3D Theory of the Ion Channel Laser

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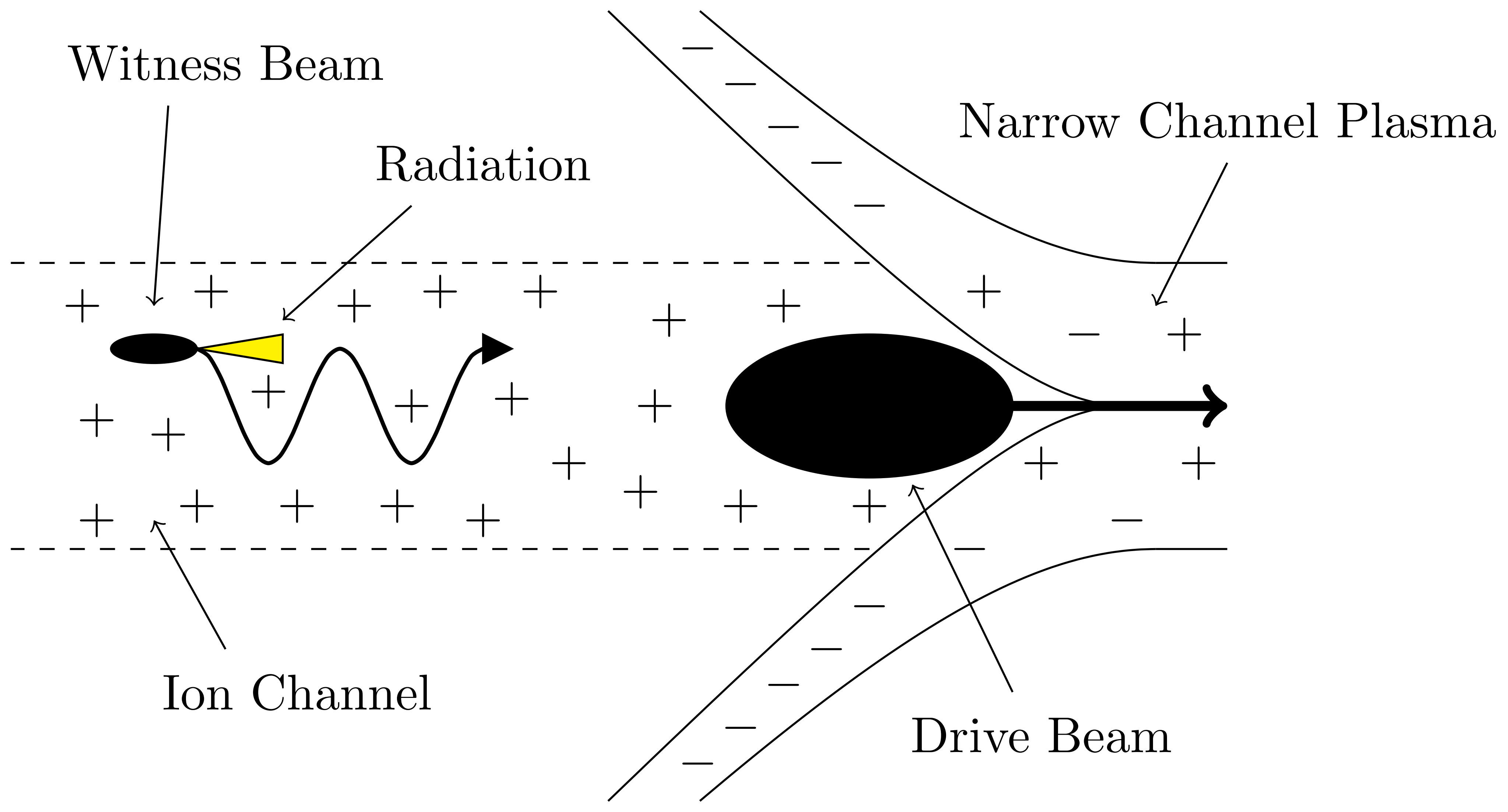
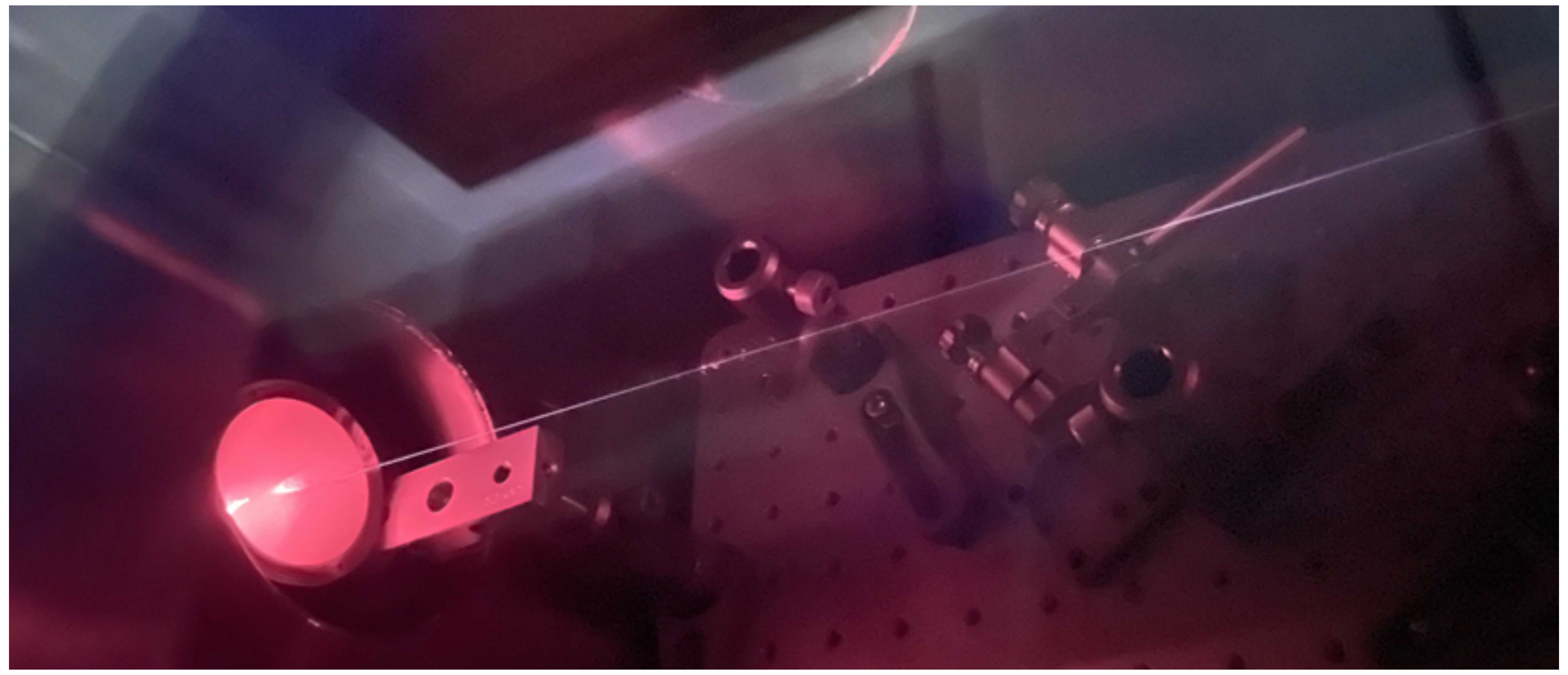
Ion Channel Laser

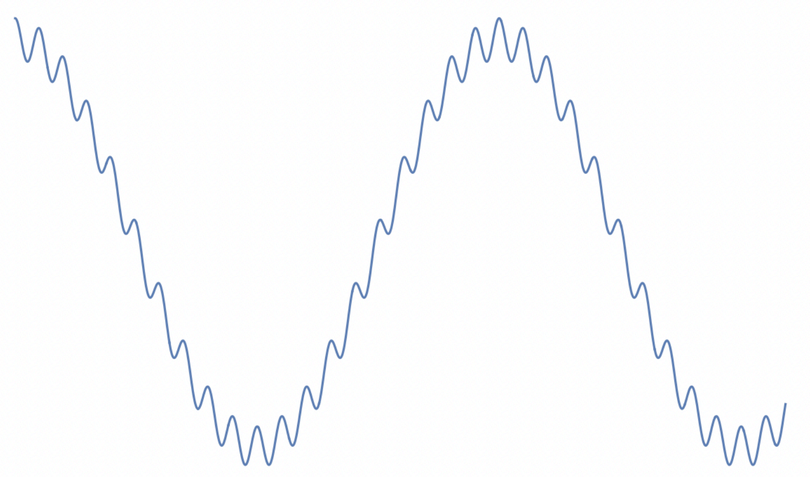
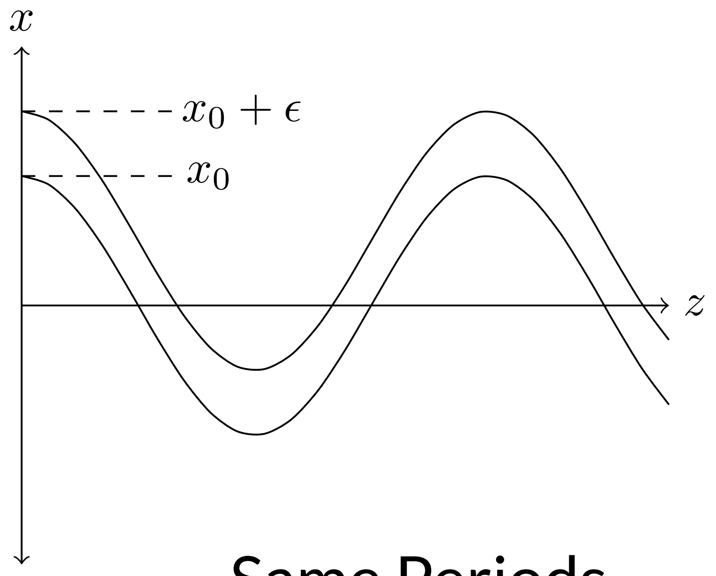
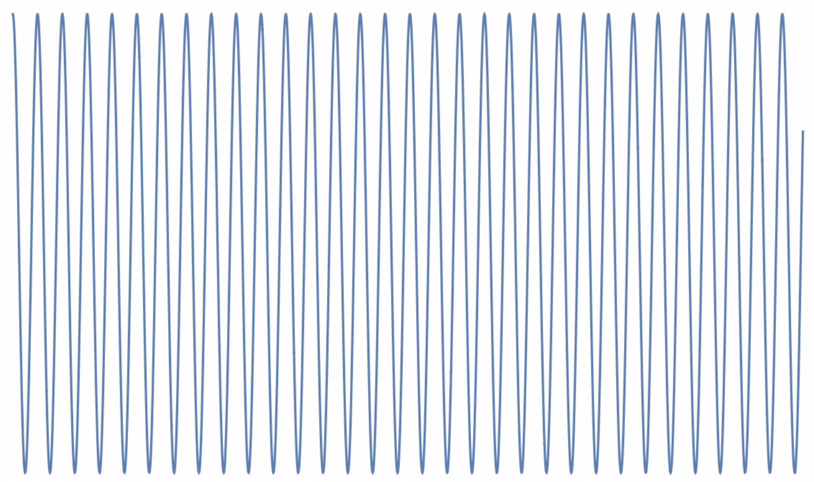
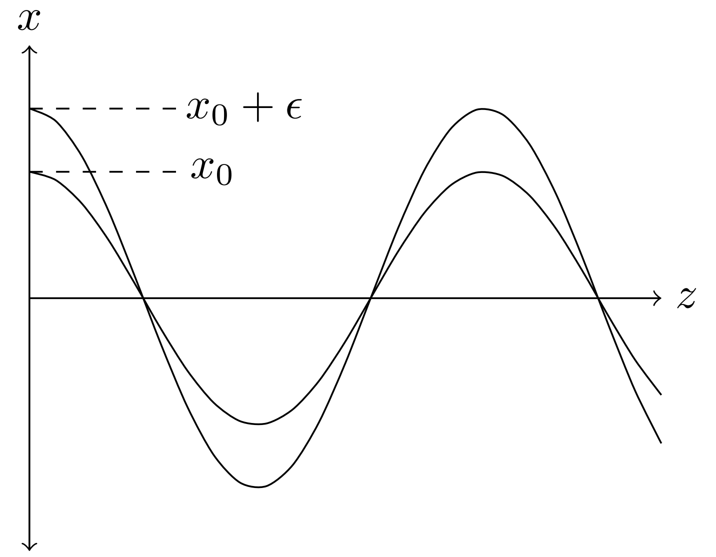
- The ICL is similar to the FEL, but uses a uniform ion channel instead of a magnetic undulator to transversely oscillate particles
- Narrow channel plasma eliminates accelerating field
- Strong ion channel focusing increases gain (ρ) and allowable energy spread by an order of magnitude, decreases gain length by the same amount.
- More stringent emittance requirements than the FEL
- **ICL physics has subtle but important differences from FEL physics**





Laser ionized plasma filament at CU Boulder. Photo: V. Lee

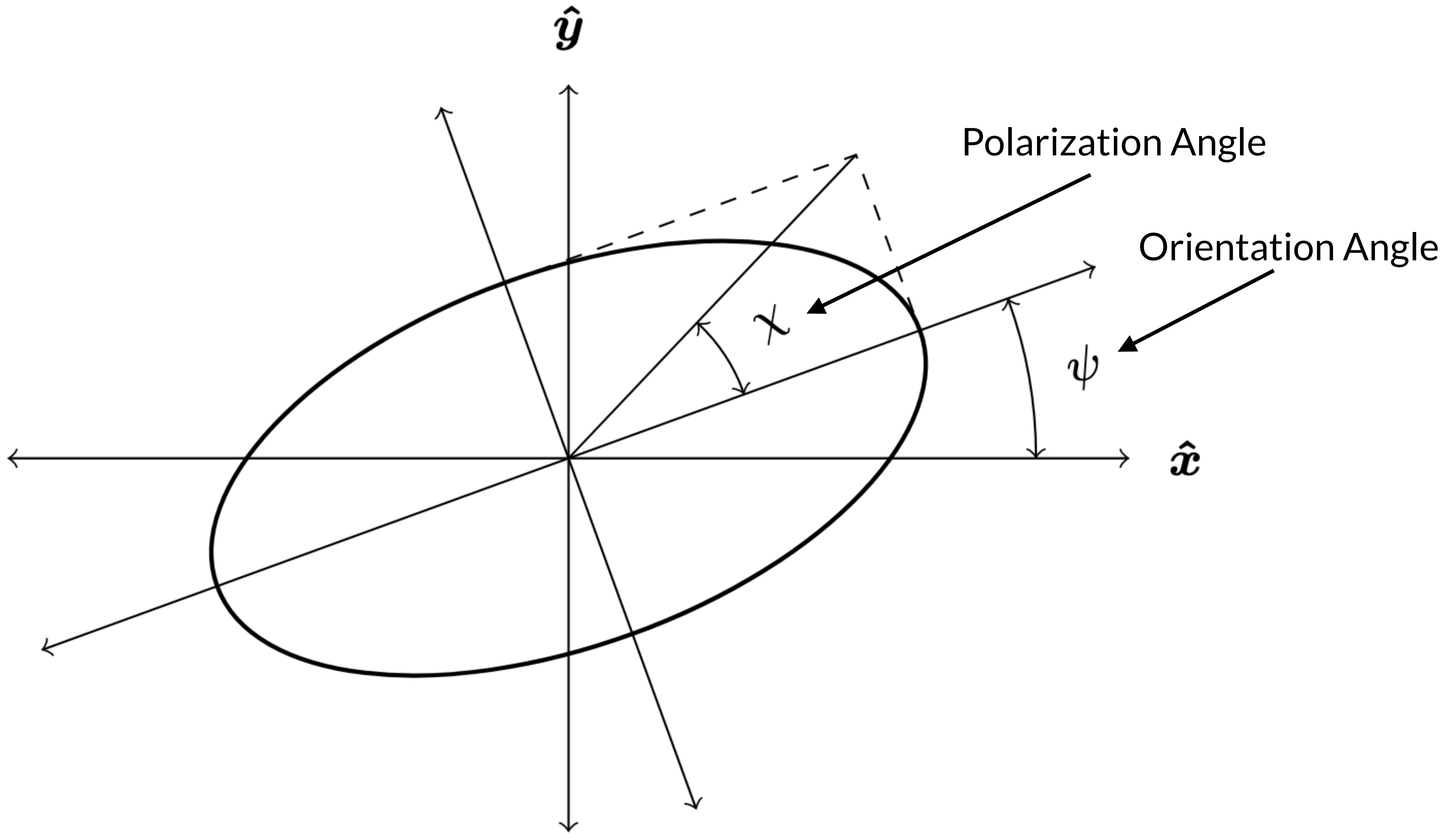
ICL vs FEL

	Field	Oscillation Period	Undulator Parameter	Emittance Constraint	Resonant Condition	Particle Motion	Effect of Offset
FEL	$\mathbf{B} = B_0 \sin(k_u z)\hat{\mathbf{y}}$	λ_u	$K = \frac{eB_0}{mck_u}$	$\epsilon_n \lesssim \frac{\gamma\lambda_1}{4\pi}\frac{\overline{\beta}}{L_{G0}}$	$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta_\beta^2\right)$	 $\mathbf{x}(z) = \mathbf{x}_u(z) + \mathbf{x}_\beta(z)$	 Same Periods Same Amplitudes Same Radiation Wavelengths
ICL	$\mathbf{E} = \frac{en_0}{2\epsilon_0}\mathbf{x}_\perp$	$\lambda_\beta = \lambda_p\sqrt{2\gamma}$	$K = \gamma k_\beta r_m$ Betatron Oscillation Amplitude	$\frac{\sigma_{r_m}}{r_m} \lesssim \frac{1 + \frac{K^2}{2}}{K^2}\rho$	$\lambda = \frac{\lambda_\beta}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$	 $\mathbf{x}(z) = \mathbf{x}_\beta(z)$	 Same Periods Different Amplitudes Different Radiation Wavelengths

Further Peculiarities

- Multiple possible ICL "configurations"; this work focuses on the off-axis configuration (see Ersfeld *et al.* (2014))
- Unlike the undulator period, the betatron period depends on γ and thus is slightly different for each particle
- Particle oscillation phase is not fixed by the field but can be different for each particle, which changes microbunching physics
- Particles transversely oscillate across the radiation mode every oscillation

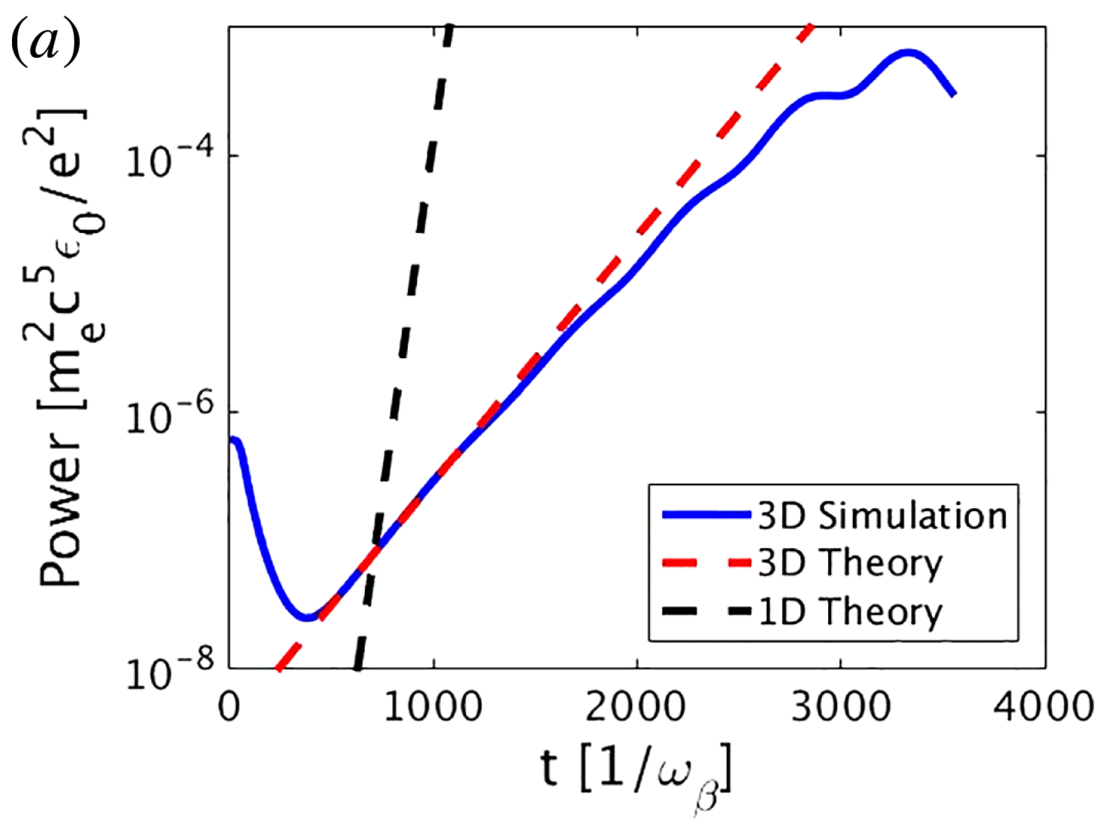
Transverse Oscillation Ellipse

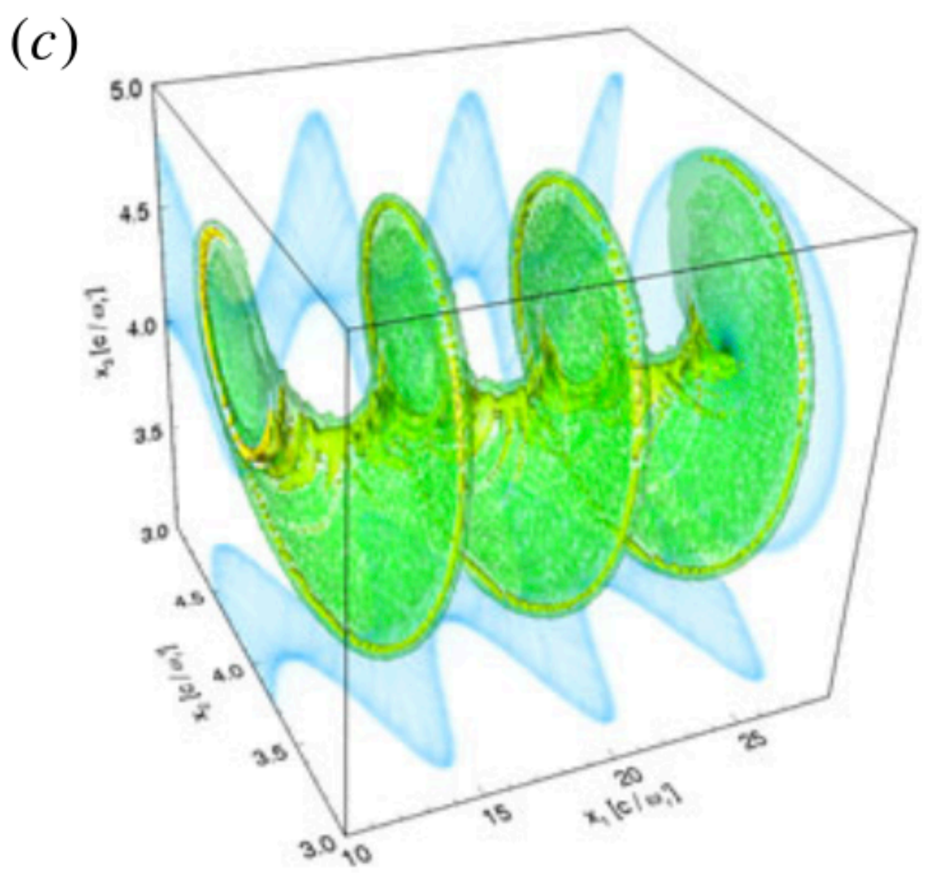


	Field	Particle Motion	Oscillation Phase ϑ	Oscillation Amplitude	Beam Size σ_{beam}
FEL	$\mathbf{B} = B_0 \sin(k_u z + \vartheta)\hat{\mathbf{y}}$	$x(z) = r_m \sin(k_u z + \vartheta)$	Fixed by field phase, same for each particle	$r_m \ll \sigma_{\text{radiation mode}}$	$\sigma_{\text{beam}} \sim \sigma_{\text{radiation mode}}$
ICL	$\mathbf{E} = \dfrac{e n_0}{2\epsilon_0}\mathbf{x}_\perp$	$x(z) = r_m \sin(k_\beta z + \vartheta)$	Independent for each particle	$r_m \sim \sigma_{\text{radiation mode}}$	$\sigma_{\text{beam}} \ll \sigma_{\text{radiation mode}}$

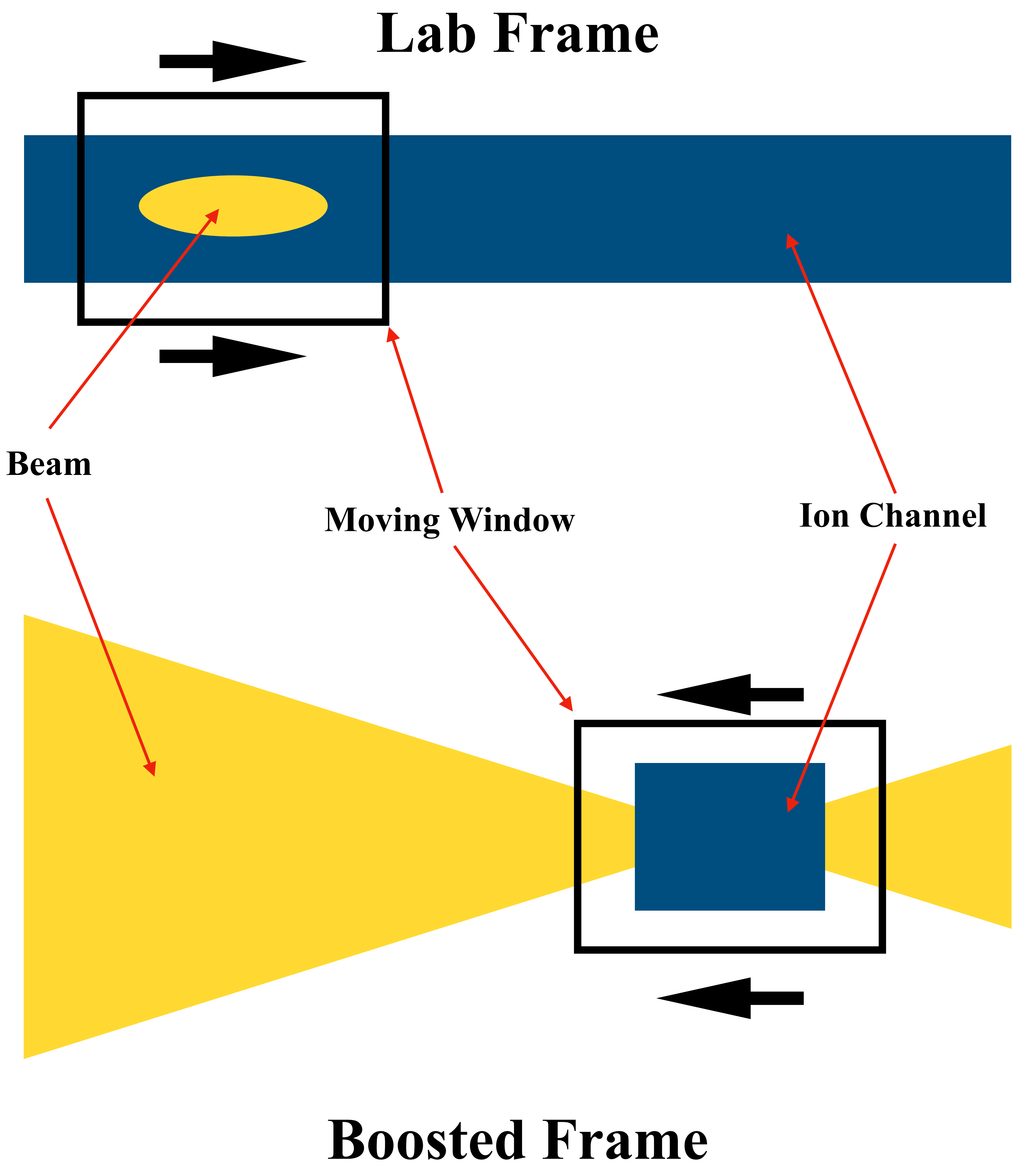
Simulation Challenges

- ICLs have extreme parameters and substantially different physics which makes simulating them using most FEL codes impossible or prohibitively computationally expensive
- The off-axis ICL is a multiscale problem in both the transverse (beam size vs oscillation amplitude) and longitudinal (beam length vs radiation wavelength) dimensions
- In general the ICL must be simulated using boosted frame PIC simulations, nobody has successfully simulated a physical ICL configuration in 3D PIC.





X. Davoine *et al.* (2018)



ICL Theory I: Slowly Varying Quantities

Equations of Motion

$$\mathcal{H} = \sum_{j=1}^{N_e} \frac{1 + (p_{c,x,j} + a(\boldsymbol{x}_{\perp,j}, \zeta_j, z))^2}{2\gamma_j} + \frac{1}{4}x_j^2$$

Turn off Fields and Solve

Single Particle Motion

$$x_j(z) = r_{m,j}\cos(k_{\beta,j}z + \varphi_j)$$
$$p_{x,j}(z) = -K_j \sin(k_{\beta,j}z + \varphi_j)$$
$$\zeta_j(z) = \zeta_{0,j} - \frac{1 + \frac{K_j^2}{2}}{2\gamma_j^2}z - \frac{K_j^2}{8\gamma_j^2 k_{\beta,j}}\sin(2\varphi_j) + \frac{K_j^2}{8\gamma_j^2 k_{\beta,j}}\sin(2(k_{\beta,j}z + \varphi_j))$$
$$\gamma_j(z) = \gamma_j$$

Construct Slowly Varying Quantities

Energy Detuning

$$\eta_j \equiv \frac{\gamma_j - \gamma_r}{\gamma_r}$$

Undulator Parameter Detuning

$$\delta_j \equiv \frac{K_j - K_r}{K_r}$$

Pondermotive Phase

$$\theta_j \equiv k_{\beta,r}z + k_{1,r}\left(\zeta_{0,j} - \frac{1 + \frac{K_j^2}{2}}{2\gamma_j^2}z - \frac{K_j^2}{8\gamma_j^2 k_{\beta,j}}\sin(2\varphi_j)\right)$$

Betatron Phase Detuning

$$\vartheta_j \equiv \varphi_j + k_{\beta,j}z - k_{\beta,r}z$$

Turn fields back on, get equations for slowly varying quantities

$$\eta_j' = \frac{K_r}{\gamma_r^2}\sin(k_{\beta,r}z + \vartheta_j)\left.\frac{\partial a}{\partial \zeta}\right|_j$$
$$\delta_j' = \frac{1 + K_r^2}{2K_r^2}\eta_j'$$
$$\theta_j' = 2k_{\beta,r}\left(\eta_j - \frac{K_r^2}{2 + K_r^2}\delta_j\right)$$
$$\vartheta_j' = -\frac{1}{2}k_{\beta,r}\eta_j$$

Where

a : normalized magnetic vector potential

$\zeta = z - t(z)$

$K_j = \gamma_j k_{\beta,j} r_{m,j}$

$k_{\beta,j} = 1/\sqrt{2\gamma}$

j subscript : jth particle in beam

r subscript : "resonant" / "reference particle" value

ICL Theory II: Period Averaging

Period Averaging of the Energy Detuning Equation

$$\eta_j' = \sum_{h\in\mathbb{N}^+}\int_{\nu\approx h} d\nu\,\frac{i\nu k_{1,r}K_r}{\gamma_r^2}e^{i\Delta\nu k_{\beta,r}z}e^{i\Delta\nu\theta_j}\overline{\mathcal{A}_h(\boldsymbol{x}_{\perp,j}(z),\nu,z)\sin(k_{\beta,r}z+\vartheta_j)e^{i\nu k_{1,r}\zeta_j(z)}} + \text{c.c.}$$

Period Averaging of the Field Equation

$$\left[\frac{\partial}{\partial z} + i\Delta\nu k_{\beta,r} - \frac{i}{2\nu k_{1,r}}\nabla_\perp^2\right]\mathcal{A}_h(\boldsymbol{x}_\perp,\nu,z) = \frac{e^{-i\Delta\nu k_{\beta,r}z}}{i\nu I_A}\sum_{j=1}^{N_e}\overline{x_j'(z)\delta^2(\boldsymbol{x}_\perp - \boldsymbol{x}_{\perp,j}(z))e^{-i\nu k_{1,r}\zeta_j}}$$

Where

$\mathcal{A}(\boldsymbol{x}_\perp,\nu,z)$: Fourier transform of slowly varying field envelope
$\nu = k_1/k_{1,r}$: radiation frequency normalized to fundamental
h : integer harmonic number
$\Delta\nu = \nu - h$: frequency detuning
$I_A = 17$ kA : Alfvén Current

Introduce particle distribution function and write Maxwell-Klimontovich Equations → Separate distribution function into background and perturbation → Perform Van Kampen normal mode expansion to obtain dispersion relation

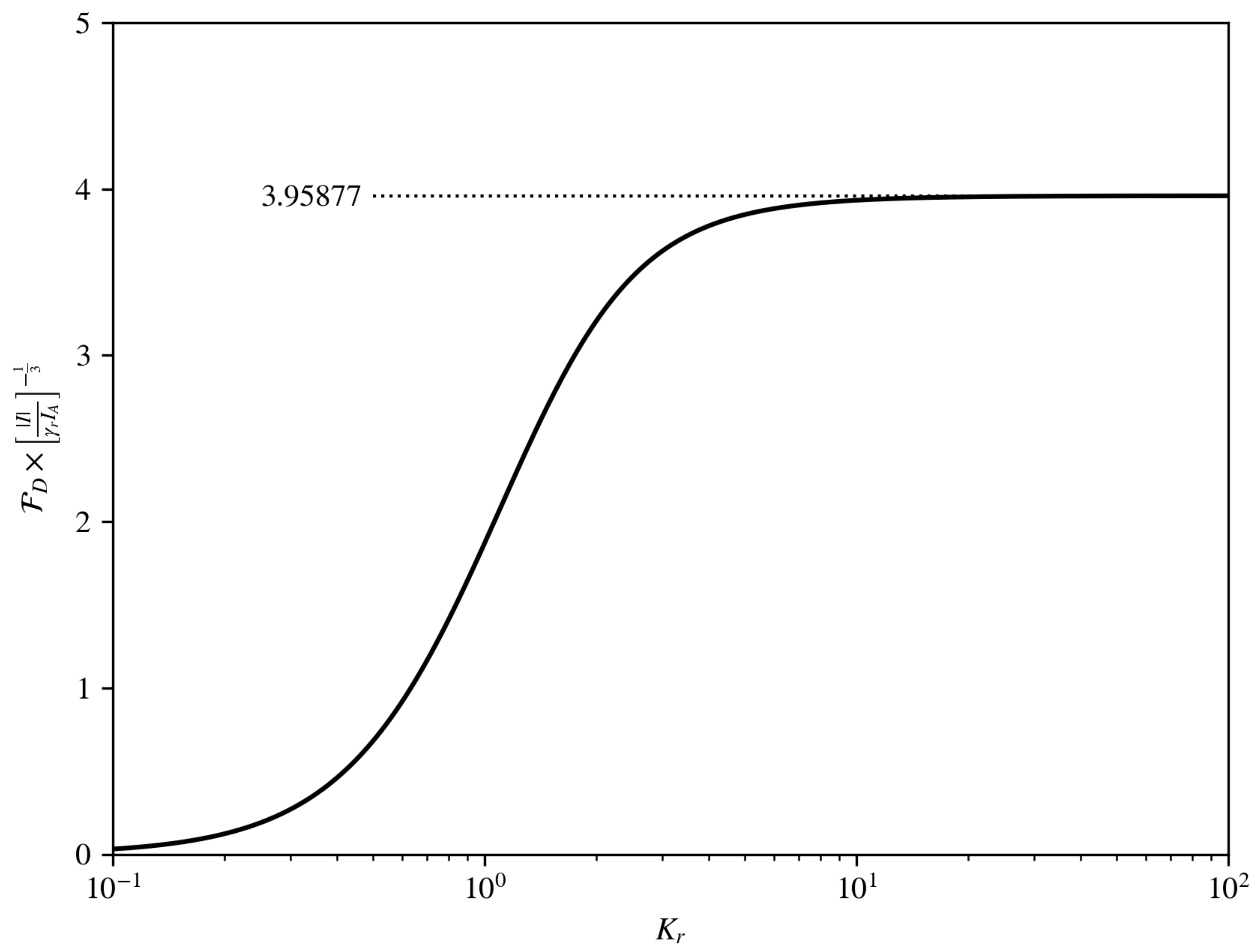
	Oscillation	Particle Motion	Period Average
FEL	$r_m \ll \sigma_{\text{radiation mode}}$	Oscillates locally, explores radiation mode over time	$\mathbf{E}(\mathbf{x}_{\perp,j}(z),z)\times\overline{\cos(k_u z)e^{ihk_1\zeta_j(z)}}$
ICL	$r_m \sim \sigma_{\text{radiation mode}}$	Oscillates across radiation mode every oscillation	$\overline{\mathbf{E}(\mathbf{x}_{\perp,j}(z),z)\cos(k_{\beta,j}z)e^{ihk_1\zeta_j(z)}}$

ICL Theory III: Dispersion Relation

Dispersion Relation

$$\left[\mu_\ell - \hat{\Delta\nu} + \mathcal{F}_D^{-1}\hat{\boldsymbol{\nabla}}_\perp^2\right]\hat{A}_\ell(\hat{\boldsymbol{x}}_\perp) - \pi\mathcal{V}(\mu_\ell)\hat{\mathcal{W}}(\hat{\boldsymbol{x}}_\perp)\int d^2\hat{\boldsymbol{x}}'_\perp \hat{\mathcal{W}}(\hat{\boldsymbol{x}}'_\perp)\hat{A}_\ell(\hat{\boldsymbol{x}}'_\perp) = 0$$

Fresnel Parameter: strength of diffraction



$$\mathcal{F}_D = 16\nu\frac{K^2}{2+K^2}\rho_0 = \frac{\sqrt{3}}{4}\frac{z_r}{L_{G,0}}$$

***V* function:** finite energy and undulator parameter spread

$$\mathcal{V}(\mu) = \frac{A}{\mu^2}I\left(\frac{\Sigma}{\mu}\right)$$

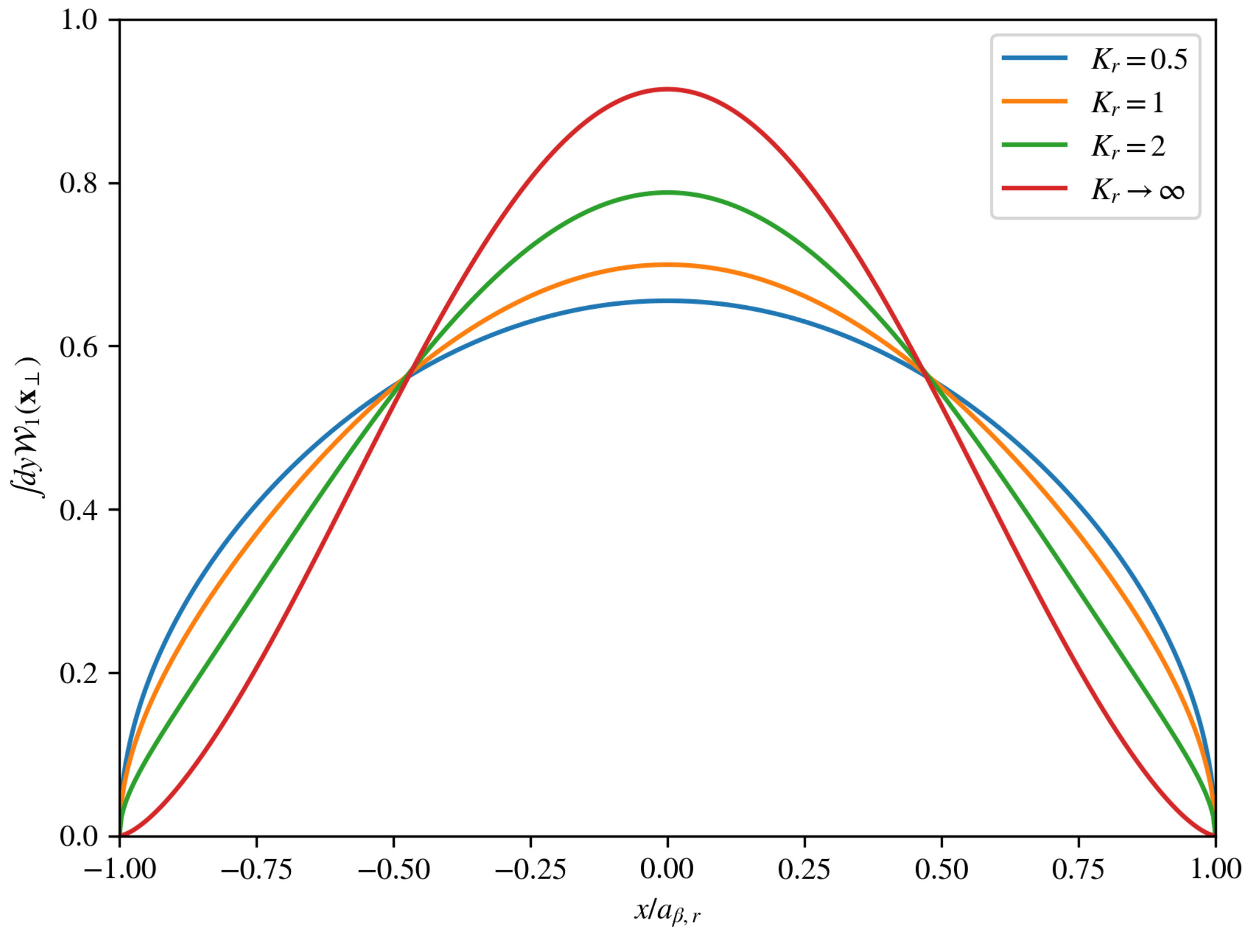
$$A = \left(1 + \frac{2(3+K_r^2)}{4+K_r^2}\Delta\nu\right)$$

$$\Sigma \equiv \sqrt{\left(\frac{3}{4}+\Delta\nu\right)^2\frac{\sigma_\eta^2}{\rho_0^2} + \frac{(1+\Delta\nu)^2 K_r^4}{(2+K_r^2)^2}\frac{\sigma_\delta^2}{\rho_0^2}}$$

$$I(y) = \frac{1}{\sqrt{2\pi}}\int dx \frac{e^{-\frac{1}{2}x^2}}{(1-xy)^2}$$

$$I(0) = 1$$

***W* function:** spacial emission/ interaction strength profile



$$\hat{\mathcal{W}}_h(\hat{\boldsymbol{x}}_\perp) = \sum_{\substack{n\in\mathbb{Z} \\ h-n \text{ odd}}} \frac{[\mathrm{JJ}]_{h-n}}{[\mathrm{JJ}]_h}\mathcal{C}_n(\hat{x})\,\delta(\hat{y})$$

$$\mathcal{C}_n(x) = \begin{cases} \frac{T_n(x)}{\pi\sqrt{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$$[\mathrm{JJ}]_n = J_{\frac{n-1}{2}}(h\xi_r) - J_{\frac{n+1}{2}}(h\xi_r)$$

$$\xi = \frac{K^2}{2(2+K^2)}$$

ICL Theory IV: Solving the ICL Equations

- We tried a number of approaches
  - ❌ ~~Variational Principle~~ → Some Results Nonphysical
  - ❌ ~~Matrix Method~~ → Extremely Slow Convergence
  - ❌ ~~P. Baxevanis *et al.* (2013) Method~~ → Quite Complicated to Implement
- So we came up with our own!

ICL Initial Value Problem Differo-Integral Equation

$$\left[\frac{\partial}{\partial \hat{z}} + i\hat{\Delta}\nu - i\mathcal{F}_D^{-1}\hat{\boldsymbol{\nabla}}_\perp^2\right]\hat{\mathcal{A}}_h(\hat{\boldsymbol{x}}_\perp,\nu,\hat{z}) = \hat{\mathcal{W}}_h(\hat{\boldsymbol{x}}_\perp)\int_0^{\hat{z}} d\hat{z}'\mathcal{X}_h(\hat{z},\hat{z}')\int d^2\hat{\boldsymbol{x}}'_\perp\hat{\mathcal{W}}_h(\hat{\boldsymbol{x}}'_\perp)\hat{\mathcal{A}}_h(\hat{\boldsymbol{x}}'_\perp,\nu,\hat{z}')$$

$$\mathcal{X}_h(\hat{z},\hat{z}') = \frac{\pi}{\mathcal{I}}\frac{[\mathrm{JJ}]_h^2}{[\mathrm{JJ}]_1^2}\iiint d\hat{\eta}\,d\hat{\delta}\,d\vartheta e^{-i\left(\left(\nu-\frac{h}{4}\right)\hat{\eta}-\nu\frac{K_r^2}{2+K_r^2}\hat{\delta}\right)(\hat{z}-\hat{z}')}\times\left[\frac{\partial}{\partial\hat{\eta}}+\frac{1+K_r^2}{2K_r^2}\frac{\partial}{\partial\hat{\delta}}\right]\hat{f}_0(\hat{\eta},\hat{\delta},\vartheta,\hat{z}')$$

This source term turns out to be much more efficient to compute than its FEL counterpart.

Crank Nicholson

Matrix Equation

$$\frac{\mathcal{B}_{i,j}^{n+1}-\mathcal{B}_{i,j}^{n}}{d\hat{z}} - \frac{i\mathcal{F}_D^{-1}}{2}\left[\frac{1}{d\hat{x}^2}\left(\mathcal{B}_{i-1,j}^{n}-2\mathcal{B}_{i,j}^{n}+\mathcal{B}_{i+1,j}^{n}+\mathcal{B}_{i-1,j}^{n+1}-2\mathcal{B}_{i,j}^{n+1}+\mathcal{B}_{i+1,j}^{n+1}\right)\right.$$

$$\left.+\frac{1}{d\hat{y}^2}\left(\mathcal{B}_{i,j-1}^{n}-2\mathcal{B}_{i,j}^{n}+\mathcal{B}_{i,j+1}^{n}+\mathcal{B}_{i,j-1}^{n+1}-2\mathcal{B}_{i,j}^{n+1}+\mathcal{B}_{i,j+1}^{n+1}\right)\right] = \mathcal{W}_{i,j}d\hat{z}\sum_{m=0}^{n-1}\Gamma^{n,m}d\hat{x}d\hat{y}\sum_{p,q}\mathcal{W}_{p,q}\mathcal{B}_{p,q}^{m}$$

ICL Theory V: Alternating Direction Implicit Method

Simplify and Rewrite

Matrix Equation

$$(1 - i\mu_x \boldsymbol{\Delta}_x - i\mu_y \boldsymbol{\Delta}_y)\boldsymbol{\mathcal{B}}^{n+1} = (1 + i\mu_x \boldsymbol{\Delta}_x + i\mu_y \boldsymbol{\Delta}_y)\boldsymbol{\mathcal{B}}^n + \boldsymbol{\mathcal{C}}^n$$

$$(\boldsymbol{\Delta}_x)_{i,j,i',j'} \equiv (\delta_{i-1,i'} - 2\delta_{i,i'} + \delta_{i+1,i'})\delta_{j,j'}$$

$$(\boldsymbol{\Delta}_y)_{i,j,i',j'} \equiv \delta_{i,i'}(\delta_{j-1,j'} - 2\delta_{j,j'} + \delta_{j+1,j'})$$

$$(\boldsymbol{\mathcal{C}}^n)_{i,j} \equiv \delta_{j,j_0} \mathcal{W}_{x,i} \frac{d\hat{z}d\hat{x}}{d\hat{y}} \sum_{m=0}^{n-1} \Gamma^{n,m} \left(\sum_p \hat{\mathcal{W}}_{x,p} \mathcal{B}^m_{p,j_0} \right)$$

These would be tridiagonal in 1D, but aren't in 2D

$$\mu_x = d\hat{z}/(2\mathcal{F}_D d\hat{x}^2)$$

$$\mu_y = d\hat{z}/(2\mathcal{F}_D d\hat{y}^2).$$

Approximate Matrix Equation:
First order is the same, but second order terms differ

$$(1 - i\mu_x \boldsymbol{\Delta}_x)(1 - i\mu_y \boldsymbol{\Delta}_y)\boldsymbol{\mathcal{B}}^{n+1}$$
$$= (1 + i\mu_x \boldsymbol{\Delta}_x)(1 + i\mu_y \boldsymbol{\Delta}_y)\boldsymbol{\mathcal{B}}^n + (1 + i\mu_x \boldsymbol{\Delta}_x)(1 - i\mu_x \boldsymbol{\Delta}_x)\boldsymbol{\mathcal{C}}^n$$

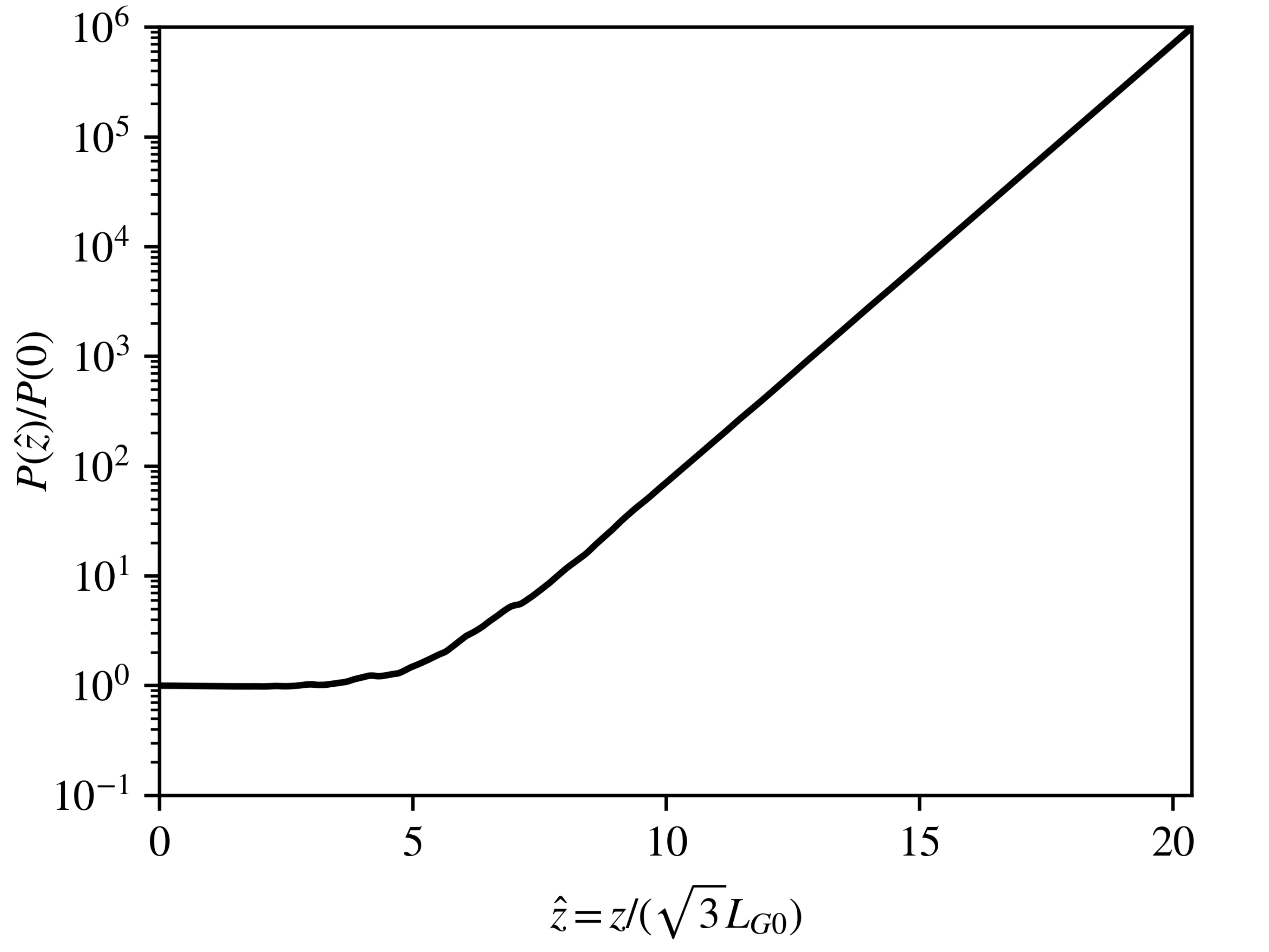
Introduce Intermediate Step:
Now we have two tridiagonal Matrix Problems

$$(1 - i\mu_x \boldsymbol{\Delta}_x)\boldsymbol{\mathcal{B}}^* = (1 + i\mu_y \boldsymbol{\Delta}_y)\boldsymbol{\mathcal{B}}^n$$
$$(1 - i\mu_y \boldsymbol{\Delta}_y)\boldsymbol{\mathcal{B}}^{n+1} = (1 + i\mu_x \boldsymbol{\Delta}_x)(\boldsymbol{\mathcal{B}}^* + \boldsymbol{\mathcal{C}}^n).$$

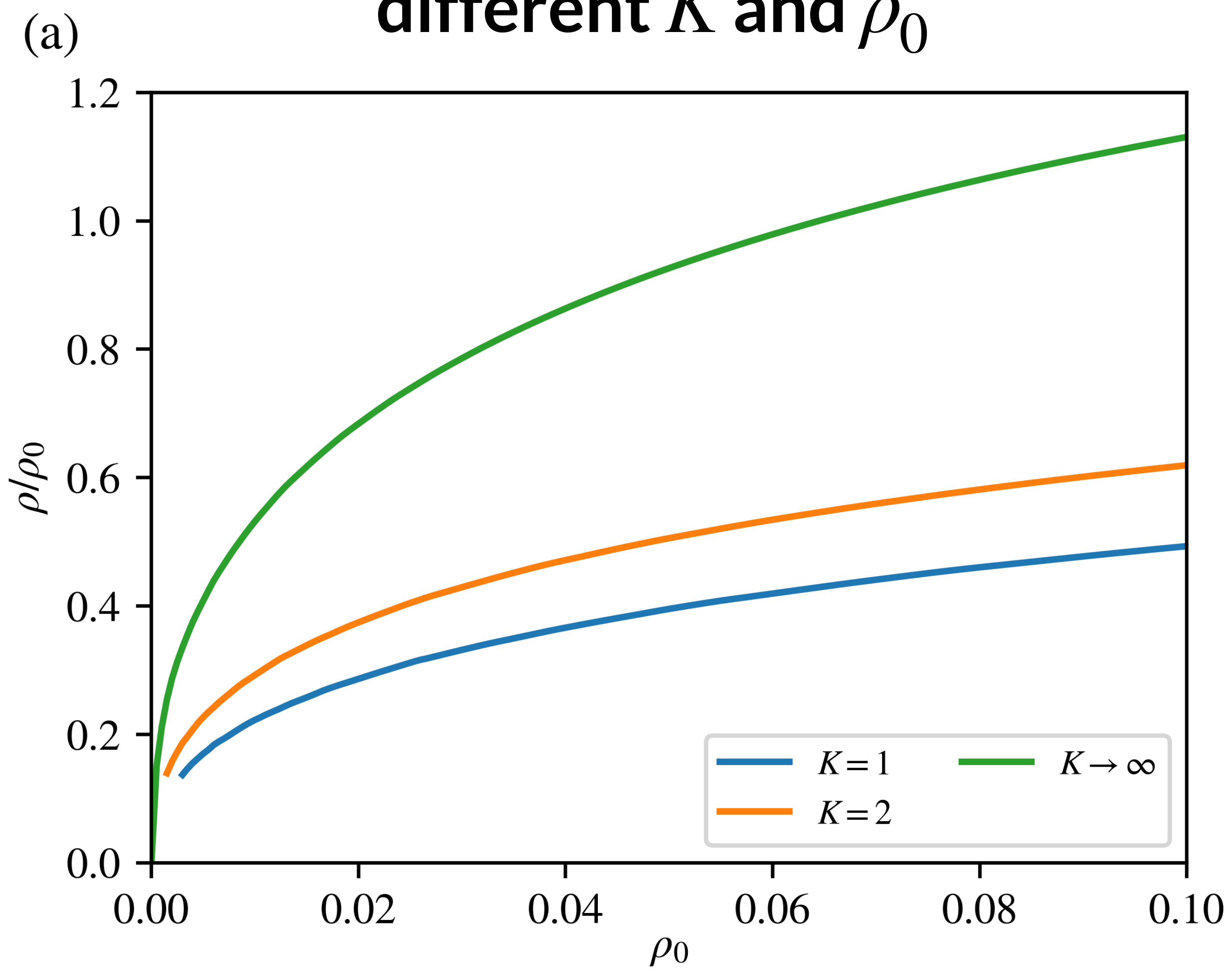
- We used the Alternating Direction Implicit (ADI) Method to simplify the numerical solution of the ICL equations.
- Instead of solving one general linear matrix equation [O(N^3)], we solve two tridiagonal linear matrix equation [O(N)].

Results

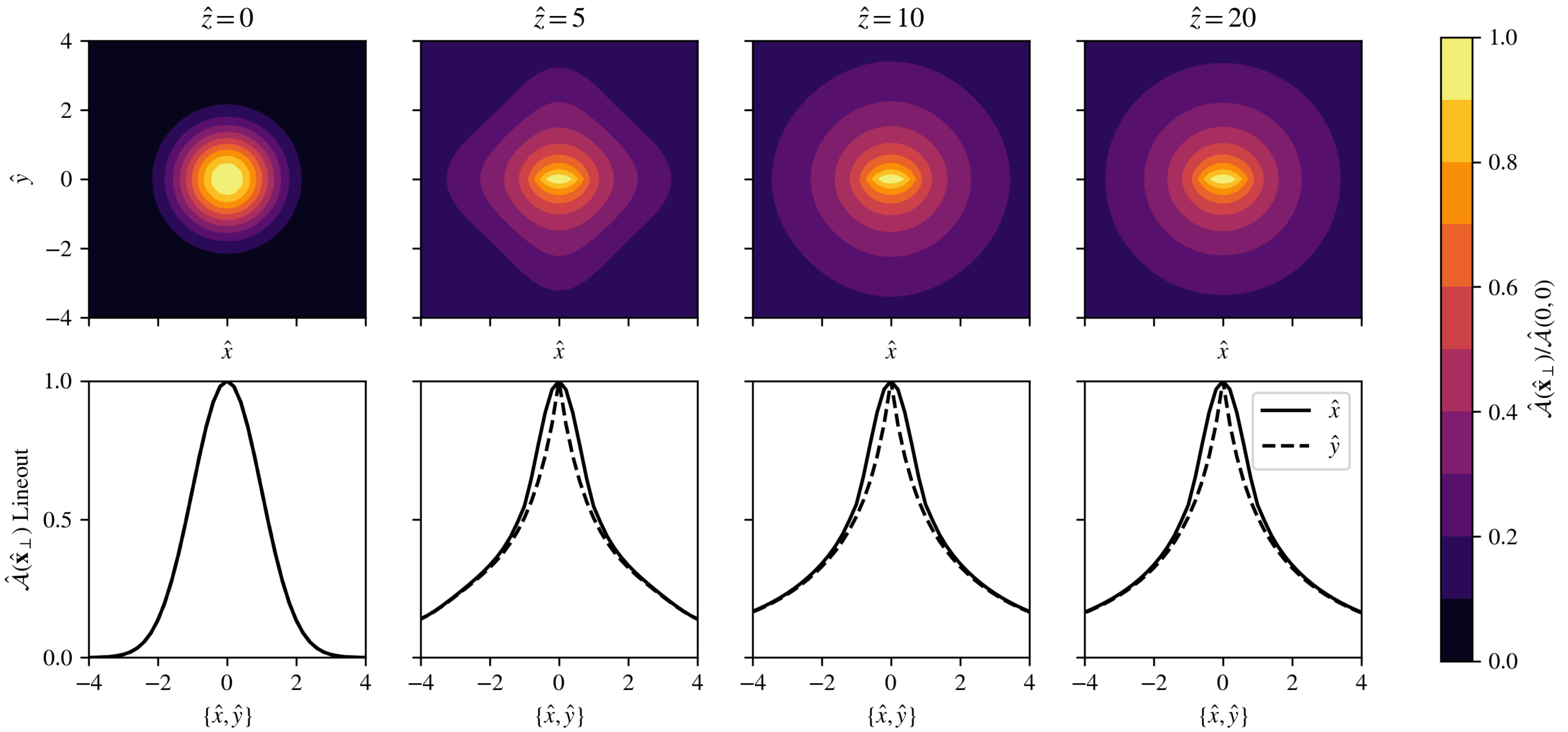
Exponential gain of ICL Radiation Power



Gain reduction due to 3D effects for different K and ρ_0

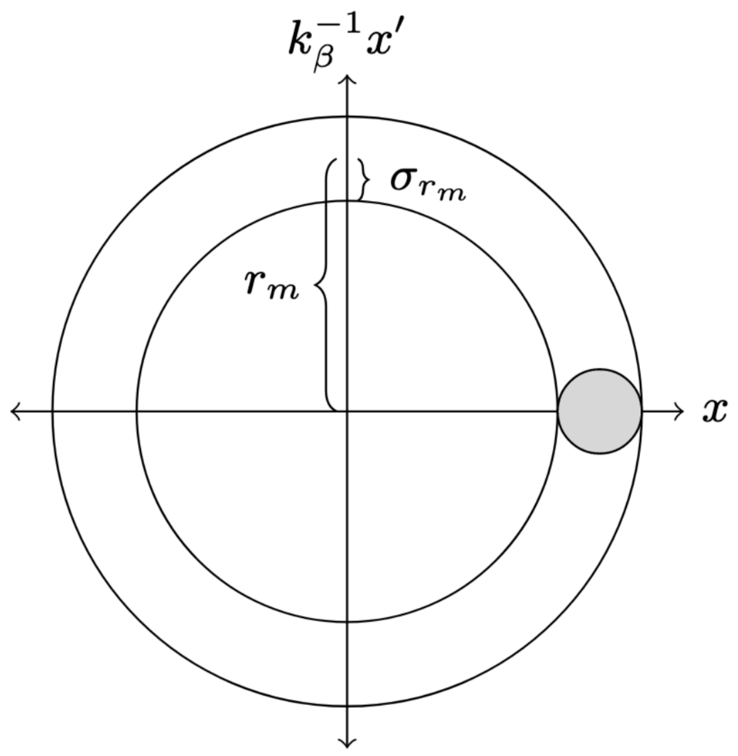
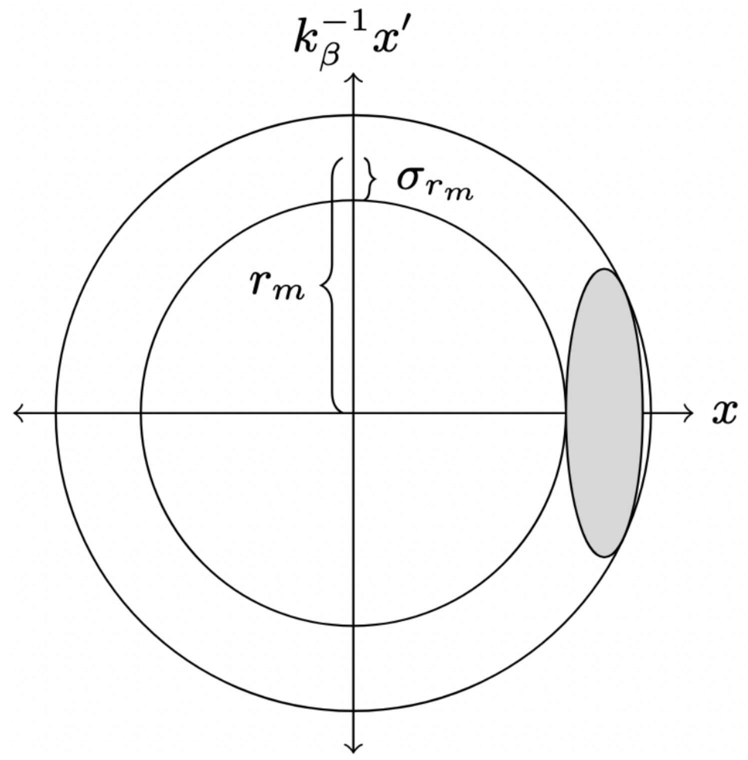
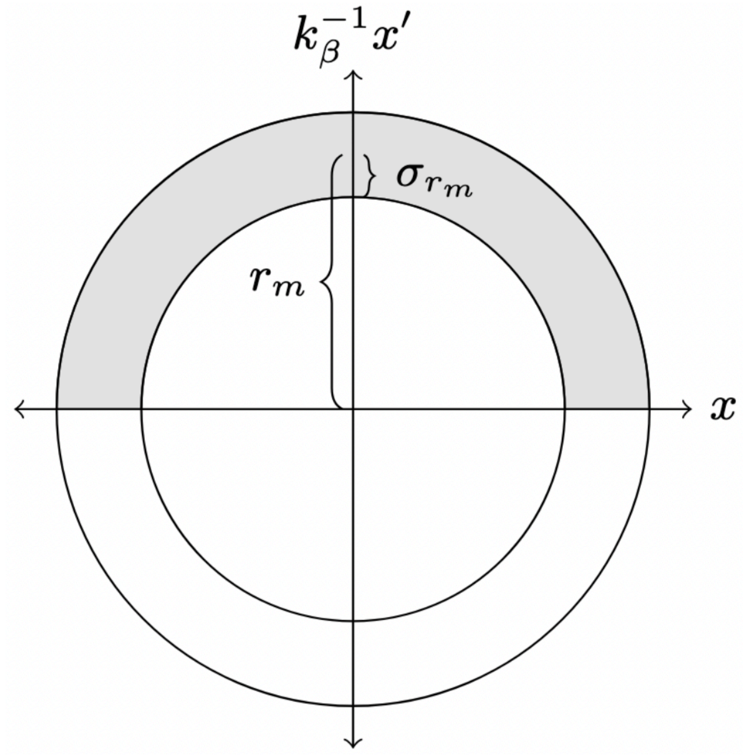


Formation of Asymmetric Guided Radiation Mode from Gaussian Seed Radiation Pulse



ICL Emittance Constraints

- $$\frac{\sigma_{r_m}}{r_m} \lesssim \frac{1+\frac{K^2}{2}}{K^2}\rho$$
- Not an *emittance* constraint but a *transverse oscillation amplitude spread* constraint
- Phase space manipulation can somewhat soften the emittance requirement
- Still, **ICLs have extremely stringent emittance requirements**

Type	FEL	Matched ICL	Optimally Overfocused ICL	"Half-Annulus" ICL
Phase Space				
Emittance Constraint	$\frac{\epsilon_n}{\gamma\lambda} < \frac{1}{4\pi}$	$\epsilon_{n,x}, \epsilon_{n,y} \lesssim \frac{\gamma\lambda_1}{\pi}\frac{1+\frac{K^2}{2}}{K^2}\rho^2$	$\epsilon_{n,x} \lesssim \frac{\gamma\lambda_1}{\pi}\left(\frac{2}{5}\right)^{\frac{3}{4}}\left(\frac{1+\frac{K^2}{2}}{K^2}\right)^{\frac{1}{2}}\rho^{\frac{3}{2}}$ $\epsilon_{n,y} \lesssim \frac{\gamma\lambda_1}{\pi}\sqrt{\frac{2}{5}}\rho$	$\epsilon_{n,x} \lesssim \frac{\gamma\lambda_1}{\pi}\sqrt{6}\rho$ $\epsilon_{n,y} \lesssim \frac{\gamma\lambda_1}{\pi}\frac{1}{\sqrt{2}}\rho$
λ = 10 nm, E = 3 GeV I = 20 kA, K = 10.89	$\epsilon_n \lesssim 4.7\mu m \times 4.7\mu m$	$\epsilon_n \lesssim 0.32\text{nm} \times 0.32\text{nm}$	$\epsilon_n \lesssim 3.0\text{nm} \times 67\text{nm}$	$\epsilon_n \lesssim 20\text{nm} \times 77\text{nm}$

Summary & Future Work

- **Paper uploaded to ArXiv**!
- We developed a 3D theory of the ion channel laser
  - Accounts for diffraction, transverse guided mode shape, frequency and Betatron phase detuning, and nonzero spread in energy and undulator parameter
  - Obtained a dispersion relation and initial value problem differo-integral equation
  - Developed an efficient algorithm to calculate growth rates and transverse guided mode profiles.
- Ongoing work to simulate the ICL using boosted frame PIC
- E306 experiment at FACET-II aims to lase at optical wavelengths in an ICL (See Litos *et al.* (2018))

Questions?

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