Ginzburg-Landau approach to inhomogeneous chiral phases of QCD

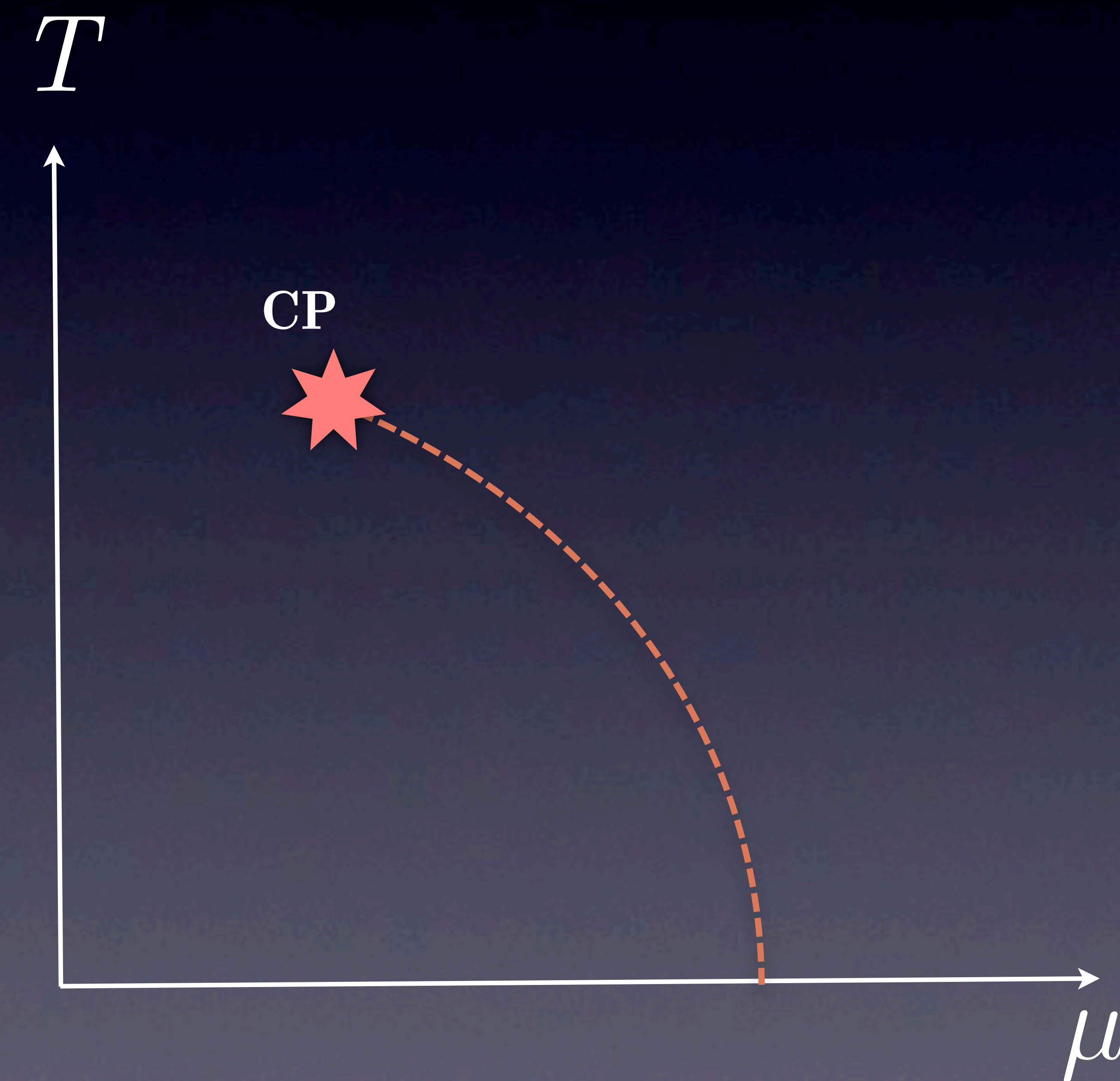
Hiroaki Abuki[a]

thanks to *Daisuke Ishibashi*[b] and *Katsuhiko Suzuki*[a]

(a) Tokyo University of Science,
(b) Water Service of Tokyo City Office

Ref: Phys. Rev. D85, 074002 (2012)

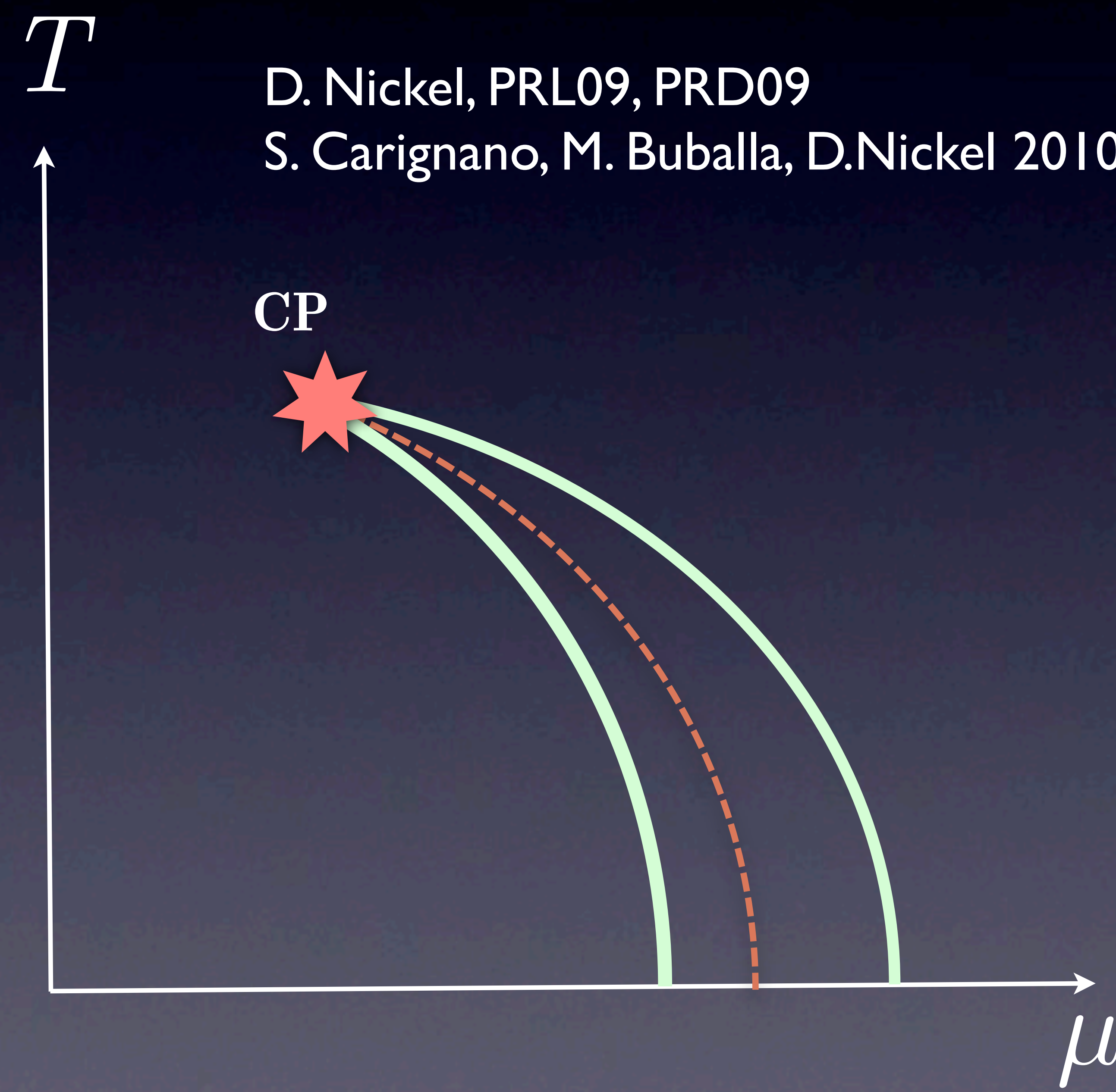
Inhomogeneous chiral condensate?



Inhomogeneous chiral condensate?

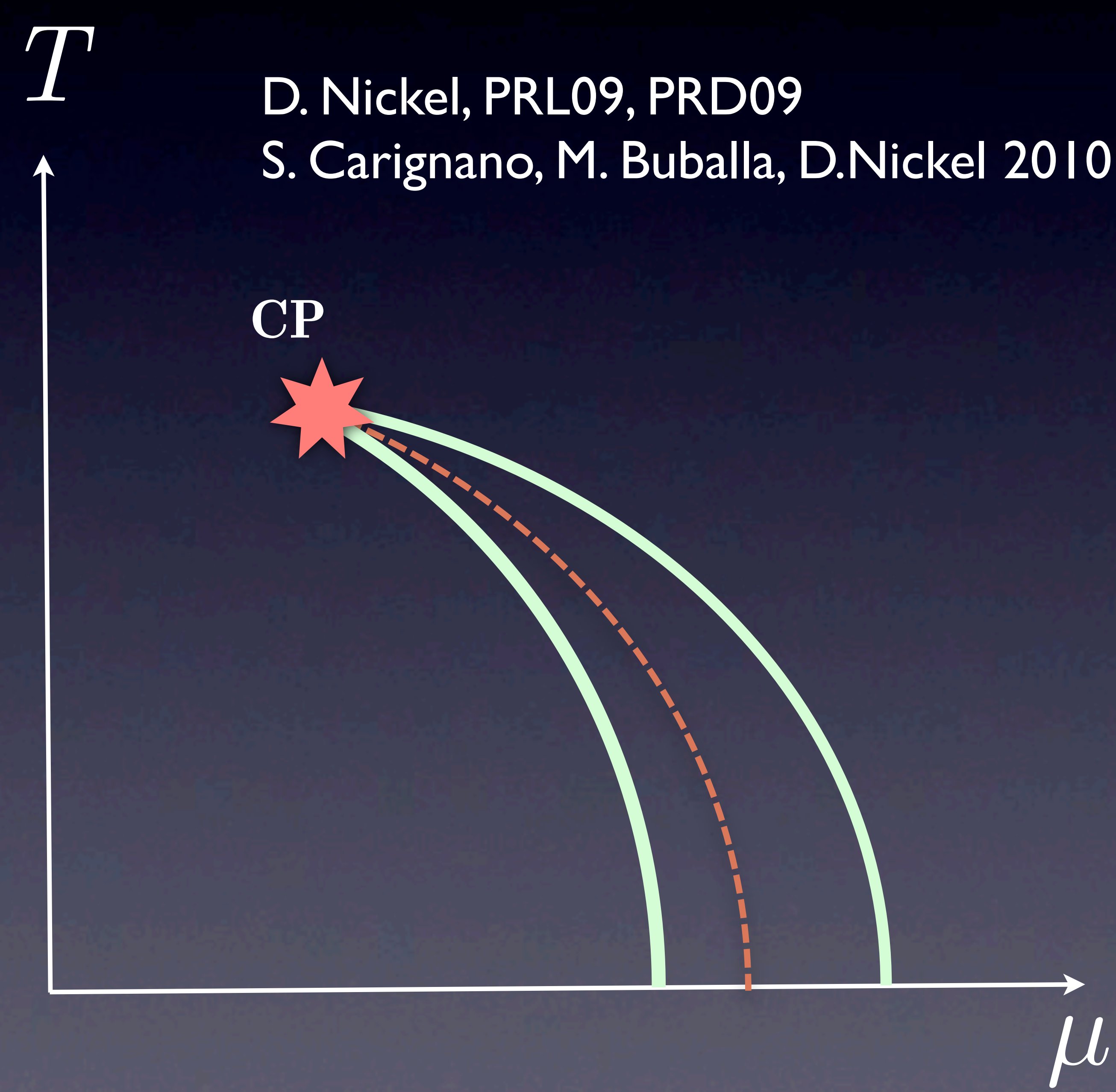
- What?: "*Halfway*" state = *femtoscale "ordered"* phase separation)

D. Nickel, PRL09, PRD09
S. Carignano, M. Buballa, D.Nickel 2010



Inhomogeneous chiral condensate?

- What?: "*Halfway*" state = *femtoscale "ordered"* phase separation)
- Why?: Compromise of two conflicting effects:
  - ★ net quark density $\langle q^+ q \rangle$
  - ★ q-qbar pair density $\langle \bar{q} q \rangle$



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- *"Hierarchical" chiral restoration at T=0*

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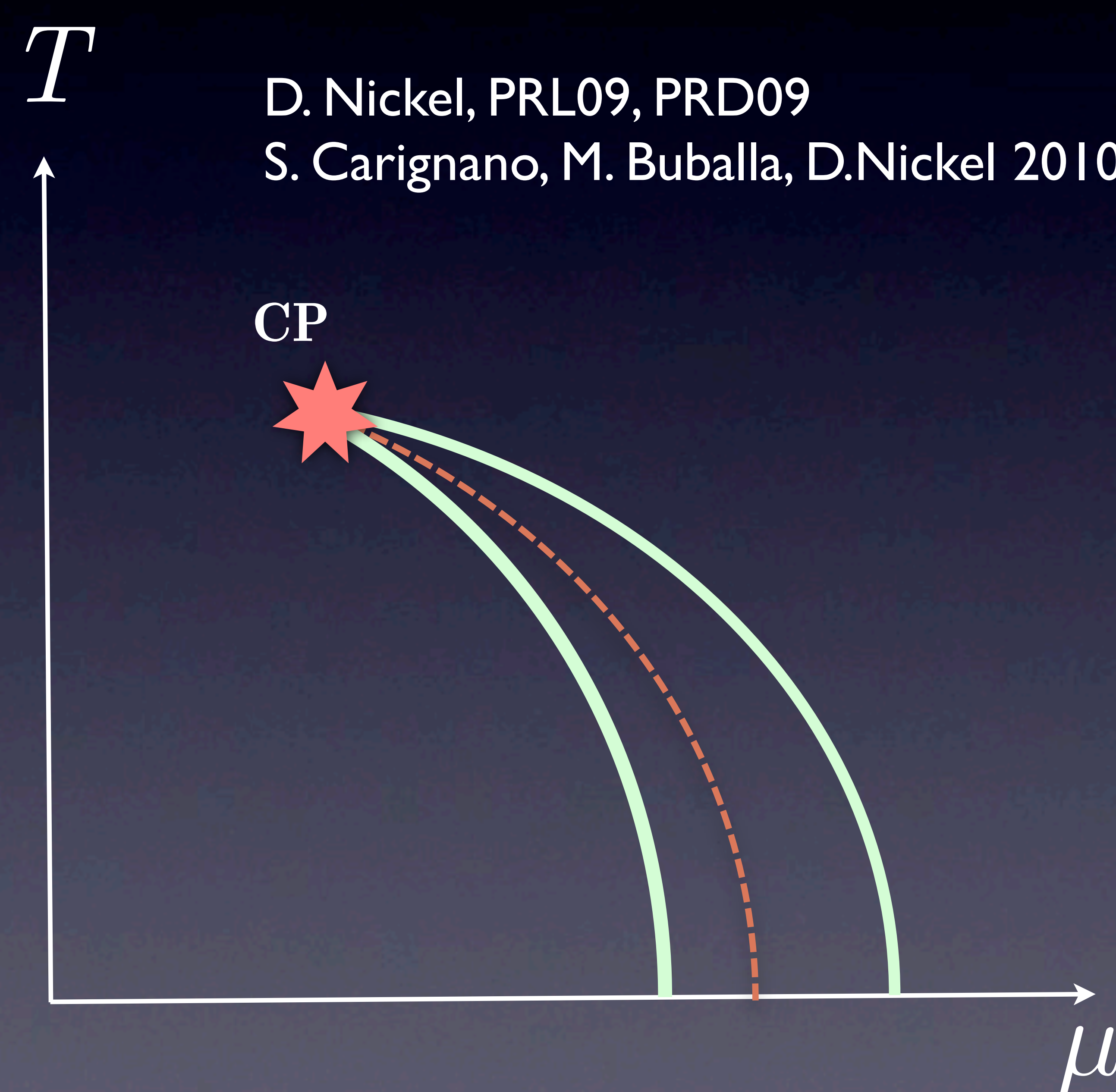

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- Similar example: FFLO state
  - ★ *Pauli paramagnetism vs superconductivity*

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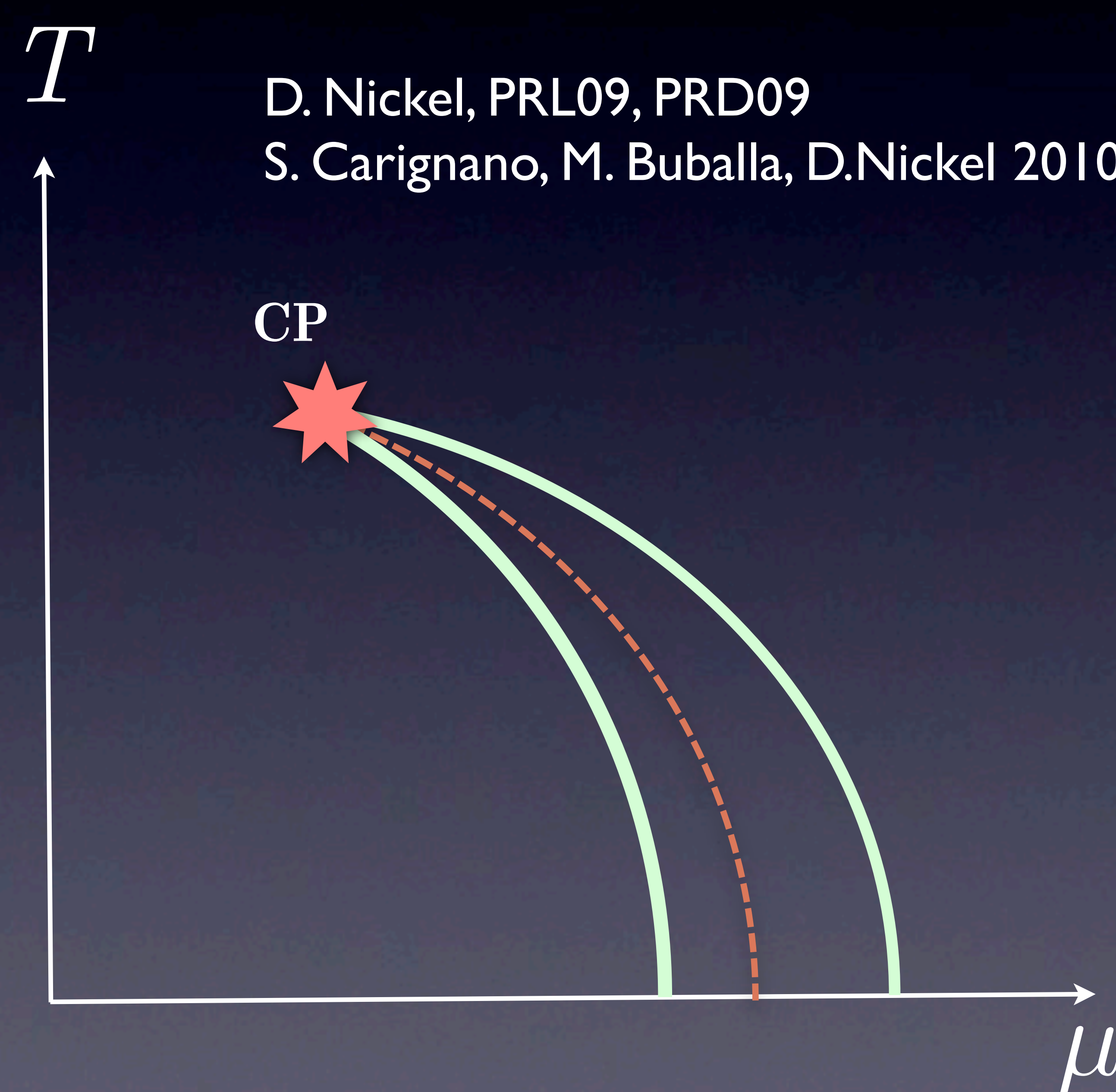


Two types of 1D Chiral Crystal

- Spatial modulation in the real space

  ★ Larkin-Ovchinnikov (LO) type

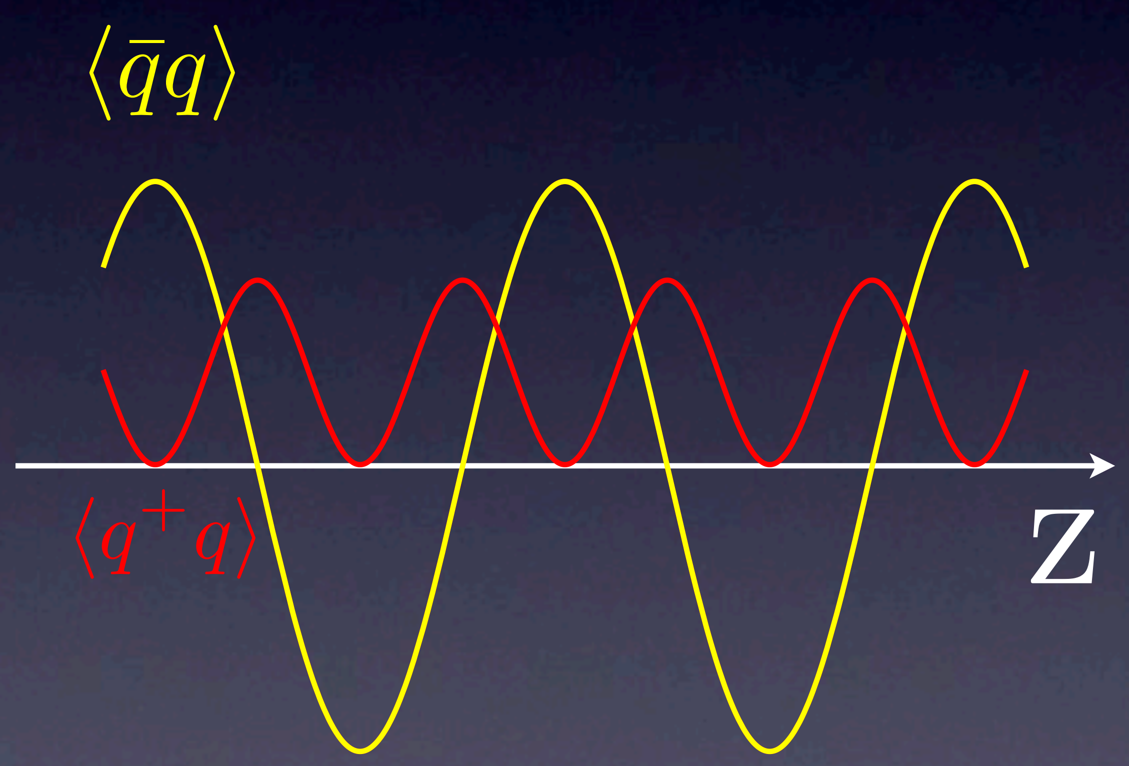
  Thies 2006, Nickel 2009

- Anisotropy in the momentum space

  ★ Flude-Ferrel (FF) type

  Dual chiral density wave; Nakano, Tatsumi (2004)
  Quarkyonic chiral spiral; Kojo, Hidaka, et al. (2010)

Spatial modulation (LO)



c.f. Carignano, Buballa, Nickel 2010

Aims

1. Multidimensional modulations near CP?
   - Low dimensional modulation in 3D might be unstable at finite T

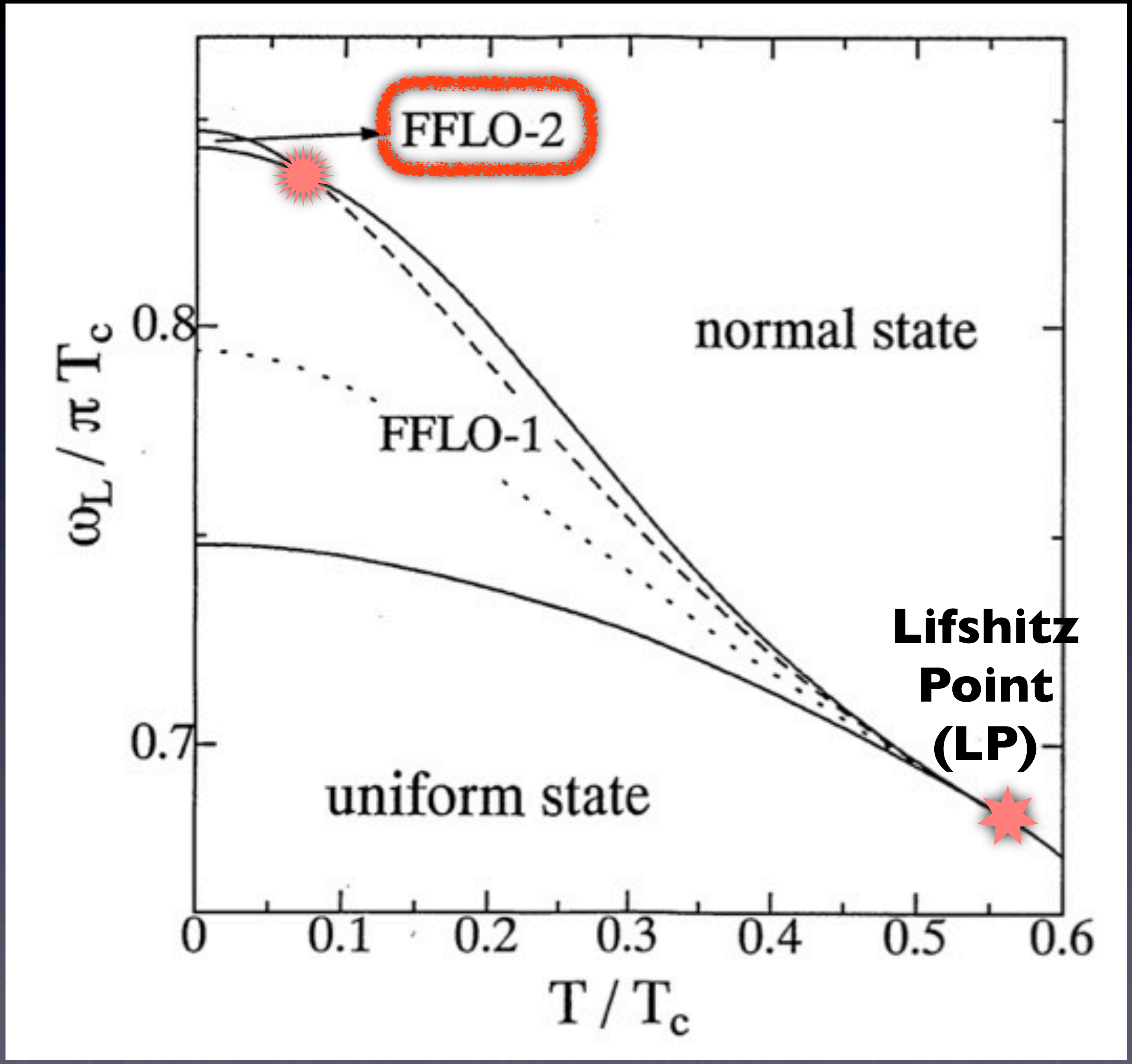
     c.f. Landau-Peierls Theorem: Baym, Friman, Grinstein, NPB1982

     Several works related:
     Quarkyonic chiral spiral; Kojo *et al.*(2011),
     NJL model analysis; Carignoano, Buballa(2011).

2. How does the phase structure change going away from the critical point?
   - Rich phase structure even in the chiral limit?

Rich phase structure?

- New state may appear at low T region far from LP
- Two types of FFLO states & New critical point
- 8th order terms in GL are needed!



Matsuo, Higashitani, Nagato, Nagai, JPSJ 1997

Aims

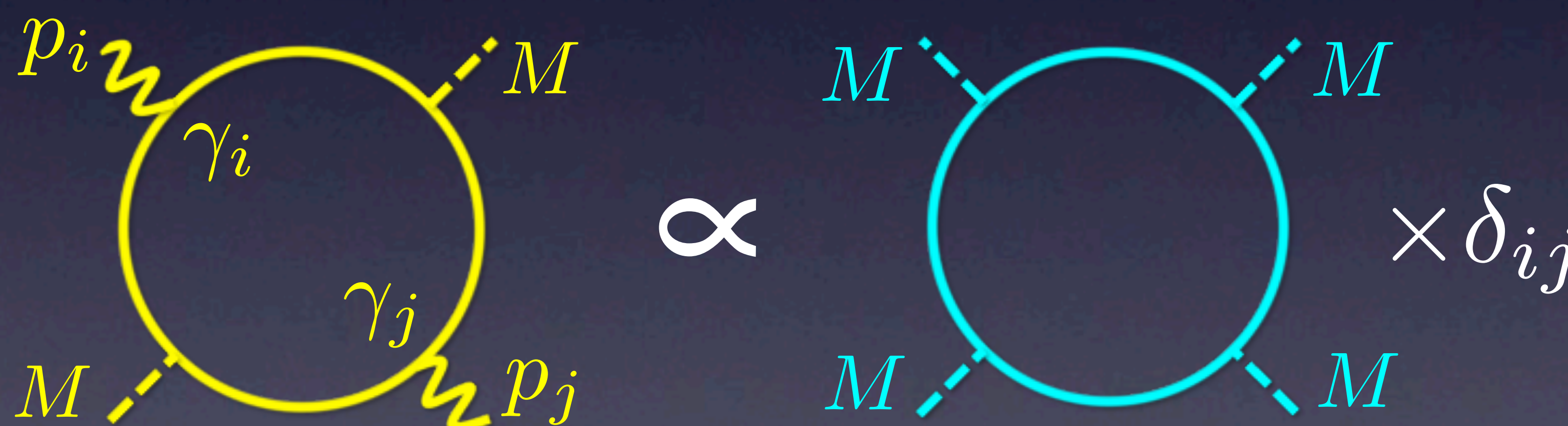
1. Multidimensional modulations near CP?

Ginzburg-Landau approach

- Expansion of Free energy w.r.t. the condensate and its spatial derivative

D. Nickel, PRL09

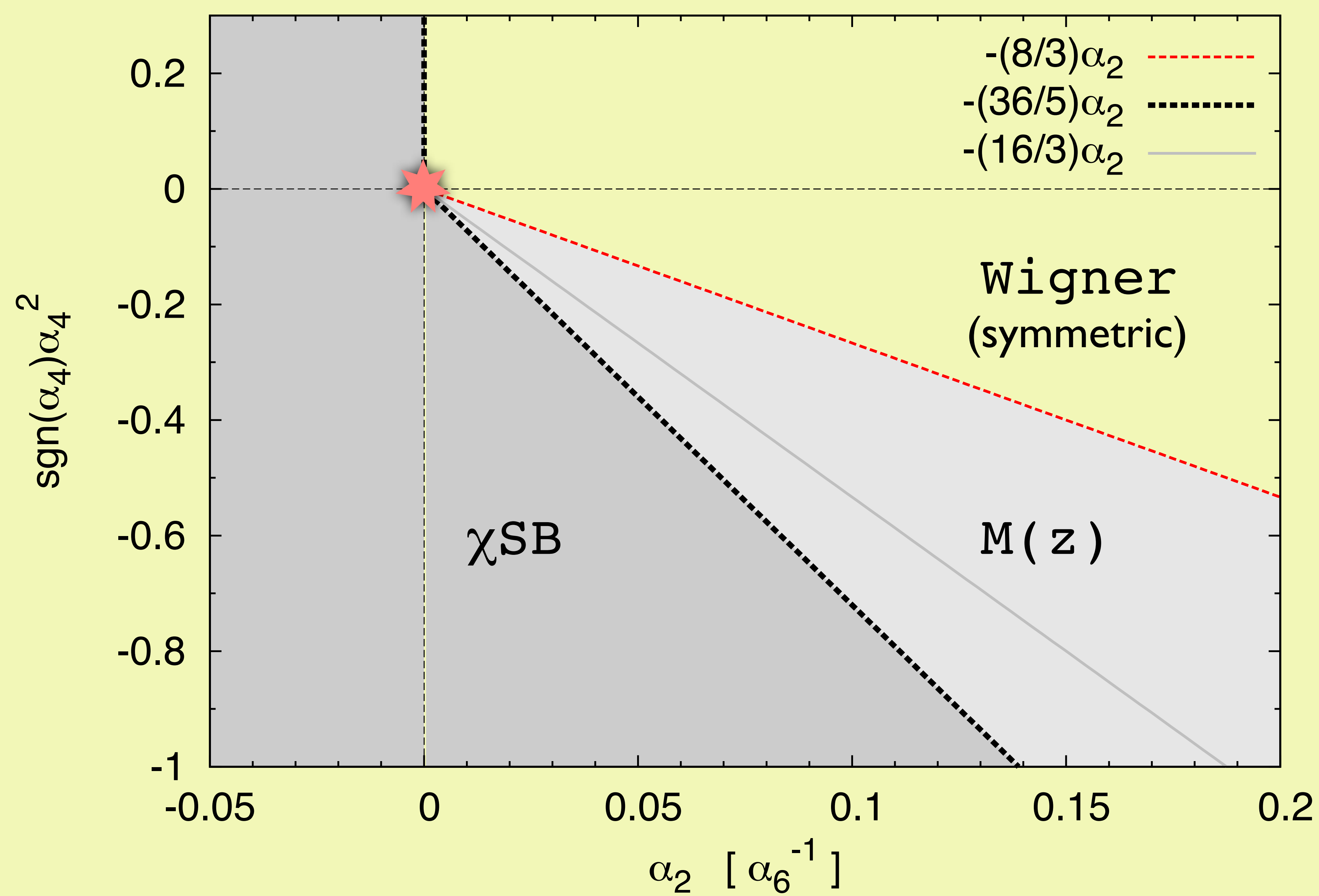
$$\Omega_{\rm GL} = \frac{\alpha_2}{2}M(\mathbf{x})^2 + \frac{\alpha_4}{4}M(\mathbf{x})^4 + \frac{\alpha_6}{6}M(\mathbf{x})^6$$
$$\boxed{+\frac{\alpha_4}{4}(\nabla M)^2 + \frac{5\alpha_6}{6}M^2(\nabla M)^2 + \frac{\alpha_6}{12}(\nabla\Delta M)^2}$$



★ Based on the symmetry & model independent in the vicinity of CP; Also applicable for multi-dimensional modulations

GL phase diagram (1D structure)

D. Nickel, PRL09



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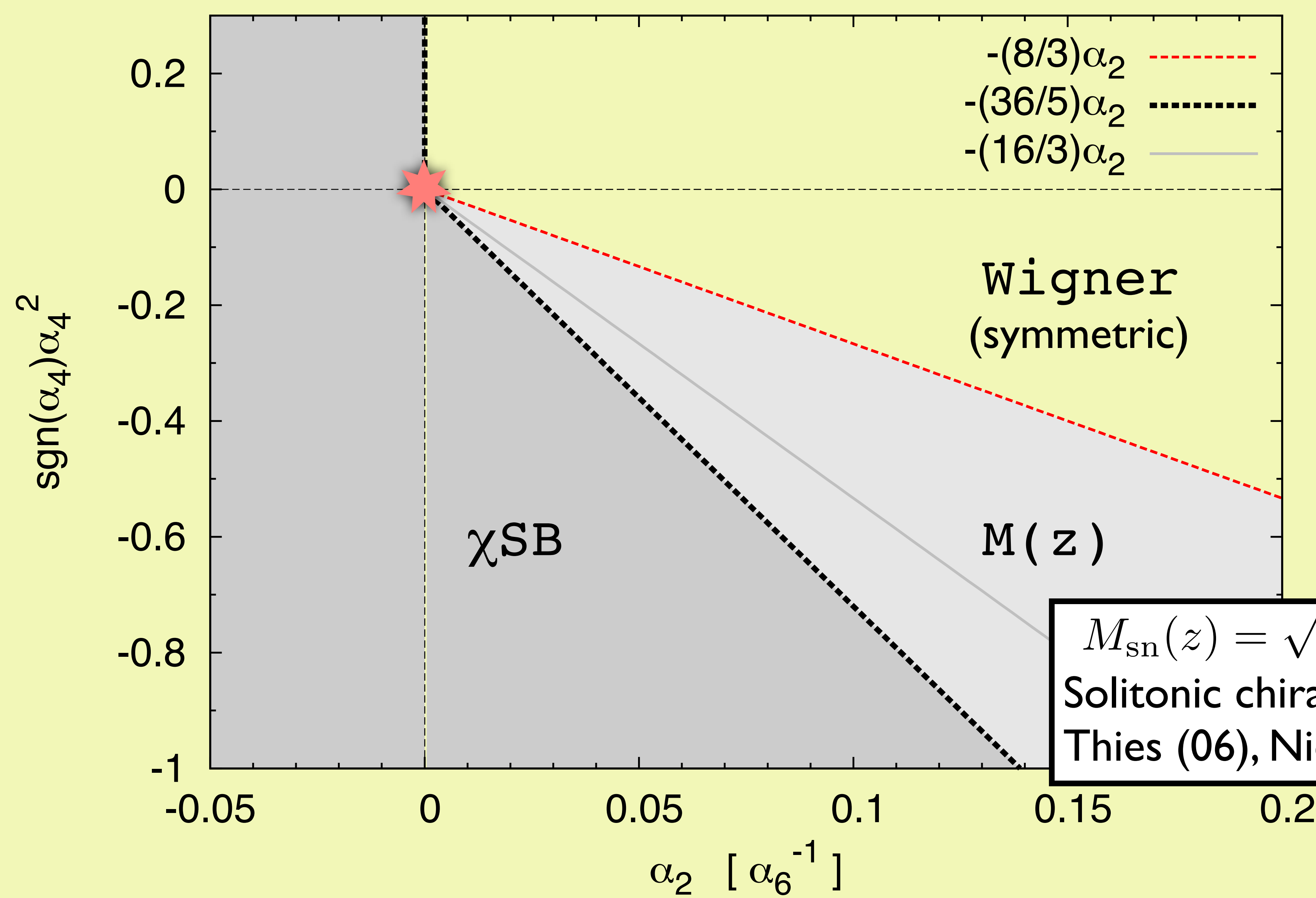


$M_{\rm sn}(z) = \sqrt{\nu} q \,{\rm sn}(qz, \nu)$

Solitonic chiral condensate

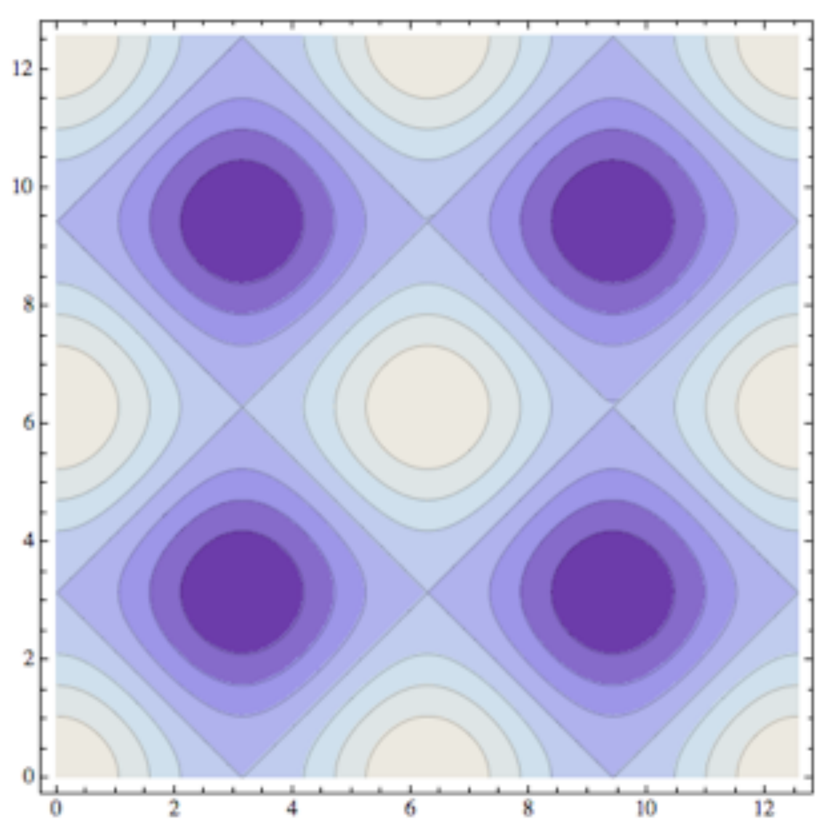
Thies (06), Nickel (09)

Multidimensional modulations?

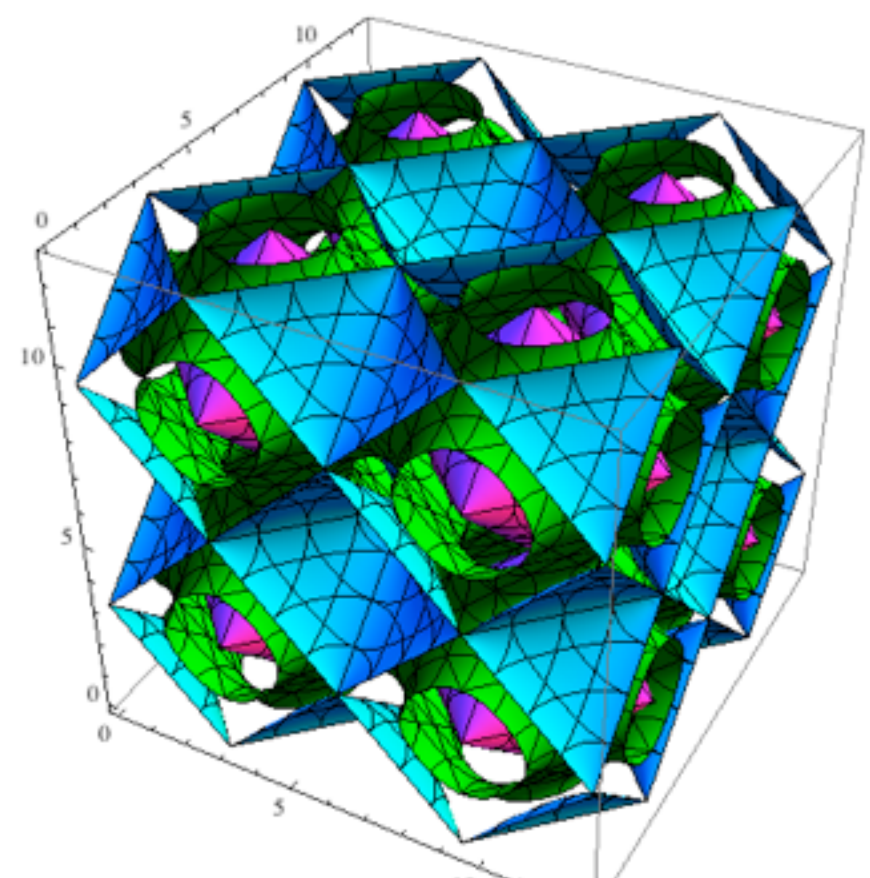
2D square lattice : $M_{\text{2D-LO}} = M_{ave}(\sin(qx) + \sin(qy))$

"egg-carton ansatz"; Carignano, Buballa (2011)

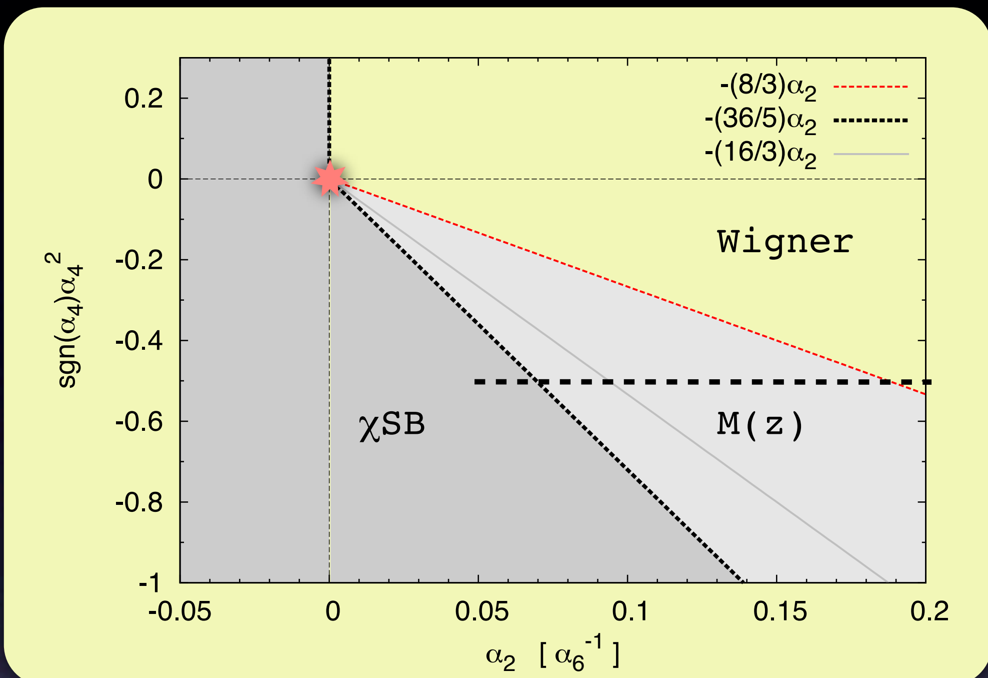
3D cubic lattice : $M_{\text{3D-LO}} = \sqrt{\frac{2}{3}} M_{ave}(\sin(qx) + \sin(qy) + \sin(qz))$

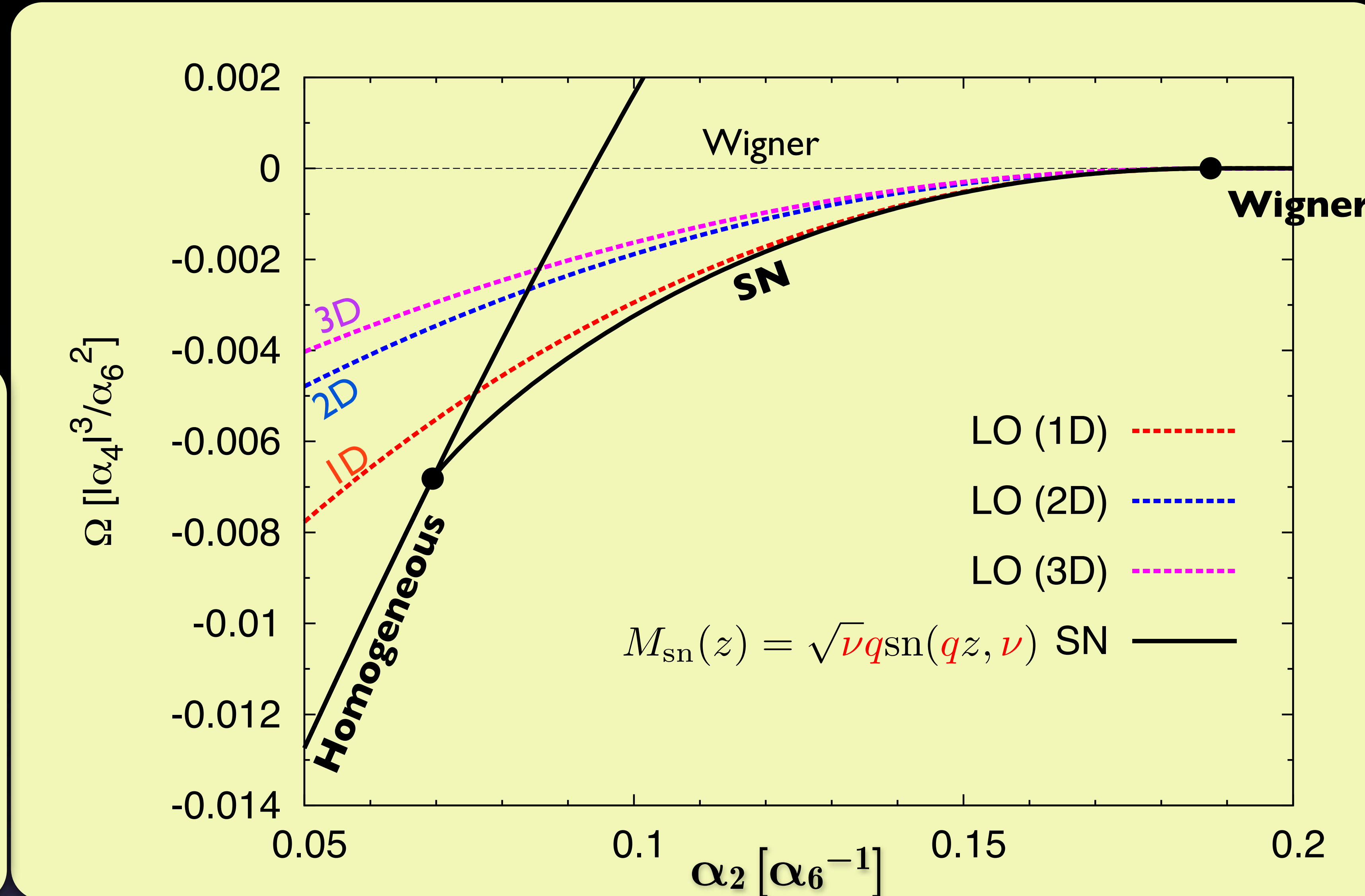


2D square lattice contour plot



3D cubic lattice contour plot

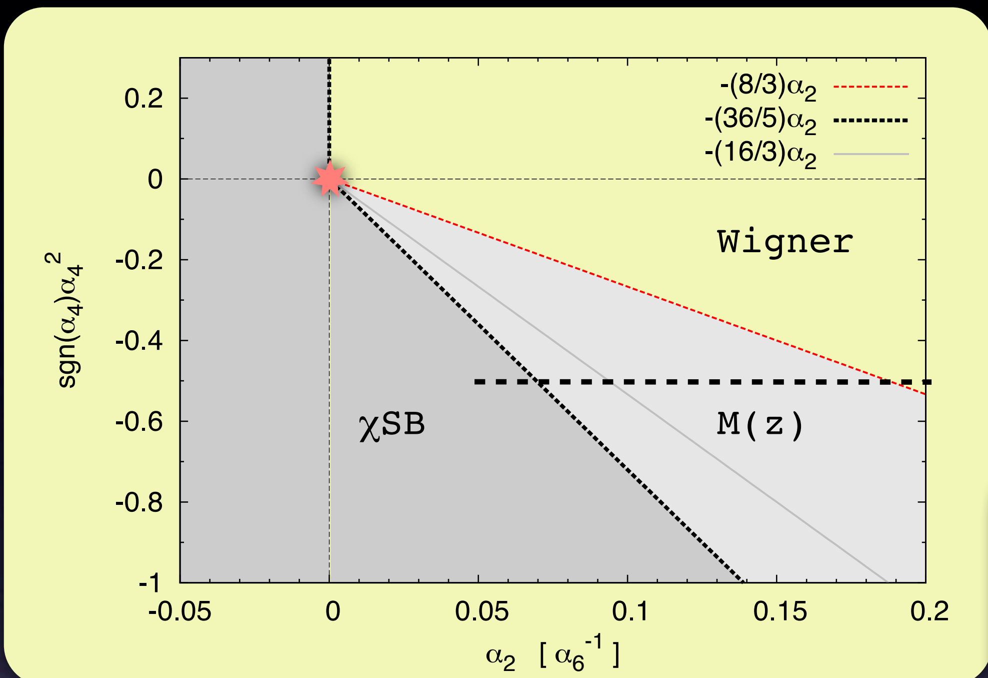


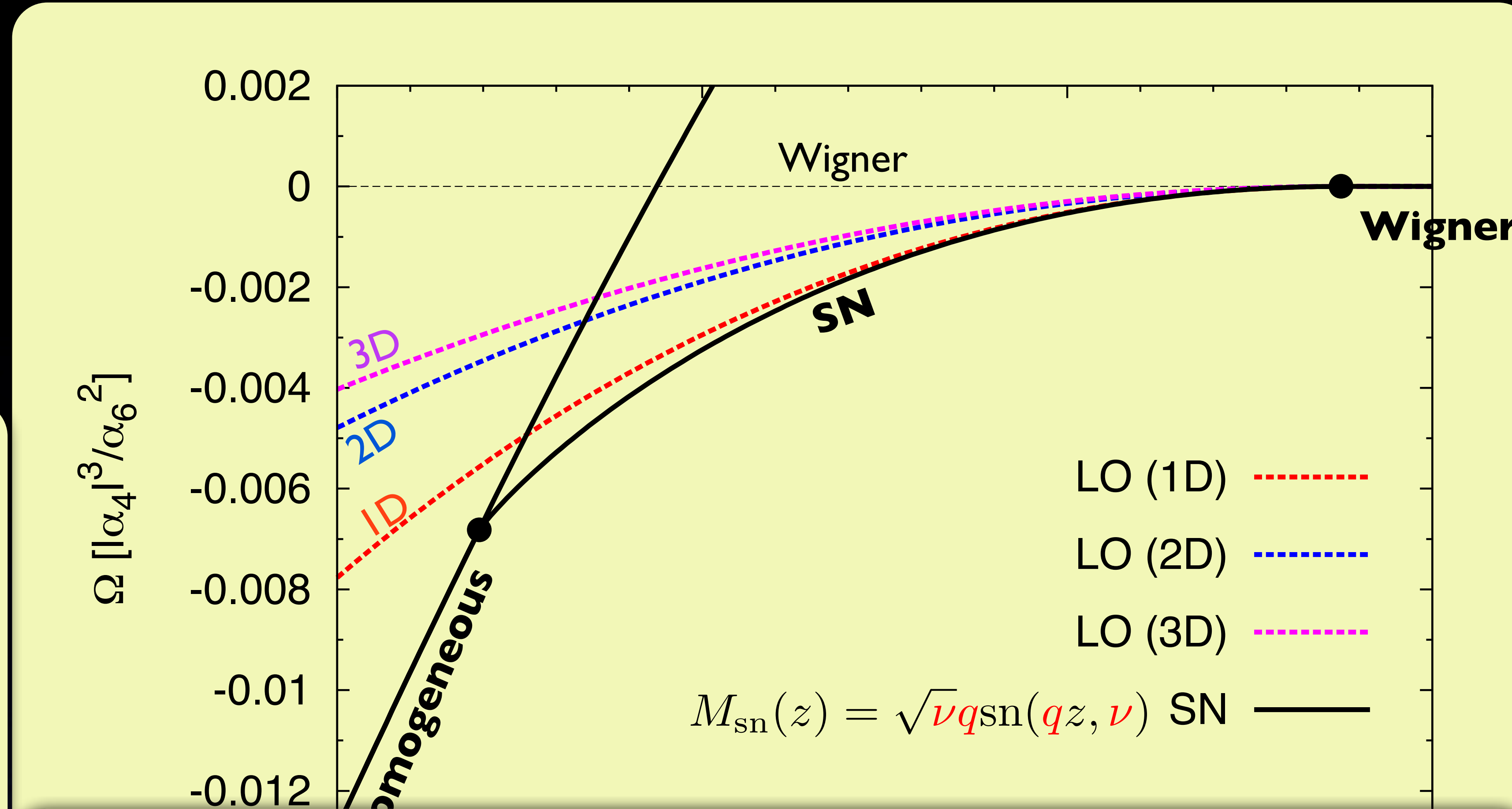


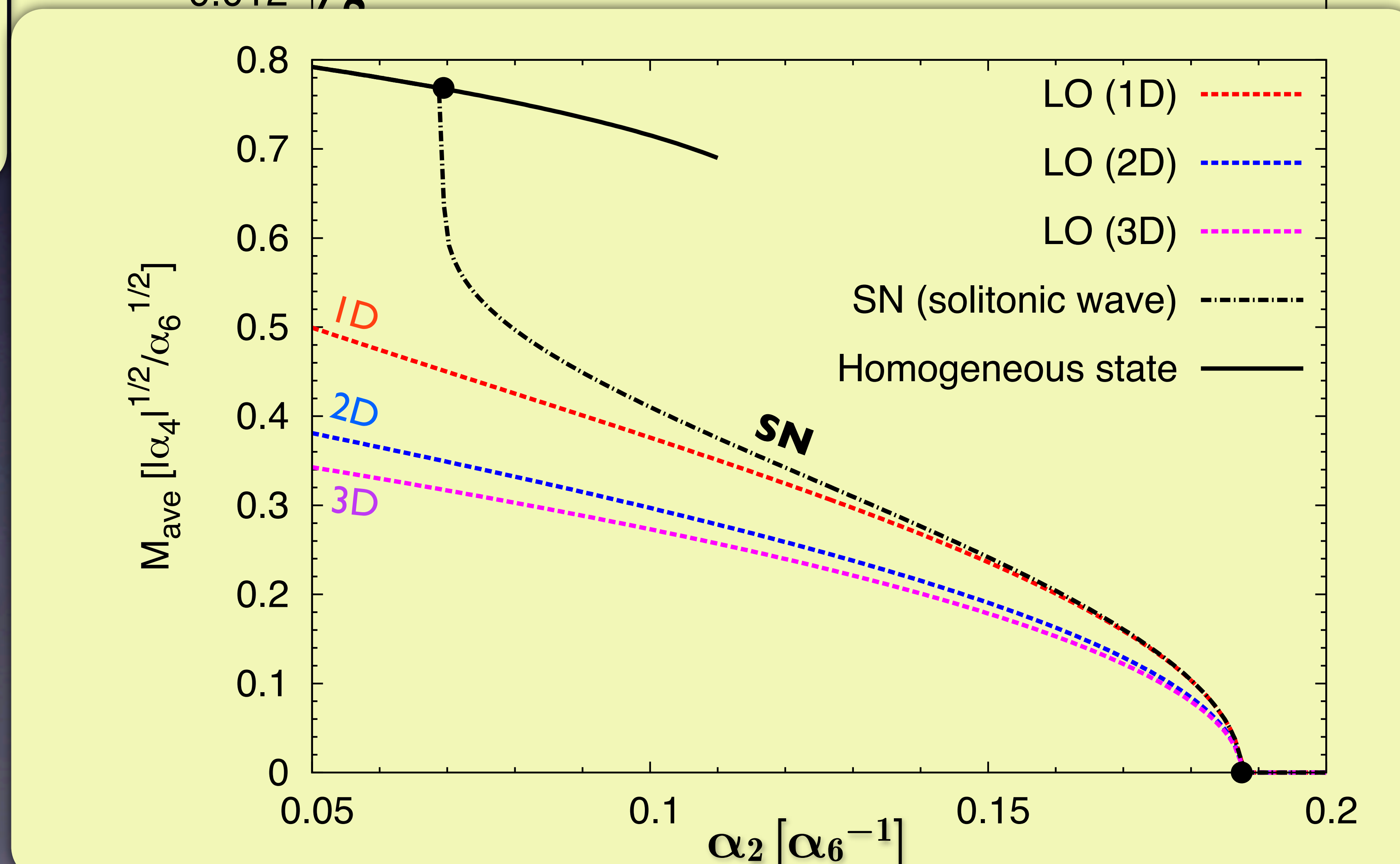
$$M_{\text{1D-LO}} = 2M_{ave}\sin(qx)$$

$$M_{\text{2D-LO}} = M_{ave}(\sin(qx) + \sin(qy))$$

$$M_{\text{3D-LO}} = \sqrt{\frac{2}{3}} M_{ave}(\sin(qx) + \sin(qy) + \sin(qz))$$







Averaged mass as
an order parameter

$$M_{\rm ave} = \sqrt{\langle M(\mathbf{x})^2 \rangle}$$

GL expansion at the critical point

- Optimizing over wavevector q, the potential can be expanded in powers of M

$$\Omega_{1\mathrm{D}} = \left(\frac{a_2}{2} - \frac{3a_4^2}{16}\right) M_{ave}^2 + \frac{6|a_4|}{24} M_{ave}^4 + \mathcal{O}(M_{ave}^6)$$

$$\Omega_{2\mathrm{D}} = \left(\frac{a_2}{2} - \frac{3a_4^2}{16}\right) M_{ave}^2 + \frac{9|a_4|}{24} M_{ave}^4 + \mathcal{O}(M_{ave}^6)$$

$$\Omega_{3\mathrm{D}} = \left(\frac{a_2}{2} - \frac{3a_4^2}{16}\right) M_{ave}^2 + \frac{10|a_4|}{24} M_{ave}^4 + \mathcal{O}(M_{ave}^6)$$

GL expansion at the critical point

- Optimizing over wavevector q, the potential can be expanded in powers of M

$$
\begin{aligned}
\Omega_{1\mathrm{D}} &= \boxed{\left(\frac{a_2}{2} - \frac{3a_4^2}{16}\right) M_{ave}^2} + \frac{6|a_4|}{24} M_{ave}^4 + \mathcal{O}(M_{ave}^6) \\
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$$

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quadratic coeff. $\rightarrow$ critical point is shared

GL expansion at the critical point

- Optimizing over wavevector q, the potential can be expanded in powers of M

$$\Omega_{1D} = \left(\frac{a_2}{2} - \frac{3a_4^2}{16}\right) M_{ave}^2 + \frac{6|a_4|}{24} M_{ave}^4 + \mathcal{O}(M_{ave}^6)$$

$$\Omega_{2D} = \left(\frac{a_2}{2} - \frac{3a_4^2}{16}\right) M_{ave}^2 + \frac{9|a_4|}{24} M_{ave}^4 + \mathcal{O}(M_{ave}^6)$$

$$\Omega_{3D} = \left(\frac{a_2}{2} - \frac{3a_4^2}{16}\right) M_{ave}^2 + \frac{10|a_4|}{24} M_{ave}^4 + \mathcal{O}(M_{ave}^6)$$

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GL expansion at the critical point

- Optimizing over wavevector q, the potential can be expanded in powers of M

$$\Omega_{1\mathrm{D}} = \left(\frac{a_2}{2} - \frac{3a_4^2}{16}\right) M_{ave}^2 + \frac{6|a_4|}{24} M_{ave}^4 + \mathcal{O}(M_{ave}^6)$$

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quartic coeff. $\rightarrow$ $\Omega_{1\mathrm{D}} < \Omega_{2\mathrm{D}} < \Omega_{3\mathrm{D}}$

Aims

2. How does the phase structure change going away from the critical point?

Need to go beyond 6th order

- Derivative expansion straightforwardly applied Abuki, Ishibashi, Suzuki (2011)

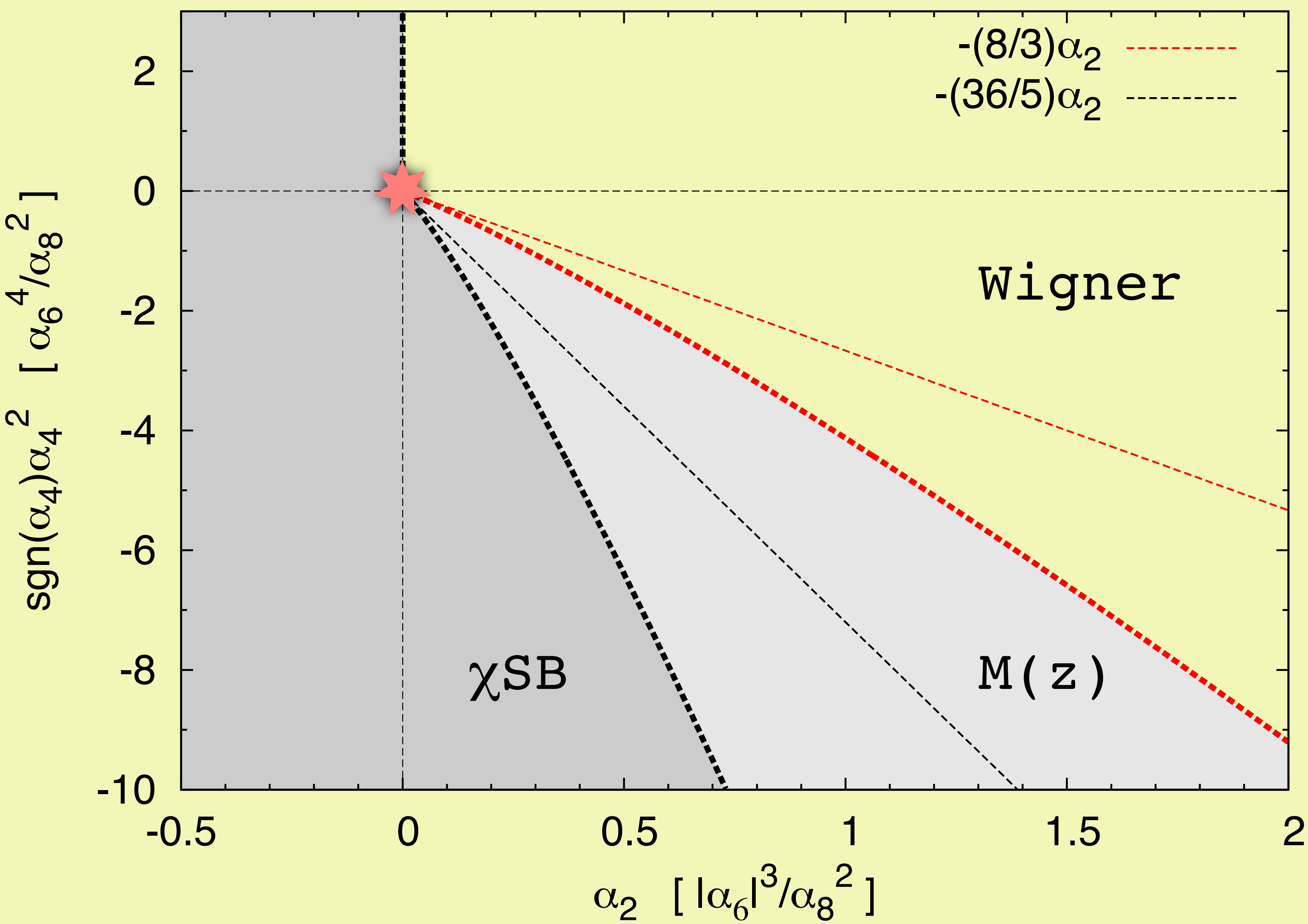
$$
\begin{aligned}
\Omega_{\mathrm{GL}} &= \frac{\alpha_2}{2} M(\mathbf{x})^2 + \frac{\alpha_4}{4} M(\mathbf{x})^4 + \frac{\alpha_6}{6} M(\mathbf{x})^6 + \boxed{\frac{\alpha_8}{8} M(\mathbf{x})^8} \\
&\quad + \frac{\alpha_4}{4} (\nabla M)^2 + \frac{5\alpha_6}{6} M^2 (\nabla M)^2 + \frac{\alpha_6}{12} (\nabla \Delta M)^2 \\
&\quad \boxed{\begin{aligned} &+ \frac{\alpha_8}{8} \left(\xi_2 M^4 (\nabla M)^2 + \xi_{4a} (\nabla M)^4 + \xi_{4b} M \Delta M (\nabla M)^2 \right. \\ &\left. + \xi_{4c} M^2 (\Delta M)^2 + \xi_6 (\nabla \Delta M)^2 \right) \end{aligned}}
\end{aligned}
$$

★ Quark loop diagrams yield:

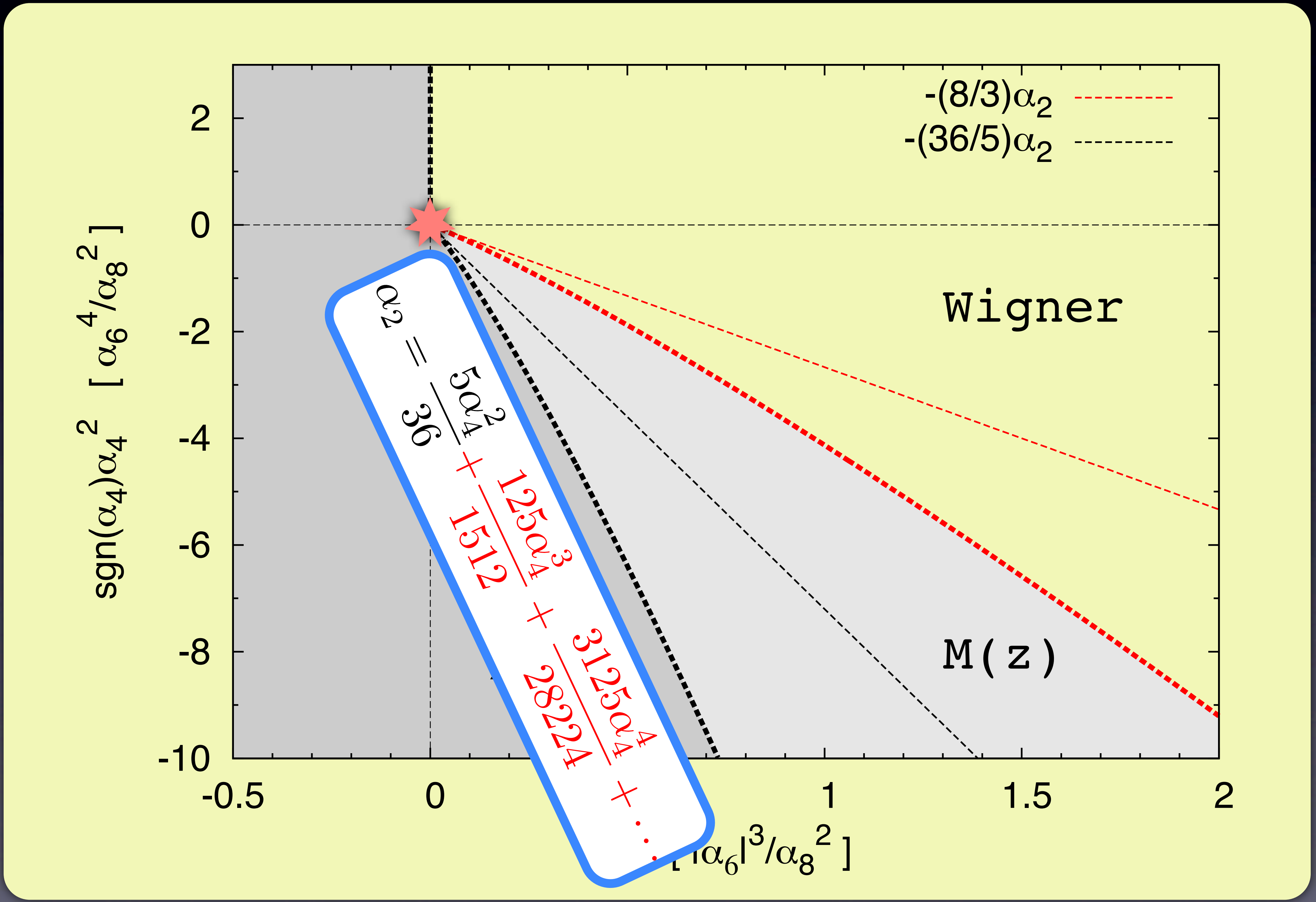
$$
\xi_2 = 14, \quad \xi_{4a} = -\frac{1}{5}, \quad \xi_{4b} = \frac{18}{5}, \quad \xi_{4c} = \frac{14}{5}, \quad \xi_6 = \frac{1}{5}
$$

★ Only one additional parameter $\alpha_8 (>0)$!

Departing the critical point ($\alpha_6 > 0$)



Departing the critical point ($\alpha_6 > 0$)



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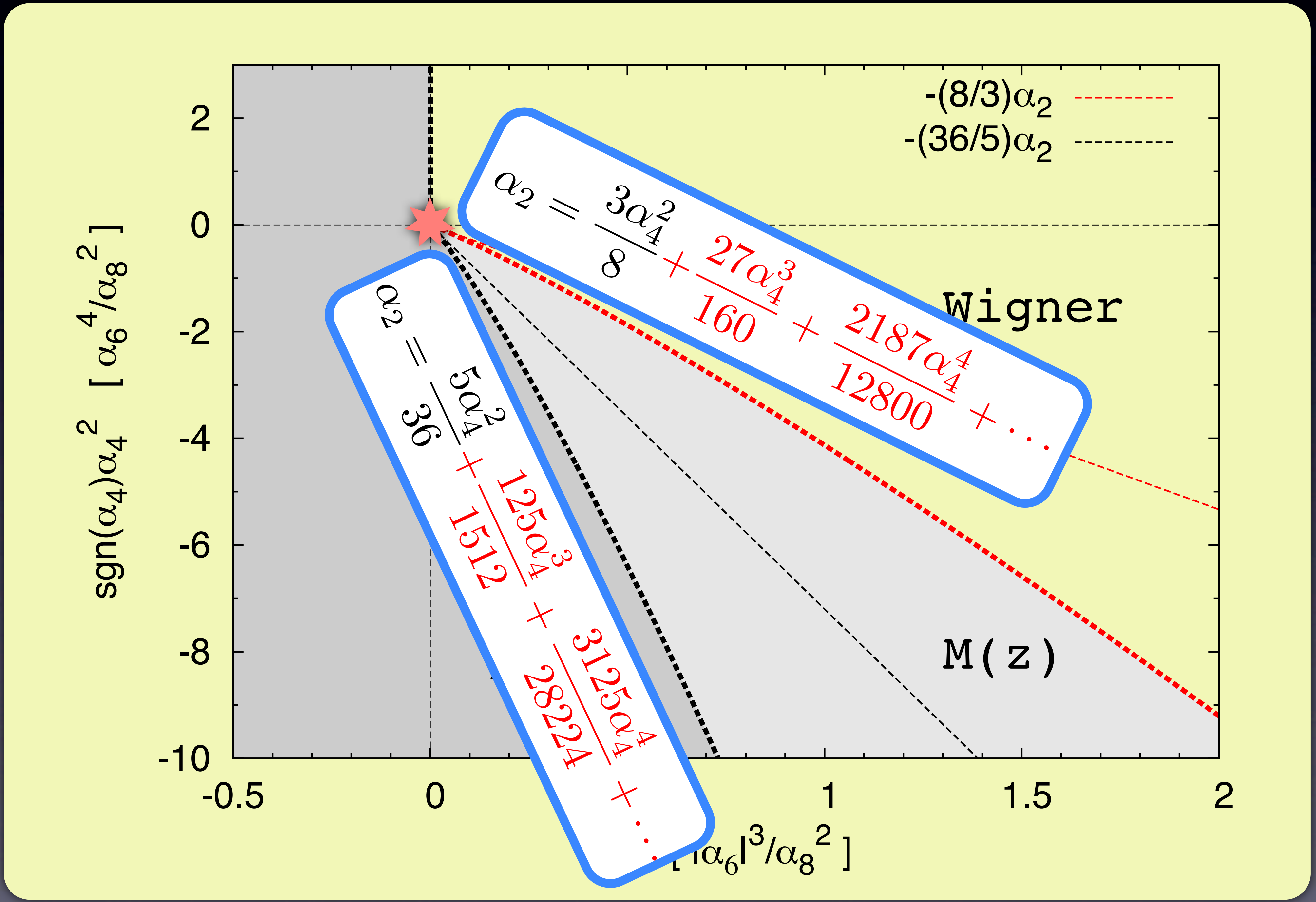
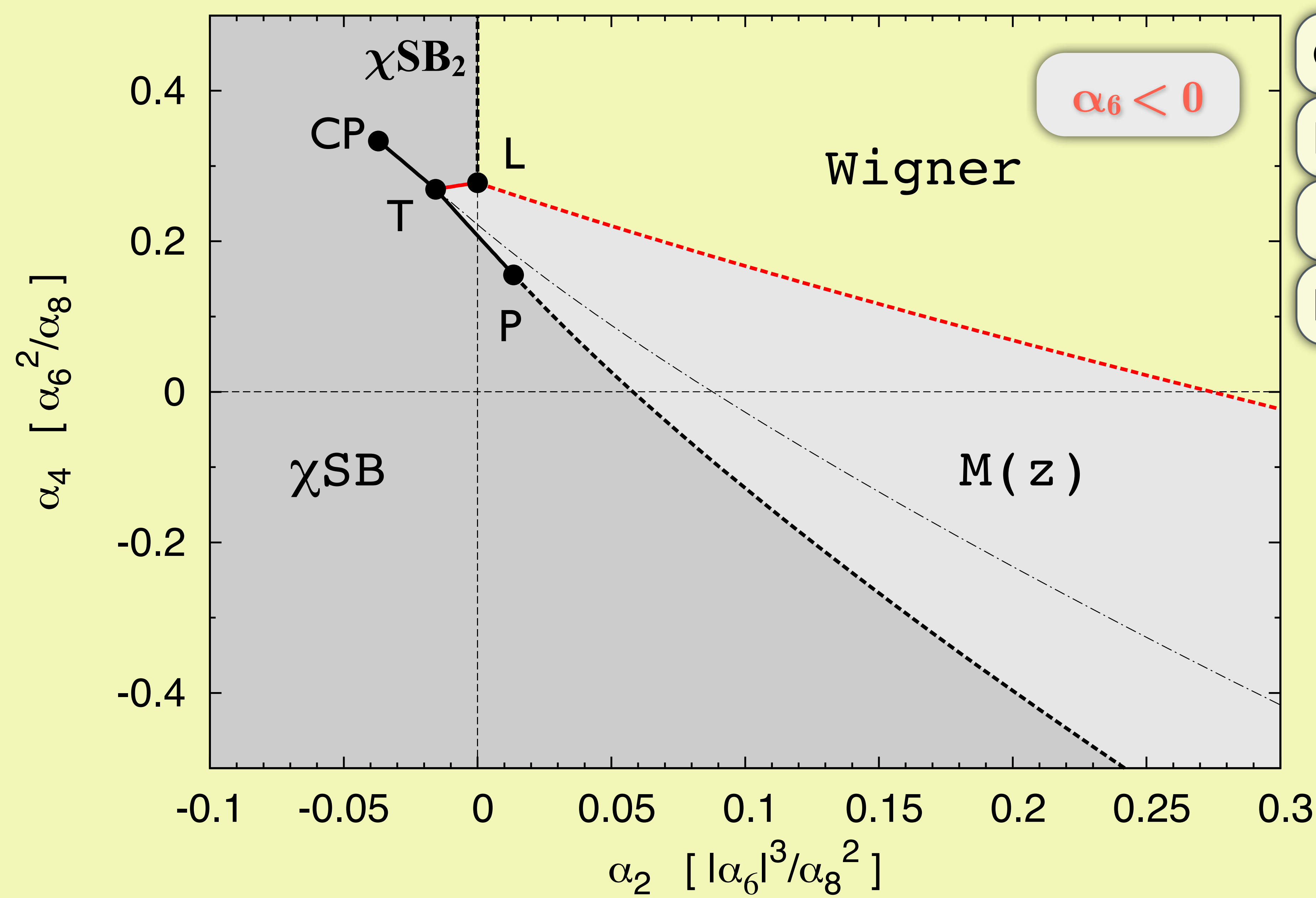


Going to new regime: $\alpha_6 < 0$



CP: (-1/27, 1/3)

L: (0, 5/18)

T: (-0.016, 0.27)

P: (0.014, 0.16)

Characterizing the triple point

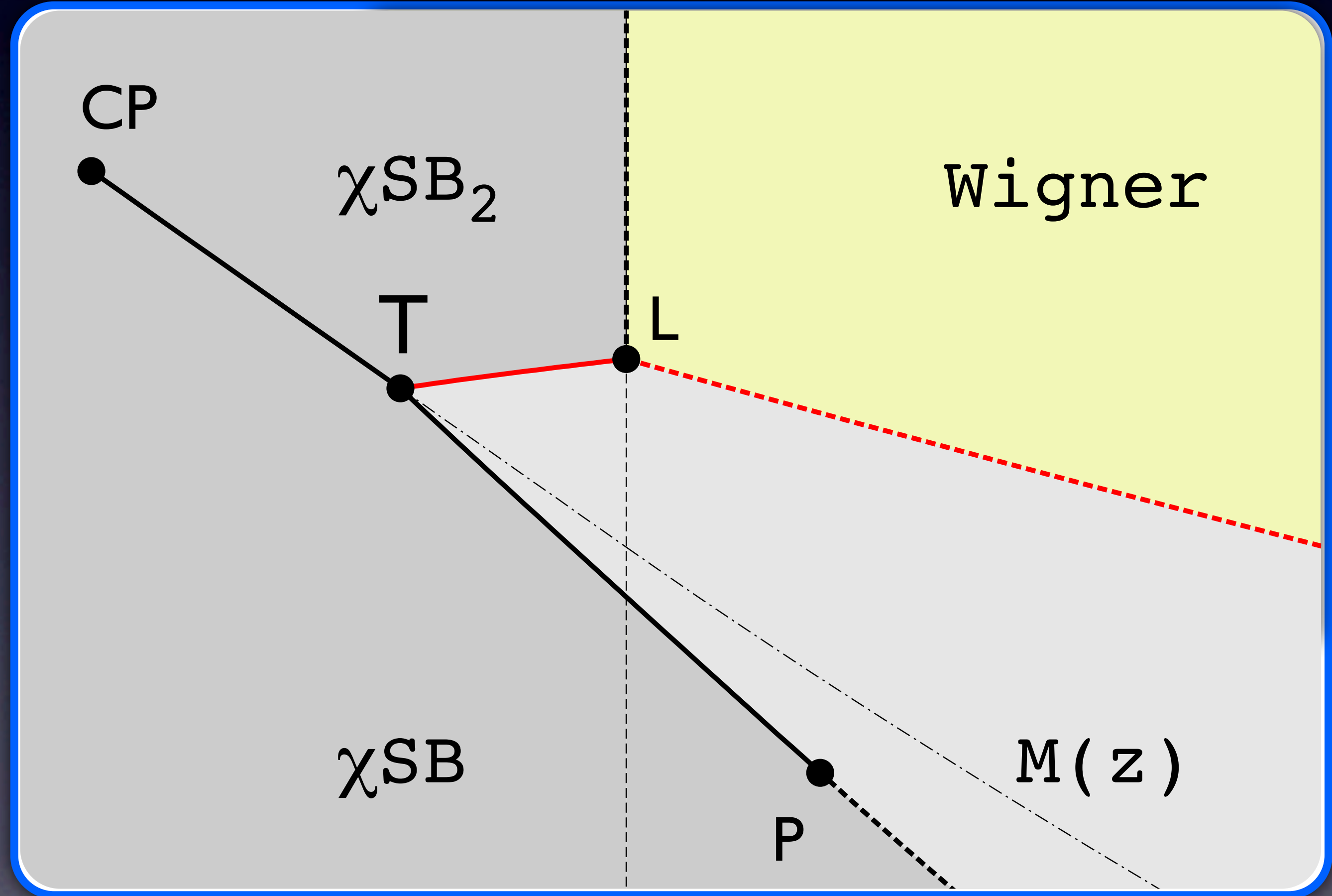
- Three different forms of sym.-broken phase compete with each other and coexist at the triple point (T)!

$$\frac{M_{\chi_2}}{M_\chi} \to 0.369..$$

$$\frac{\sqrt{\langle M(z)^2\rangle}}{M_\chi} \to 0.308..$$

$$\frac{q}{\sqrt{\langle M(z)^2\rangle}} \to 5.001..$$

$$\left(q \equiv \frac{2\pi}{L}\right)$$



- Model independent universal ratios associated with T

Summary

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- Multidimensional crystals, not favored!
- Extended the previous GL analysis to higher order and explored the phase structure away from the Lifshitz point.
  - ★ if $\alpha_6 > 0$, non-linear effects; but qualitative phase structure unchanged.
  - ★ If $\alpha_6 < 0$, phase structure becomes rather rich;
    Existence of Triple Point (T)!

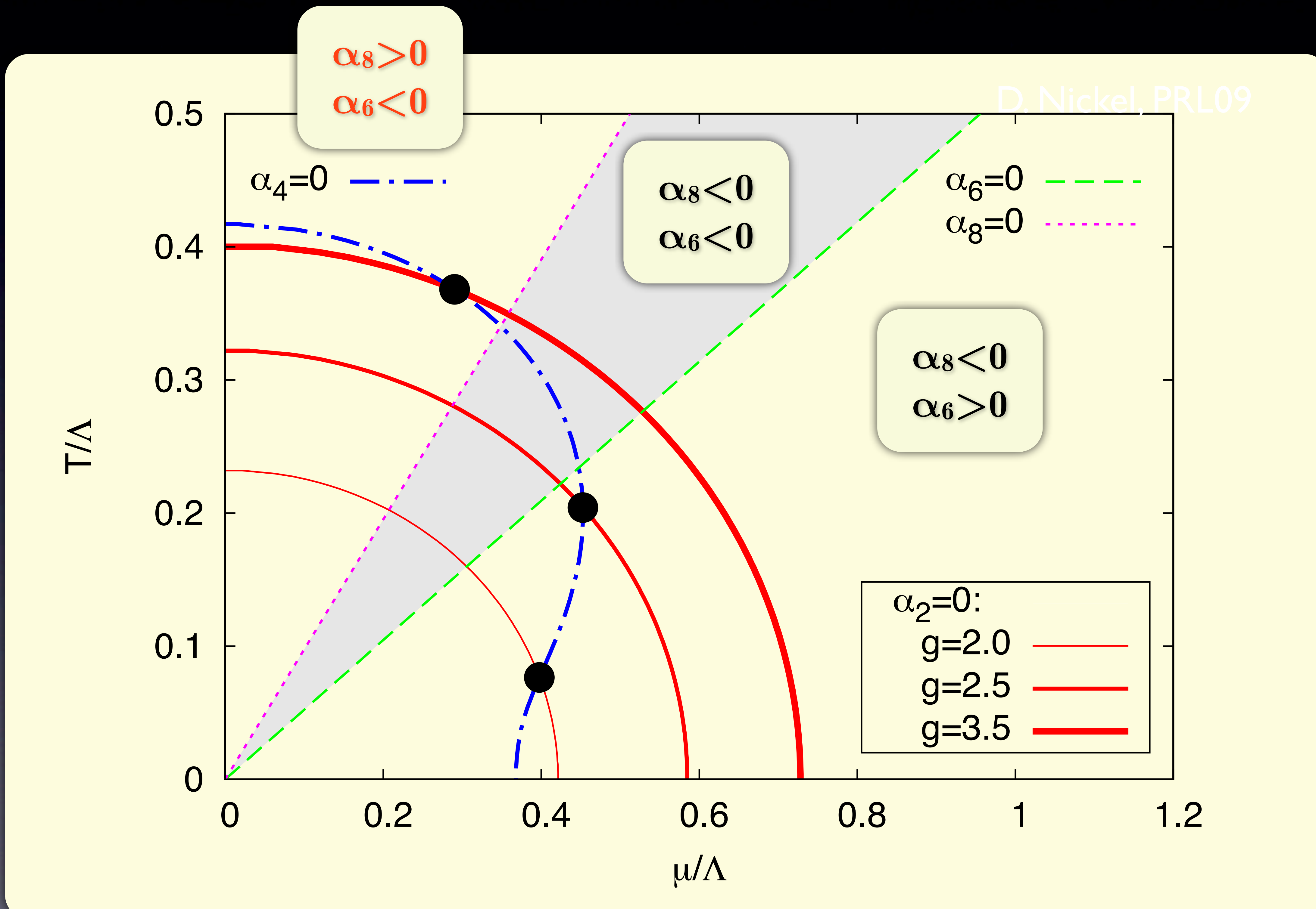
Further questions

- Do we really have no suitable multi-dimensional chiral crystal? If no, how could one dimensional structure be stabilized against thermal fluctuations? c.f. Landau-Peierls Theorem
- Triple point in model? How Polyakov loop and/or vector interaction change α_6 at TCP?
- How does *isospin asymmetry* affect the TCP and phase structure of its neighborhood?

[see poster contribution by *Yuhei Iwata*]

Backups

How does GL map onto NJL phase diagram



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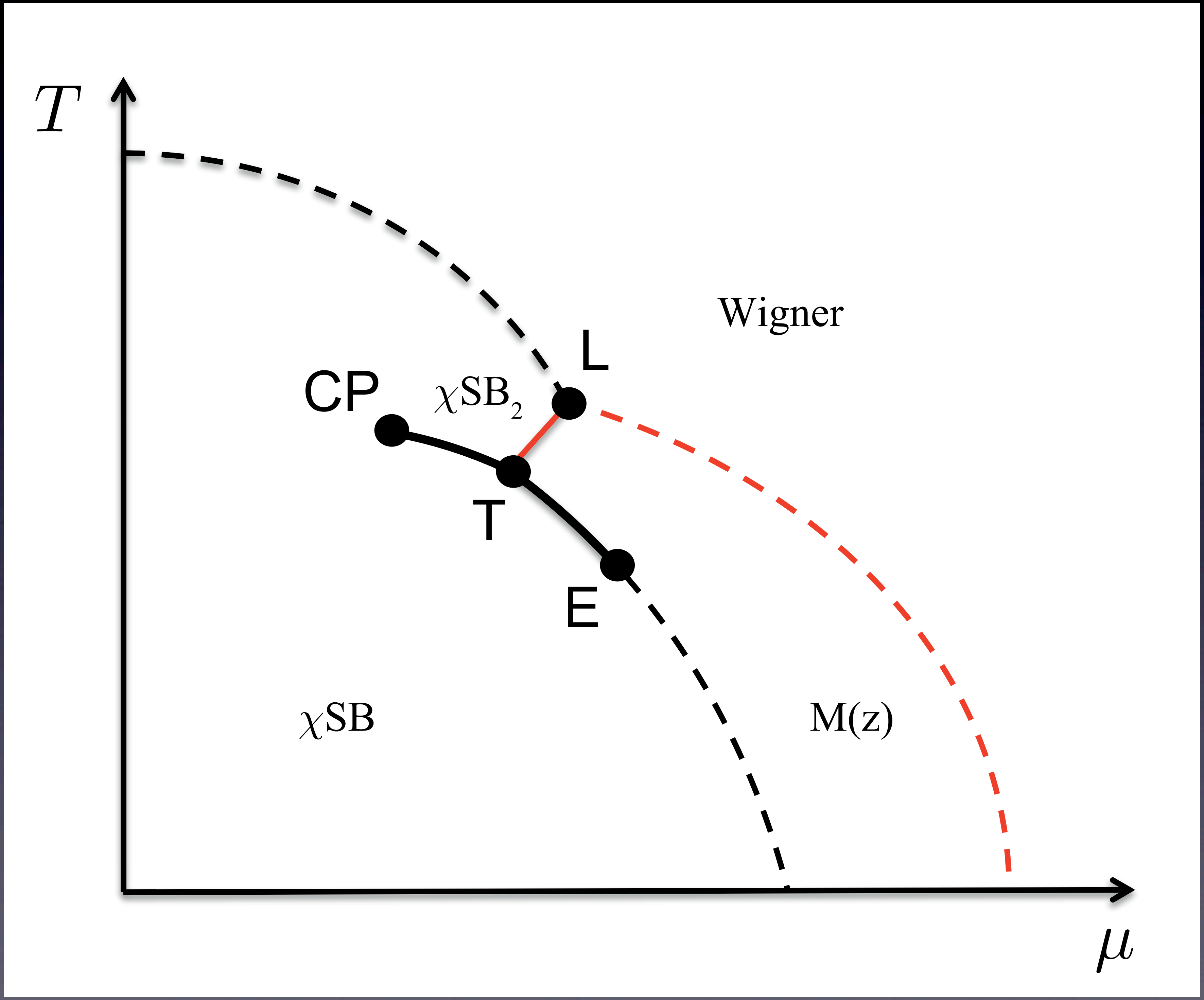


Dimensional analysis and Scaling

- Introducing dimensionless variables:

$$\begin{aligned} \alpha_2 &= \eta_2\,[\alpha_4^2/\alpha_6] & \Omega &= \omega\,[|\alpha_4|^3/\alpha_6^2] \\ M &= m\,[\sqrt{|\alpha_4|/\alpha_6}] & \mathbf{x} &= \tilde{\mathbf{x}}\,[\sqrt{\alpha_6/|\alpha_4|}] \end{aligned}$$

- Relevant parameters are reduced:

$$\begin{aligned} \omega &= \frac{\eta_2}{2} m^2 + \frac{\mathrm{sgn}(\alpha_4)}{4}(m^4 + (\tilde{\nabla} m)^2) \\ &\quad + \tfrac{1}{6}\left(m^6 + 5m^2(\tilde{\nabla} m)^2 + \tfrac{1}{2}(\tilde{\nabla}\tilde{\Delta} m)^2\right) \end{aligned}$$

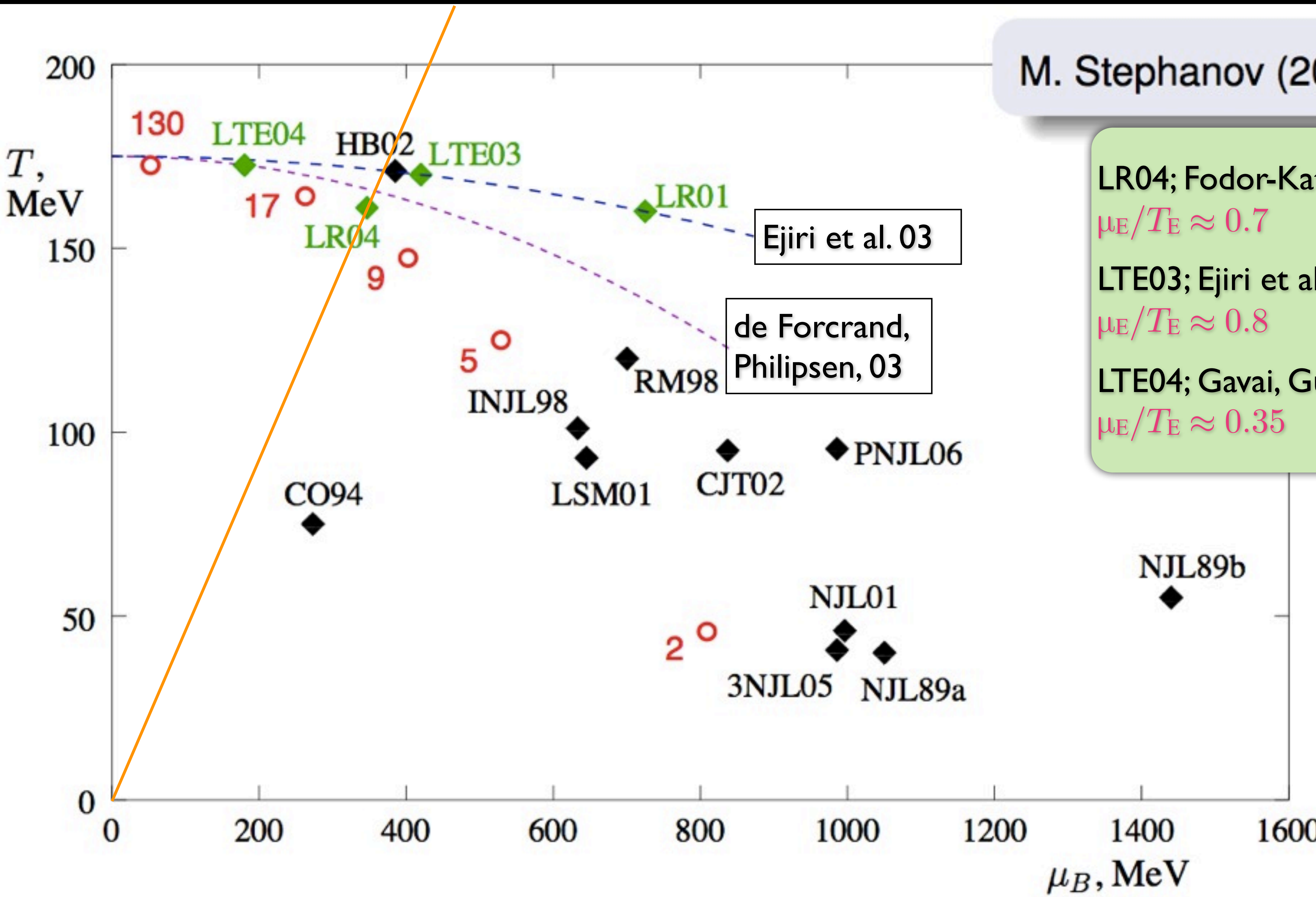
- Inhomogeneous phase appears when α_4 becomes negative

One dimensional modulations

Chiral Condensate: $M(\mathbf{x}) = -2G\left(\langle\bar{q}q\rangle + i\langle\bar{q}i\gamma_5\tau_3 q\rangle\right)$

Fulde-Ferrell (1964) Chiral spiral Nakano-Tatsumi (2005)	$M_{\mathrm{FF}}(z) = \Delta e^{iqz}$	**Complex**
Larkin-Ovchinnikov (1964) CDW; Nickel (2009)	$M_{\mathrm{LO}}(z) = \Delta\sin(qz)$	**Real**
Solitonic chiral condensate Buzdin, Kachkachi (1997) Thies (2006), Nickel (2009)	$M_{\mathrm{sn}}(z) = \sqrt{\nu}q\,\mathrm{sn}(qz,\nu)$	**Real**
Higher harmonics truncated at N=5	$M_{\mathrm{hh}}(z) = \sum_{-N}^{N}\Delta_n e^{inqz}$	**Complex**

M. Stephanov (2007)

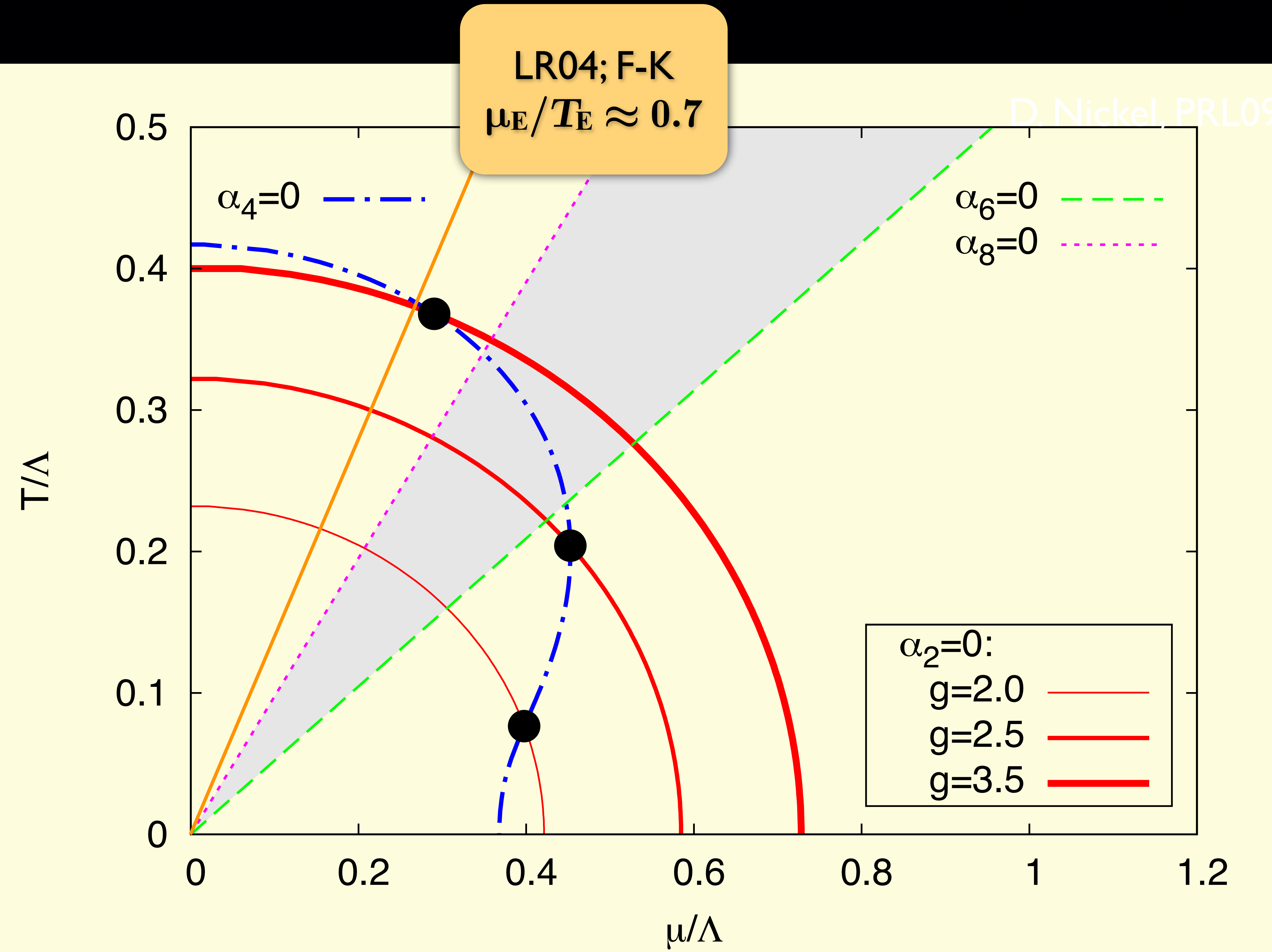


LR04; Fodor-Katz;
$\mu_E/T_E \approx 0.7$

LTE03; Ejiri et al.,
$\mu_E/T_E \approx 0.8$

LTE04; Gavai, Guputa
$\mu_E/T_E \approx 0.35$

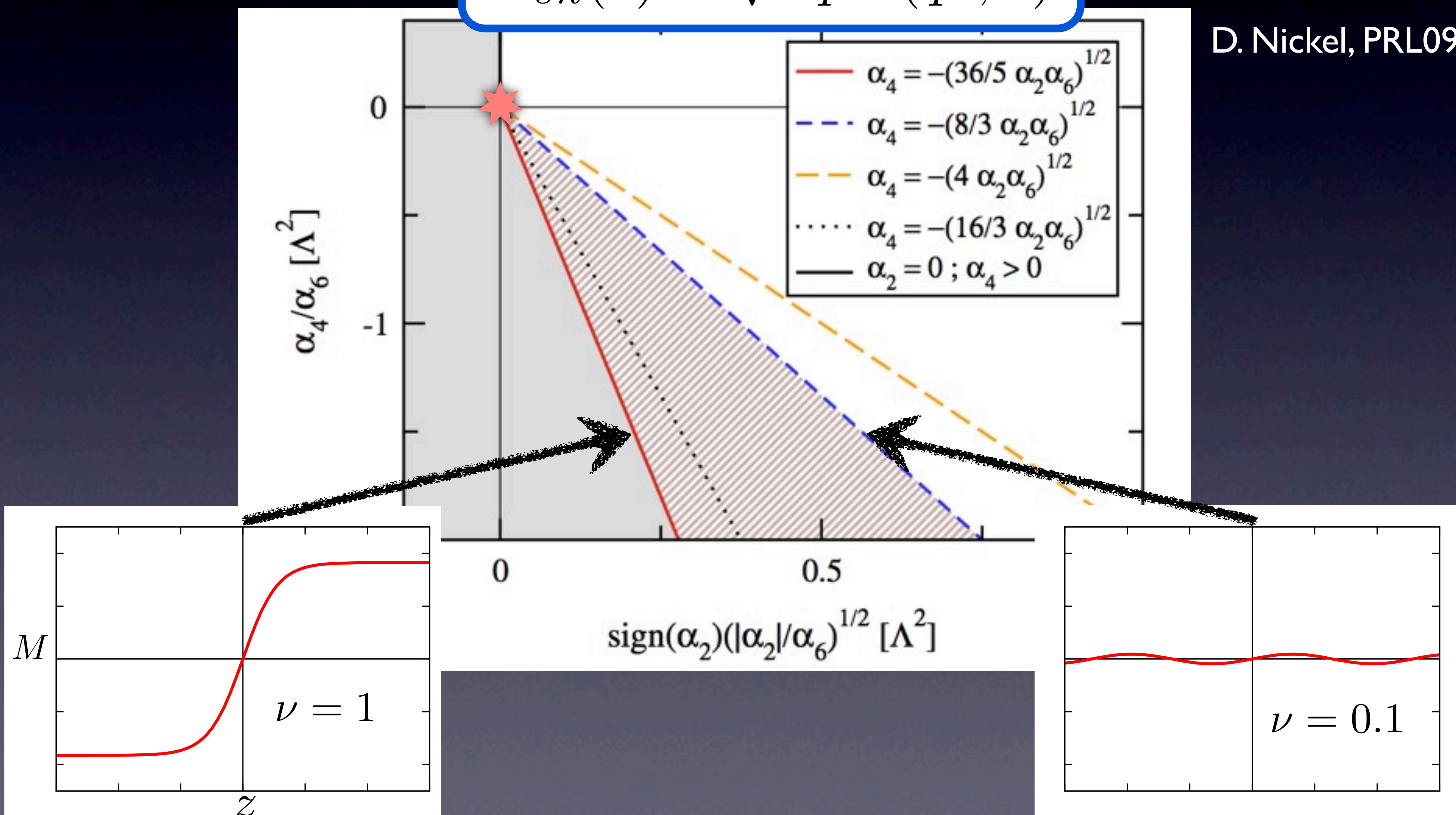
How does GL map onto (μ, T)-phase diagram



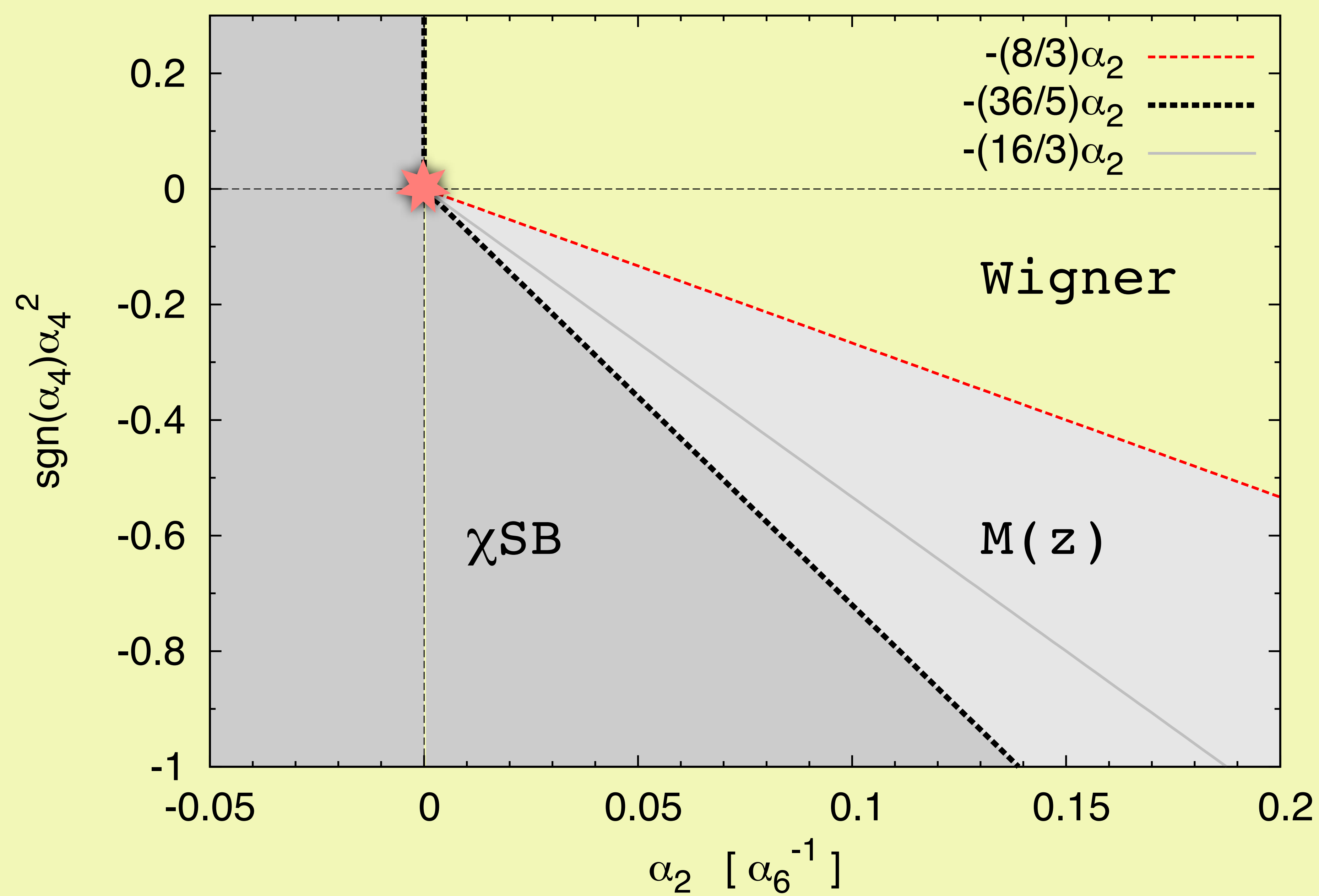
GL in the vicinity of critical point

$$M_{sn}(z) = \sqrt{\nu} q \, \mathrm{sn}(qz, \nu)$$

D. Nickel, PRL09



Soliton formation



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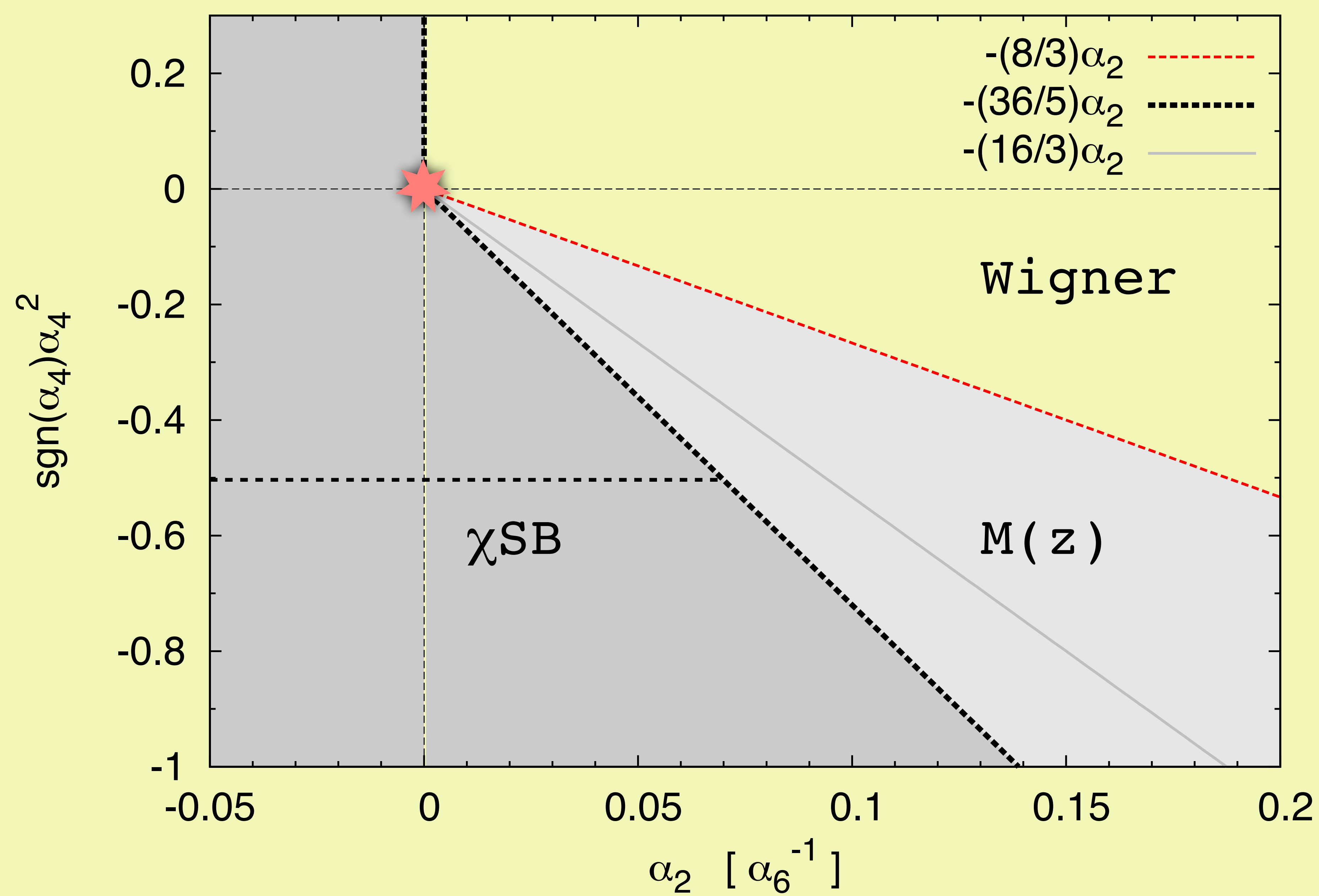
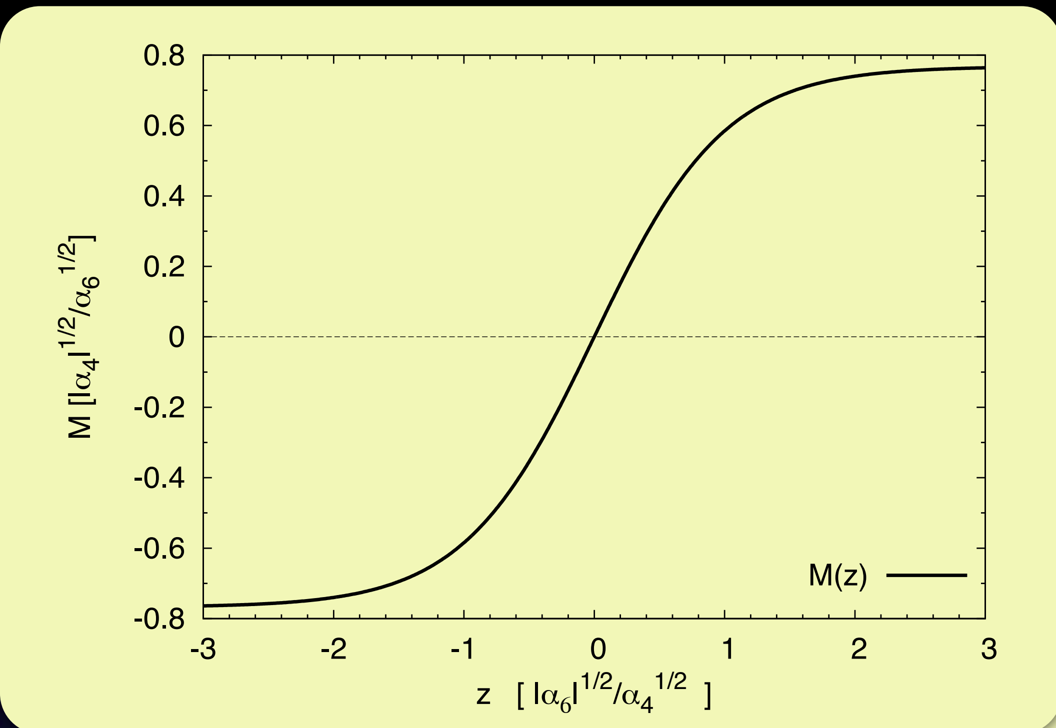
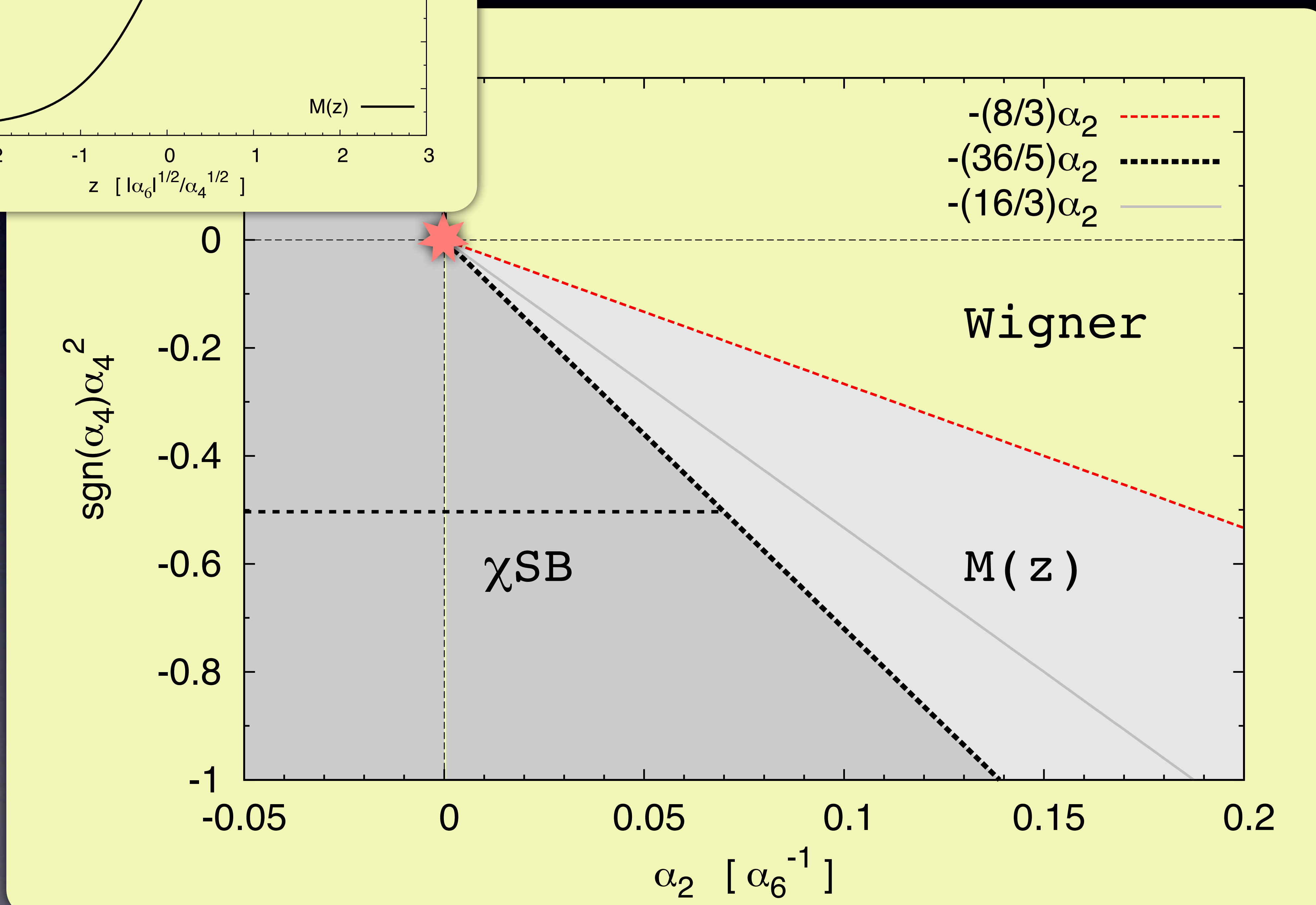
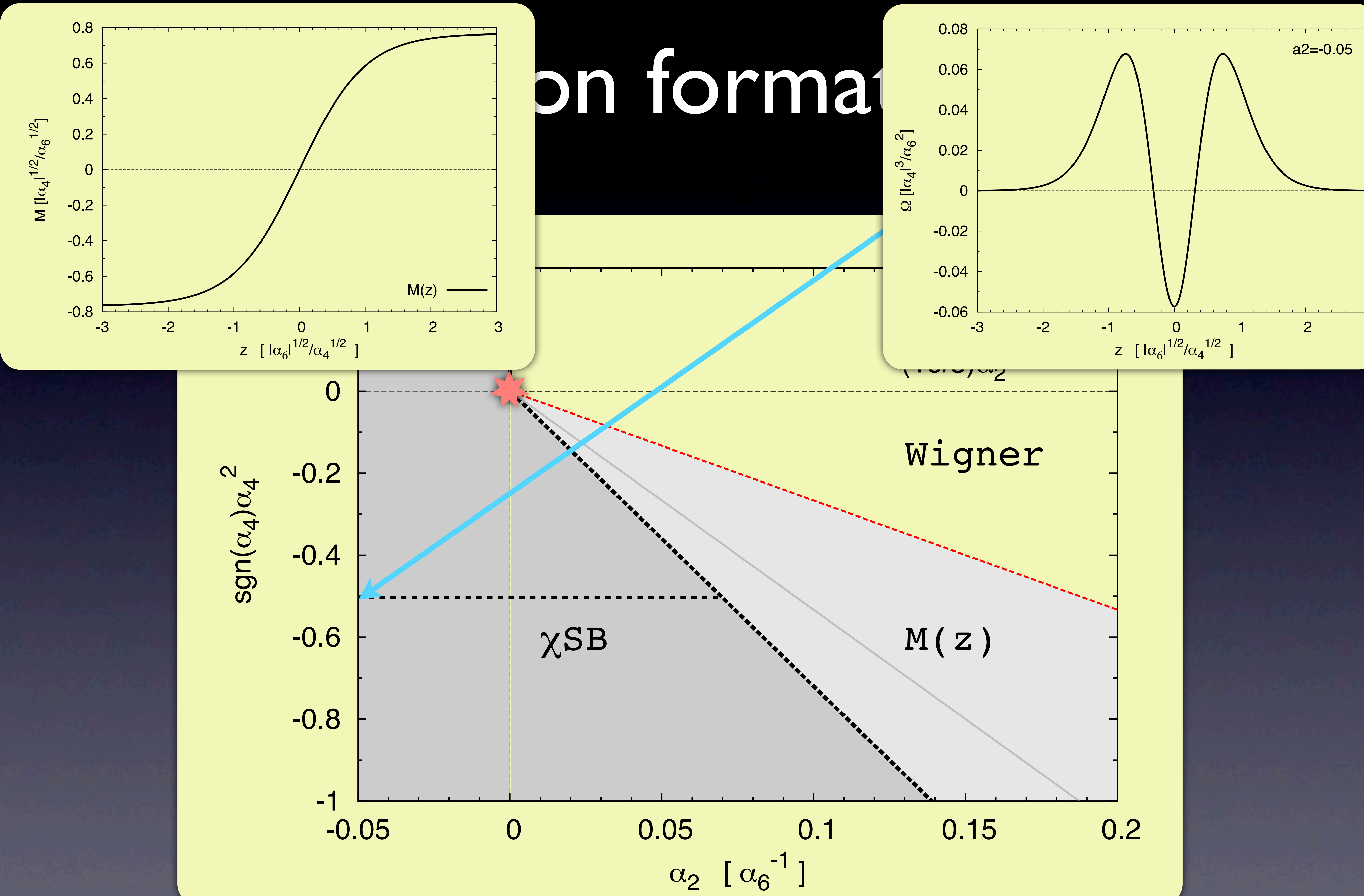


on formation



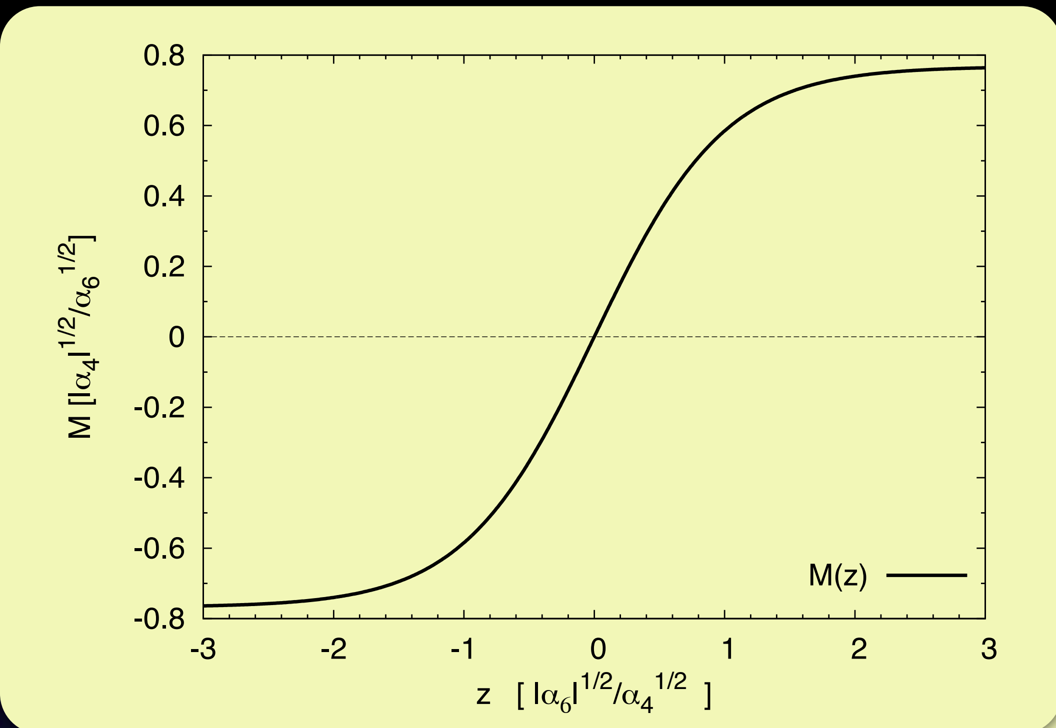


...on format...



M(z)
M [$|\alpha_4|^{1/2}/\alpha_6^{1/2}$]
z [$|\alpha_6|^{1/2}/\alpha_4^{1/2}$]
a2=-0.05
Ω [$|\alpha_4|^3/\alpha_6^2$]
z [$|\alpha_6|^{1/2}/\alpha_4^{1/2}$]
Wigner
χSB
M(z)
sgn(α_4)α_4^2
α_2 [α_6^{-1}]

on format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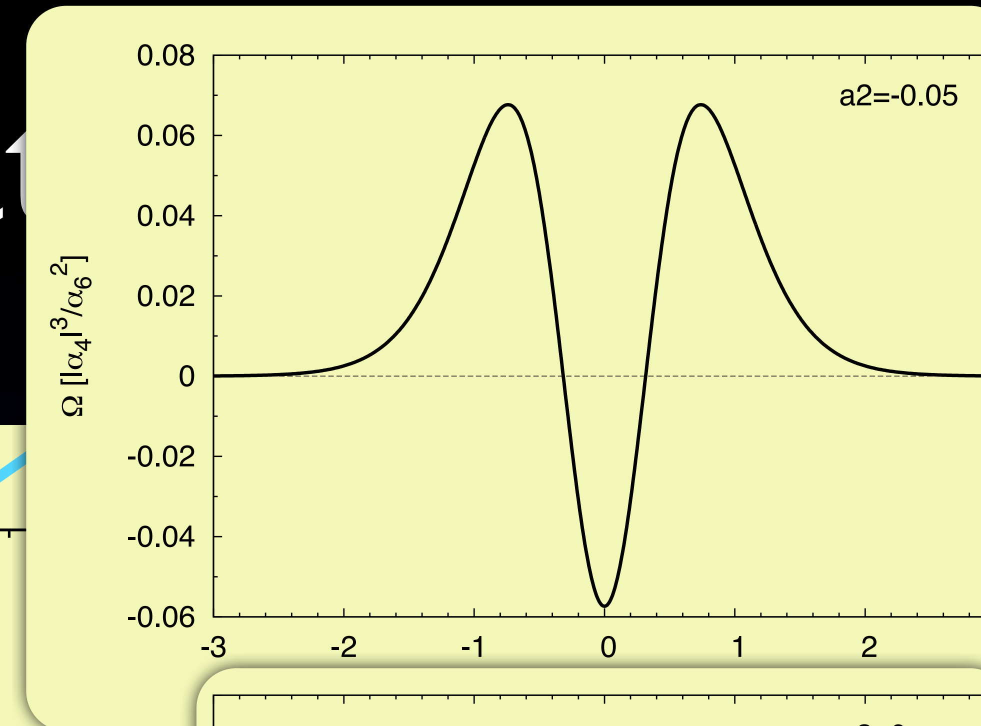
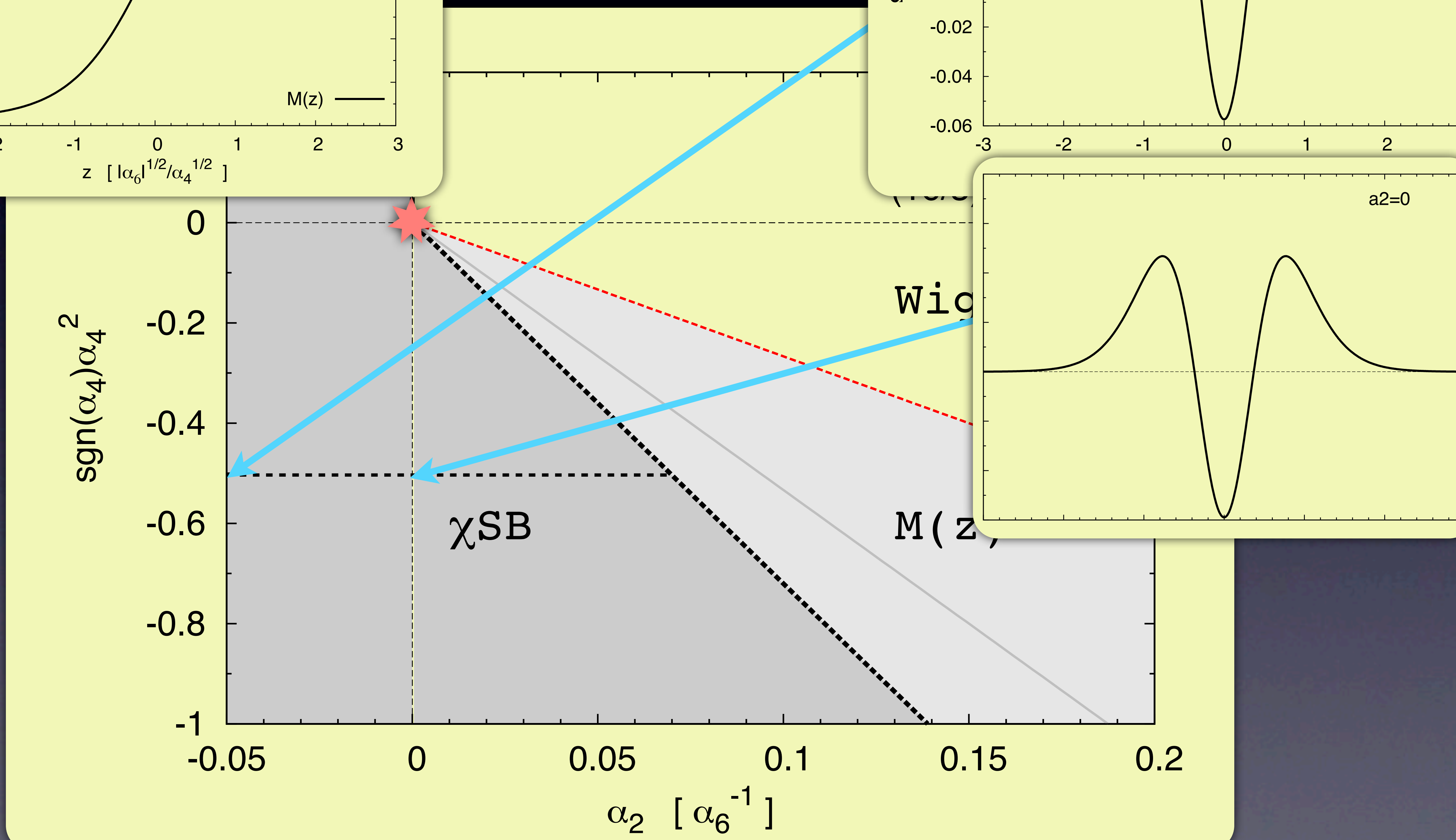
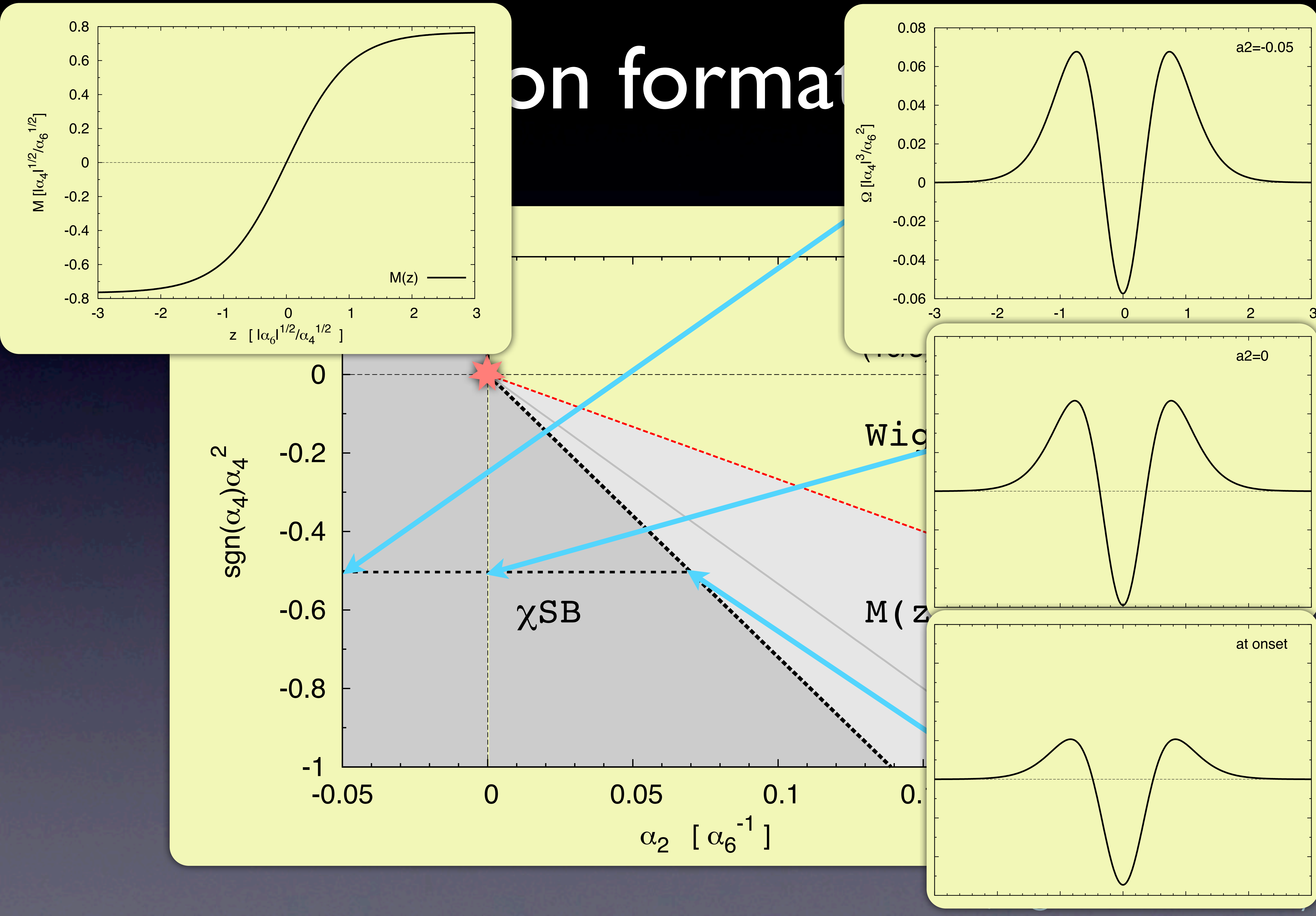
M $[|\alpha_4|^{1/2}/\alpha_6^{1/2}]$
z $[|\alpha_6|^{1/2}/\alpha_4^{1/2}]$


a2=-0.05
$\Omega\ [|\alpha_4|^3/\alpha_6^2]$


a2=0
χSB
M(z
Wig
$\mathrm{sgn}(\alpha_4)\alpha_4^2$
$\alpha_2\ [\alpha_6^{-1}]$


on format
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-3
-2
-1
0
1
2
3
M [|α4|^1/2/α6^1/2]
z [|α6|^1/2/α4^1/2]
M(z)
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
Ω [|α4|^3/α6^2]
a2=-0.05
a2=0
at onset
sgn(α4)α4^2
-0.2
-0.4
-0.6
-0.8
-1
-0.05
0
0.05
0.1
α2 [α6^-1]
χSB
Wig
M(z

Energy density profile in detail

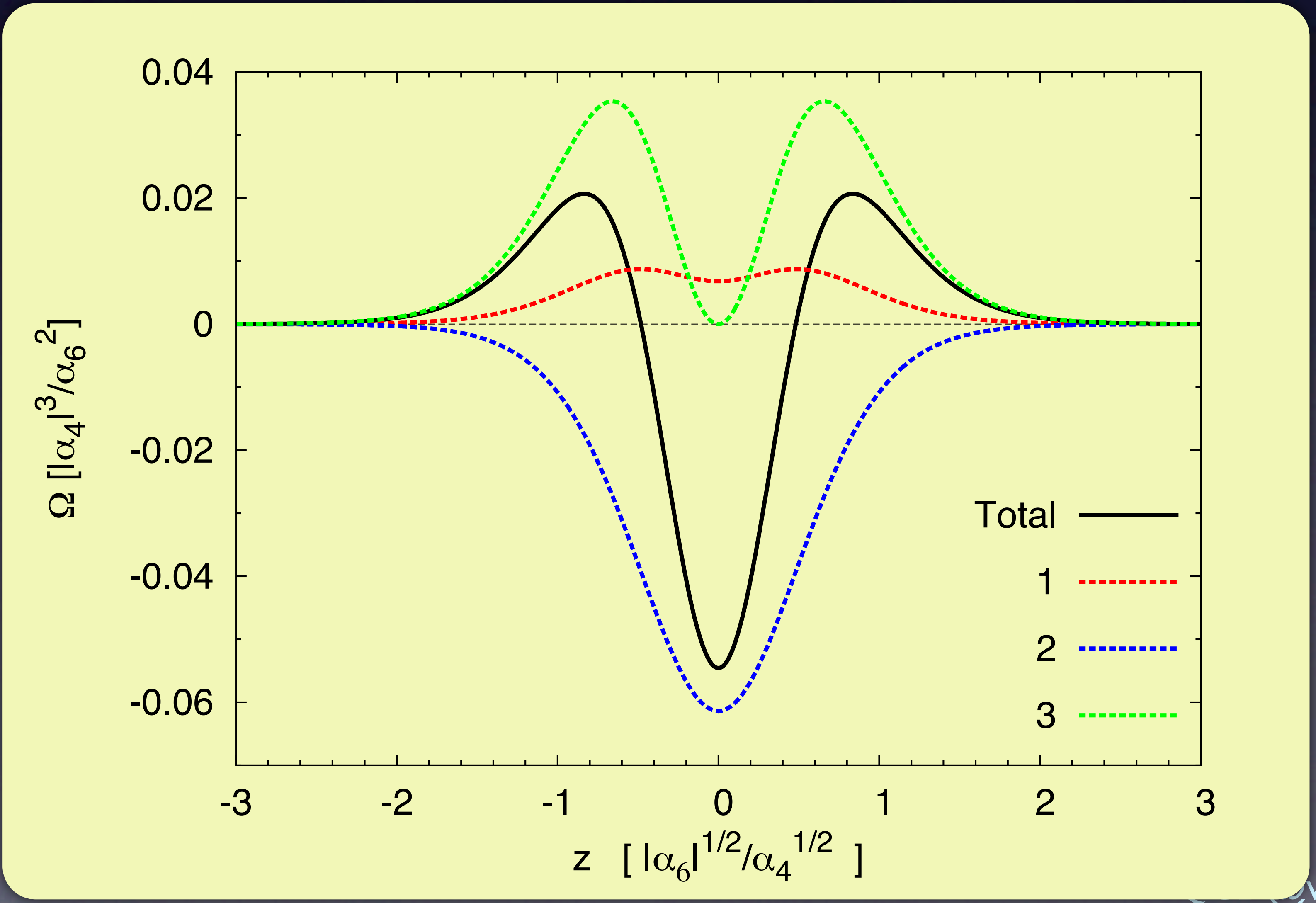
$$\Omega(z) = \underbrace{\left(\frac{\alpha_2}{2}M(z)^2 + \frac{\alpha_4}{4}M(z)^4 + \frac{1}{6}M(z)^6\right) - \Omega_0}_{\text{1: no derivative terms}}$$

$$\underbrace{+\,\alpha_4\frac{\nabla M^2}{4}}_{\text{2: Derivative term at 4 th order}} + \underbrace{\left(\frac{5M^2(\nabla M)^2}{6} + \frac{(\Delta M)^2}{12}\right)}_{\text{3: Derivative terms at 6 th order}}$$

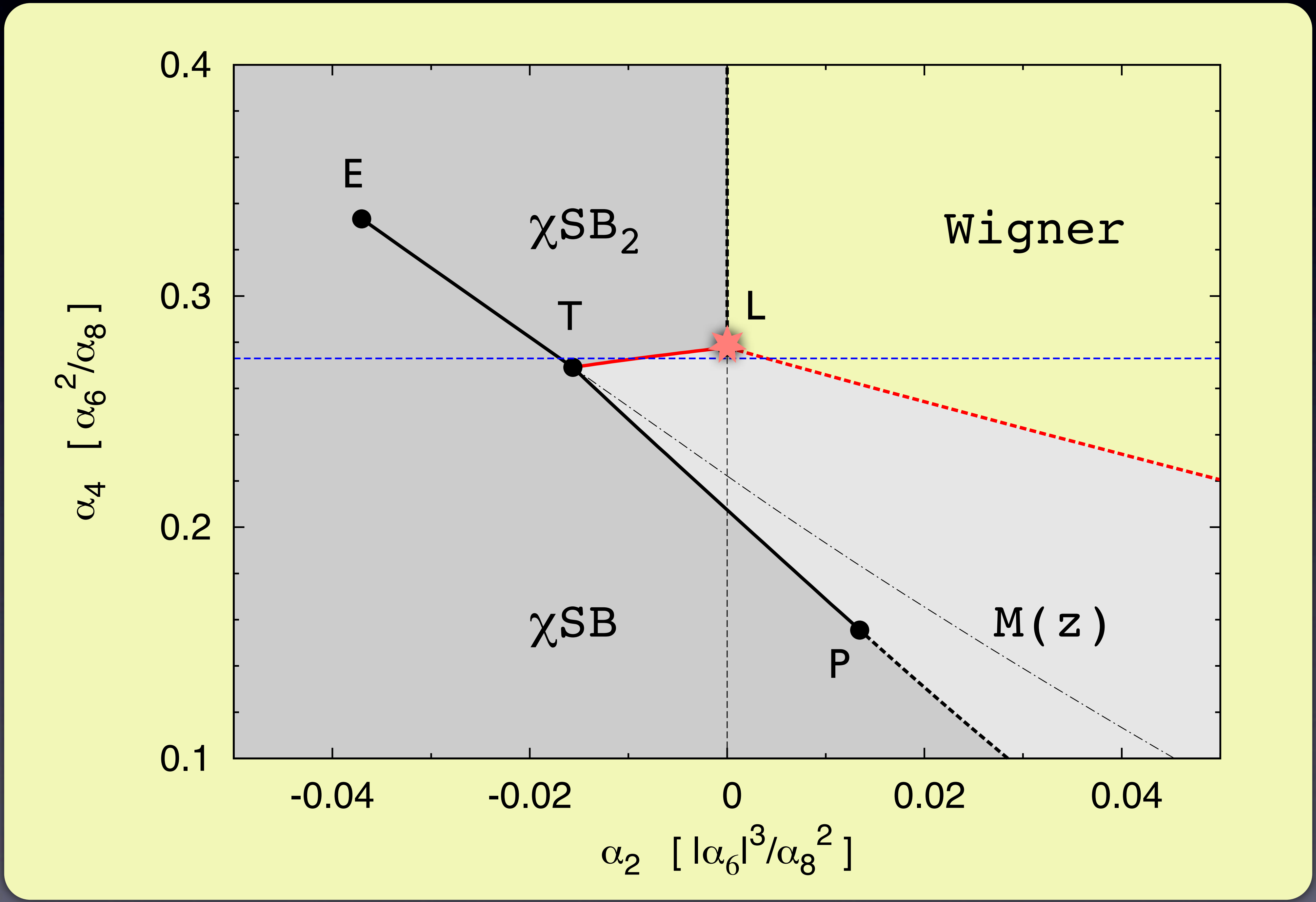
1: no derivative terms

2: Derivative term at 4 th order

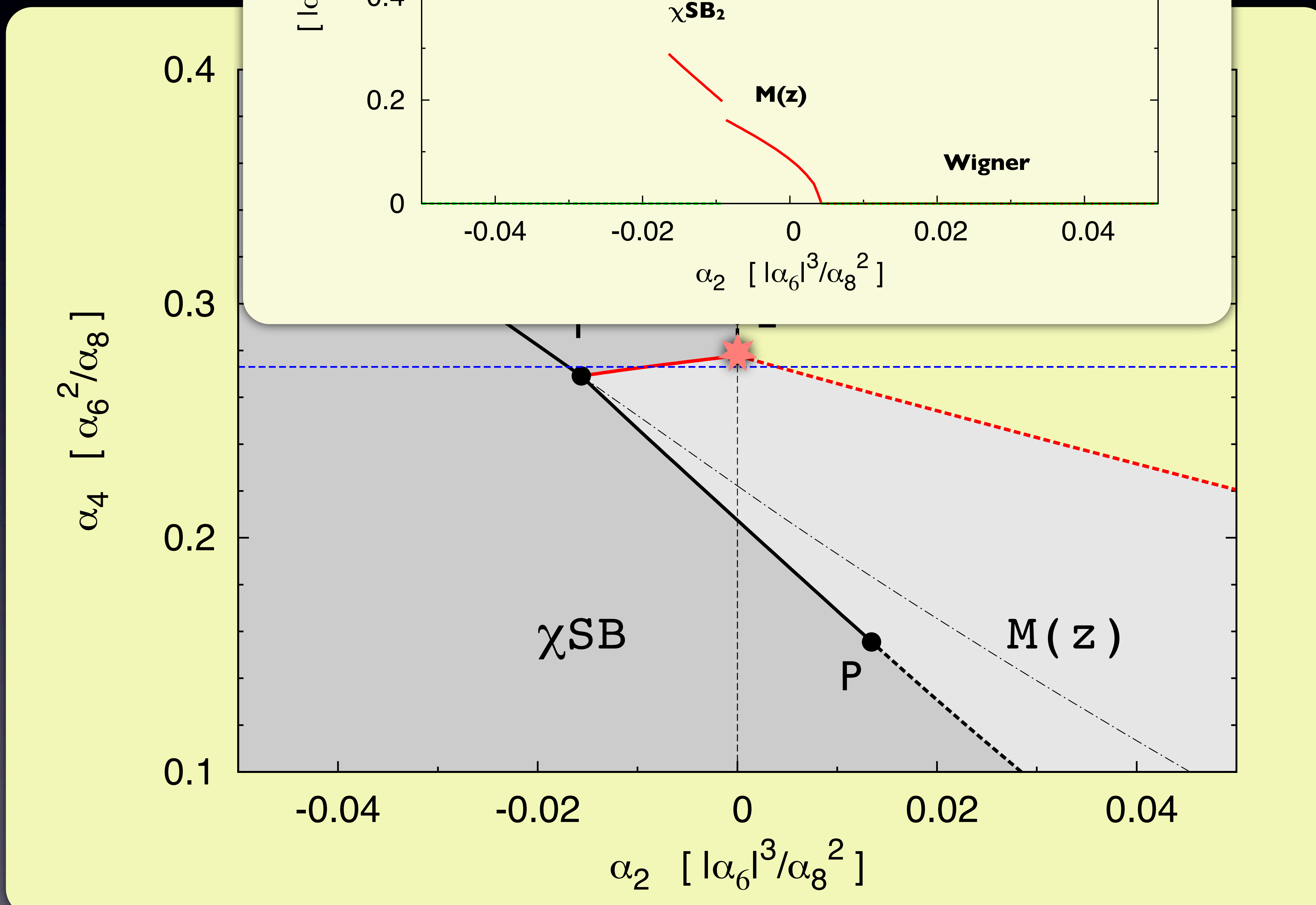
3: Derivative terms at 6 th order



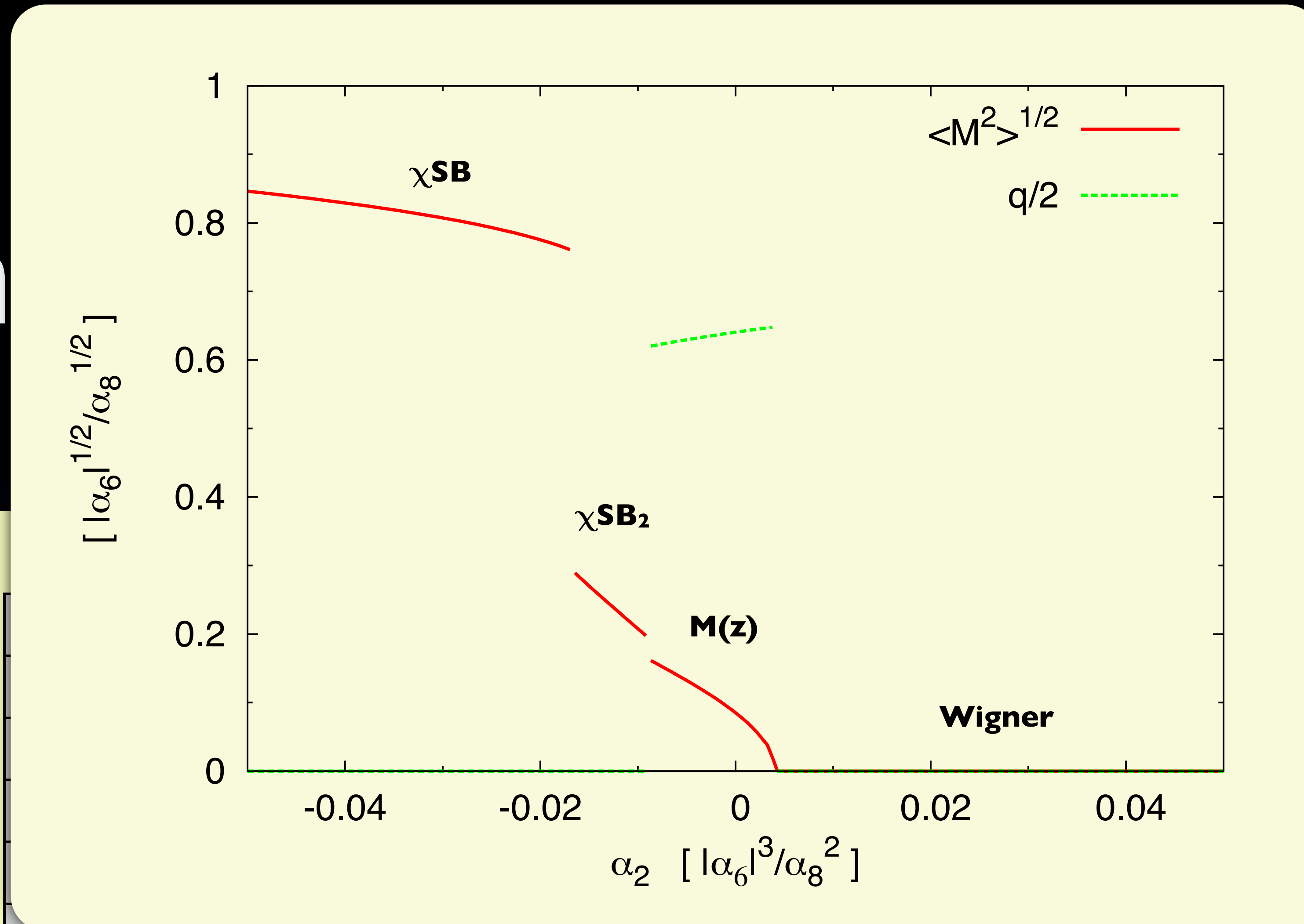
Going to new regime: $\alpha_6 < 0$



Goin <0



0.4
0.3
0.2
0.1
-0.04
-0.02
0
0.02
0.04
α_4 [$\alpha_6{}^2/\alpha_8$]
α_2 [$|\alpha_6|^3/\alpha_8{}^2$]
χSB
M(z)
P



$\langle M^2\rangle^{1/2}$
q/2
χSB
χSB₂
M(z)
Wigner
1
0.8
0.6
0.4
0.2
0
-0.04
-0.02
0
0.02
0.04
[$|\alpha_6|^{1/2}/\alpha_8{}^{1/2}$]
α_2 [$|\alpha_6|^3/\alpha_8{}^2$]

More on the dimensionality

- d-dimensional extension of 3D-cubic LO

$$M_{\text{dD-LO}} = \sqrt{\frac{2}{d}} M_{ave} (\sin(qx_1) + \sin(qx_2) + \cdots + \sin(qx_d))$$

- Effective potential

$$\Omega_{\text{dD}} = \left(\frac{a_2}{2} - \frac{3a_4^2}{16} \right) M_{ave}^2 + \left(\frac{1}{2} - \frac{1}{4d} \right) |a_4| M_{ave}^4 + \mathcal{O}(M_{ave}^6)$$

- Taking the minimum

$$\Omega_{\text{dD}} = -\frac{1}{|a_4|} \left(\frac{a_2}{2} - \frac{3a_4^2}{16} \right)^2 \frac{1}{2 - 1/d} + \cdots \to \lim_{d \to \infty} \frac{\Omega_{\text{dD}}}{\Omega_{\text{1D}}} \cong 0.5$$

Dimensional & Scaling analysis again

- Introducing dimensionless variables:

$$\alpha_2 = \eta_2 \left[|\alpha_6|^3/\alpha_8^2\right] \quad M = m \left[\sqrt{\alpha_6|/\alpha_8}\right]$$

$$\alpha_4 = \eta_4 \left[\alpha_6^2/\alpha_8\right] \quad \mathbf{x} = \tilde{\mathbf{x}} \left[\sqrt{\alpha_8/|\alpha_6|}\right]$$

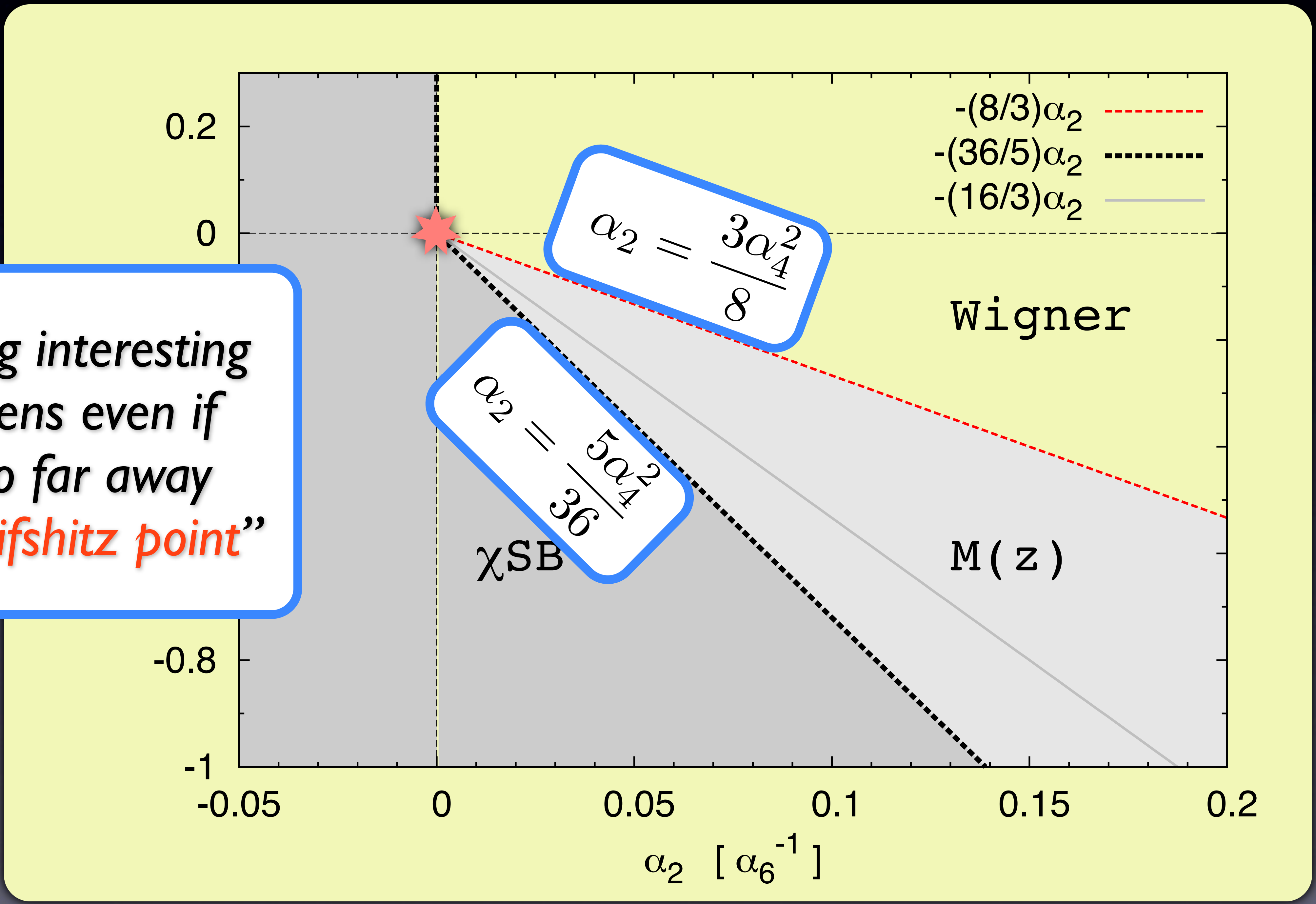
$$\Omega = \omega \left[\alpha_6^4/\alpha_8^3\right]$$

- Relevant parameters are reduced:

$$\begin{aligned}\omega &= \frac{\eta_2}{2} m^2 + \frac{\eta_4}{4}(m^4 + (\tilde{\nabla} m)^2) \\ &+ \frac{\mathrm{sgn}(\alpha_6)}{6}\left(m^6 + 5m^2(\tilde{\nabla} m)^2 + \tfrac{1}{2}(\tilde{\nabla}\tilde{\Delta} m)^2\right) \\ &+ \tfrac{1}{8}\left(m^8 + 14m^4(\tilde{\nabla} m)^2 - \tfrac{1}{5}(\tilde{\nabla} m)^4\right. \\ &\left. + \tfrac{18}{5} m\tilde{\Delta} m(\tilde{\nabla} m)^2 + \tfrac{14}{5} m^2(\tilde{\Delta} m)^2 + \tfrac{1}{5}(\tilde{\nabla}\tilde{\Delta} m)^2\right)\end{aligned}$$

GL in the vicinity of critical point

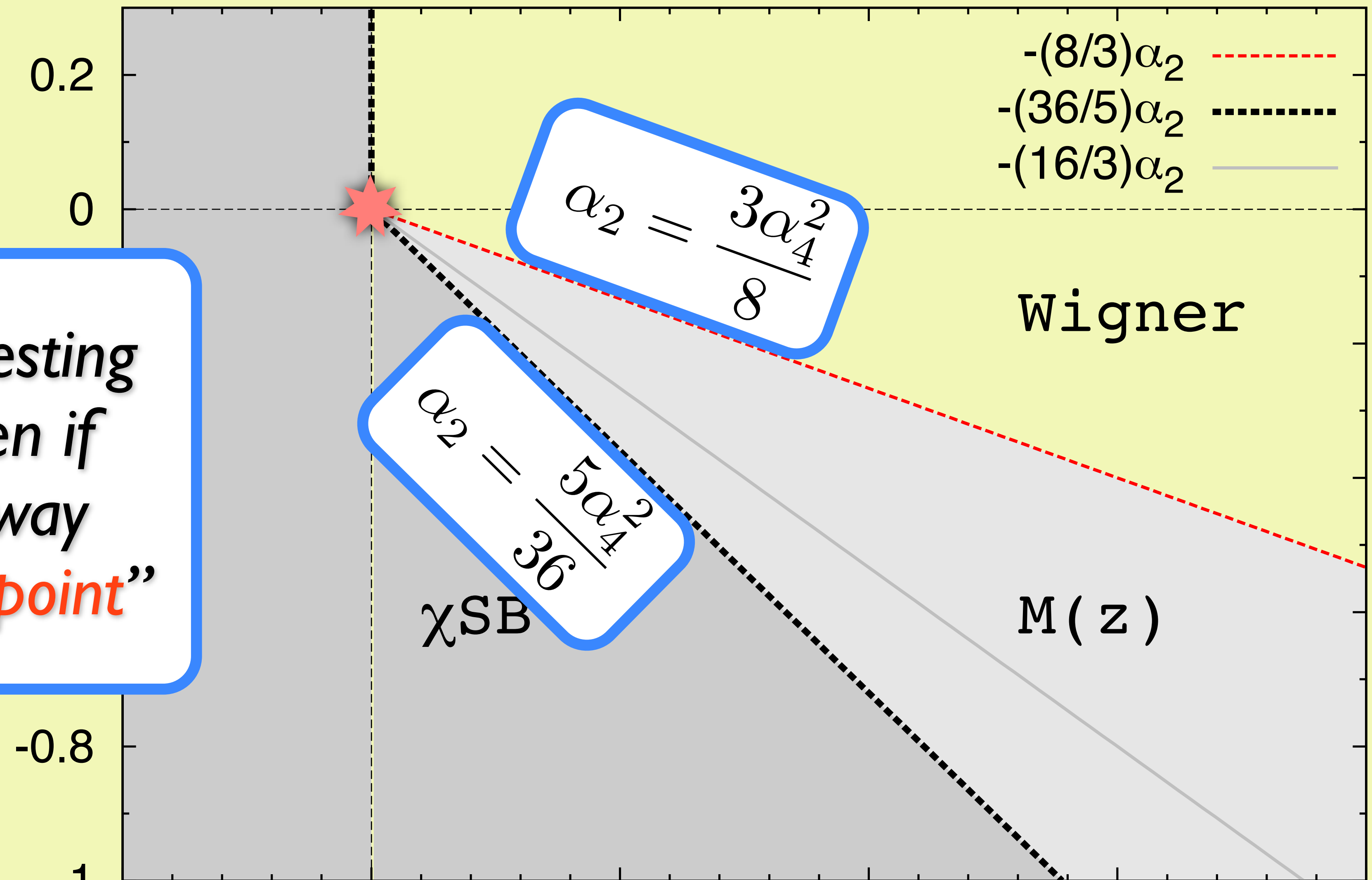
D. Nickel, PRL09



Nothing interesting happens even if we go far away from "Lifshitz point"

GL in the vicinity of critical point

D. Nickel, PRL09



Nothing interesting happens even if we go far away from "Lifshitz point"

Need to go beyond the minimal (6-th order) GL description