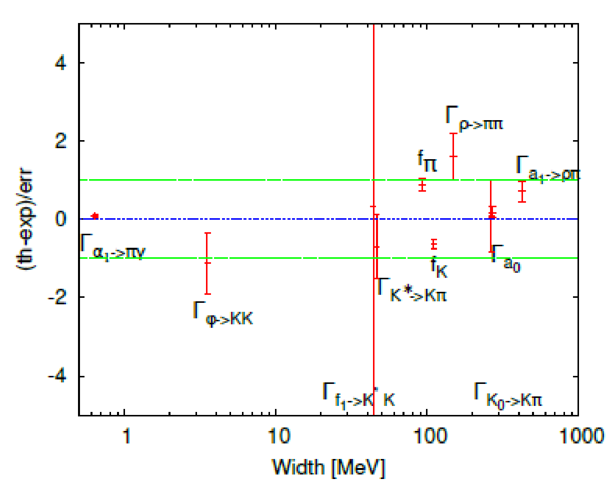
Table 3.2: Mass expressions of spin-0 mesons (scalars and pseudoscalars) within the eLSM.

eLSM, fit vacuum

Observable	Fit [MeV]	Experiment [MeV]	Observable	Fit [MeV]	Experiment [MeV]
f_π	96.3 ± 0.7	92.2 ± 4.6	f_K	106.9 ± 0.6	110.4 ± 5.5
m_π	141.0 ± 5.8	138 ± 6.9	m_K	485.6 ± 3.0	495.6 ± 24.8
m_η	509.4 ± 3.0	547.9 ± 27.4	$m_{\eta'}$	962.5 ± 5.6	957.8 ± 47.9
m_ρ	783.1 ± 7.0	775.5 ± 38.8	$m_{K^\star}$	885.1 ± 6.3	893.8 ± 44.7
m_ϕ	975.1 ± 6.4	1019.5 ± 51.0	m_{a_1}	1186 ± 6.0	1230 ± 62
$m_{f_1(1420)}$	1372.4 ± 5.3	1426 ± 71	m_{a_0}	1363 ± 1	1474 ± 74
$m_{K_0^\star}$	1450 ± 1	1425 ± 71	$\Gamma_{\rho\to\pi\pi}$	160.9 ± 4.4	149.1 ± 7.4
$\Gamma_{K^\star\to K\pi}$	44.6 ± 1.9	46.2 ± 2.3	$\Gamma_{\phi\to \bar{K}K}$	3.34 ± 0.14	3.54 ± 0.18
$\Gamma_{a_1\to\rho\pi}$	549 ± 43	425 ± 175	$\Gamma_{a_1\to\pi\gamma}$	0.66 ± 0.01	0.64 ± 0.25
$\Gamma_{f_1(1420)\to K^\star K}$	44.6 ± 39.9	43.9 ± 2.2	Γ_{a_0}	266 ± 12	265 ± 13
$\Gamma_{K_0^\star\to K\pi}$	285 ± 12	270 ± 80			

Table 3.4: An example of fit results from [6], together with the experimental values taken from [13].





Francesco Giacosa

The pseudoscalar glueball: predictions from the eLSM

$$\mathcal{L}^{int}_{\tilde{G}\text{-mesons}} = ic_{\tilde{G}\Phi}\tilde{G}\left(\det\Phi - \det\Phi^{\dagger}\right)$$

MG = 2.6 GeV as been used as an input.

Quantity	Value
$\Gamma_{\tilde{G}\to KK\eta}/\Gamma^{tot}_{\tilde{G}}$	0.049
$\Gamma_{\tilde{G}\to KK\eta'}/\Gamma^{tot}_{\tilde{G}}$	0.019
$\Gamma_{\tilde{G}\to\eta\eta\eta}/\Gamma^{tot}_{\tilde{G}}$	0.016
$\Gamma_{\tilde{G}\to\eta\eta\eta'}/\Gamma^{tot}_{\tilde{G}}$	0.0017
$\Gamma_{\tilde{G}\to\eta\eta'\eta'}/\Gamma^{tot}_{\tilde{G}}$	0.00013
$\Gamma_{\tilde{G}\to KK\pi}/\Gamma^{tot}_{\tilde{G}}$	0.46
$\Gamma_{\tilde{G}\to\eta\pi\pi}/\Gamma^{tot}_{\tilde{G}}$	0.16
$\Gamma_{\tilde{G}\to\eta'\pi\pi}/\Gamma^{tot}_{\tilde{G}}$	0.094

Quantity	Value
$\Gamma_{\tilde{G}\to KK_S}/\Gamma^{tot}_{\tilde{G}}$	0.059
$\Gamma_{\tilde{G}\to a_0\pi}/\Gamma^{tot}_{\tilde{G}}$	0.083
$\Gamma_{\tilde{G}\to\eta\sigma_N}/\Gamma^{tot}_{\tilde{G}}$	0.028
$\Gamma_{\tilde{G}\to\eta\sigma_S}/\Gamma^{tot}_{\tilde{G}}$	0.012
$\Gamma_{\tilde{G}\to\eta'\sigma_N}/\Gamma^{tot}_{\tilde{G}}$	0.019

$$\Gamma_{\tilde{G}\to\pi\pi\pi} = 0$$

X(2370)and X(2600) found at BESIII possible candidate.

Future experimental search, e.g. at BES and PANDA

Details in:

W. Eshraim, S. Janowski, F.G., D. Rischke, **Phys.Rev. D87 (2013) 054036. arxiv: 1208.6474** .
W. Eschraim, S. Janowski, K. Neuschwander, A. Peters, F.G., **Acta Phys. Pol. B**, Prc. Suppl. 5/4, **arxiv: 1209.3976**

Pseudoscalar glueball/2

Thanks to DIG it was possible to estimate the coupling constant, see 2309.00086
Then not only ratio possible, but actual widths!

$$\Gamma(\tilde{G}_0 \to K\bar{K}\pi) \approx 0.24\ \text{GeV} \quad \text{and} \quad \Gamma(\tilde{G}_0 \to \pi\pi\eta') \approx 0.05\ \text{GeV}$$

Recent experimental results:

PHYSICAL REVIEW LETTERS **129,** 042001 (2022)

Observation of a State $X(2600)$ in the $\pi^+\pi^-\eta'$ System in the Process $J/\psi \to \gamma\pi^+\pi^-\eta'$

PHYSICAL REVIEW LETTERS **132,** 181901 (2024)

Editors' Suggestion

Determination of Spin-Parity Quantum Numbers of $X(2370)$ as 0^{-+} from $J/\psi \to \gamma K^0_S K^0_S \eta'$

M. Ablikim *et al.*[*]
(BESIII Collaboration)

Average over instantons

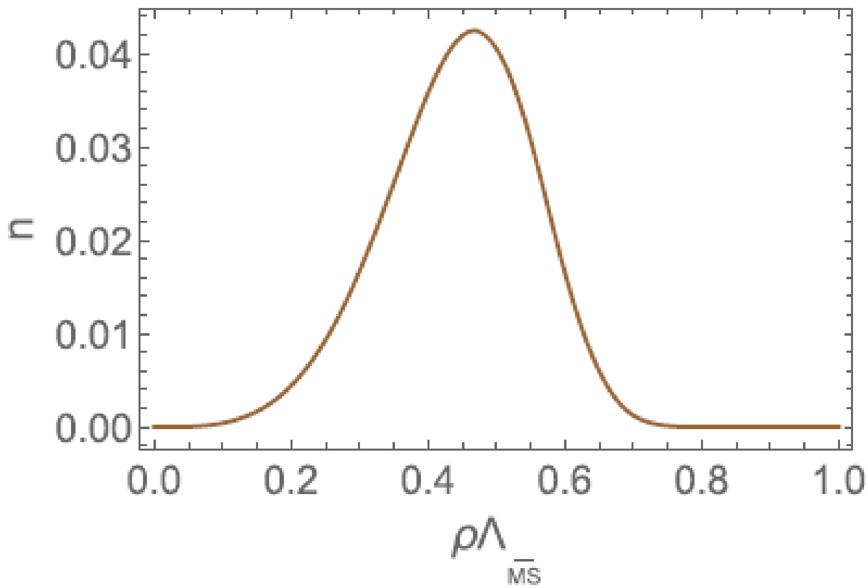


FIG. 1. The density of instantons for $N_c = N_f = 3$.

$$\mathcal{L}_{\text{eff}}^{J=0} = -a_0(\det\Phi + \det\Phi^\dagger)$$

$$k_J = (8\pi^2)^3 \int_0^{\Lambda_{\overline{MS}}^{-1}} d\rho\, n(\rho)\rho^{9+2J},$$

$J = 0$

$$a_0 = k_0 M_0^6/48 > 0.$$

$$a_0 = 1.3 \text{ GeV}$$

$$M_0 = 170 \text{ MeV}$$

Polydeterminant properties

$$\epsilon(A_1, A_2, ..., A_N) = \frac{1}{N!}\epsilon^{i_1 i_2 ... i_N}\epsilon^{i'_1 i'_2 ... i'_N} A_1^{i_1 i'_1} A_2^{i_2 i'_2} ... A_N^{i_N i'_N}$$

$$\epsilon(A, A, \cdots, A) = \det(A)$$

$$\epsilon(A_1, \cdots, A_i, \cdots, A_j, \cdots A_N) = \epsilon(A_1, \cdots, A_j, \cdots, A_i, \cdots A_N)$$

$$\epsilon(A_1, \cdots, A_i = \alpha B_i + \beta C_i, \cdots A_N) = \alpha\epsilon(A_1, \cdots, B_i, \cdots A_N) + \beta\epsilon(A_1, \cdots, C_i, \cdots A_N)$$

$$\epsilon(A, \mathbf{1}, \cdots, \mathbf{1}) = \frac{1}{N}\mathrm{Tr}(A)$$

$$\epsilon(A'_1, A'_2, \cdots A'_N) = \epsilon(A_1, A_2, \cdots A_N)$$

$$A'_i = U A_i U^{-1}$$

$$\epsilon(A_1, A_2, \cdots, A_N) = \sum_{\substack{n_1,..,n_N \geq 0 \\ n_1+2n_2+\cdots+Nn_N=N}} C_{n_1 n_2 \cdots n_N} X^{n_1 n_2 \cdots n_N}$$

$$\begin{aligned} X^{n_1 n_2 \cdots n_N} = \frac{1}{N!} \sum_{\sigma} & \mathrm{Tr}\left(A_{\sigma(1)}\right) \mathrm{Tr}\left(A_{\sigma(2)}\right) \cdot \ldots \cdot \mathrm{Tr}\left(A_{\sigma(n_1)}\right) \\ & \mathrm{Tr}\left(A_{\sigma(n_1+1)} A_{\sigma(n_1+2)}\right) \mathrm{Tr}\left(A_{\sigma(n_1+3)} A_{\sigma(n_1+4)}\right) \cdot \ldots \cdot \mathrm{Tr}\left(A_{\sigma(n_1+2n_2-1)} A_{\sigma(n_1+2n_2)}\right) \\ & \mathrm{Tr}\left(A_{\sigma(n_1+2n_2+1)} A_{\sigma(n_1+2n_2+2)} A_{\sigma(n_1+2n_2+1)}\right) \cdot \ldots . \end{aligned}$$

Generalized determinant for 3x3 matrices

Determinant of a 3×3 Matrix

$$\det\Phi = \frac{1}{3!}\epsilon^{ijk}\epsilon^{i'j'k'}\Phi^{ii'}\Phi^{jj'}\Phi^{kk'}$$

One can write determinant like a product for matrices $\Phi_{1,2,3}$

$$\epsilon[\Phi_1, \Phi_2, \Phi_3] := \frac{1}{3!}\epsilon^{ijk}\epsilon^{i'j'k'}\Phi_1^{ii'}\Phi_2^{jj'}\Phi_3^{kk'}$$

It has the following properties

$$\epsilon[\Phi_1, \Phi_1, \Phi_1] = \det\Phi_1 \,, \quad \epsilon[1, 1, \Phi_1] = \frac{1}{3}\mathrm{Tr}[\Phi_1]$$

By using the epsilon product, we can construct anomalous lagrangians

Mixing angle for pseudotensor mesons

$$\begin{pmatrix} \eta_2(1645) \\ \eta_2(1870) \end{pmatrix} = \begin{pmatrix} \cos\beta_2 & \sin\beta_2 \\ -\sin\beta_2 & \cos\beta_2 \end{pmatrix} \begin{pmatrix} \eta_{2,N} = \sqrt{1/2}(\bar{u}u + \bar{d}d) \\ \eta_{2,S} = \bar{s}s \end{pmatrix}$$

$$\beta_2 \approx -(1^\circ, 10^\circ) < 0$$

Instanton-based result,
GPJ 2309.00086

$$\beta_2 \approx -40^\circ$$

Phenomenology results,
FG & A. Koenigstein 1608.08777
V. Shastry, E. Trotti, FG: 2107.13501

Anomalous Lagrangian for heterochiral tensors

$$\mathscr{L}_{\Phi_{\mu\nu}}^{\text{anomaly}} = c_{\text{A}}^{(3)}(\varepsilon^{ijk}\varepsilon^{i'j'k'}\Phi^{ii'}\Phi^{jj'}\Phi_{\mu\nu}^{kk'} - h.c.)^2 + \ldots,$$

Again, the various terms are $SU(3)_R \times SU(3)_L$ invariant but break $U(1)_A$.

First term generates mixing for pseudotensors and also for their chiral partners.
Second term generates decays of pseudotensor (and partners) into (pseudo)scalars.
Third term generates mixing for pseudotensors only.

Columbia plot/4

FRG calculation: multiple fixed point at $N_f = 2$ and $N_f = 3$ that gives first or second order transition in the presence of the axial anomaly

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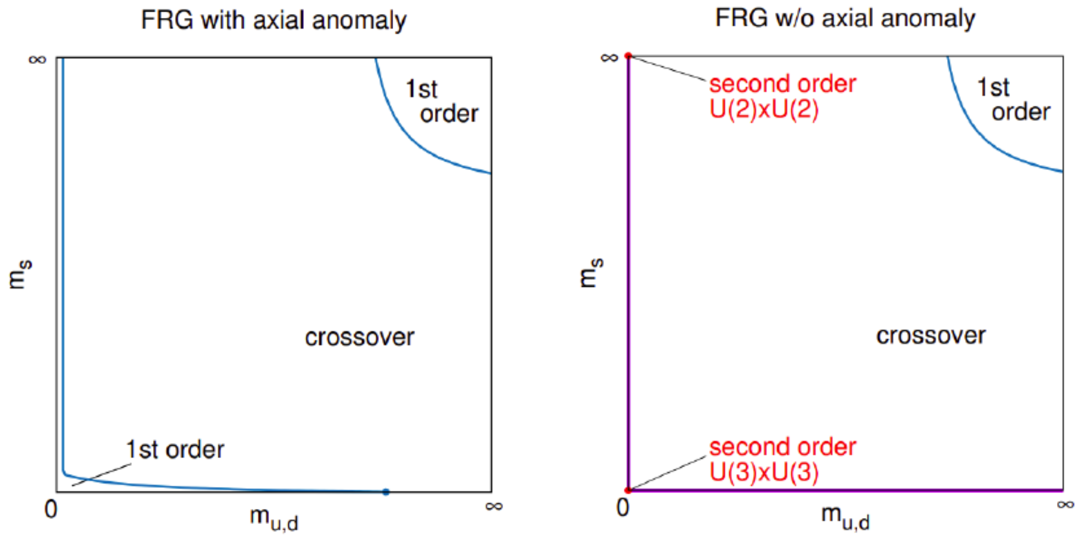
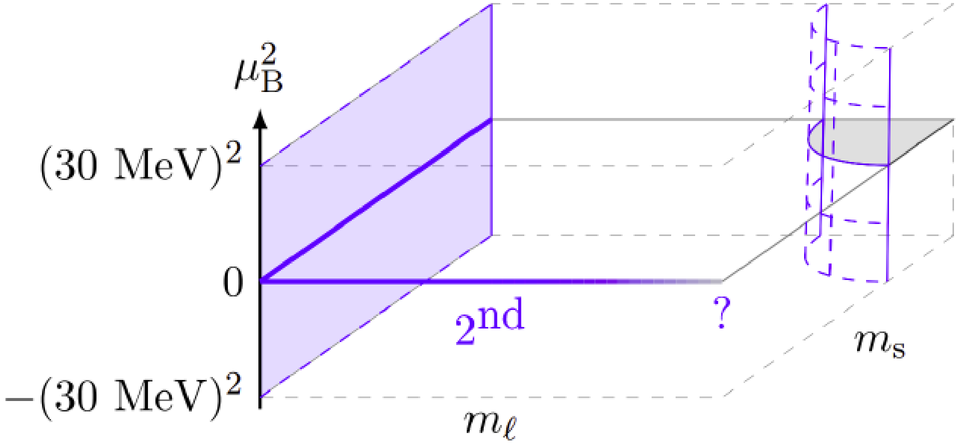


In 3d the only relevant operator is $(\det \Phi + \det \Phi^\dagger)$!

Slide von G. Kovacs, seminar at UJK, march 2025

Dyson-Schwinger approach:

$U(1)_A$ anomaly through η meson
(but only 2-point function)



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