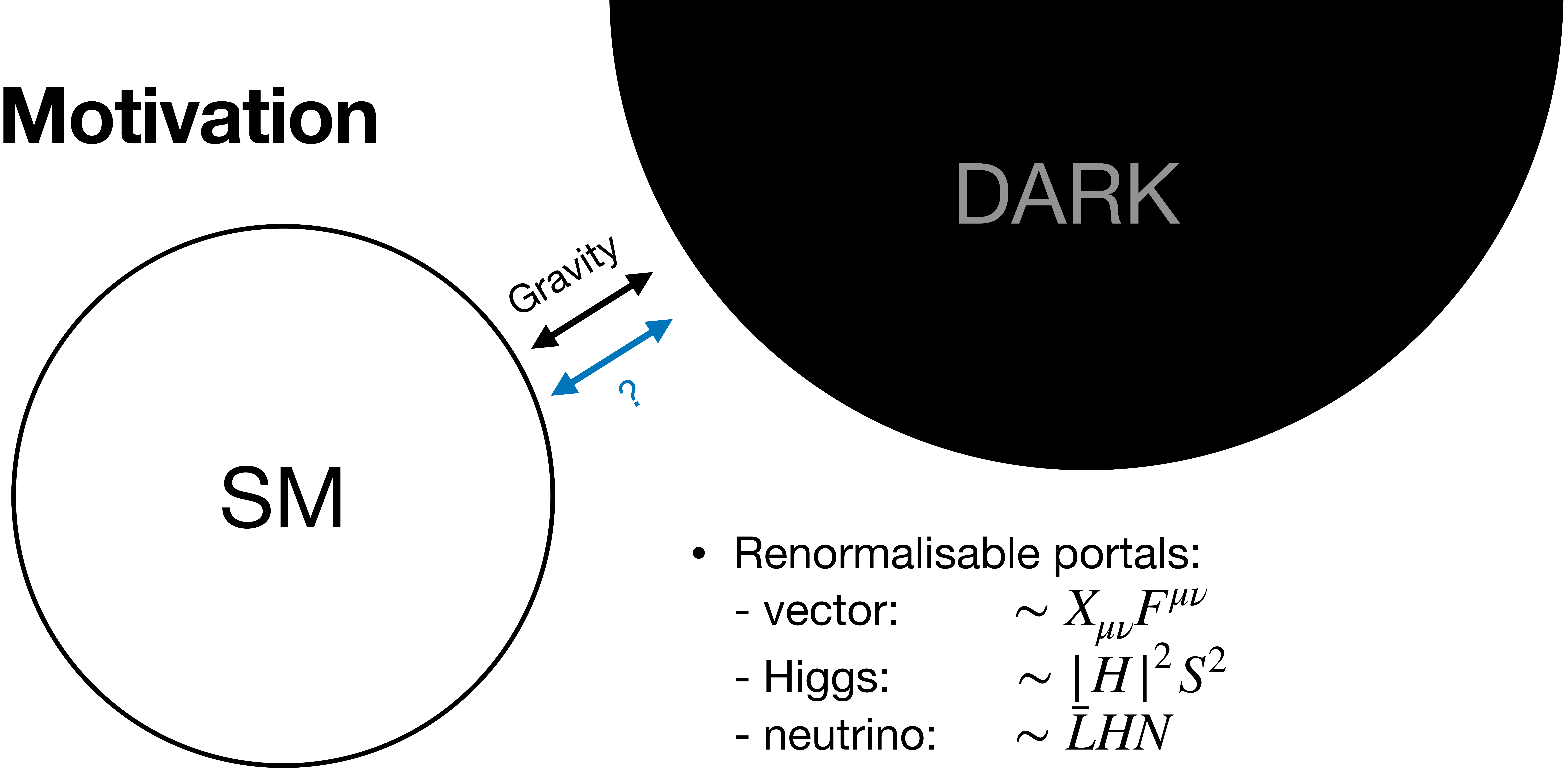
3rd General Meeting of COST Action COSMIC WISPers (CA21106)
Sofia 09.09.2025.

Topological Portals to the Dark Sector

Nudžeim Selimović, INFN Padova

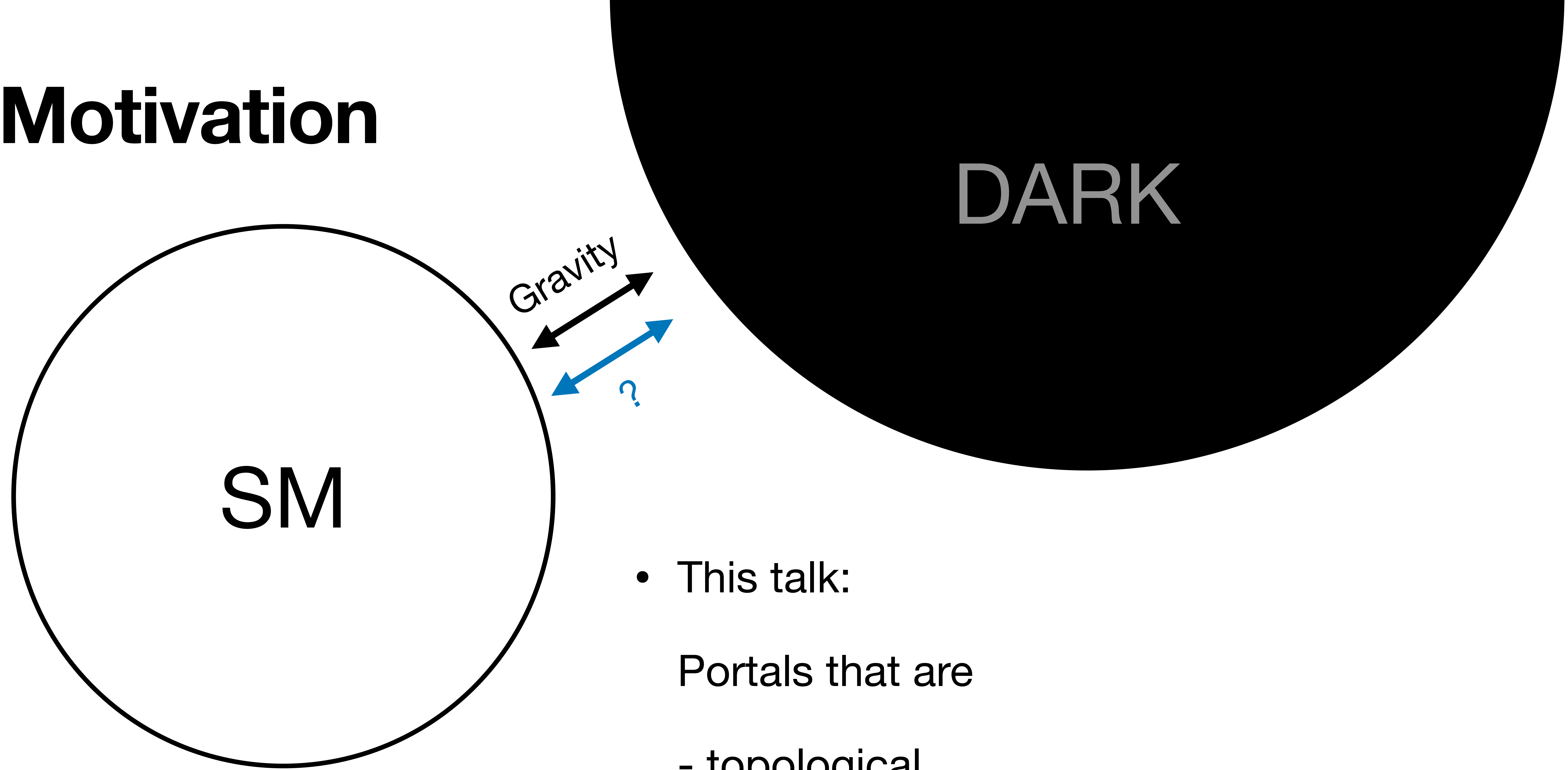
Motivation



- Renormalisable portals:
  - vector: $\sim X_{\mu\nu}F^{\mu\nu}$
  - Higgs: $\sim |H|^2 S^2$
  - neutrino: $\sim \bar{L}HN$

- Non-renormalisable portals:
  - ALPs: $\sim a\tilde{F}_{\mu\nu}F^{\mu\nu}$ ...

Motivation



- This talk:

  Portals that are

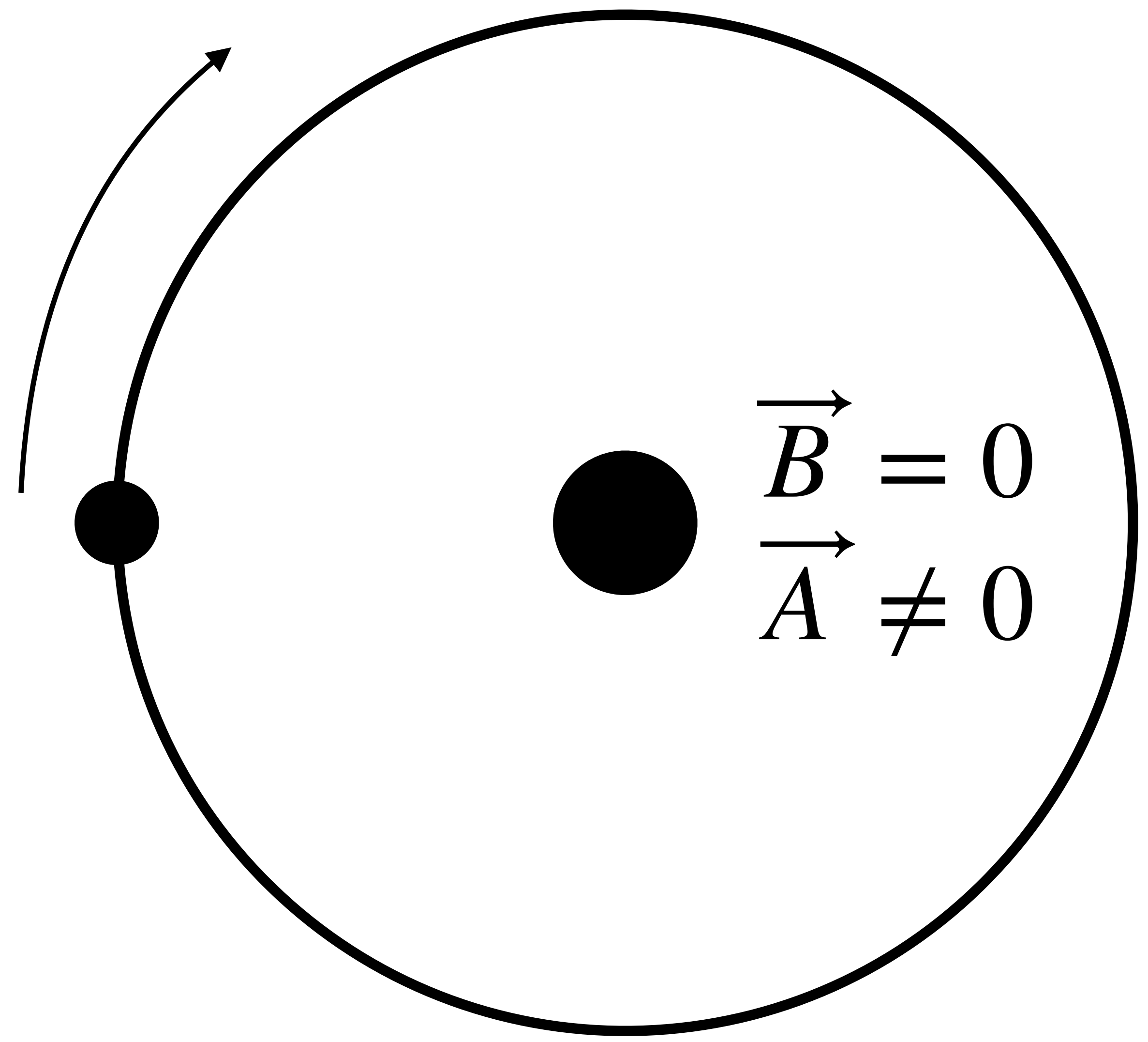
  - topological
  - non-renormalisable

Topological actions

Topological terms

- *Class 1: Theta terms*
- Example: Aharonov-Bohm effect

Particle on a circle: $q(t)$



$$I = \underbrace{\int \mathcal{D}q(t)}_{\text{only one path}} \exp(iS) = \mathcal{N} \exp\left(ieA \int dq\right) = \mathcal{N} \exp\left(i\theta \int dq\right)$$

differential form*

theta-coefficient: $\mathbb{R}$ - valued.

Common source of topological actions:

$$S_{\text{top}} \sim \int_{\Sigma} (\text{diff. form})$$

Terms not involving the spacetime metric.

* *Differential forms are mathematical objects - smooth antisymmetric tensors - ready to be integrated over curves, surfaces, volumes... E.g. a field strength tensor $F = F_{\mu\nu} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}$ is a 2-form.*

Topological terms

- *Class 2: Wess-Zumino-Witten terms*

- Example: Low-energy QCD

$$\left.\begin{array}{c} G = SU(3)_L \times SU(3)_R \\ \downarrow \\ H = SU(3)_{L+R} \end{array}\right\} \quad U(x) = e^{\frac{2i}{f_\pi}\pi(x)^a T^a} : \Sigma_4 \to X = \frac{SU(3)_L \times SU(3)_R}{SU(3)_{L+R}} \simeq SU(3)$$

Chiral perturbation theory:

$$\mathscr{L} = \frac{f_\pi^2}{4} \operatorname{tr}\left(D_\mu U^\dagger D^\mu U\right) + \mathcal{O}(D_\mu^4)$$

Weinberg '68, CCWZ '69

Symmetries:

$P_0 : \vec{x} \to -\vec{x}$

$(-1)^{N_\pi} : U \to U^\dagger \implies \pi^a \to -\pi^a$

- CCWZ terms are invariant under both
- QCD preserves only $P = P_0(-1)^{N_\pi}$
- Are there terms missing?

Witten '83

Topological terms

- *Class 2: Wess-Zumino-Witten terms*

- Example: Low-energy QCD

Are there terms missing? Can one construct Lagrangian that is P_0 - odd?

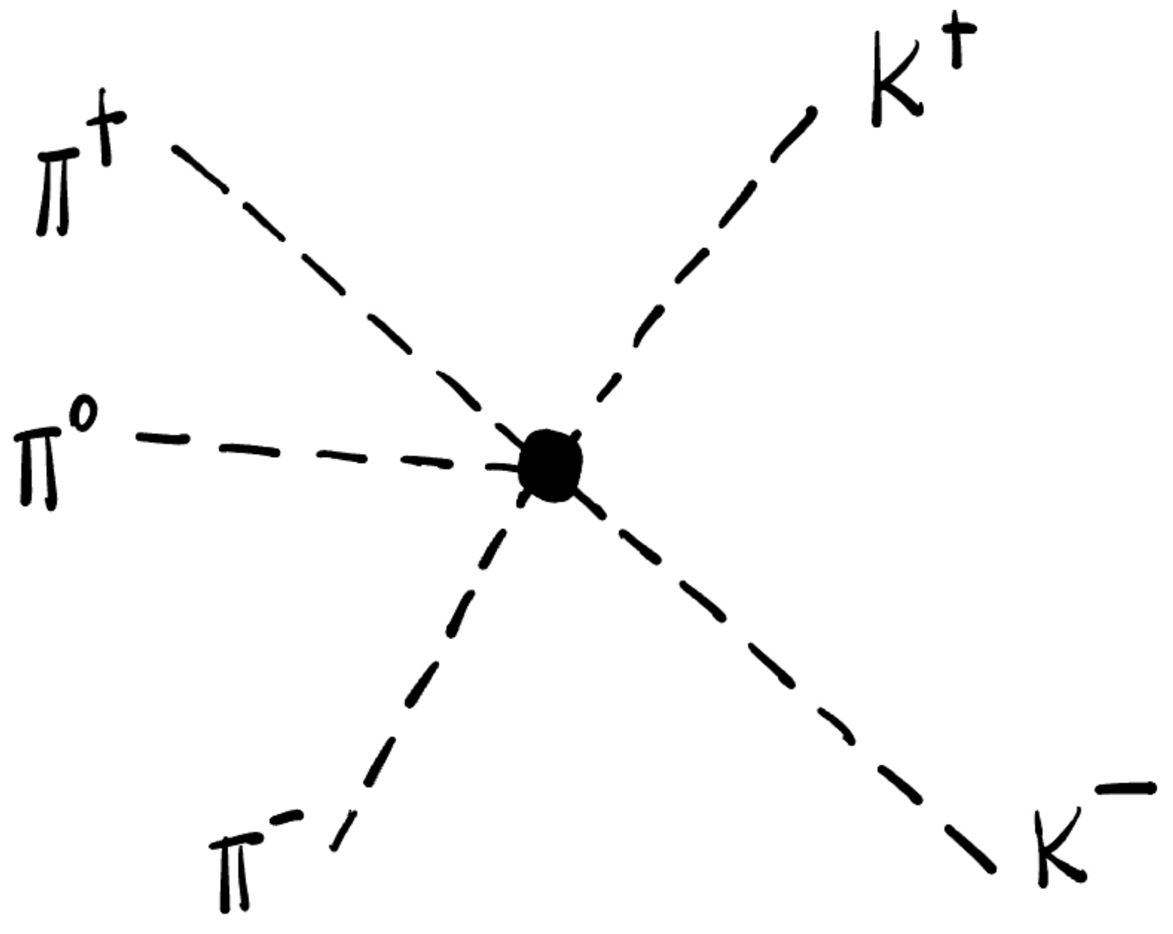
$$\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(U^\dagger\,(\partial_\mu U)U^\dagger\,(\partial_\nu U)U^\dagger\,(\partial_\rho U)U^\dagger\,(\partial_\sigma U)\right) = 0 \quad ?$$

However, equation of motion can be written: Witten '83

$$\frac{1}{2}f_\pi^2\,\partial_\mu(U^\dagger\partial^\mu U) = \frac{k}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\,U^\dagger\,(\partial_\mu U)U^\dagger\,(\partial_\nu U)U^\dagger\,(\partial_\rho U)U^\dagger\,(\partial_\sigma U)$$

Topological terms

- *Class 2: Wess-Zumino-Witten terms*



- Example: Low-energy QCD

The solution was to go to the higher dimension:

To write the action that gives the correct EOM, Witten noted the existence a 5-form:

$$\left.\begin{aligned}\omega_5 &= -\frac{1}{480\pi^3}\mathrm{Tr}\left[\left(U^{-1}\mathrm{d}U\right)^5\right] \\ &= -\frac{1}{480\pi^3}\,\mathrm{d}x^5\,\epsilon^{\mu\nu\rho\sigma\tau}\,\mathrm{Tr}\left[U^\dagger\left(\partial_\mu U\right)U^\dagger\left(\partial_\nu U\right)U^\dagger\left(\partial_\rho U\right)U^\dagger\left(\partial_\sigma U\right)U^\dagger\left(\partial_\tau U\right)\right]\end{aligned}\right\}$$

$$\boxed{S_{\mathrm{WZW}} = ik\int_{\Sigma_5}\omega_5}$$

Wess-Zumino '71
Witten '83

- Expanding locally:

$$U^\dagger\partial_\mu U = \frac{2i}{f_\pi}\partial_\mu\pi + \mathcal{O}(\pi^2)$$

WZW-coefficient: $\mathbb{Z}$ - valued

- Results in new interactions: $S_{\mathrm{WZW}} = \dfrac{2k}{45\pi^2 f_\pi^5}\displaystyle\int_{\Sigma_4=\partial\Sigma_5}\mathrm{d}^4x\,\epsilon^{\nu\rho\sigma\tau}\,\mathrm{Tr}\left[\pi\left(\partial_\nu\pi\right)\left(\partial_\rho\pi\right)\left(\partial_\sigma\pi\right)\left(\partial_\tau\pi\right)\right] + \mathcal{O}(\pi^6)$

QCD: $k = N_c = 3$

Topological terms

- *Summary:*

Common source of topological terms: $S_{\text{top}} \sim \int_\Sigma (\text{diff . form})$

Theta terms

- Obtained from closed d-forms, for d-dimensional theories.
- $\mathbb{R}$-valued coefficients.
- Do not affect classical EOMs.
- No perturbative effects.

WZW terms

- Obtained from closed d+1-forms, for d-dimensional theories.
- $\mathbb{Z}$-valued coefficients.
- Affect classical EOMs.
- Seen as perturbative effects in QFT.

Our topological portals/terms are of the WZW type.

Construction of the portal

Joe Davighi, Admir Greljo, Nudžeim Selimović Phys.Rev.Lett. 134 (2025)

Invariant forms in QCD

$$\left.\begin{array}{c} G = SU(3)_L \times SU(3)_R \\ \downarrow \\ H = SU(3)_{L+R} \end{array}\right\} \quad U(x) = e^{\frac{2i}{f_\pi}\pi(x)^a T^a} \; : \Sigma_4 \to X = \frac{SU(3)_L \times SU(3)_R}{SU(3)_{L+R}} \simeq SU(3)$$

There are *only* two G-invariant (closed) forms on X.

$$\omega_5 \sim \mathrm{Tr}\,(U^{-1}dU)^5 \qquad \omega_3 \sim \mathrm{Tr}\,(U^{-1}dU)^3$$

WZW term

What is this?

Invariant forms in QCD

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There are *only* two G-invariant (closed) forms on X.

$$\omega_3 \sim \mathrm{Tr}\,(U^{-1}dU)^3$$

What is this?

Appears as a charge of the topologically conserved current:

$$B^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,(U^\dagger(\partial_\nu U)\,U^\dagger(\partial_\rho U)\,U^\dagger\,\partial_\sigma U) \quad \Big| \quad \partial_\mu B^\mu = 0 \quad \Big| \quad B = \int_{\Sigma_3}\omega_3 = \int d^3x\, B^0$$

Invariant 3-form

$$\omega_3 \sim \mathrm{Tr}\,(U^{-1}dU)^3$$

- What else?

Can be used to construct:

$$\omega_5^{\mathrm{portal}} \sim \omega_3^{\mathrm{QCD}} \wedge \omega_2^{\mathrm{D}}$$

The main idea.

- ω_2^D : invariant 2-form on dark coset
- ω_3^{QCD}: already provided by QCD
- Which dark cosets provide ω_2^D? ⟶ Unique* dark coset $\Longrightarrow$ unique portal!

QCD × Dark Topological Portal

- Collective non-linear sigma model on a product coset:

$$X = \frac{SU(3)_L \times SU(3)_R \times G_{\rm D}}{SU(3)_{L+R} \times H_{\rm D}} \simeq SU(3) \times \frac{G_{\rm D}}{H_{\rm D}}$$

$$\omega_5^{\rm portal} \sim \omega_3^{\rm QCD} \wedge \omega_2^{\rm D} \quad \text{on} \quad X = \frac{SU(3)_L \times SU(3)_R \times G_{\rm D}}{SU(3)_{L+R} \times H_{\rm D}} \simeq SU(3) \times \frac{G_{\rm D}}{H_{\rm D}}$$

- If dynamical assumption is the chiral symmetry breaking in the dark sector:

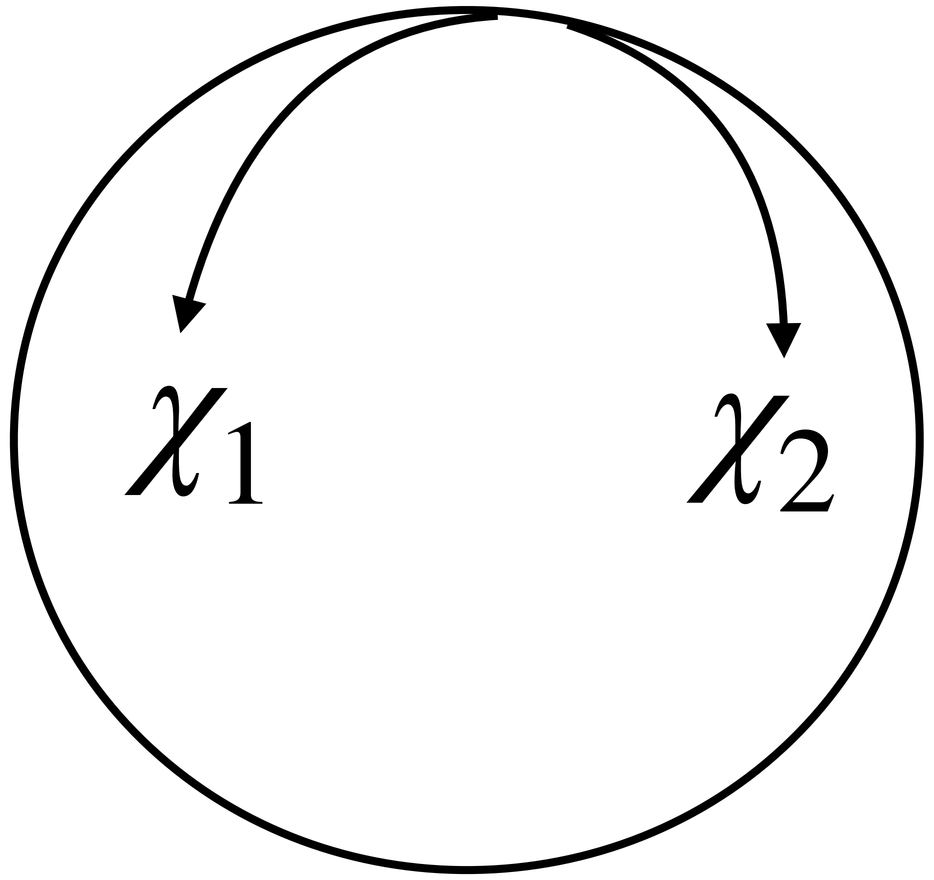
uniquely ➡ $G_{\rm D}/H_{\rm D} = SU(2)/SO(2) \simeq S^2$!

C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, Trans. Am. Math. Soc. 63 (1948) 85–124

H. Cartan, D´emonstration homologique des th´eoremes de p´eriodicit´e de bott, ii. homologie et cohomologie des groupes classiques et de leurs espaces homogenes, S´eminaire Henri Cartan 12 (1959) 1–32.

J. Davighi and B. Gripaios, *Homological classification of topological terms in sigma models on homogeneous spaces*, JHEP 09 (2018) 155

QCD $\times$ Dark Topological Portal



- So $G_D/H_D = SU(2)/SO(2) \simeq S^2$ → a sphere.

$$\omega_2^{\mathrm{D}} = \frac{1}{4\pi f_D^2}\epsilon_{ij}\,\mathrm{d}\chi_i\,\mathrm{d}\chi_j$$ → a volume form, with χ_1/f_D and χ_2/f_D coordinates

$$\omega_{\text{portal}} = \frac{n}{96\pi^3 f_\pi^3 f_D^2} f_{abc}\epsilon_{ij} d\pi_a d\pi_b d\pi_c d\chi_i d\chi_j$$

- Using the Stoke's theorem:

$$\mathcal{L}_{\text{portal}}^{e=0} = \frac{in\epsilon^{\mu\nu\rho\sigma}}{48\pi^2 f_\pi^3 f_D^2} f_{abc}\epsilon_{ij}\pi_a\partial_\mu\pi_b\partial_\nu\pi_c\partial_\rho\chi_i\partial_\sigma\chi_j$$

Topological Portal

- Main phenomenology after gauging (same as for $\pi^0 \to \gamma\gamma$)

- Prescription (Yonekura: General anomaly matching by Goldstone bosons, JHEP 03 (2021) 057):

$$\frac{1}{f_\pi^2}\partial_\mu \pi^+ \partial_\nu \pi^- \to e F_{\mu\nu}$$

$$\boxed{\tilde{\mathcal{L}}_{\rm portal} - \mathcal{L}^{e=0}_{\rm portal} = \frac{n e\, \epsilon^{\mu\nu\rho\sigma}}{16\pi^2 f_\pi f_D^2}\left(\pi^0 + \frac{\eta}{\sqrt{3}}\right) F_{\mu\nu}\partial_\rho \chi_1 \partial_\sigma \chi_2}$$

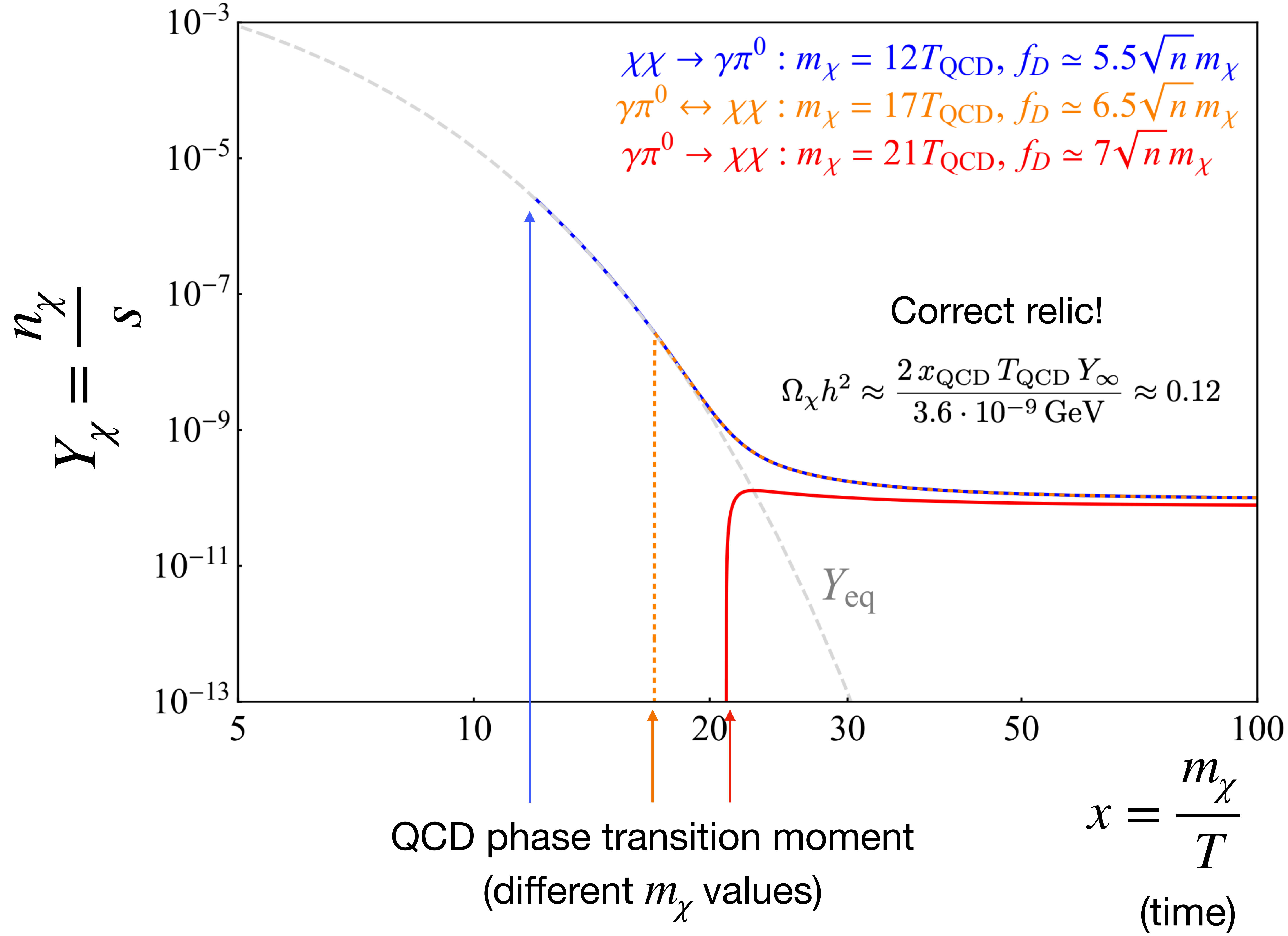
- The leading term in the EFT power counting, the next one being:

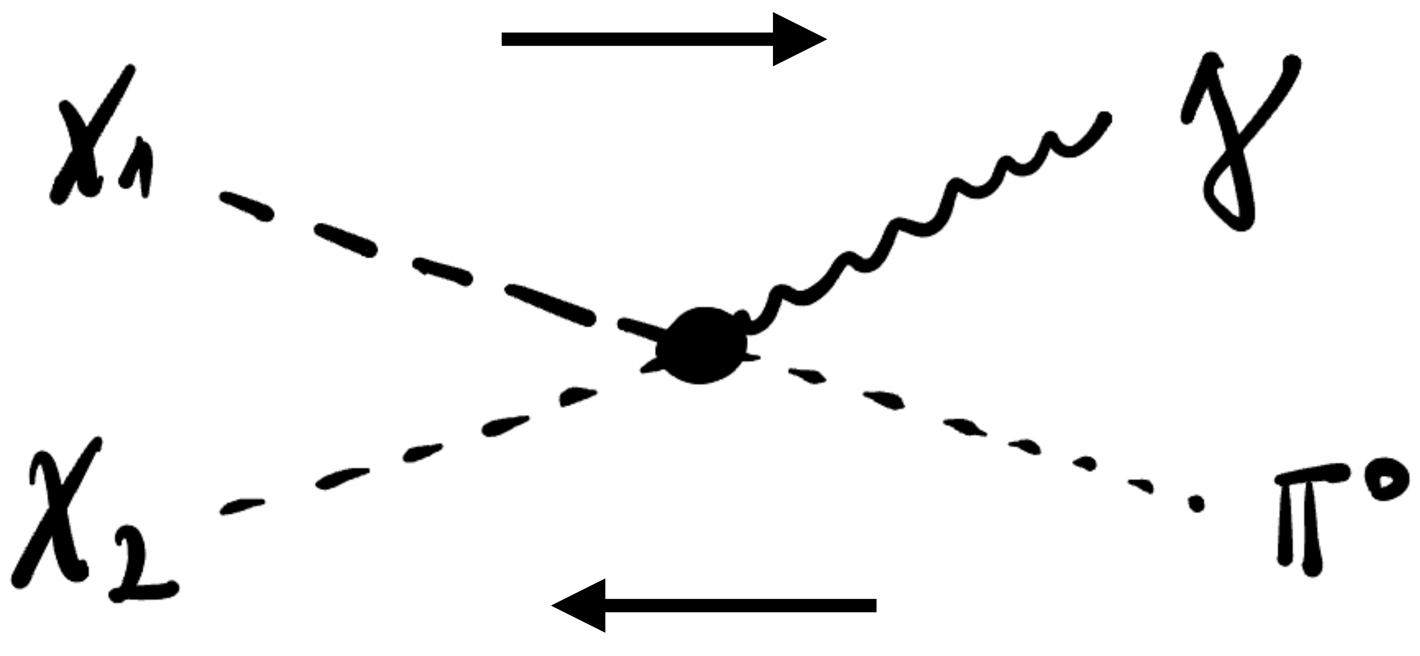
$$\frac{1}{f_\pi^2 f_D^2}(D_\mu \pi_a D^\mu \pi^a)(\partial_\nu \chi_i \partial^\nu \chi^i)$$

Phenomenology

Relic Abundance

Can be robustly set.





- *No dependence* on the cosmological history of the dark pions.
- Everything is fixed by m_χ and f_D .
- QCD phase transition should happen *no later* than $x = x_{\rm max} \approx 23$:

$$\boxed{m_\chi \lesssim 3.7\ {\rm GeV}}$$

$$f_D \sim \mathcal{O}(5-7)\sqrt{n}\, m_\chi$$

- Dark pions can be light thermal DM, and the mass range is such that pion-EFT description is reasonable!

How to test this?

Indirect/Direct detection.

- Topological operator (a differential form) couples two *different* dark pions:

  1. *Indirect detection*

  Annihilation of $\chi_1\chi_1$ highly suppressed.
  No late-time DM annihilation.

  Light thermal inelastic DM scenario

  David Tucker-Smith, Neal Weiner,
  Inelastic dark matter, *Phys.Rev.D* 64 (2001) 043502

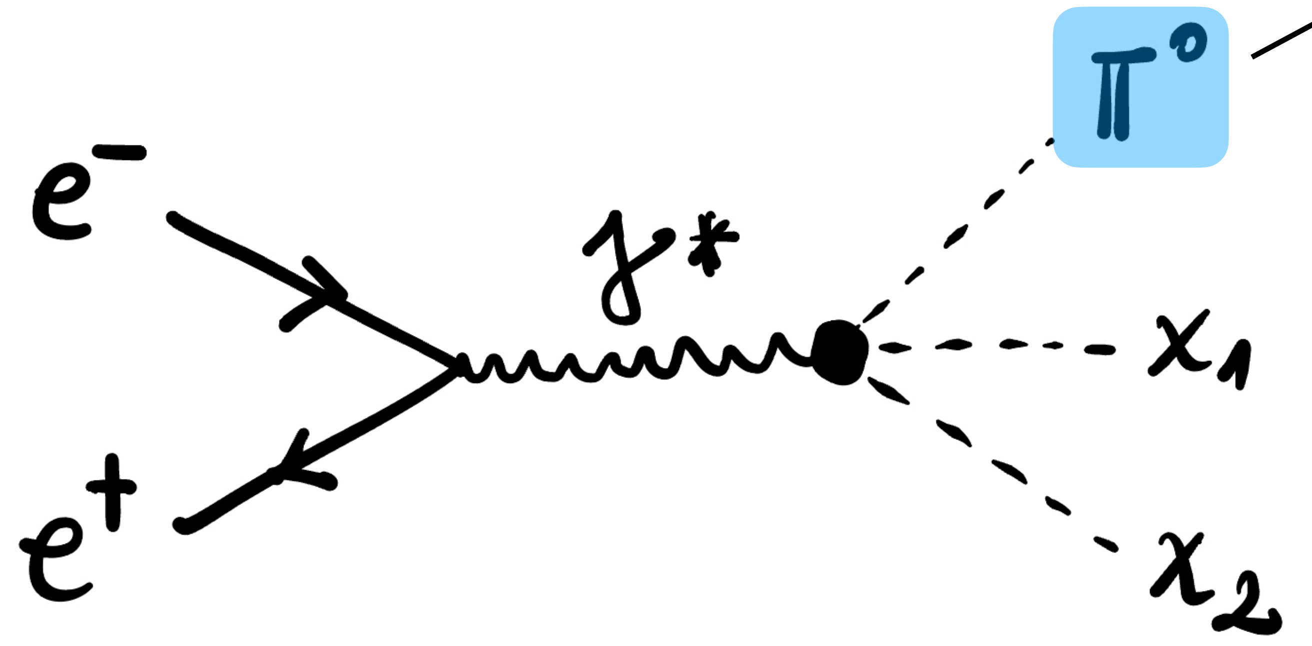
  2. *Direct detection*

  As we need $\Delta m_\chi > m_{\pi^0}$ for consistent BBN:
  No sufficient energy for χ_1 to up-scatter: $\chi_1 \to \chi_2$, huge suppression $(\partial/f_D)^3$ for $v \ll 1$.
  Direct detection cross-section effectively zero.

How to test this?

Novel collider signatures.

- *Belle II* could tell us a lot.



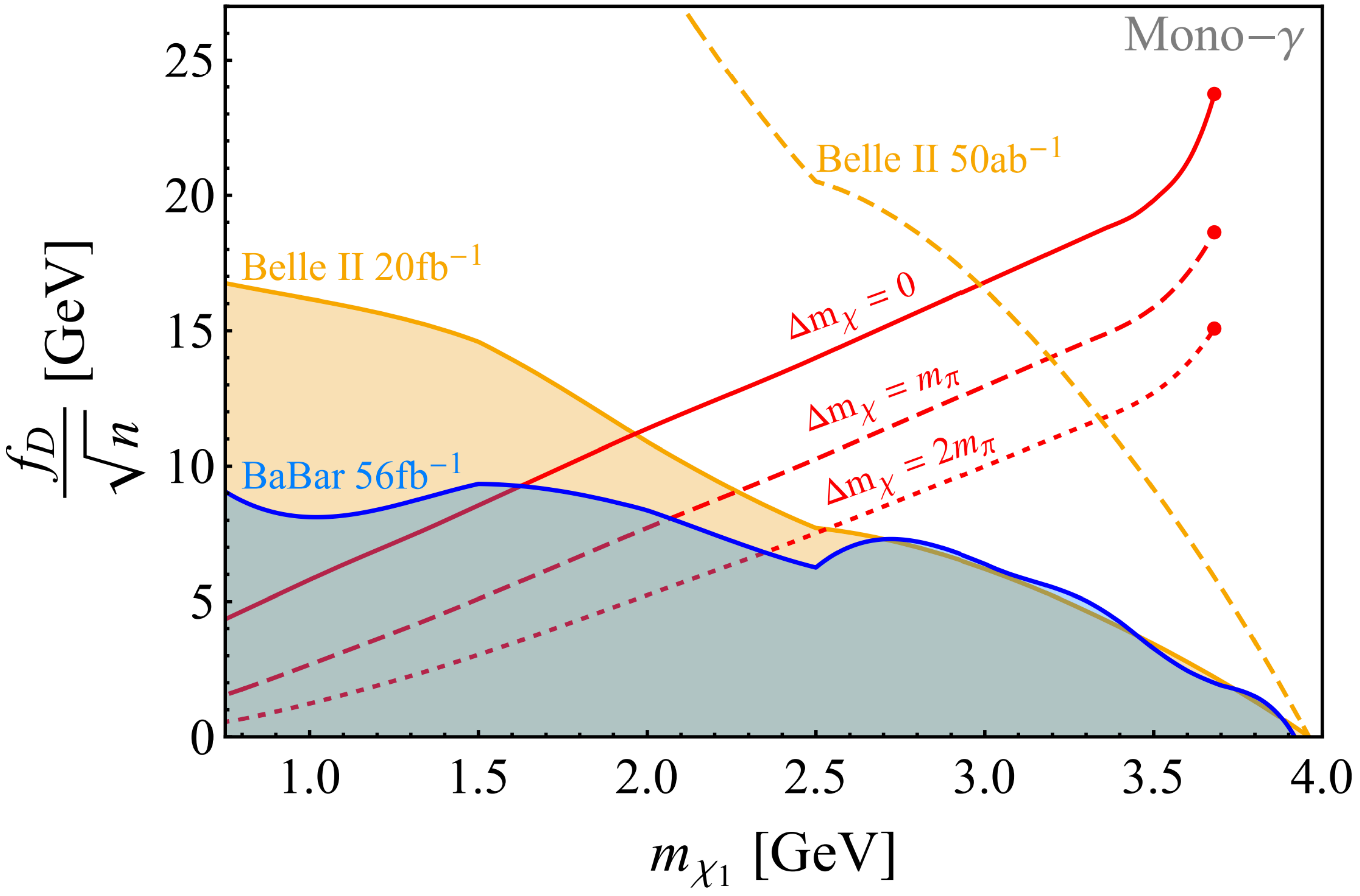
- Two regimes:
  1. χ_2 long lived (detector stable $c\tau > 10$ m)
  2. χ_2 "medium" lived (displaced vertex)

Δm_χ	$\lesssim 1.7 m_{\pi^0}$	$\gtrsim 1.7 m_{\pi^0}$
Signature	$\pi^0 + \not{E}_T$	$\pi^0 + \not{E}_T + \mathrm{DV}(\pi^0\gamma\not{E}_T)$

Belle II is working on this now: work with Christopher Hearty and Guorui Lui

Belle II Physics Briefing Book 1808.10567

Boosted to a few GeV, reconstructed as a single photon at Belle II: Recast of the mono-γ proposed search.



(long-lived χ_2)

UV completion?

Joe Davighi and Nakarin Lohitsiri: SciPost Phys. 17 (2024) 168

Joe Davighi, Admir Greljo, Nudžeim Selimović: work in progress

A puzzle

Topological term in QCD ($\omega_5 \sim \mathrm{Tr}\left(U^{-1}dU\right)^5$) matches all t'Hooft anomalies related to gauging any subgroup of $SU(3)_L$ and $SU(3)_R$: tells us about underlying quark content.

Here, however, there is no mixed anomaly between $SU(3)_{L/R}$ and $SU(2)_D$:

$$\mathrm{Tr}(F_{SU(3)}\, F_{SU(3)}\, f_{SU(2)}) = 0$$

regardless of the fermion content.

A hint from a generalised symmetry

Pointed out in: *WZW terms without anomalies: generalised symmetries in chiral Lagrangians, 2407.20340 SciPost Phys.* 17 (2024) 168, Joe Davighi and Nakarin Lohitsiri

There is an additional 1-form symmetry related to a conserved 2-form ω_2:

$$d\omega_2 = 0 \qquad\qquad \partial^\mu j^{(2)}_{\mu\nu} \sim \partial^\mu(\partial_\mu \chi^i \partial_\nu \chi^j) = 0$$

Moreover, the topological portal mixes 0-form (global QCD) and 1-form symmetry in a *2-group structure.*

E.g. this can be captured, following the tradition of current algebra, by a Ward identity of the form:

$$\langle i\partial^\mu j^{(1)a}_\mu(x) j^{(1)b}_\nu(y)\rangle = \langle -\delta(x-y) f^{abc} j^{(1)c}_\nu + \frac{n}{8\pi^2}\delta^{ab}\partial^\mu \delta(x-y) j^{(2)}_{\mu\nu}(y)\rangle\,, \quad n \in \mathbb{Z}$$

A hint from a generalised symmetry

Crucially, 2-group structure is integer quantised (Cordova, Dumitrescu, Intriligator, 1802.04790): preserved along RG flow.

The topological portal provides the IR with a non-vanishing 2-group algebra:
Same must be true for the UV!

Study UV completions that at least have 1-form symmetries, Abelian theories, even weakly coupled.

Work in progress with Joe and Admir.

An example

Work in progress with Joe and Admir.

UV

1. UV: SM x $U(1)_{X=B-3L_\tau}$ + $\phi_{1,2} \sim (\mathbf{1},\mathbf{1},0,x_\phi)$

$$\mathcal{L} = \mathcal{L}_{\rm SM} - \frac{1}{4e_X^2} X_{\mu\nu}X^{\mu\nu} + J^\mu_{X,\rm SM} X_\mu + \sum_{a=1}^{2} |D_\mu \phi_a|^2 - V\left(\sum_a |\phi_a|^2\right)$$

$$J^\mu_{X,\rm SM} = x_q \sum_{i=1}^{3} (\bar{q}_i\gamma^\mu q_i + \bar{u}_i\gamma^\mu u_i + \bar{d}_i\gamma^\mu d_i) - 9x_q(\bar{\ell}_3\gamma^\mu \ell_3 + \bar{e}_3\gamma^\mu e_3)$$

2. EWSB, above QCD confinement: $J^\mu_{X,\rm SM} \longrightarrow x_q(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d + \bar{s}\gamma^\mu s)$

   Dark phase transition: $v_D \gg \Lambda_{\rm QCD}$

$$\phi(x) = e^{\frac{i}{2v_D}(\chi_1\sigma^1 + \chi_2\sigma^2)} \begin{bmatrix} 0 \\ v_D + \frac{\rho(x)}{\sqrt{2}} \end{bmatrix} \rightarrow$$

$$\mathcal{L} \supset (J^\mu_{X,\rm SM} + J^\mu_{X,\phi})X_\mu + v_D^2 X_\mu X^\mu$$

$$\boxed{J^\mu_{X,\phi} = \frac{1}{2} x_\phi \epsilon^{ab}\chi_a \partial_\mu \chi_b}$$

3. Integrate out X_μ:

$$\mathcal{L}_{\rm int} = -\frac{x_q}{2x_\phi v^2}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d + \bar{s}\gamma^\mu s)\epsilon^{ab}\chi_a\partial_\mu\chi_b$$

IR

An example

Work in progress with Joe and Admir.

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UV

1. UV: SM x $U(1)_{X=B-3L_\tau}$ + $\phi_{1,2} \sim (\mathbf{1},\mathbf{1},0,x_\phi)$

4. QCD phase transition:

$$J^\mu_{\mathrm{X,SM}} = x_q(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d + \bar{s}\gamma^\mu s) \to N_c j^\mu_{\mathrm{B}}$$

$$J^\mu_{\mathrm{B}} = \frac{\epsilon^{\mu\nu\rho\sigma}}{12\pi^2 f_\pi^3} f^{abc}\partial_\nu\pi^a\partial_\rho\pi^b\partial_\sigma\pi^c + \dots$$

$$L_{\mathrm{int}} \sim \frac{\epsilon^{\mu\nu\rho\sigma}}{12\pi^2 f_\pi^3} f^{abc}\epsilon^{ij}\pi^a\partial_\mu\pi^b\partial_\nu\pi^c\partial_\rho\chi_i\partial_\sigma\chi_j$$

3. Integrate out X_μ:

$$\mathcal{L}_{\mathrm{int}} = -\frac{x_q}{2x_\phi v^2}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d + \bar{s}\gamma^\mu s)\epsilon^{ab}\chi_a\partial_\mu\chi_b$$

IR

More topological portals

Joe Davighi, Serah Moldovsky, Hitoshi Murayama, Christiane Scherb, Nudžeim Selimović arXiv:2506.05468

Dark Sectors with gauged $U(1)_B$

- A simple $SU(N_c)$ gauge theory with N_f Dirac fermions Q_i in the fundamental:

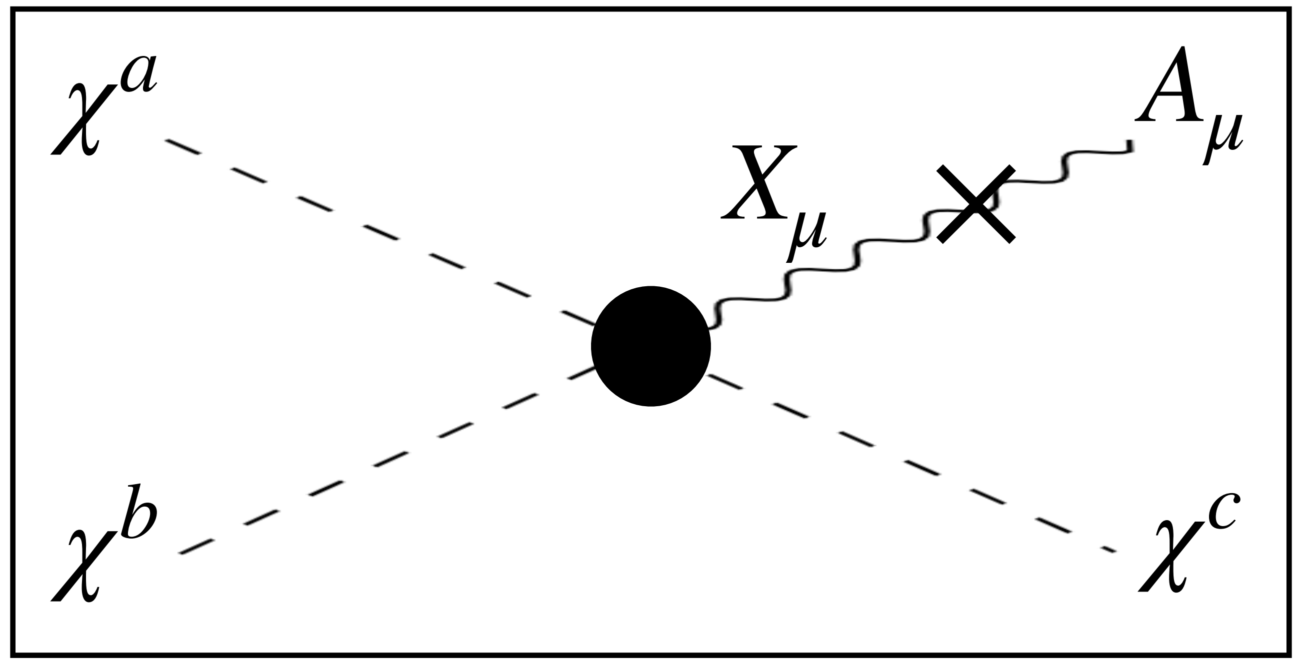
$$\mathcal{L}_{\rm UV} = \mathcal{L}_{\rm SM} - \frac{1}{2}{\rm tr}G_{\mu\nu}G^{\mu\nu} - \frac{1}{4}X_{\mu\nu}X^{\mu\nu} - m_X^2 X_\mu X^\mu + \sum_i \bar{Q}_i(i\not{D} - m_Q)Q_i - \frac{\epsilon}{2\cos\theta_w}X_{\mu\nu}B^{\mu\nu}$$

- Again, low-energy baryon number current:

$$j_B^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}{\rm tr}(U^{-1}\partial_\nu U)(U^{-1}\partial_\rho U)(U^{-1}\partial_\sigma U)$$

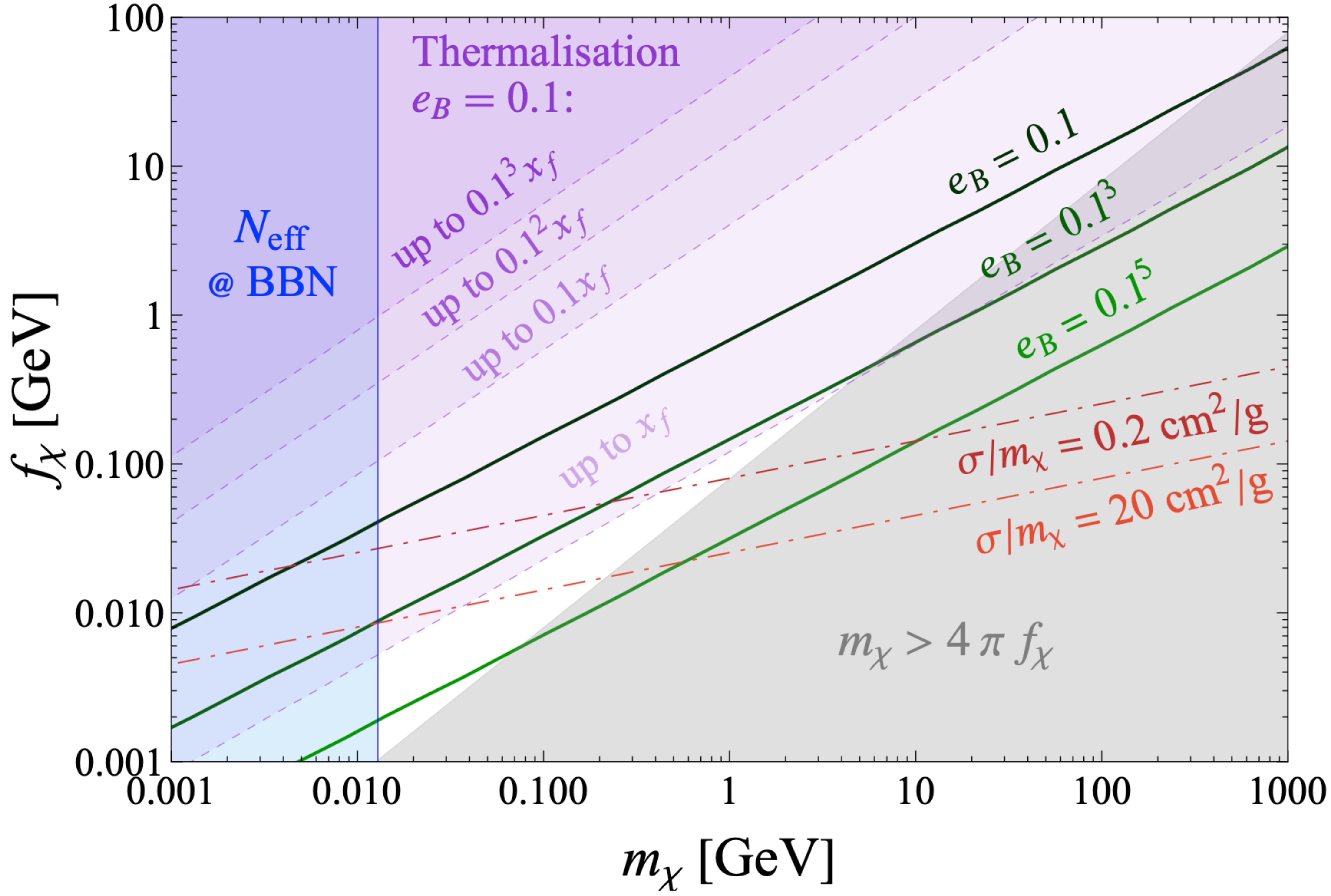
Minimally coupled to the gauge field results in natural model of semi-annihilating DM:

$$\mathcal{L}_\chi = \frac{1}{2}(\partial^\mu\chi^a\partial_\mu\chi^a) - \frac{1}{2}m_\chi^2\chi^a\chi^a + \frac{e_B}{12\pi^2 f_\chi^3}\epsilon^{\mu\nu\rho\sigma}f^{abc}X_\mu\partial_\nu\chi^a\partial_\rho\chi^b\partial_\sigma\chi^c + O(\chi^4)$$

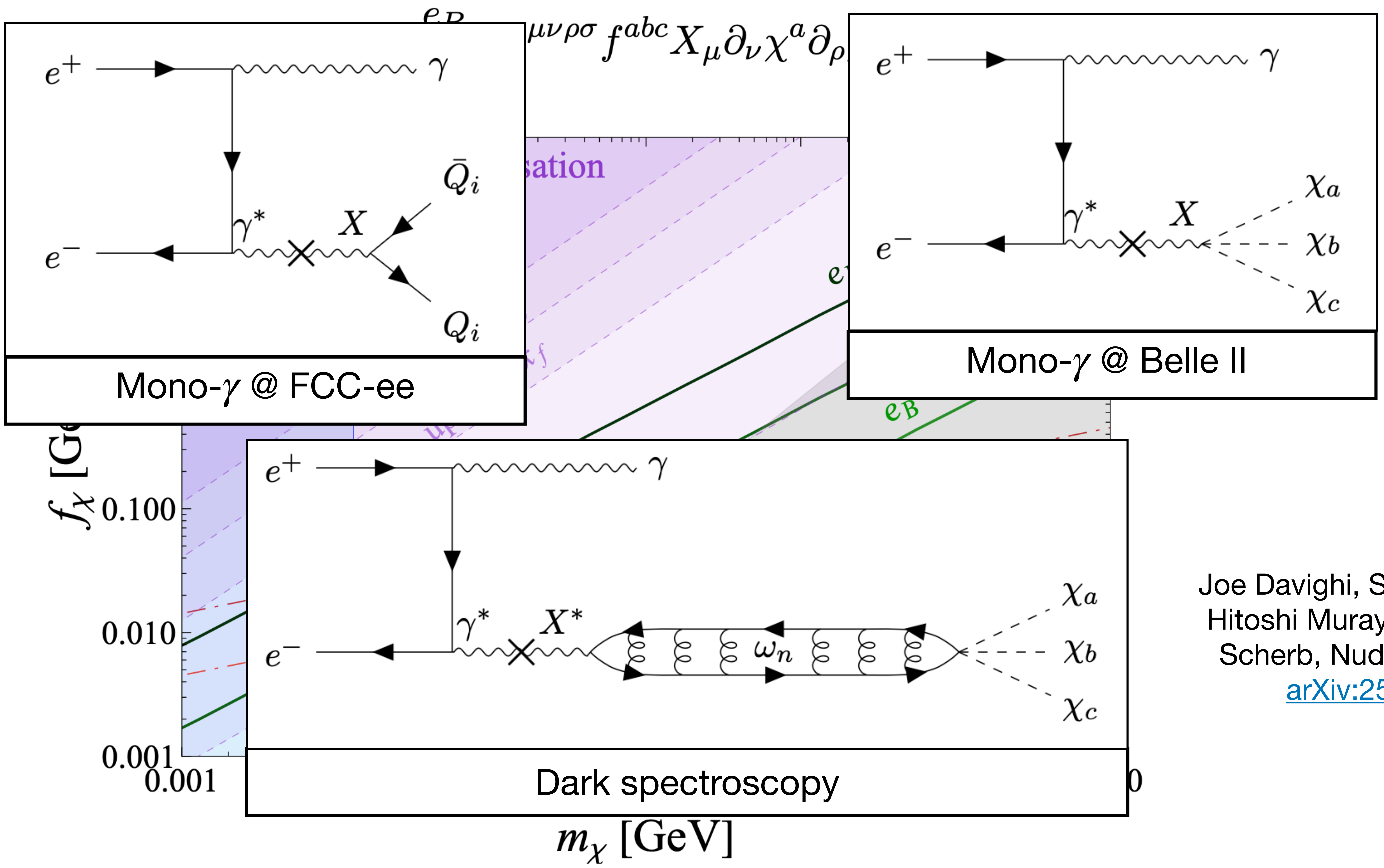


Rich phenomenology

$$\mathcal{L}_\chi = \frac{e_B}{12\pi^2 f_\chi^3}\epsilon^{\mu\nu\rho\sigma} f^{abc} X_\mu \partial_\nu \chi^a \partial_\rho \chi^b \partial_\sigma \chi^c \,.$$



Rich phenomenology



Joe Davighi, Serah Moldovsky, Hitoshi Murayama, Christiane Scherb, Nudžeim Selimović
arXiv:2506.05468

Conclusions

- There is a novel portal (with a unique dark coset) resulting in an elegant light thermal inelastic DM scenario.
- Topology could explain why we haven't observed DM in direct/indirect detection experiments.
- Belle II could give us a definite answer!

  Relative rate for $e^+e^- \to \gamma* \to \pi^0\chi_1\chi_2$ *and* $e^+e^- \to \gamma* \to \eta\chi_1\chi_2$ *completely fixed.*
- Topological interactions offer new directions for DM model building.

  Non-trivial consequences for dark-sectors from topological interactions.

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  Non-trivial consequences for dark-sectors from topological interactions.

Thank you!

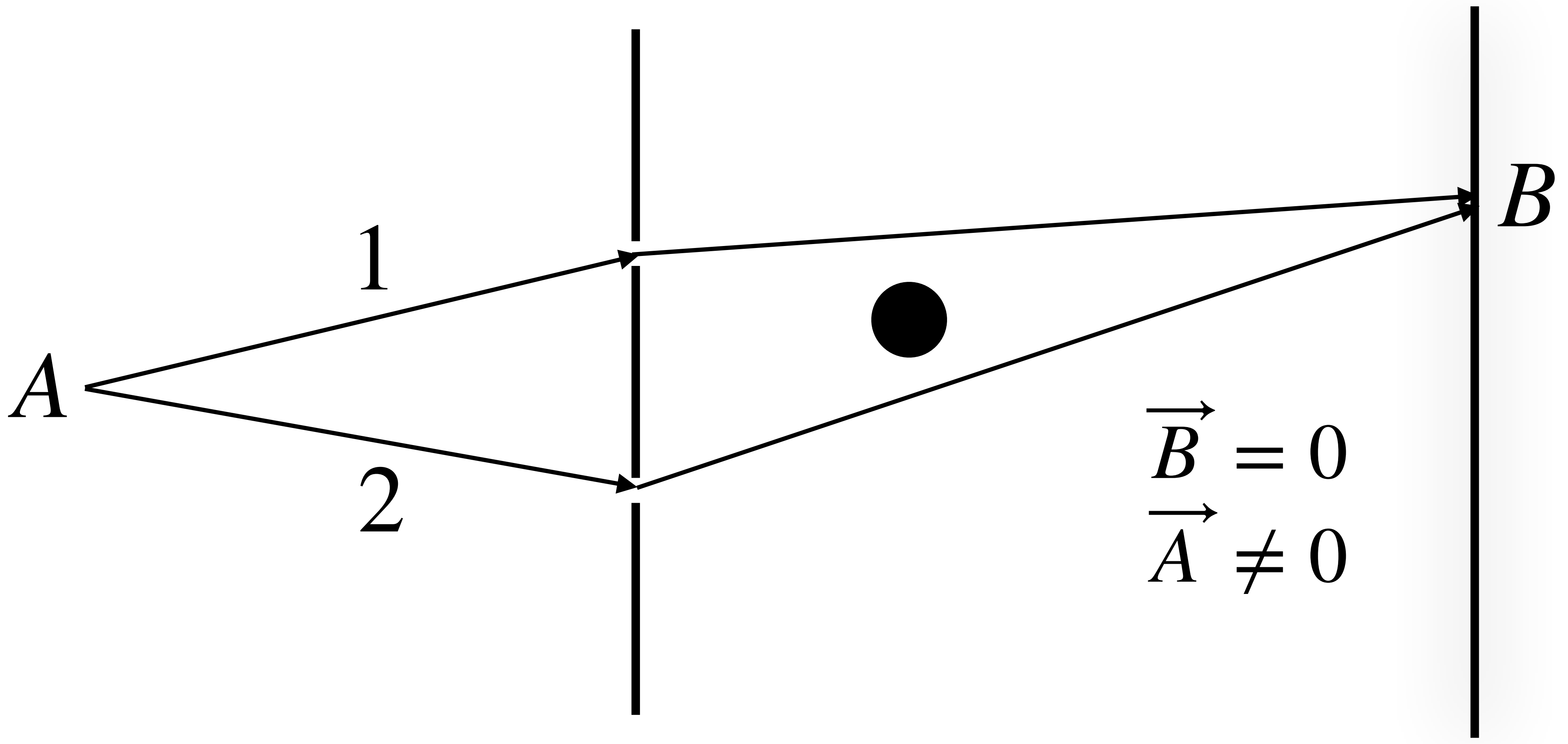
Backup

Topological terms

- *Class 1: Theta terms*

- Example 1: Aharonov-Bohm effect

Has physical consequences: double slit experiment with the solenoid



Akira Tonomura et.al '86

$$I^{(1)}_{A\to B} + I^{(2)}_{A\to B} = \mathcal{N}\, \exp\left(iq \oint A_i\, dx^i \right)$$

Stoke's theorem:

$$\oint A_i\, dx^i = \iint \vec{\nabla} \times \vec{A}\, ds = \iint \vec{B} \cdot \vec{n}\, ds = \Phi_B$$

Extra phase changes the interference pattern!

Nudžeim Selimović, INFN Padova

Topological terms

- *Class 1: Theta terms*

- Example 2: Instantons in 4d gauge theory

$$S_\theta = \theta \int d^4x \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = \theta N_{\text{inst}} \longrightarrow \theta \in \mathbb{R}$$

$$S_\theta = \theta \int d^4x \frac{g^2}{32\pi^2} \partial_\mu \mathrm{Tr} \left(A_\nu \partial_\alpha A_\beta + \frac{2}{3} A_\nu A_\alpha A_\beta \right) \longrightarrow \text{Total derivative}$$

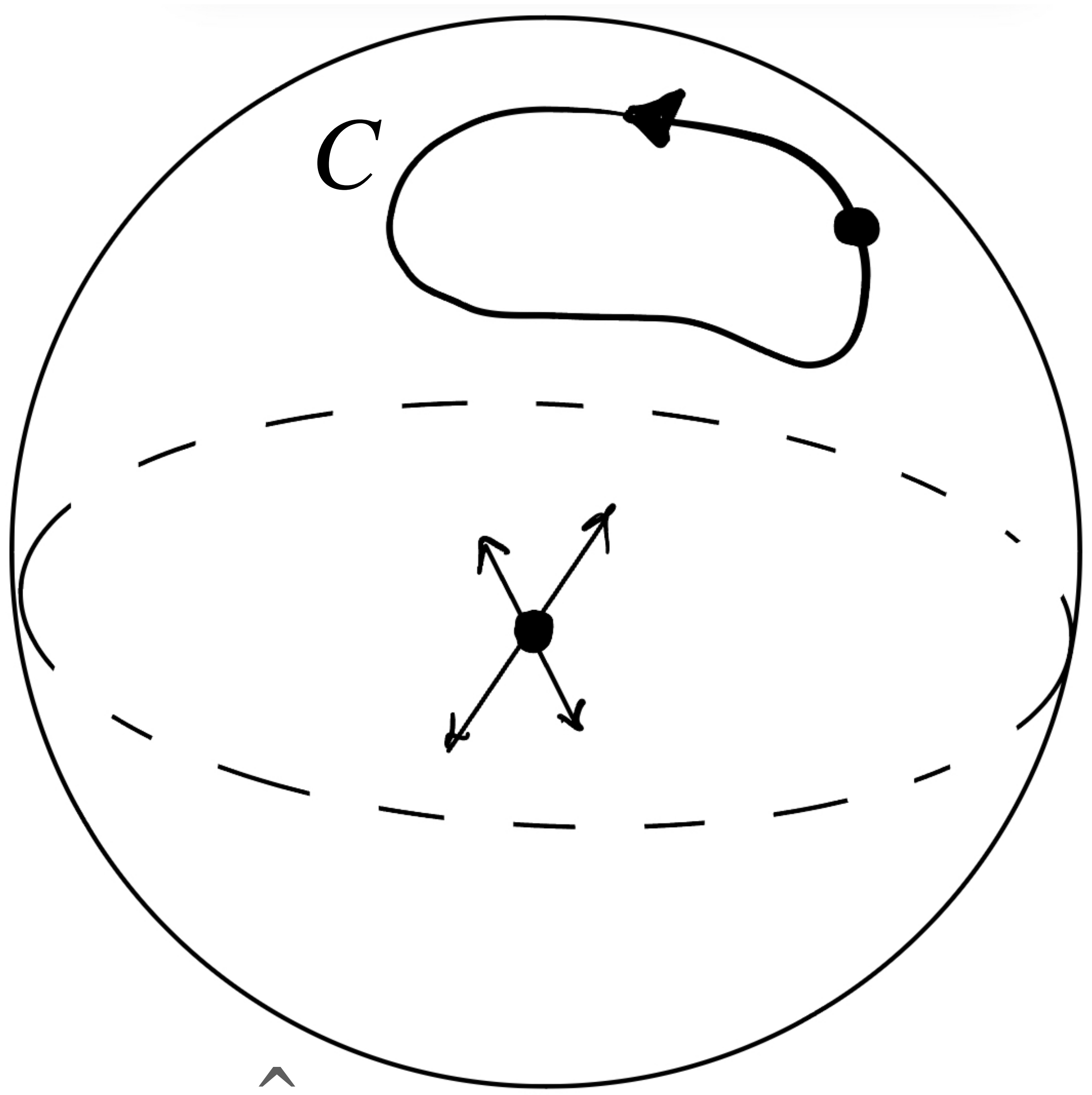
Effects not seen in perturbation theory: no Feynman diagrams.

Topological terms

- *Class 2: Wess-Zumino-Witten terms*

Symmetries:
SO(3) rotations
LHS: ***P*** and ***T*** invariant
RHS: ***PT*** invariant

- Example 1: Particle on a sphere coupled to magnetic monopole.
  (More intuitive?)



$$\vec{B} = \frac{g\hat{r}}{4\pi r^2}$$

$$m\vec{a} = q\vec{v} \times \vec{B} \longrightarrow m\ddot{x}_i = \lambda \epsilon_{ijk} x_k \dot{x}_k$$

Can we construct the action which would give this?

$$L \sim \epsilon_{ijk} x_i x_j \dot{x}_k = 0 \quad ?$$

First possibility: introduce the gauge potential $A_i(x)$

$$S = \int_C dt \left(\frac{1}{2} m \dot{x}_i^2 + \lambda A_i(x) \dot{x}^i \right)$$

$$A_\phi^N = \frac{g}{4\pi r} \frac{1 - \cos\theta}{\sin\theta}$$

- No manifest SO(3) symmetry - violated by A_ϕ^N
- Dirac string along $\theta = \pi$

Witten '83
- Global aspects of current algebra

Topological terms

- *Class 2: Wess-Zumino-Witten terms*

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LHS: ***P*** and ***T*** invariant
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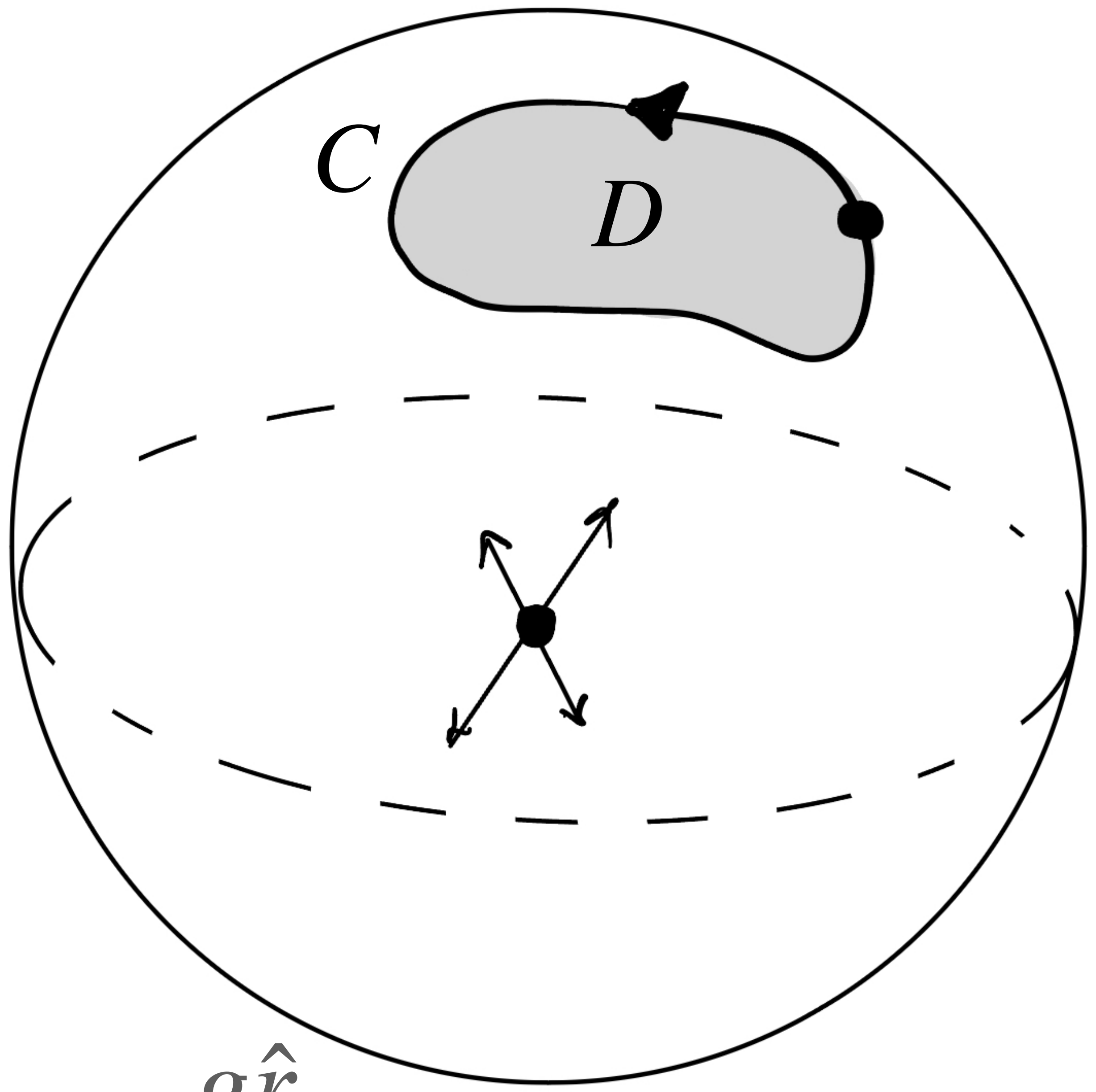


$$\overrightarrow{B} = \frac{g\hat{r}}{4\pi r^2}$$

$$m\vec{a} = q\vec{v} \times \overrightarrow{B} \longrightarrow m\ddot{x}_i = \lambda\epsilon_{ijk}x_k\dot{x}_k$$

Can we construct the action which would give this?

Second possibility: go one dimension higher

$$\int_C dtA_i(x)\dot{x}^i = \int_D dS^{ij}F_{ij}(x) \qquad C = \partial D$$

$$F_{ij} = \epsilon_{ijk}x^k/|x|^3$$

- Manifestly SO(3) symmetric
- Non-singular away from the origin

Witten '83
- Global aspects of current algebra

Topological terms

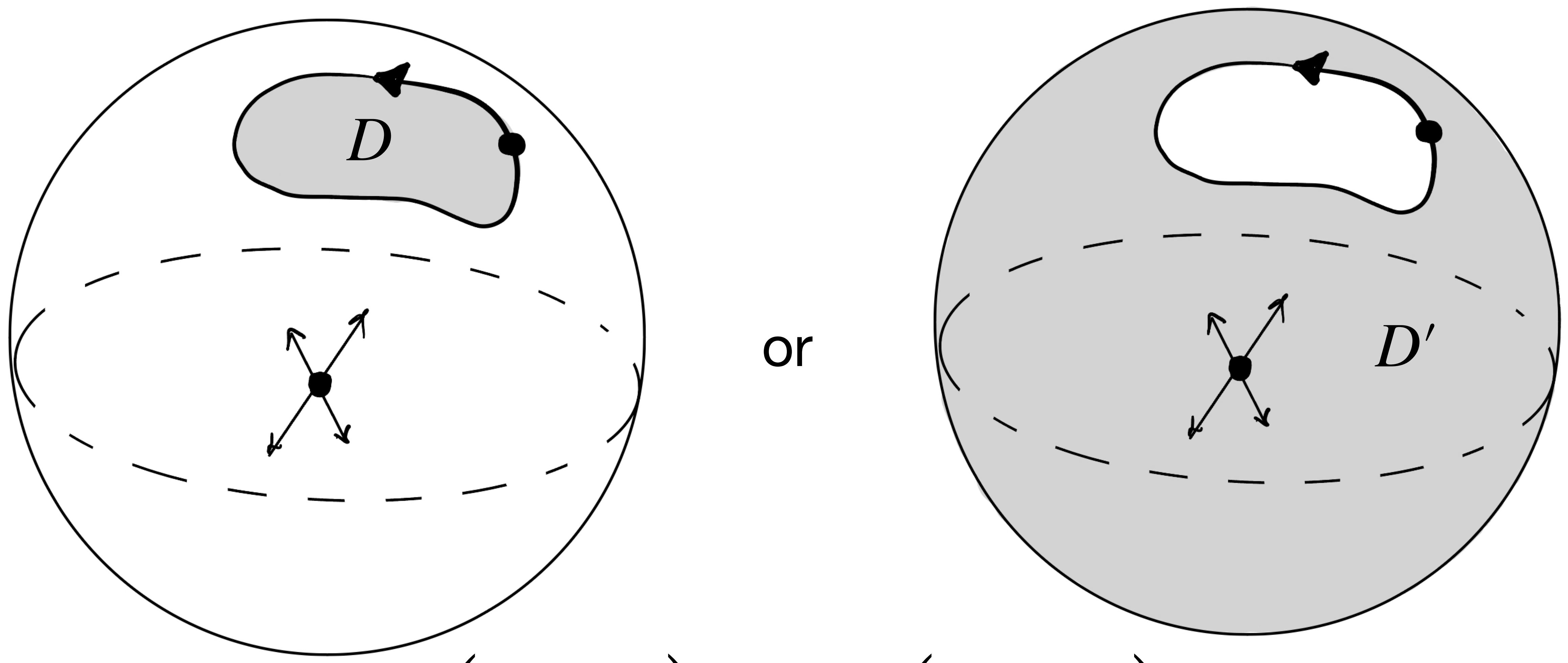
- *Class 2: Wess-Zumino-Witten terms*

- Example 1: Particle on a sphere coupled to magnetic monopole.
  (More intuitive?)

Again, a topological term is an integral of a differential form: 2-form ω_2

$$S = \exp\left(i\lambda \int_D dS^{ij} F_{ij}\right)$$

$$i\lambda \int_D \epsilon_{ijk} dx^i dx^j x^k / |x|^3 = i\lambda \int_D \omega_2$$



$$\exp\left(i\lambda \int_D \omega_2\right) = \exp\left(i\lambda \int_{D'} \omega_2\right)$$

$$\exp\left(i\lambda \int_{DUD'=S^2} \omega_2\right) = 1 \quad \text{and} \quad \underbrace{\int_{S^2} \omega_2 = 2\pi n}_{} \longrightarrow \lambda \in \mathbb{Z}$$

The magnetic flux through any closed surface is quantised

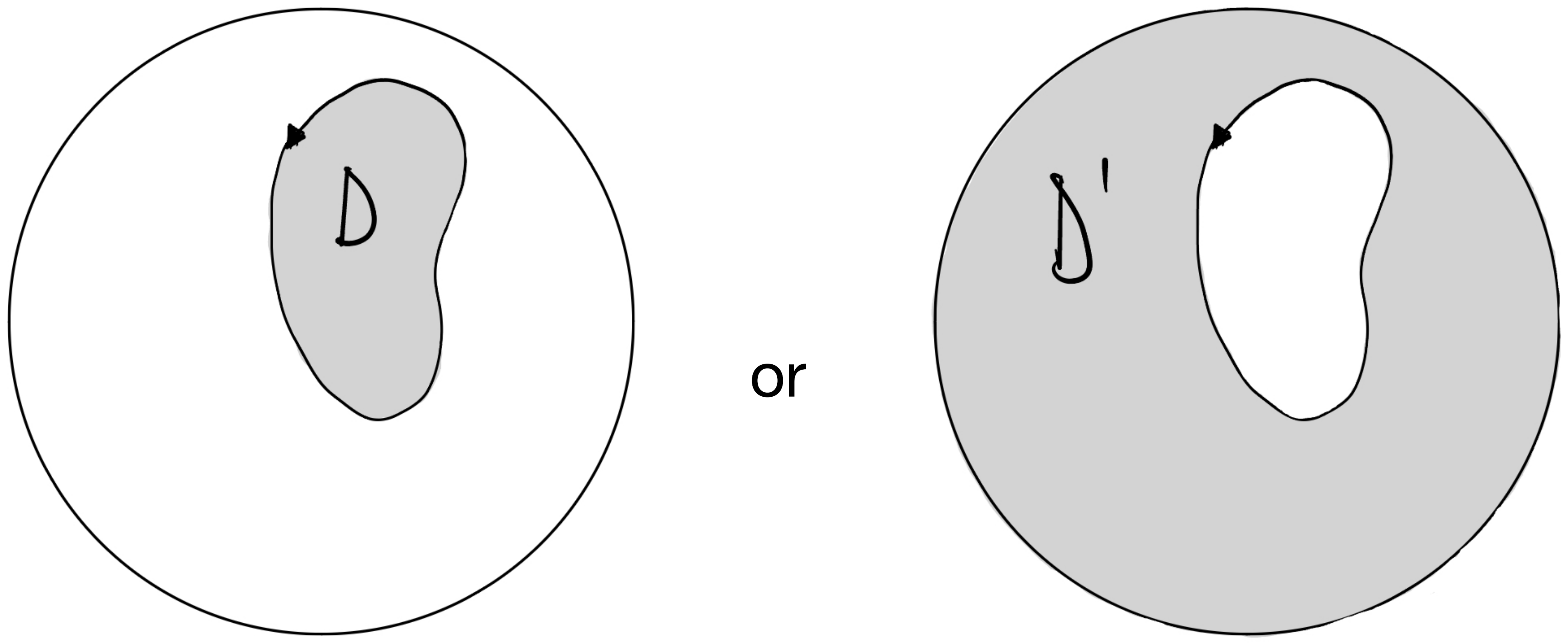
Topological terms

Witten '83
- Global aspects of current algebra

- *Class 2: Wess-Zumino-Witten terms*

- Example 2: Low-energy QCD

Integer coefficients:



Mathematically, ω_5 is:

1. $G = SU(3)_L \times SU(3)_R$ - invariant;
2. *Closed*, $d\omega_5 = 0$;
3. *Integral*, integrates to $(\text{normalisation}) \times n$.

Complete analogy with Example 1.

$$\exp\left(ik\int_D \omega_5\right) = \exp\left(ik\int_{D'} \omega_5\right) \qquad \exp\left(ik\int_{DUD'=\Sigma_5} \omega_5\right) = 1 \text{ and } \underbrace{\int_{\Sigma_5} \omega_5 = 2\pi n}_{} \longrightarrow k \in \mathbb{Z}$$

Integrality ($\Pi_5(SU(3)) = \mathbb{Z}$)

Topological terms

Witten '83
- Global aspects of current algebra

- *Class 2: Wess-Zumino-Witten terms*

- Example 2: Low-energy QCD

Chiral anomalies: gauging the WZW term $U(1)_{\text{QED}} \supset SU(3)_{L+R}$

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}$$

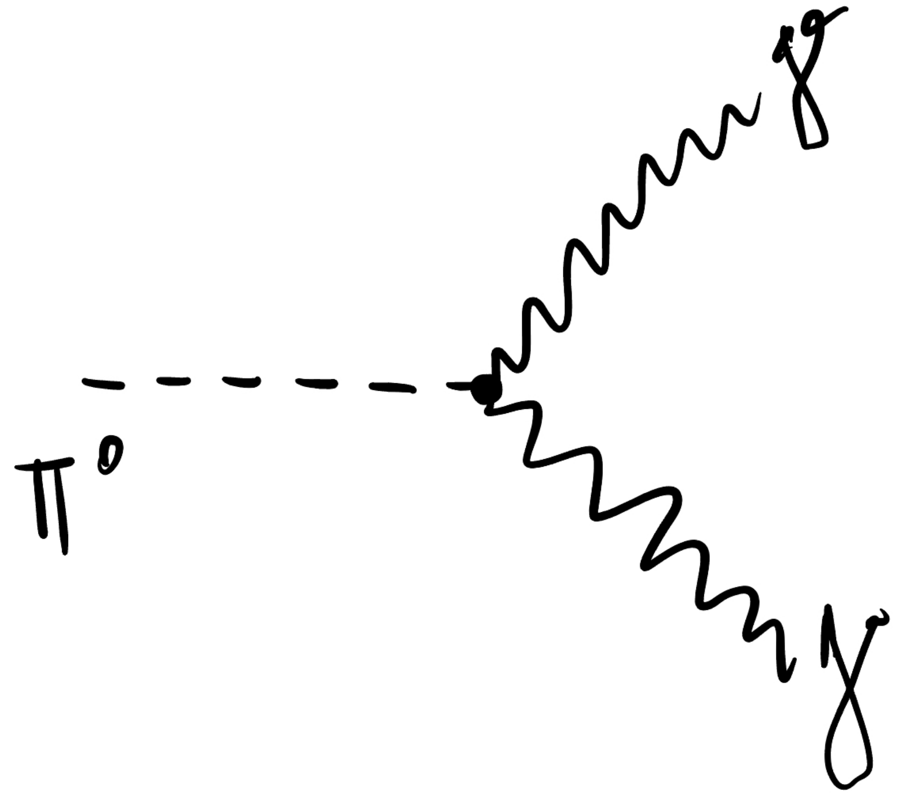
Non-trivial derivation using Noether method:

$$J^\mu = \frac{1}{48\pi^2}\epsilon^{\mu\rho\sigma\tau}\text{tr}\left(\{Q, U^\dagger\}\,\partial_\rho U\, U^\dagger\partial_\sigma U\, U^\dagger\partial_\lambda U\right)$$

$$S_{WZW} = k\left[-e\int d^4x\; A_\mu(x)J^\mu + \frac{ie^2}{24\pi^2}\int d^4x\;\epsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu)A_\rho\text{tr}\left(\{Q^2, U^\dagger\}\,\partial_\sigma U + U^\dagger Q U Q U^\dagger \partial_\sigma U\right)\right]$$

$$L_{\text{WZW}}^{\text{gauged}} \supset \frac{ke^2}{96\pi^2 f_\pi}\pi^0\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$k = N_c$$



Invariant 3-form

$$\omega_3 \sim \mathrm{Tr}\,(U^{-1}dU)^3$$

David Tong
- Gauge theory

- Does not appear in the QCD action.
- However, it appears as the *topologically conserved* current:

Consider static field configurations in the chiral Lagrangian: $U(x) \to 1$ as $|x| \to \infty$

Fields are maps: $U(x) : S^3 \to SU(3)$

Characterised by the winding number: $\Pi_3(SU(3)) = \mathbb{Z}$, which can be computed as

$$B = \int_{\Sigma_3} \omega_3 = \frac{1}{24\pi^2}\int d^3x\, \epsilon_{ijk}\mathrm{Tr}\left(U^\dagger(\partial_i U)U^\dagger(\partial_j U)U^\dagger(\partial_k U)\right)$$

Also a charge of the conserved current:

$$B^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,(U^\dagger(\partial_\nu U)\,U^\dagger(\partial_\rho U)\,U^\dagger\,\partial_\sigma U) \quad \Big| \quad \partial_\mu B^\mu = 0 \quad \Big| \quad B = \int d^3x\, B^0$$

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Same properties as ω_5:

1. $G = SU(3)_L \times SU(3)_R$ - invariant;
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QCD $\times$ Dark Topological Portal

EFT description.

- Collective non-linear sigma model on a product coset:

$$X = \frac{SU(3)_L \times SU(3)_R \times G_{\rm D}}{SU(3)_{L+R} \times H_{\rm D}} \simeq SU(3) \times \frac{G_{\rm D}}{H_{\rm D}}$$

$$\omega_5^{\rm portal} \sim \omega_3^{\rm QCD} \times \omega_2^{\rm D} \quad \text{on} \quad X = \frac{SU(3)_L \times SU(3)_R \times G_{\rm D}}{SU(3)_{L+R} \times H_{\rm D}} \simeq SU(3) \times \frac{G_{\rm D}}{H_{\rm D}}$$

1. Closed: $\mathrm{d}\omega_5^{\rm portal} = 0$ implies $\mathrm{d}\omega_2^{\rm D} = 0$ since $\mathrm{d}\omega_3^{\rm QCD} = 0$.
2. $SU(3)_L \times SU(3)_R \times G_{\rm D}$ - invariance: Product structure implies $\omega_2^{\rm D}$ is $G_{\rm D}$ - invariant.
3. Integrality: Cycles factorise; normalise $\omega_3^{\rm QCD}$ and $\omega_2^{\rm D}$ separately.

J. Davighi and B. Gripaios, *Homological classification of topological terms in sigma models on homogeneous spaces*, JHEP 09 (2018) 155

QCD × Dark Topological Portal

Which dark coset fits?

- Consider QCD-like theories*:

$$\frac{G_D}{H_D} = \left\{ \frac{SU(N)_L \times SU(N)_R}{SU(N)_{L+R}}, \frac{SU(N)}{SO(N)}, \frac{SU(2N)}{Sp(2N)} \right\}$$

- All are symmetric spaces:
G_D-invariant forms on G_D/H_D are in 1-to-1 with cohomology classes (C. Chevalley and S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, Trans. Am. Math. Soc. 63 (1948) 85–124.)

The portal exists iff $H^2(G_D/H_D) \neq 0$.

(de Rham cohomology $H^k(M)$ - the set of closed ***k***-forms on ***M.***)

**One could, in principle, consider other options, such as a complex projective space $\mathbb{C}P^n$, that go beyond the chiral symmetry-breaking dynamical assumption.*

QCD × Dark Topological Portal

Which dark coset fits?

H. Cartan, D´emonstration homologique des th´eoremes de p´eriodicit´e de bott, ii. homologie et cohomologie des groupes classiques et de leurs espaces homogenes, S´eminaire Henri Cartan 12 (1959) 1–32.

		Portal			SIMP
p	1	2	3	4	5
$H^p(SU(2))$	0	0	$\mathbb{R}$	–	–
$H^p(SU(n)),\ n \geq 3$	0	0	$\mathbb{R}$	0	$\mathbb{R}$
$H^p(SU(2)/SO(2))$	0	$\mathbb{R}$	–	–	–
$H^p(SU(3)/SO(3))$	0	0	0	0	$\mathbb{R}$
$H^p(SU(4)/SO(4))$	0	0	0	$\mathbb{R}$	$\mathbb{R}$
$H^p(SU(n)/SO(n)),\ n \geq 5$	0	0	0	0	$\mathbb{R}$
$H^p(SU(2n)/Sp(2n)),\ n \geq 2$	0	0	0	0	$\mathbb{R}$

QCD → $H^p(SU(n)),\ n \geq 3$

Unique choice! → $H^p(SU(2)/SO(2))$, $p = 2$

5 dark pions - no portal → SIMP, $p = 5$

Hochberg et al, 1402.5143, 1411.3727

Relic Abundance

Mass splitting effects.

- Define: $\Delta m_\chi = m_{\chi_2} - m_{\chi_1}$ and $\Delta := \Delta m_\chi / m_{\chi_1}$: χ_1 is the sole dark matter candidate
- Affects the thermally averaged cross section:

D'Agnolo, Mondino, Ruderman, Wang, Exponentially Light Dark Matter from Coannihilation, *JHEP* 08 (2018) 079

$$\langle \sigma v \rangle \to \langle \sigma v \rangle \exp(-x\Delta)$$

- Translates to f_D needed for the correct relic abundance:

$$f_D(\Delta) \to f_D(0) \exp\left(-\frac{x_{\max}\Delta}{4}\right)$$

- If $\Delta m_\chi \neq 0$, then $\Delta m_\chi > m_{\pi^0}$, otherwise $\chi_2 \to \chi_1 \gamma\gamma\gamma$ through π^0 with long lifetime $\tau > 1$ sec: problematic for BBN

Kawasaki et.al, Revisiting Big-Bang Nucleosynthesis Constraints on Long-Lived Decaying Particles *Phys.Rev.D* 97 (2018) 2, 023502