Amplitudes, Wilson loops, Symbols and Coproducts in N=4 Super Yang-Mills

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Motivation

- in gauge field theories, one-loop calculations are in general quite involved
- over 30 years since first non trivial computations — K. Ellis Ross Terrano 81
- progress has been very slow
  (adding one more parton would take ~10 years)
- yet, in the last ~5 years, one-loop calculations have undergone tremendous progress, so-called NLO revolution

  various causes:
  - generalised unitarity — Bern Dixon Dunbar Kosower 94; Britto Cachazo Feng 04
  - Witten's twistor string theory
  - OPP method — Ossola Papadopoulos Pittau 2006
- two-loop calculations are much younger — Smirnov Tausk 99-00
  obviously they are much more difficult
- can we envisage a similar leap forward ?

N=4 Super Yang-Mills

- maximal supersymmetric theory (without gravity)
  conformally invariant, β fn. = 0
  - spin 1 gluon
    4 spin 1/2 gluinos
    6 spin 0 real scalars

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- AdS/CFT duality Maldacena 97
  - large-λ limit of 4dim CFT $\leftrightarrow$ weakly-coupled string theory
    (aka weak-strong duality)

AdS/CFT duality, amplitudes & Wilson loops

- planar scattering amplitude at strong coupling — Alday Maldacena 07

$$\mathcal{M} \sim \exp\left[i \frac{\sqrt{\lambda}}{2\pi} (Area)_{cl} \right]$$

area of string world-sheet (classical solution, neglect $O(1/\sqrt{\lambda})$ corrections)

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- amplitude has same form as ansatz for MHV amplitudes at weak coupling

$$M_n = M_n^{(0)} \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon)\, m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

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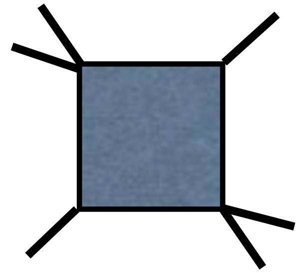
- computation ``formally the same as ... the expectation value of a Wilson loop given by a sequence of light-like segments"

MHV amplitudes in planar N=4 SYM

- at any order in the coupling, colour-ordered MHV amplitude in N=4 SYM can be written as tree-level amplitude times helicity-free loop coefficient $M_n^{(L)} = M_n^{(0)} m_n^{(L)}$

- at 1 loop Bern Dixon Dunbar Kosower 94

$$m_n^{(1)} = \sum_{pq} F^{2\mathrm{me}}(p,q,P,Q) \qquad n \geq 6$$



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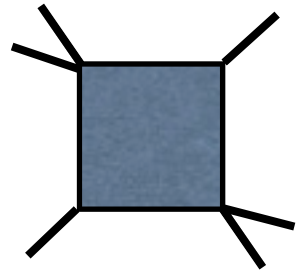

- at 2 loops, iteration formula for the *n*-pt amplitude

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + R$$

Anastasiou Bern Dixon Kosower 03

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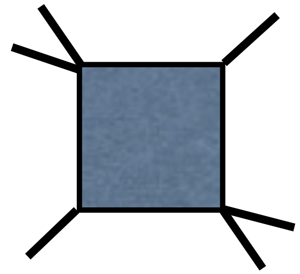

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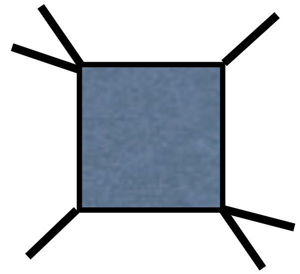

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remainder function

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ansatz for MHV amplitudes in planar N=4 SYM

Bern Dixon Smirnov 05

$$M_n = M_n^{(0)} \left[1 + \sum_{L=1}^{\infty} a^L m_n^{(L)}(\epsilon) \right]$$

$$= M_n^{(0)} \exp \left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right]$$

coupling $a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}$ $\lambda = g^2 N$ ‘t Hooft parameter

$$f^{(l)}(\epsilon) = \frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon \frac{l}{2} \hat{G}^{(l)} + \epsilon^2 f_2^{(l)}$$

$$E_n^{(l)}(\epsilon) = O(\epsilon)$$

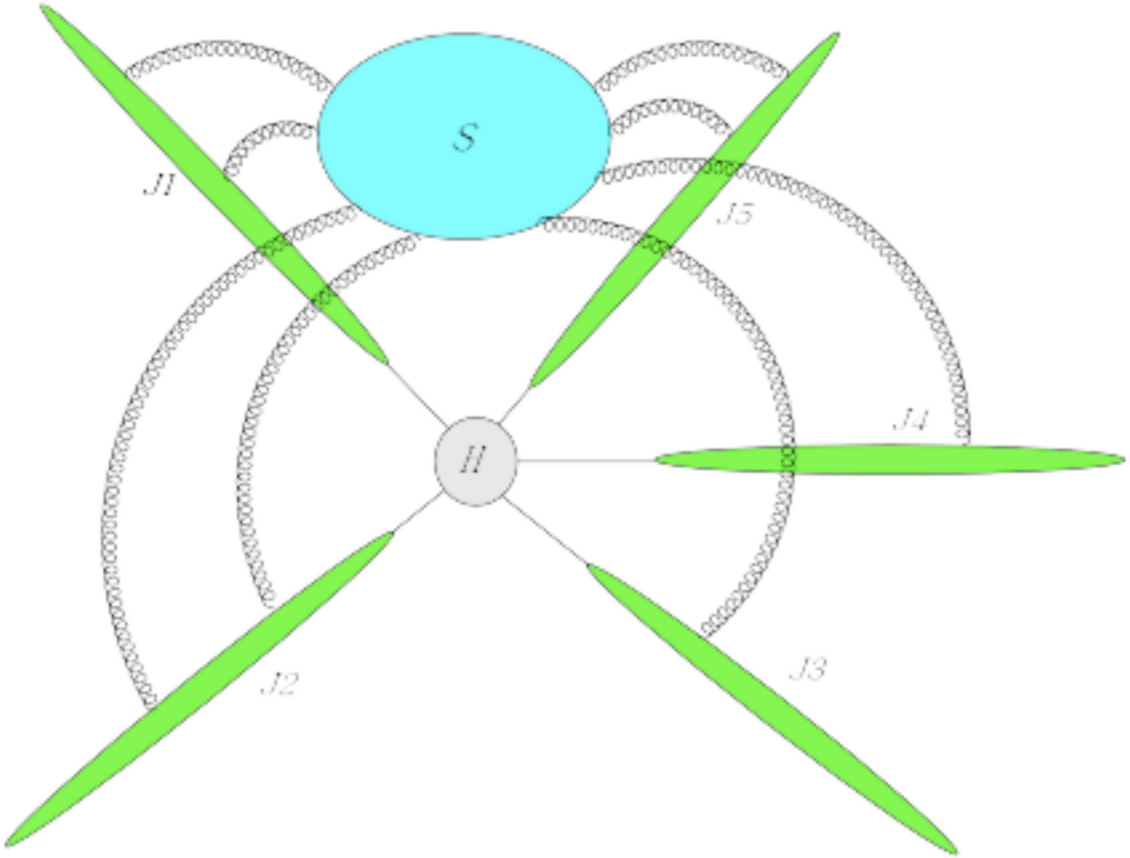
$\hat{\gamma}_K^{(l)}$ cusp anomalous dimension, known to all orders of *a*

Korchemsky Radyuskin 86
Beisert Eden Staudacher 06

$\hat{G}^{(l)}$ collinear anomalous dimension, known through O(a^4)

Bern Dixon Smirnov 05
Cachazo Spradlin Volovich 07

Factorisation of a multi-leg amplitude in QCD



Mueller 1981
Sen 1983
Botts Sterman 1987
Kidonakis Oderda Sterman 1998
Catani 1998
Tejeda-Yeomans Sterman 2002
Kosower 2003
Aybat Dixon Sterman 2006
Becher Neubert 2009
Gardi Magnea 2009

$$\mathcal{M}_N(p_i/\mu,\epsilon) = \sum_L \mathcal{S}_{NL}(\beta_i\cdot\beta_j,\epsilon)\, H_L\left(\frac{2p_i\cdot p_j}{\mu^2}, \frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}\right) \prod_i \frac{J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

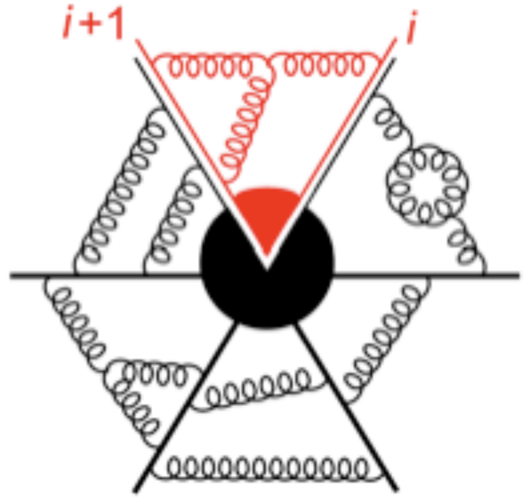
$p_i = \beta_i Q_0/\sqrt{2}$ value of Q_0 is immaterial in *S*, *J*

to avoid double counting of soft-collinear region (IR double poles),
$\mathcal{J}_i$ removes eikonal part from J_i, which is already in *S*
$J_i/\mathcal{J}_i$ contains only single collinear poles

$N = 4$ SYM in the planar limit

- colour-wise, the planar limit is trivial:
  can absorb S into J_i

- each slice is square root
  of Sudakov form factor



$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} h_n(\{p_i\}, \mu^2, \alpha_s, \epsilon)$$

- β fn = 0 ⇒ coupling runs only through dimension $\bar{\alpha}_s(\mu^2)\mu^{2\epsilon} = \bar{\alpha}_s(\lambda^2)\lambda^{2\epsilon}$

  Sudakov form factor has simple solution

$$\ln\left[\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right] = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \left(\frac{-Q^2}{\mu^2}\right)^{-n\epsilon}\left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon}\right]$$

Magnea Sterman 90
Bern Dixon Smirnov 05

⇒ IR structure of $N = 4$ SYM amplitudes

- the ansatz checked for the 3-loop 4-pt amplitude
  2-loop 5-pt amplitude

Bern Dixon Smirnov 05

Cachazo Spradlin Volovich 06
Bern Czakon Kosower Roiban Smirnov 06

- the ansatz fails on 2-loop 6-pt amplitude

Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08
Alday Maldacena 07; Bartels Lipatov Sabio-Vera 08

- at 2 loops, the remainder function characterises the deviation from the ansatz

$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 - f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) - Const^{(2)}$$

  - for $n = 4, 5$, R is a constant
    for $n \geq 6$, R is a function of conformally invariant cross ratios

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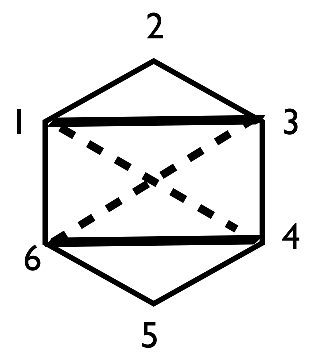
  - for $n = 6$, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$



x_i are variables in a dual space s.t. $p_i = x_i - x_{i+1}$

thus $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$

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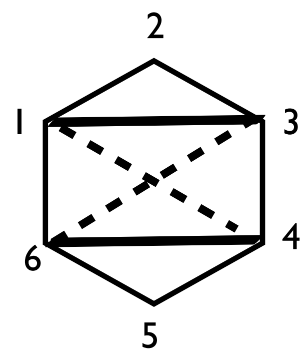

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$R_6^{(2)}$ known

numerically — Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08; Drummond Henn Korchemsky Sokatchev 08; Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

analytically — Duhr Smirnov VDD 09

Wilson loops

- $$W[\mathcal{C}_n] = \text{Tr}\ \mathcal{P}\ \exp\left[ig \oint \mathrm{d}\tau \dot{x}^\mu(\tau) A_\mu(x(\tau))\right]$$

closed contour $\mathcal{C}_n$ made by light-like external momenta

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- non-Abelian exponentiation theorem: vev of Wilson loop as an exponential, allows us to compute the log of W

  Gatheral 83
  Frenkel Taylor 84

$$\langle W[\mathcal{C}_n] \rangle = 1 + \sum_{L=1}^{\infty} a^L W_n^{(L)} = \exp \sum_{L=1}^{\infty} a^L w_n^{(L)}$$

through 2 loops $\quad w_n^{(1)} = W_n^{(1)} \qquad w_n^{(2)} = W_n^{(2)} - \frac{1}{2}\left(W_n^{(1)}\right)^2$

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- relation between 1 loop amplitudes & Wilson loops

$$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} m_n^{(1)} = m_n^{(1)} - n\frac{\zeta_2}{2} + \mathcal{O}(\epsilon)$$

Brandhuber Heslop Travaglini 07

Wilson loops & Ward identities

Drummond Henn Korchemsky Sokatchev 07

- *N*=4 SYM is invariant under *SO*(2,4) conformal transformations
- the Wilson loops fulfill conformal Ward identities
- the solution of the Ward identity for special conformal boosts is given by the finite parts of the BDS ansatz + *R*
  - at 2 loops

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n,WL}^{(2)} + \mathcal{O}(\epsilon)$$

with $f_{WL}^{(2)}(\epsilon) = -\zeta_2 + 7\zeta_3\epsilon - 5\zeta_4\epsilon^2$

(to be compared with $f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2$ for the amplitudes)

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- duality Wilson loop ⇔ MHV amplitude is expressed by

$$R_{n,WL}^{(2)} = R_n^{(2)}$$

MHV amplitudes ⇔ Wilson loops

- agreement between n-edged Wilson loop and n-point MHV amplitude at weak coupling (aka weak-weak duality)
  - verified for n-edged 1-loop Wilson loop — Brandhuber Heslop Travaglini 07
    up to 6-edged 2-loop Wilson loop — Drummond Henn Korchemsky Sokatchev 07
    Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08
- n-edged 2-loop Wilson loops computed (numerically)
  Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09
- no amplitudes are known beyond the 6-point 2-loop amplitude!

2-loop 6-edged remainder function $R_6^{(2)}$

Duhr Smirnov VDD 09

- the remainder function $R_6^{(2)}$ is explicitly dependent on the cross ratios u_1, u_2, u_3
- it is symmetric in all its arguments
  (for $n > 6$, it is symmetric under cyclic permutations and reflections)
- it is of uniform transcendental weight 4

  transcendental weights: $w(\ln x) = w(\pi) = 1$ $\quad w(\mathrm{Li}_2(x)) = w(\pi^2) = 2$
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↓ finite answer, but in intermediate steps many divergences
output is punishingly long

Analytic 2-loop 6-edged Wilson loop

- in MB representation of the integrals in general kinematics, get up to 8-fold integrals

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- after procedure in qmR limit, at most 3-fold integrals
  in fact, only one 3-fold integral, which comes from $f_H(p_1, p_3, p_5; p_4, p_6, p_2)$

$$\int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} (z_1 z_2 + z_2 z_3 + z_3 z_1) u_1^{z_1} u_2^{z_2} u_3^{z_3}$$
$$\times \Gamma(-z_1)^2 \Gamma(-z_2)^2 \Gamma(-z_3)^2 \Gamma(z_1 + z_2) \Gamma(z_2 + z_3) \Gamma(z_3 + z_1)$$

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$$\times\,\Gamma\,(-z_1)^2\,\Gamma\,(-z_2)^2\,\Gamma\,(-z_3)^2\,\Gamma\,(z_1+z_2)\,\Gamma\,(z_2+z_3)\,\Gamma\,(z_3+z_1)$$

the result is in terms of multiple polylogarithms

$$G(a,\vec{w};z) = \int_0^z \frac{\mathrm{d}t}{t-a}\,G(\vec{w};t)\,, \qquad G(a;z) = \ln\left(1-\frac{z}{a}\right)$$

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$$\int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}z_1}{2\pi i} \frac{\mathrm{d}z_2}{2\pi i} \frac{\mathrm{d}z_3}{2\pi i} (z_1 z_2 + z_2 z_3 + z_3 z_1)\, u_1^{z_1} u_2^{z_2} u_3^{z_3}$$
$$\times \Gamma(-z_1)^2 \, \Gamma(-z_2)^2 \, \Gamma(-z_3)^2 \, \Gamma(z_1 + z_2) \, \Gamma(z_2 + z_3) \, \Gamma(z_3 + z_1)$$

  the result is in terms of multiple polylogarithms

$$G(a, \vec{w}; z) = \int_0^z \frac{\mathrm{d}t}{t - a} G(\vec{w}; t) \,, \qquad G(a; z) = \ln\left(1 - \frac{z}{a}\right)$$

- the remainder function $R_6^{(2)}$ is given in terms of $O(10^3)$ multiple polylogarithms $G(u_1, u_2, u_3)$

Duhr Smirnov VDD 09

the remainder $R_6^{(2)}$ has been simplified and given in terms of polylogarithms

Goncharov Spradlin Vergu Volovich 10

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

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where

$$x_i^\pm = u_i x^\pm \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3} \qquad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$

$$L_4(x^+,x^-) = \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-)) + \frac{1}{8!!}\log(x^+x^-)^4$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\mathrm{Li}_n(1/x)\right) \qquad J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-))$$

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↑ answer is short and simple
introduces *symbols* in TH physics

Symbols

- take a fn. defined as an iterated integral of logs of rational functions R_i

$$T^{(k)} = \int_a^b \mathrm{d}\, lnR_1 \circ \cdots \circ \mathrm{d}\, lnR_k = \int_a^b \left(\int_a^t \mathrm{d}\, lnR_1 \circ \cdots \circ \mathrm{d}\, lnR_{k-1} \right) \mathrm{d}\, lnR_k(t)$$

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Goncharov

as such, the symbol is defined on the tensor product of the group of rational functions, modulo constants

$$\cdots \otimes R_1 R_2 \otimes \cdots = \cdots \otimes R_1 \otimes \cdots + \cdots \otimes R_2 \otimes \cdots$$

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- if T is a multiple polylogarithm G, then

$$dG(a_{n-1}, \ldots, a_1; a_n) = \sum_{i=1}^{n-1} G(a_{n-1}, \ldots, \hat{a}_i, \ldots, a_1; a_n) d \ln \left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}} \right)$$

the symbol is

$$\mathrm{Sym}\left(G(a_{n-1}, \ldots, a_1; a_n) \right) = \sum_{i=1}^{n-1} \mathrm{Sym}\left(G(a_{n-1}, \ldots, \hat{a}_i, \ldots, a_1; a_n) \right) \otimes \left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}} \right)$$

- Euler and Nielsen polylogarithms are multiple polylogarithms with special arguments

$$G(\vec{0}_n;x) = \frac{1}{n!}\ln^n x \qquad G(\vec{a}_n;x) = \frac{1}{n!}\ln^n\left(1-\frac{x}{a}\right)$$

$$G(\vec{0}_{n-1},a;x) = -\,\mathrm{Li}_n\left(\frac{x}{a}\right) \qquad G(\vec{0}_n,\vec{a}_m;x) = (-1)^m\,S_{n,m}\left(\frac{x}{a}\right) \qquad S_{n-1,1}(x) = \mathrm{Li}_n(x)$$

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- when the root equals +1,-1,0 multiple polylogarithms become harmonic polylogarithms (HPLs)

$$H(a,\vec{w};z) = \int_0^z \mathrm{d}t\, f(a;t)\, H(\vec{w};t) \qquad f(-1;t) = \frac{1}{1+t}\,, \quad f(0;t) = \frac{1}{t}\,, \quad f(1;t) = \frac{1}{1-t}$$

with $\{a,\vec{w}\} \in \{-1,0,1\}$

Remiddi Vermaseren

when the root equals +1,0 HPLs reduce to Euler and Nielsen polylogarithms

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- ... on to symbols

$$\mathrm{Sym}[\ln x] = x \qquad \mathrm{Sym}\left[\frac{1}{n!}\ln^n x\right] = \underbrace{x\otimes\cdots\otimes x} \equiv x^{\otimes n}$$

$$\mathrm{Sym}[\mathrm{Li}_n(x)] = -(1-x)\otimes x^{\otimes(n-1)}$$

$$\mathrm{Sym}[S_{n,m}(x)] = (-1)^m(1-x)^{\otimes m}\otimes x^{\otimes n}$$

$$\mathrm{Sym}[H(a_1,\ldots,a_n;x)] = (-1)^k(a_n-x)\otimes\cdots\otimes(a_1-x) \qquad \{a_i\} \in \{\,0,1\}$$

k is the number of *a*'s equal to 1

- the symbol knows about the discontinuities of *T*; if

$$\mathrm{Sym}[T^{(k)}] = R_1 \otimes \cdots \otimes R_k$$

then *T* has a branch cut at $R_I = 0$, and the symbol of the discontinuity is

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- $Disc(\ln x \; ln\, y) = \begin{cases} 2\pi i \; ln\, x & \text{along the } y \text{ cut } [-\infty, 0] \\ 2\pi i \; ln\, y & \text{along the } x \text{ cut } [-\infty, 0] \end{cases}$

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- in general, if $Disc(f\,g) = Disc(f)\,g + f\,Disc(g)$

and $\mathrm{Sym}[f] = \otimes_{i=1}^{n} R_i \qquad \mathrm{Sym}[g] = \otimes_{i=n+1}^{m} R_i$

then $\mathrm{Sym}[f\,g] = \sum_\sigma \otimes_{i=1}^{n} R_{\sigma(i)}$

where σ denotes the set of all shuffles of *n*+(*m*-*n*) elements

e.g. $\mathrm{Sym}[f] = R_1 \otimes R_2 \qquad \mathrm{Sym}[g] = R_3 \otimes R_4$

$$\begin{aligned}\mathrm{Sym}[f\,g] = {} & R_1 \otimes R_2 \otimes R_3 \otimes R_4 + R_1 \otimes R_3 \otimes R_2 \otimes R_4 + R_1 \otimes R_3 \otimes R_4 \otimes R_2 \\ + {} & R_3 \otimes R_1 \otimes R_2 \otimes R_4 + R_3 \otimes R_1 \otimes R_4 \otimes R_2 + R_3 \otimes R_4 \otimes R_1 \otimes R_2\end{aligned}$$

- the symbol knows about the discontinuities of T; if

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- symbols form a shuffle algebra, *i.e.* a vector space with a shuffle product
(also iterated integrals and multiple polylogarithms form shuffle algebras)

polylogarithm identities satisfied by the function f
become algebraic identities satisfied by its symbol

let us prove the identity $$\mathrm{Li}_2(1-x) = -\mathrm{Li}_2(x) - \ln x \ln(1-x) + \frac{\pi^2}{6}$$

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$$\mathrm{Sym}[\ln x \ln(1-x)] = x\otimes(1-x) + (1-x)\otimes x$$

thus $\mathrm{Sym}[\mathrm{Li}_2(1-x)] = \mathrm{Sym}[-\mathrm{Li}_2(x) - \ln x \ln(1-x)]$

which determines the function up to functions of lesser degree

$$\mathrm{Li}_2(1-x) = -\mathrm{Li}_2(x) - \ln x \ln(1-x) + c\,\pi^2 + i\pi\,(c' \ln x + c'' \ln(1-x))$$

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but the equation is real for $0 < x < 1$, so c'=c''=0

at $x = 1$ $0 = -\frac{\pi^2}{6} - 0 + c\,\pi^2$ ⟹ $c = \frac{1}{6}$

- take f, g with $w(f) = w(g) = n$ and $\mathrm{Sym}[f] = \mathrm{Sym}[g]$
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Thus, we have a procedure to simplify a generic function of polylogarithms:

- find suitable variables (through momentum twistors or else) such that the arguments of the multiple polylogarithms become rational functions

- determine the symbol of the function

- through some symbol-processing procedure, find a simpler form of the integral in terms of multiple polylogarithms

Duhr Gangl Rhodes 11

Recent results on symbols

- symbol of *n*-point 2-loop MHV amplitudes/Wilson loops (in principle one can get the *n*-point 2-loop Wilson loop, but the symbol is complicated) Caron-Huot 11
- symbol of 6-point 3-loop MHV amplitude, up to 2 constants (and function in the multi-Regge limit) Dixon Drummond Henn 11
- symbol of 6-point 2-loop NMHV amplitude (and function up to a 1-dim integral) Dixon Drummond Henn 11
- symbol of non-planar massive double box (to be used in *qq*, *gg* → *ttbar*)

  von Manteuffel *presented at ACAT2011*
- symbol of 3-gluon 2-loop form factor Brandhuber Travaglini Yang 12

Coproducts

- symbols miss transcendental constants
- look for *something* with more structure

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- algebra is a vector space with a multiplication $\mu\colon A \otimes A \to A$ $\mu(a \otimes b) = a \cdot b$
  that is associative $A \otimes A \otimes A \to A \otimes A \to A$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

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- coalgebra is a vector space with a comultiplication $\Delta\colon B \to B \otimes B$
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- take a word, sum over ways to split it into two: *deconcatenation*

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$$\Delta(T) = w\,x\,y\,z \otimes 1 + w\,x\,y \otimes z + w\,x \otimes y\,z + w \otimes x\,y\,z + 1 \otimes w\,x\,y\,z$$

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iterate: sum over ways to split it into three

$$w\,x \otimes y\,z \rightarrow (w \otimes x) \otimes y\,z$$

$$w\,x \otimes y\,z \rightarrow w\,x \otimes (y \otimes z)$$

if sum over all possibilities, get to the same result

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    primitive element

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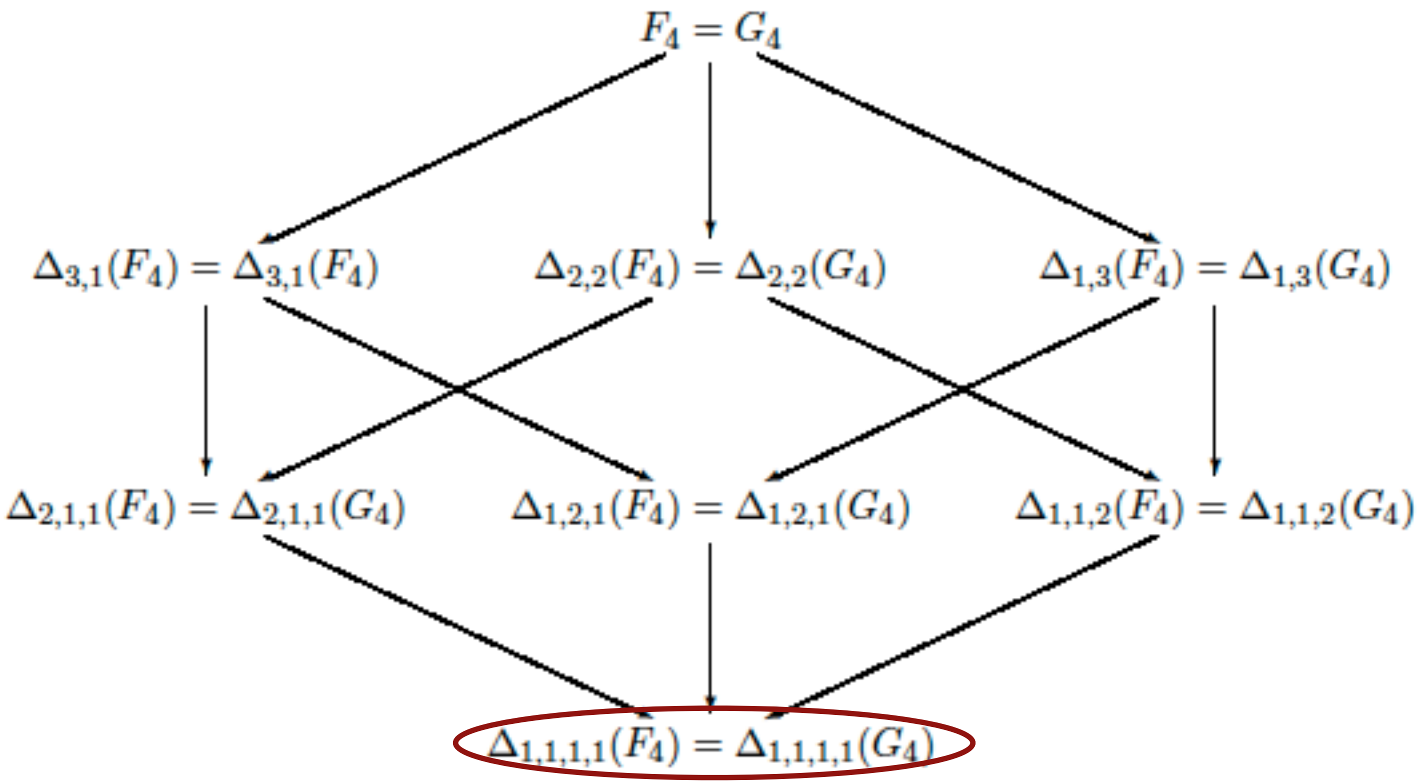
- example on a function of weight 4

Duhr 12



symbols represent the maximal iteration of a coproduct

... but there is a problem

put **z = 1** in $\Delta\big(\mathrm{Li}_n(z)\big) = 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!}$

get $\Delta(\zeta_n) = 1 \otimes \zeta_n + \zeta_n \otimes 1$

better than symbols $\mathrm{Sym}[\zeta_n] = 0$

however $\zeta_4 = \frac{1}{15}\,\zeta_2^2$

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contradiction!

- define $\Delta(\zeta_{2n}) = \zeta_{2n} \otimes 1$ Francis Brown 11

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- define also $\Delta(\pi) = \pi \otimes 1$ Duhr 12

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this allows us to account consistently for $\zeta, i\pi$ terms (which the symbol misses) so the coproduct fixes all but the primitive elements

Coproducts and inverse relations

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- weight 2 $$\Delta_{1,1}\left(\mathrm{Li}_2\left(\frac{1}{z}\right)\right) = -\ln\left(1-\frac{1}{z}\right)\otimes\ln\left(\frac{1}{z}\right)$$
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iπ more than the symbol

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- **weight 3**

$$\Delta_{1,1,1}\left(\mathrm{Li}_3\left(\frac{1}{z}\right)\right) = -\ln\left(1-\frac{1}{z}\right)\otimes\ln\left(\frac{1}{z}\right)\otimes\ln\left(\frac{1}{z}\right)$$
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$$= -\ln(1-z)\otimes\ln z\otimes\ln z + \ln z\otimes\ln z\otimes\ln z - i\pi\otimes\ln z\otimes\ln z$$
$$= \Delta_{1,1,1}\left(\mathrm{Li}_3(z) + \frac{1}{6}\ln^3 z - \frac{i\pi}{2}\ln^2 z\right)$$

one can do better

$$\Delta_{2,1}\left(\mathrm{Li}_3\left(\frac{1}{z}\right) - \left(\mathrm{Li}_3(z) + \frac{1}{6}\ln^3 z - \frac{i\pi}{2}\ln^2 z\right)\right) = -\frac{\pi^2}{3}\otimes\ln z$$
$$= \Delta_{2,1}\left(-\frac{\pi^2}{3}\ln z\right)$$

so $\mathrm{Li}_3\left(\frac{1}{z}\right) = \mathrm{Li}_3(z) + \frac{1}{6}\ln^3 z - \frac{i\pi}{2}\ln^2 z - \frac{\pi^2}{3}\ln z + c_1\zeta_3 + c_2 i\pi^3$ $\qquad z=1 \to c_1 = c_2 = 0$

Higgs + 3 gluons

- the 2-loop amplitudes for Higgs + 3 gluons have been computed in terms of 2-dim HPLs

Koukoutsakis 03
Gehrmann Jacquier Glover Koukoutsakis 11

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  Brandhuber Travaglini Yang 12

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  Brandhuber Travaglini Yang 12

- using coproducts, the whole 2-loop amplitude for Higgs + 3 gluons can be expressed in terms of classical polylogarithms up to weight 4

  Duhr 12

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- ... but symbols loose much info about the target function. Most of that info can be recovered using coproducts, which include the symbols, and much more ...

Back-up slides

Resummation: Sudakov form factor

- Sudakov (quark) form factor as matrix element of EM current

$$\Gamma_\mu(p_1,p_2;\mu^2,\epsilon) \equiv < 0|J_\mu(0)|p_1,p_2> = \bar{v}(p_2)\gamma_\mu u(p_1)\,\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)$$

obeys evolution equation

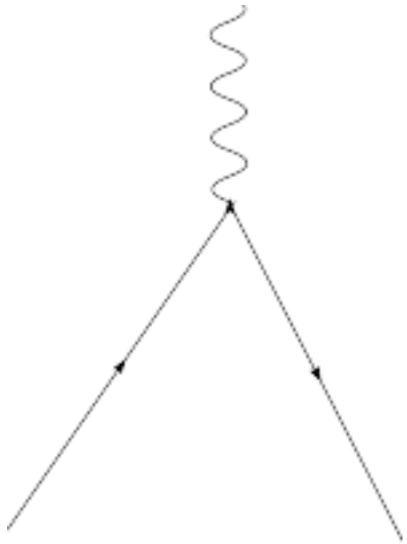
$$Q^2\frac{\partial}{\partial Q^2}\ln\left[\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right] = \frac{1}{2}\left[K\left(\alpha_s(\mu^2),\epsilon\right)+G\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)\right]$$

K is a counterterm; G is finite as $\varepsilon \to 0$

RG invariance requires

$$\mu\frac{dG}{d\mu} = -\mu\frac{dK}{d\mu} = \gamma_K(\alpha_s(\mu^2))$$

Korchemsky Radyushkin 1987



γ_K is the cusp anomalous dimension

solution is

$$\Gamma\left(Q^2,\epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\left[G\left(-1,\bar{\alpha}_s(\xi^2,\epsilon),\epsilon\right)-\frac{1}{2}\gamma_K\left(\bar{\alpha}_s(\xi^2,\epsilon)\right)\ln\left(\frac{-Q^2}{\xi^2}\right)\right]\right\}$$

Collinear limits of Wilson loops

collinear limit $a||b$

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

$R_6 \to 0$ $R_7 \to R_6$ $R_n \to R_{n-1}$

triple collinear limit $a||b||c$

$R_6 \to R_6$ $R_7 \to R_6$ $R_8 \to R_6 + R_6$ $R_n \to R_{n-2} + R_6$

quadruple collinear limit $a||b||c||d$

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Collinear limits of Wilson loops

collinear limit $a||b$ — Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

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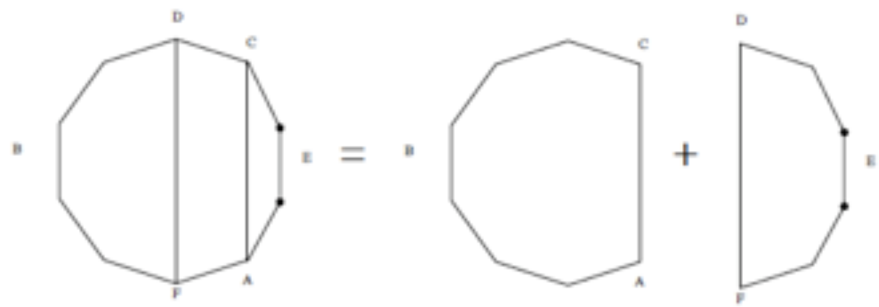
(k+1)-ple collinear limit $i_1||i_2||\cdots||i_{k+1}$

$R_n \rightarrow R_{n-k} + R_{k+4}$



(n-4)-ple collinear limit $i_1||i_2||\cdots||i_{n-4}$

$R_{n-1} \rightarrow R_{n-1} \qquad R_n \rightarrow R_{n-1}$

(n-3)-ple collinear limit $i_1||i_2||\cdots||i_{n-3}$

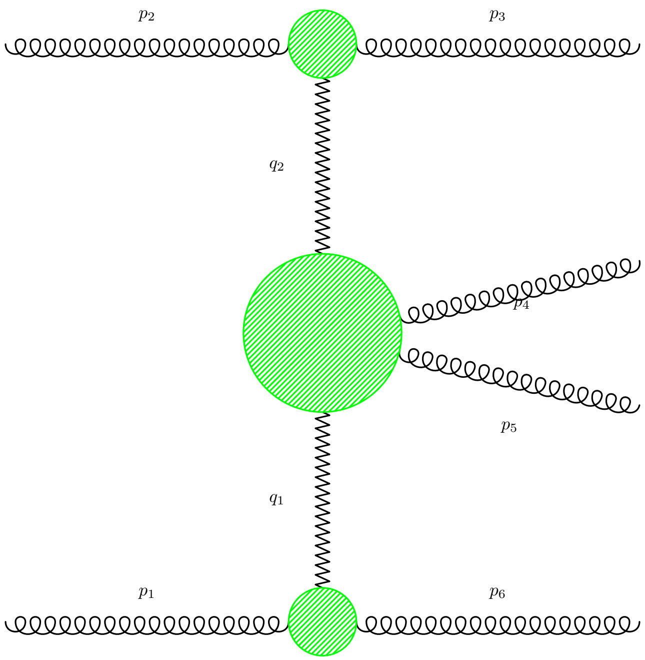
$R_n \rightarrow R_n$

- thus R_n is fixed by the (n-3)-ple collinear limit

Quasi-multi-Regge limit of hexagon Wilson loop

- 6-pt amplitude in the qmR limit of a pair along the ladder

$$y_3 \gg y_4 \simeq y_5 \gg y_6; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}|$$



the conformally invariant cross ratios are

$$u_{36} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

$$u_{14} = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

$$u_{25} = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} = \frac{s_{34}s_{61}}{s_{234}s_{345}}$$

the cross ratios are all $O(1)$

→ R_6 does not change its functional dependence on the u's

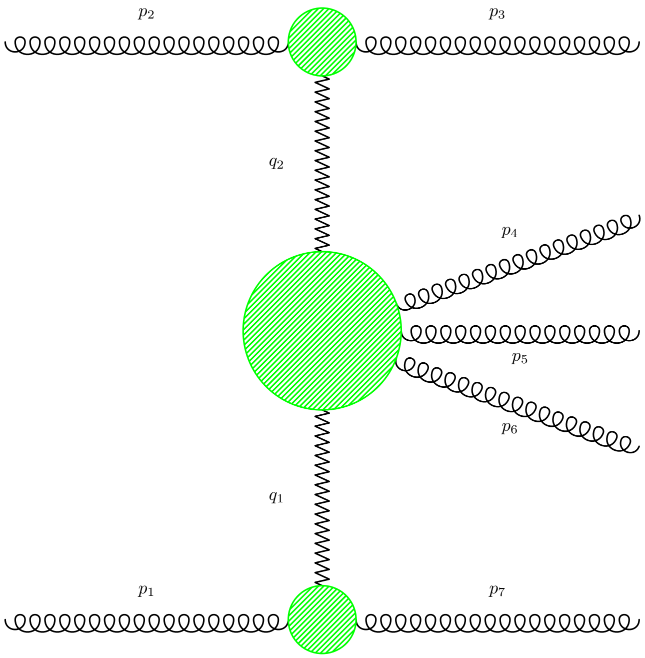
- R_6 is invariant under the qmR limit of a pair along the ladder

Duhr Glover Smirnov VDD 08

Quasi-multi-Regge limit of n-sided Wilson loop

- 7-pt amplitude in the qmR limit of a triple along the ladder

$$y_3 \gg y_4 \simeq y_5 \simeq y_6 \gg y_7; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}| \simeq |p_{7\perp}|$$



7 cross ratios, which are all $O(1)$
R_7 is invariant under the qmR limit of a triple along the ladder

Quasi-multi-Regge limit of *n*-sided Wilson loop

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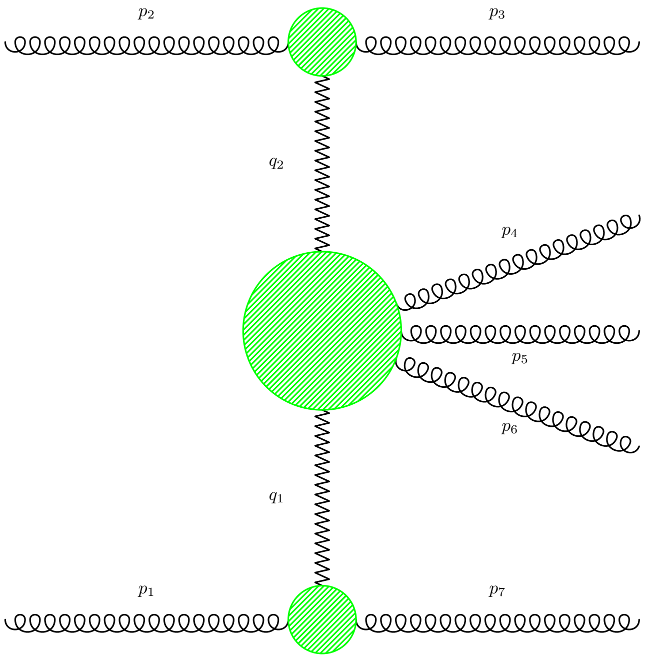

7 cross ratios, which are all $O(1)$
R_7 is invariant under the qmR limit of a triple along the ladder

- can be generalised to the *n*-pt amplitude in the qmR limit of a (*n*-4)-ple along the ladder

$$y_3 \gg y_4 \simeq \ldots \simeq y_{n-1} \gg y_n; \qquad |p_{3\perp}| \simeq \ldots \simeq |p_{n\perp}|$$

Duhr Smirnov VDD 09

Quasi-multi-Regge limit of Wilson loops

- *L*-loop Wilson loops are Regge exact

Drummond Korchemsky Sokatchev 07
Duhr Smirnov VDD 09

$$w_n^{(L)}(\epsilon) = f_{WL}^{(L)}(\epsilon)\, w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n,WL}^{(L)}(u_{ij}) + \mathcal{O}(\epsilon)$$

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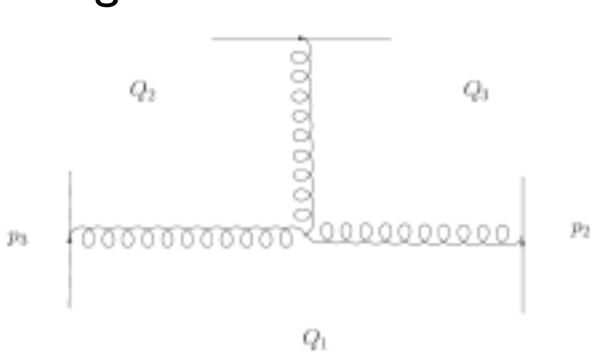
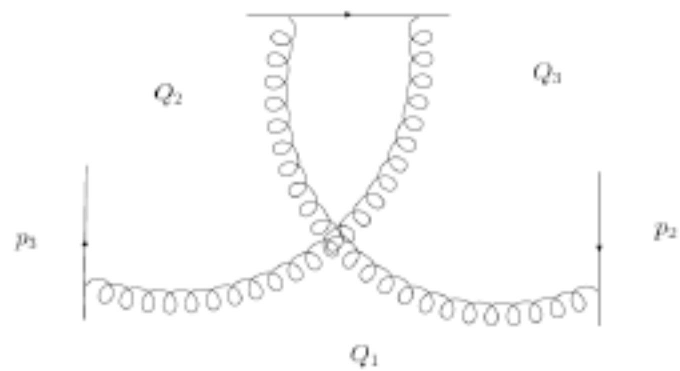
- we may compute the Wilson loop in qmRk
  the result will be correct in general kinematics !!!

Diagrams of 2-loop Wilson loops

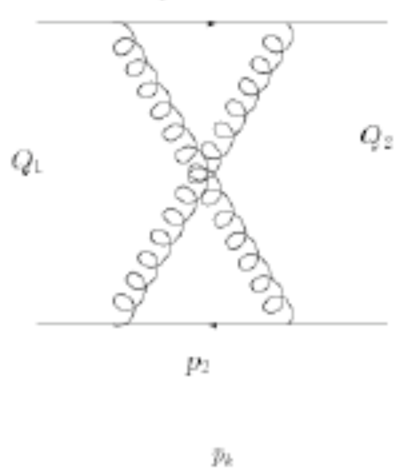
hard diagram



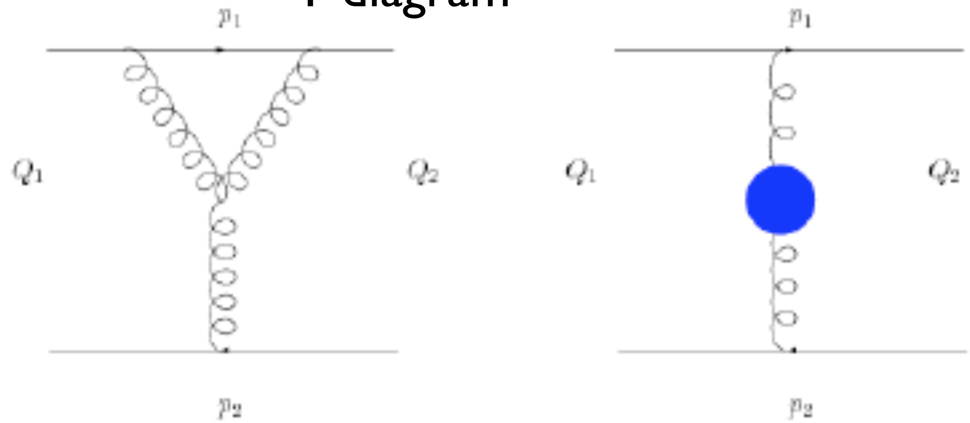
curtain diagram

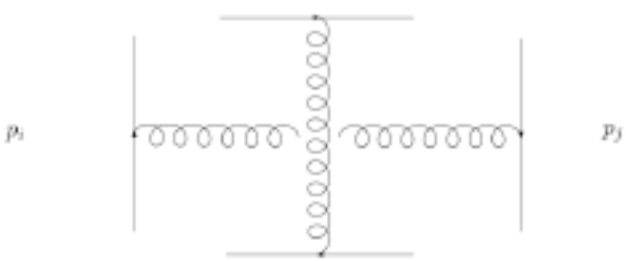


cross diagram



Y diagram





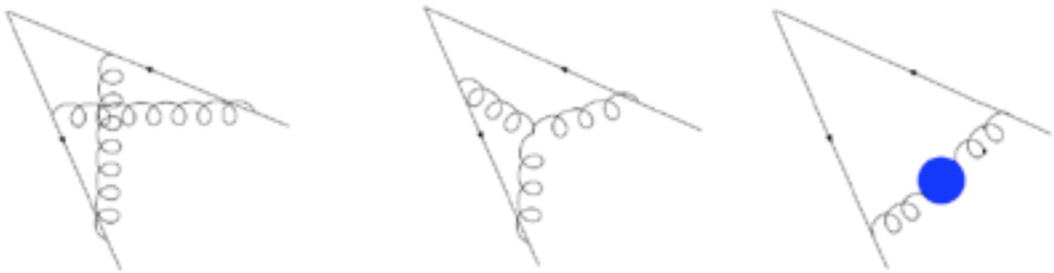
factorised cross diagram

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

each diagram yields an integral,
similar to a Feynman-parameter integral

Computing 2-loop Wilson loops

cusp diagrams are given by cross and Y diagrams with gluons attaching to consecutive sides



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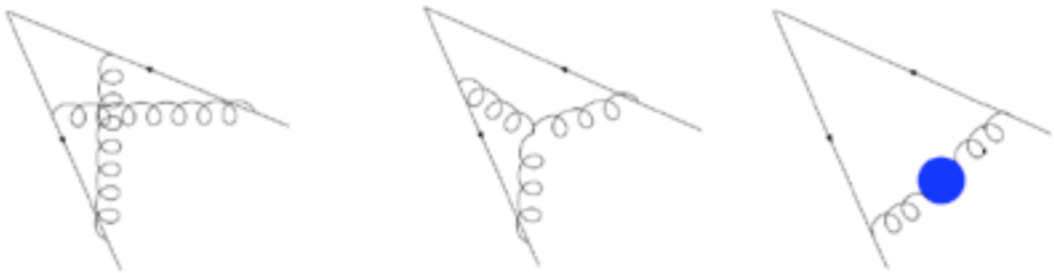
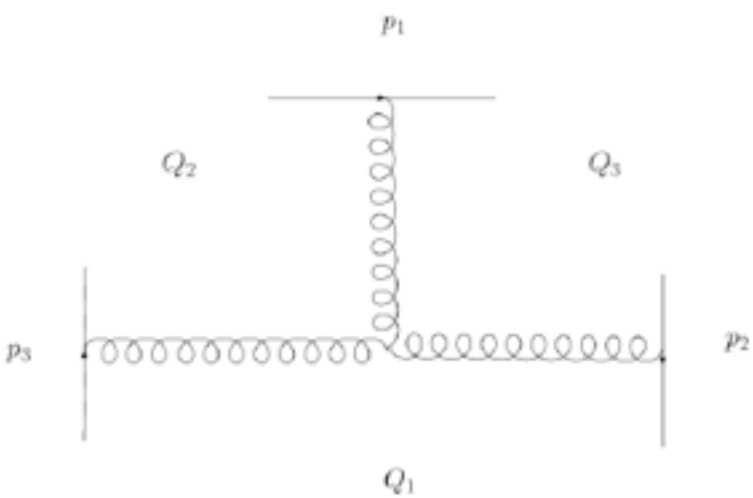


- most difficult diagrams to compute are hard diagrams



f_H has $1/\varepsilon^2$ singularities if $Q_1 = Q_2 = 0, Q_3 \neq 0$
it has $1/\varepsilon$ singularities if $Q_1 = 0, Q_2, Q_3 \neq 0$
it is finite if $Q_1, Q_2, Q_3 \neq 0$

e.g. for $n=6$, *the* most difficult diagram is

$f_H(p_1, p_3, p_5; p_4, p_6, p_2)$ which is finite

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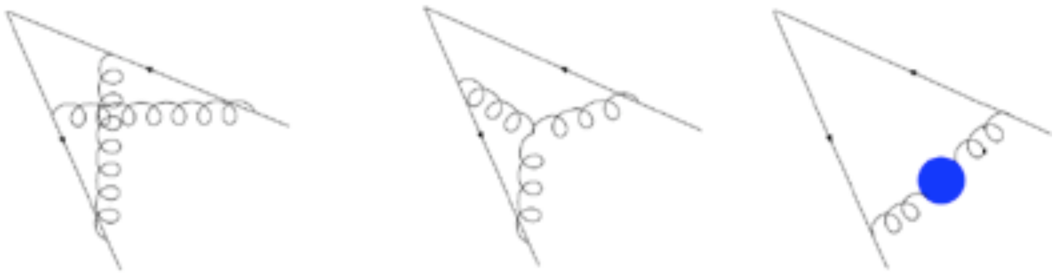

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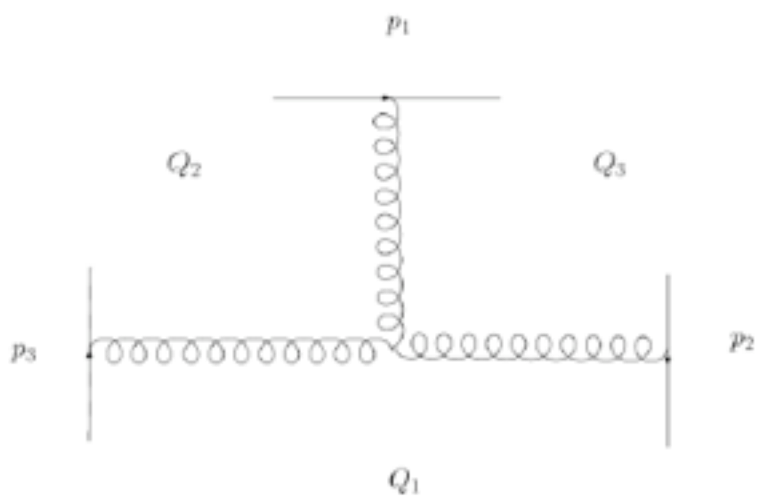

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- most general hard diagram has $Q_1^2, Q_2^2, Q_3^2 \neq 0$; it occurs for $n \geq 9$

A comment on 2-loop *n*-edged Wilson loops

- 2-loop 7-edged Wilson loop:
  in the MB repr. of the integrals in qmRk, one gets up to 4-fold integrals

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- For $10 \leq n \leq 12$, the only new contributions come from the *factorized cross diagram* topology, which is the simplest

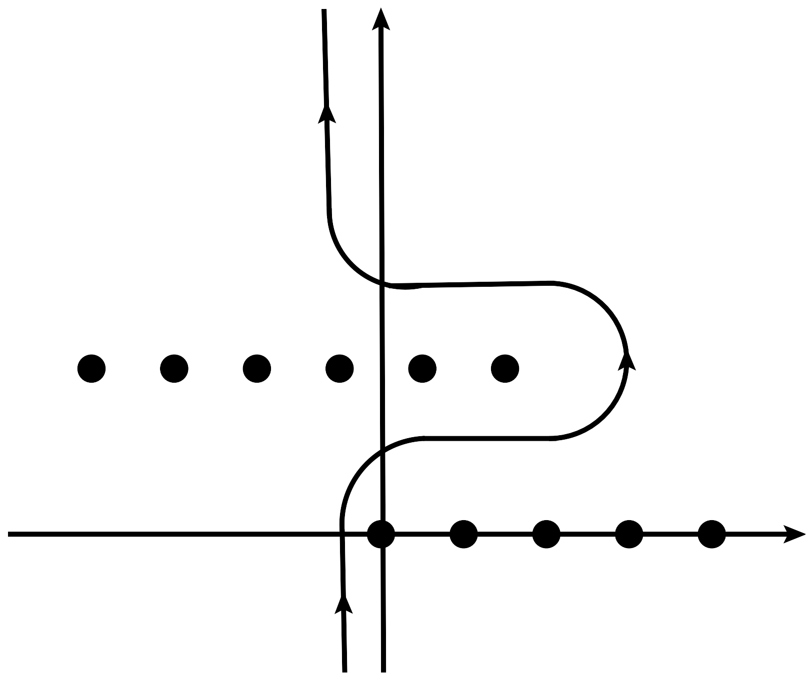
Wilson loops: analytic calc

1. Use Mellin-Barnes (MB) representation of the Feynman-parameter integrals: replace each denominator by a contour integral

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \, \Gamma(-z) \, \Gamma(\lambda+z) \, \frac{A^z}{B^{\lambda+z}}$$

integral turns into a sum of residues

$$\mathrm{Res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$$



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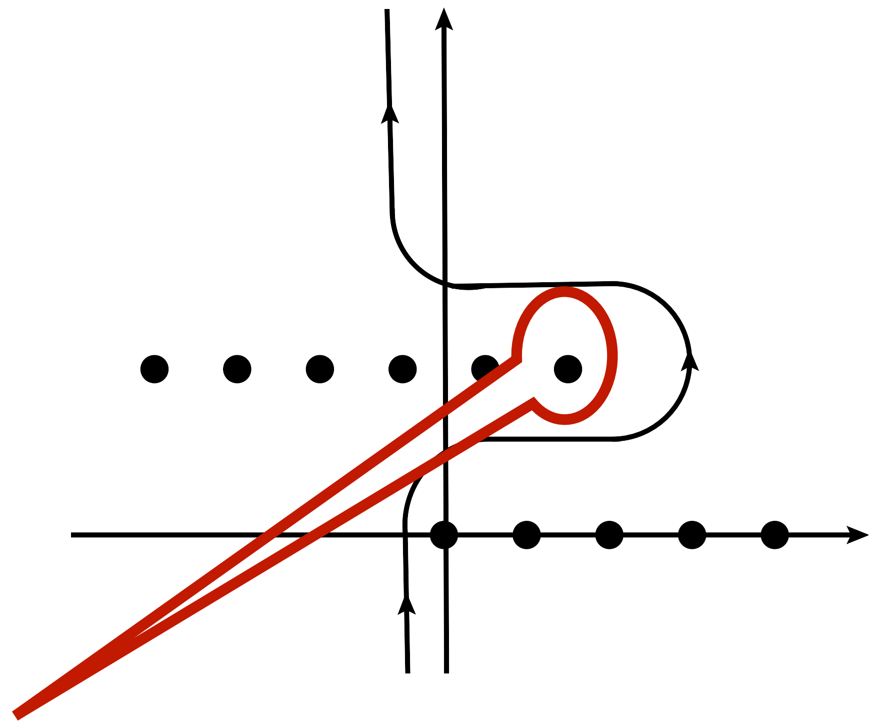
2. Use Regge exactness in the qmR limit: retain only leading behaviour (*i.e.* leading residues) of the integral



leading residue

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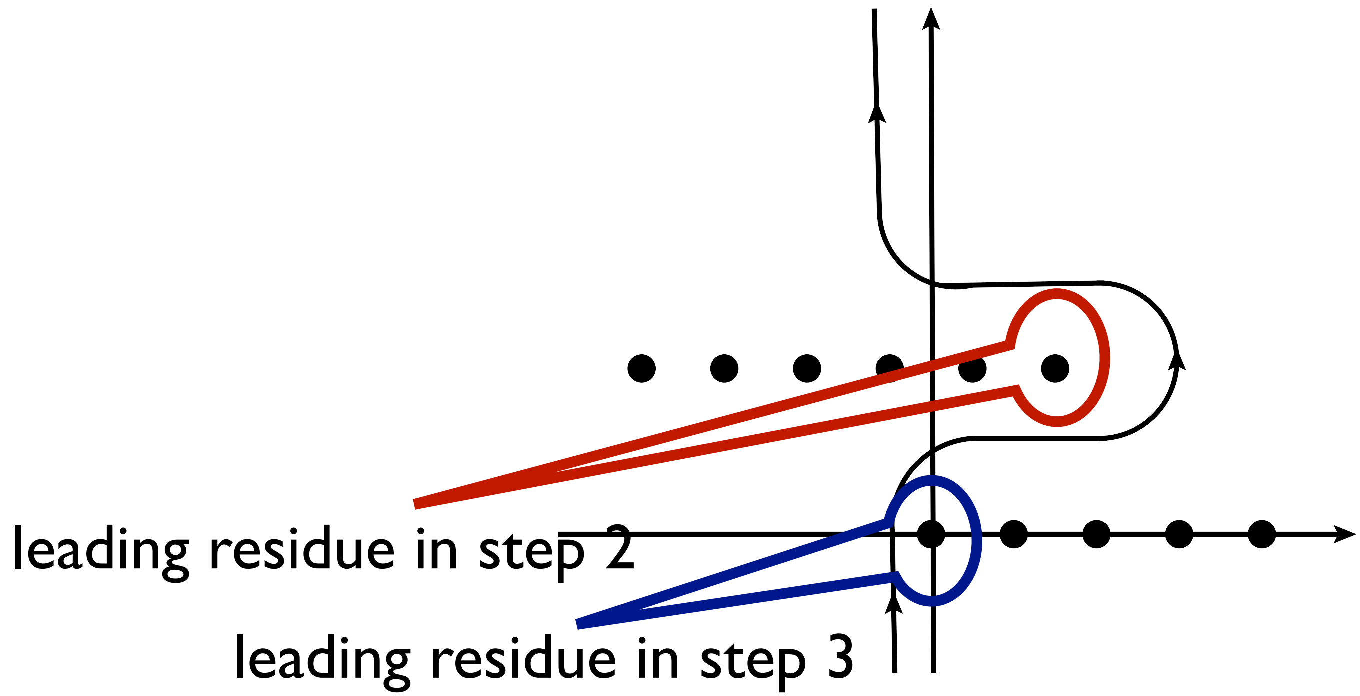

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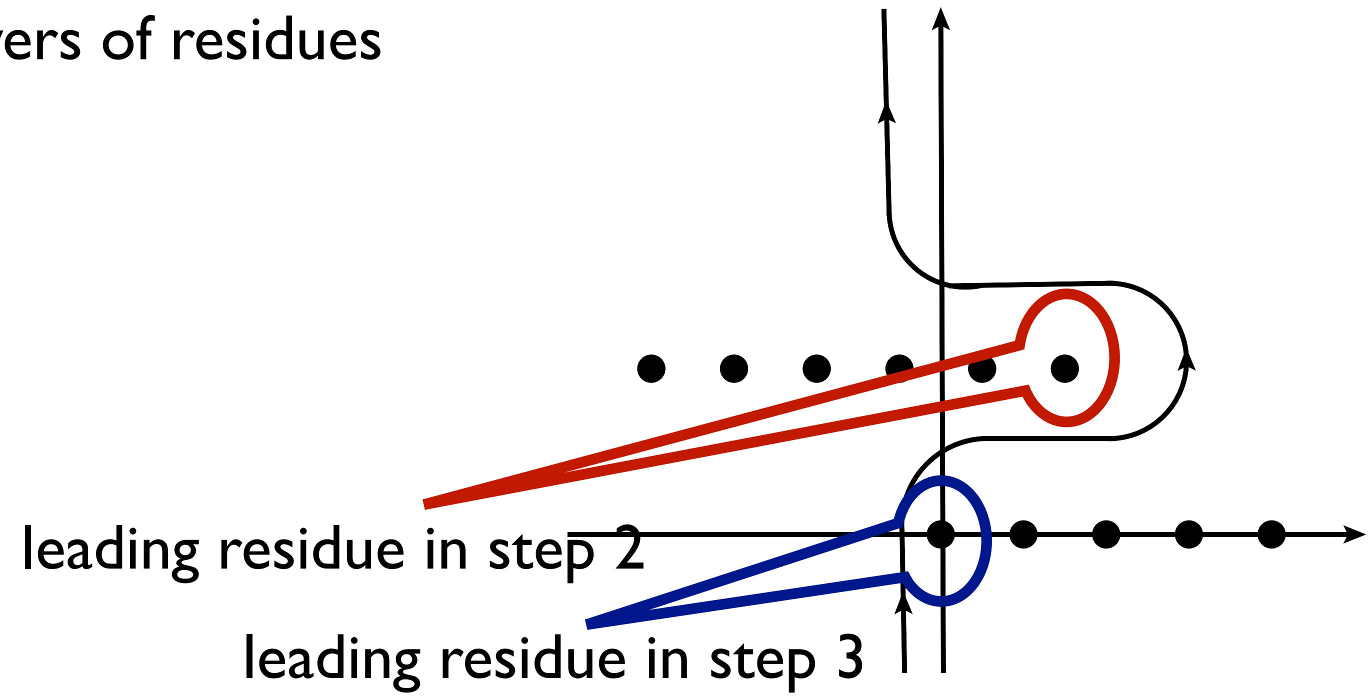
4. Sum remaining towers of residues

$$\sum_{n=1}^{\infty} \frac{u^n}{n} = -\ln(1-u)$$

$$\sum_{n=1}^{\infty} \frac{u^n}{n^k} = \mathrm{Li}_k(u)$$



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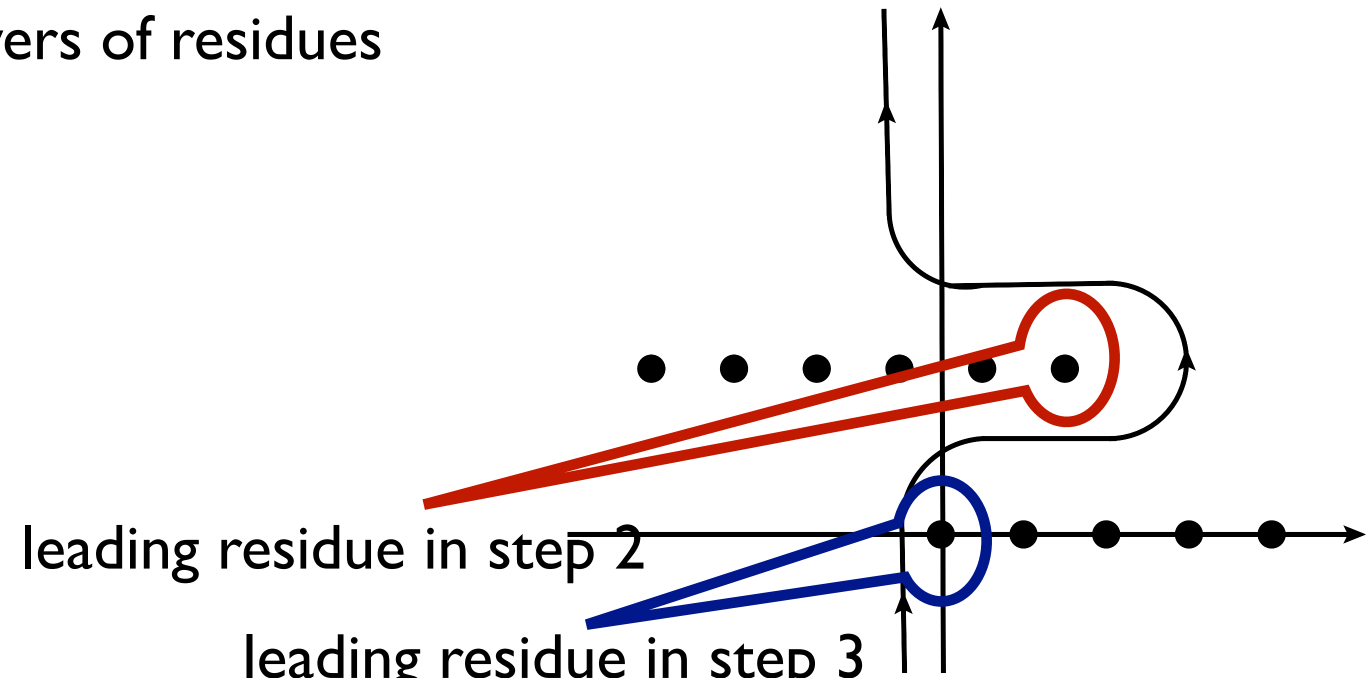

in general, get nested harmonic sums → multiple polylogarithms

$$\sum_{n_1=1}^{\infty} \frac{u_1^{n_1}}{n_1^{m_1}} \sum_{n_2=1}^{n_1-1} \cdots \sum_{n_k=1}^{n_{k-1}-1} \frac{u_k^{n_k}}{n_k^{m_k}} = (-1)^k\, G\left(\underbrace{0,\ldots,0}_{m_1-1}, \frac{1}{u_1}, \ldots, \underbrace{0,\ldots,0}_{m_k-1}, \frac{1}{u_1 \ldots u_k}; 1\right)$$

using symbols, one can reduce the HPLs to a minimal set Buehler Duhr 11

weight 1: $B_1^{(1)}(x) = \ln x\,, \qquad B_1^{(2)}(x) = \ln(1-x)\,, \qquad B_1^{(3)}(x) = \ln(1+x)$

weight 2: $B_2^{(1)}(x) = \mathrm{Li}_2(x)\,, \qquad B_2^{(2)}(x) = \mathrm{Li}_2(-x)\,, \qquad B_2^{(3)}(x) = \mathrm{Li}_2\left(\frac{1-x}{2}\right)$

weight 3: polylogarithms of type Li_3 of various arguments

weight 4: polylogarithms of type Li_4 of various arguments,
plus a few polylogarithms of type $\mathrm{Li}_{2,2}$, like $\mathrm{Li}_{2,2}(-1, x)$ etc.
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- multiple polylogarithms are also defined through nested harmonic sums

$$\mathrm{Li}_{m_1,\ldots,m_k}(u_1,\ldots,u_k) = \sum_{n_k=1}^{\infty} \frac{u_k^{n_k}}{n_k^{m_k}} \sum_{n_{k-1}=1}^{n_k-1} \cdots \sum_{n_1=1}^{n_2-1} \frac{u_1^{n_1}}{n_1^{m_1}} = (-1)^k G_{m_k,\ldots,m_1}\left(\frac{1}{u_k},\ldots,\frac{1}{u_1\cdots u_k}\right)$$

$$G_{m_1,\ldots,m_k}(u_1,\ldots,u_k) = G\Bigg(\underbrace{0,\ldots,0}_{m_1-1},u_1,\ldots,\underbrace{0,\ldots,0}_{m_k-1},u_k;1\Bigg)$$

also multiple polylogarithms can be reduced to a minimal set — Duhr Gangl Rhodes 11

weight 1: one needs functions of type $\ln x$

weight 2: $\mathrm{Li}_2(x)$

weight 3: $\mathrm{Li}_3(x)$

weight 4: $\mathrm{Li}_4(x)$, $\mathrm{Li}_{2,2}(x,y)$

weight 5: $\mathrm{Li}_5(x)$, $\mathrm{Li}_{2,3}(x,y)$

weight 6: $\mathrm{Li}_6(x)$, $\mathrm{Li}_{2,4}(x,y)$, $\mathrm{Li}_{3,3}(x,y)$, $\mathrm{Li}_{2,2,2}(x,y,z)$

let us prove the identity $$\mathrm{Li}_2\left(1-\frac{1}{x}\right) = -\mathrm{Li}_2(1-x) - \frac{1}{2}\ln^2 x$$

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$$\begin{aligned}\mathrm{Sym}\left[\mathrm{Li}_2\left(1-\frac{1}{x}\right)\right] &= -\frac{1}{x} \otimes \left(1-\frac{1}{x}\right) \\ &= x \otimes \frac{x-1}{x} \\ &= x \otimes (1-x) - x \otimes x\end{aligned}$$

$$\mathrm{Sym}[\ln^2 x] = 2\, x \otimes x$$

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thus $$\mathrm{Sym}\left[-\mathrm{Li}_2(1-x) - \frac{1}{2}\ln^2 x\right] = x \otimes (1-x) - \frac{1}{2}\,2\,x \otimes x = \mathrm{Sym}\left[\mathrm{Li}_2\left(1-\frac{1}{x}\right)\right]$$

which determines the function up to functions of lesser degree

$$\mathrm{Li}_2\left(1-\frac{1}{x}\right) = -\mathrm{Li}_2(1-x) - \frac{1}{2}\ln^2 x + c\,\pi^2$$

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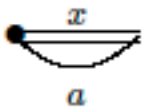
at $x = 1$ $\quad 0 = -0 - 0 + c\,\pi^2 \quad \Rightarrow \quad c = 0$

Symbols in the DGR construction

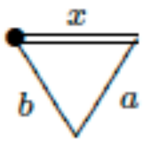
Duhr Gangl Rhodes 11

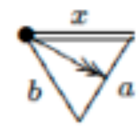
- DGR associate *decorated (n+1)-gons* to multiple polylogarithms of weight *n*

  - $G(a;x) \leftrightarrow$ 

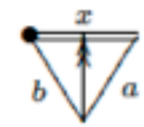
    $\mathcal{S}(G(a;x)) = \left(1 - \frac{x}{a}\right)$

    Gangl Goncharov Levin 05

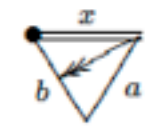
  - $G(a,b;x) \leftrightarrow$ 

$S(G(a,b;x)) \leftrightarrow$ 

$+ax|ba$

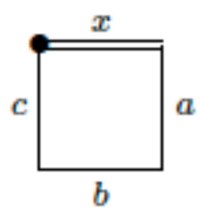


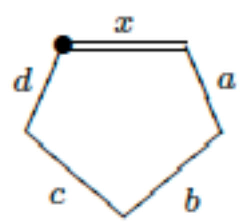
$+bx|ax$



$-bx|ab$

$$ab|cd = \left(1 - \frac{b}{a}\right) \otimes \left(1 - \frac{d}{c}\right)$$

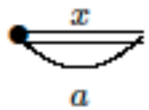
  - $G(a,b,c;x) \leftrightarrow$ 

    $G(a,b,c,d;x) \leftrightarrow$ 

Symbols in the DGR construction

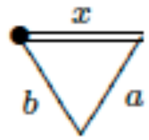
Duhr Gangl Rhodes 11

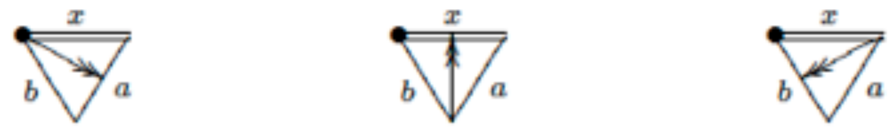
- DGR associate *decorated (n+1)-gons* to multiple polylogarithms of weight *n*

  - *G(a; x)* ↔ 

    $\mathcal{S}(G(a;x)) = \left(1 - \frac{x}{a}\right)$

    Gangl Goncharov Levin 05

  - *G(a, b; x)* ↔ 

    S(G(a, b; x)) ↔ 

    $+ax|ba$ $\quad$ $+bx|ax$ $\quad$ $-bx|ab$

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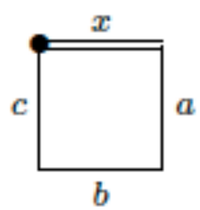
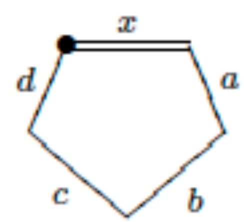
  - *G(a, b, c; x)* ↔ 

    G(a, b, c, d; x) ↔ 

- the symbol in the DGR construction is basically equivalent to GSVV's, *except* that one needs not treat *d log c* as zero

$$C \otimes 2^m\, 3^n\, x^{-5} \otimes D = m\,(C \otimes 2 \otimes D) + n\,(C \otimes 3 \otimes D) - 5(C \otimes x \otimes D)$$

6-dim one-loop 6-point integrals

- $2n$-dim one-loop $2n$-pt integrals ($n > 2$) are finite and conformal invariant
- For n=3, its symbol contributes to the symbol of two-loop Wilson loop

Caron-Huot 11

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  Caron-Huot 11

- explicit expression of massless one-loop 6-pt integral
  is reminiscent of 2-loop 6-edged Wilson loop, but it has weight 3

Duhr Smirnov VDD 11
Dixon Drummond Henn 11

$$I_6(u_1,u_2,u_3) = \frac{1}{\sqrt{\Delta}}\Bigg[-2\sum_{i=1}^{3} L_3(x_i^+,x_i^-) + \frac{1}{3}\left(\sum_{i=1}^{3}\ell_1(x_i^+)-\ell_1(x_i^-)\right)^3 + \frac{\pi^2}{3}\chi\sum_{i=1}^{3}(\ell_1(x_i^+)-\ell_1(x_i^-))\Bigg]$$

$$L_3(x^+,x^-) = \sum_{k=0}^{2}\frac{(-1)^k}{(2k)!!}\ln^k(x^+x^-)\left(\ell_{3-k}(x^+)-\ell_{3-k}(x^-)\right)$$

6-dim one-mass one-loop 6-pt integral

- hexagon with a massive side

$$x_{12}^2 = m^2 \qquad x_{23}^2 = x_{34}^2 = x_{45}^2 = x_{56}^2 = x_{61}^2 = 0$$

- the cross ratios are

$$u_1 = \frac{x_{26}^2\, x_{35}^2}{x_{25}^2\, x_{36}^2}\,, \quad u_2 = \frac{x_{13}^2\, x_{46}^2}{x_{36}^2\, x_{14}^2}\,, \quad u_3 = \frac{x_{15}^2\, x_{24}^2}{x_{14}^2\, x_{25}^2}\,, \quad u_4 = \frac{x_{12}^2\, x_{36}^2}{x_{13}^2\, x_{26}^2}$$

  - in the massless limit, $u_4 \to 0$
  - Z_2 symmetry swaps u_1 and u_2

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- after using MB integrals, the symbol map and momentum twistors, the integral is

Duhr Smirnov VDD 11

$$\mathcal{I}_{6,m}(u_1,u_2,u_3,u_4)$$

$$= \frac{1}{\sqrt{\Delta_7}}\Bigg[-\sum_{i=1}^{8}\sum_{j=1}^{2}\left(L_3(x_{i,j}^+, x_{i,j}^-) - \frac{1}{6}\,\bar\ell_1(x_{i,j}^+, x_{i,j}^-)^3 - \frac{\pi^2}{6}\,\bar\ell_1(x_{i,j}^+, x_{i,j}^-)\right)$$

$$+\frac{1}{2}\left(\bar\ell_1(x_{2,1}^+, x_{2,1}^-) + \bar\ell_1(x_{2,2}^+, x_{2,2}^-)\right)\big(2\bar\ell_1(x_{1,1}^+, x_{1,1}^-)\,\bar\ell_1(x_{1,2}^+, x_{1,2}^-)$$

$$+\bar\ell_1(x_{1,1}^+, x_{1,1}^-)\,\bar\ell_1(x_{3,1}^+, x_{3,1}^-) + \bar\ell_1(x_{1,1}^+, x_{1,1}^-)\,\bar\ell_1(x_{3,2}^+, x_{3,2}^-) + \bar\ell_1(x_{1,2}^+, x_{1,2}^-)\,\bar\ell_1(x_{3,1}^+, x_{3,1}^-)$$

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$$\bar\ell_n(x^+, x^-) = \ell_n(x^+) - \ell_n(x^-)$$

$$\Delta_7 = (u_1 + u_2 + u_3 - u_1u_2u_4 - 1)^2 - 4u_1u_2u_3\,(1-u_4)$$ reduces to Δ in the massless limit

$$x_{i,2}^{\pm}(u_1,u_2,u_3,u_4) = x_{i,1}^{\pm}(u_2,u_1,u_3,u_4)\,, \qquad i = 1,\ldots,8$$ under Z_2 symmetry

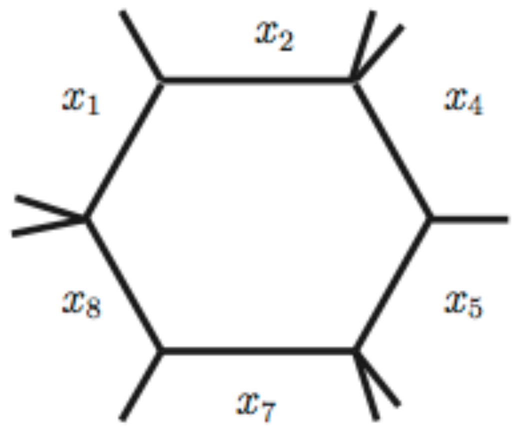
6-dim *3-mass easy* one-loop 6-pt integral

- hexagon with 3 massive sides, x_{24}, x_{57}, x_{81}

  the cross ratios are

$$u_1 = \frac{x_{25}^2 x_{17}^2}{x_{15}^2 x_{27}^2}\,, \quad u_2 = \frac{x_{58}^2 x_{41}^2}{x_{48}^2 x_{15}^2}\,, \quad u_3 = \frac{x_{82}^2 x_{74}^2}{x_{27}^2 x_{48}^2}\,,$$

$$u_4 = \frac{x_{24}^2 x_{15}^2}{x_{14}^2 x_{25}^2}\,, \quad u_5 = \frac{x_{57}^2 x_{48}^2}{x_{47}^2 x_{58}^2}\,, \quad u_6 = \frac{x_{81}^2 x_{72}^2}{x_{82}^2 x_{17}^2}$$



  - in the massless limit, $u_4, u_5, u_6 \to 0$
  - $D_3 \cong S_3$ symmetry made of cyclic rotations c and reflections r

$$u_1 \xrightarrow{c} u_2 \xrightarrow{c} u_3 \xrightarrow{c} u_1\,, u_4 \xrightarrow{c} u_5 \xrightarrow{c} u_6 \xrightarrow{c} u_4\,,$$

$$u_1 \xleftrightarrow{r} u_3\,, u_4 \xleftrightarrow{r} u_5\,,$$

$$u_2 \xleftrightarrow{r} u_2\,, u_6 \xleftrightarrow{r} u_6\,.$$

Dixon Drummond Duhr Henn Smirnov VDD 11

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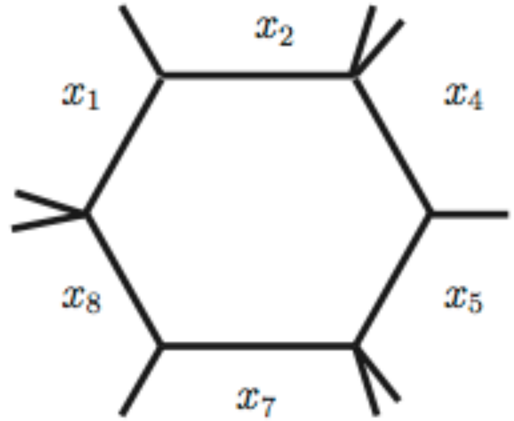

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Dixon Drummond Duhr Henn Smirnov VDD 11

- after using diff. eqs, the symbol map and momentum twistors, the integral is

$$\Phi_9(u_1,\ldots,u_6) = \frac{1}{\sqrt{\Delta_9}} \sum_{i=1}^{4} \sum_{g\in S_3} \sigma(g)\, \mathcal{L}_3(x_{i,g}^+, x_{i,g}^-)$$

$$\sigma(g) = \begin{cases} +1 \text{ for } \{1, c, c^2\} \\ -1 \text{ for } \{r, rc, rc^2\} \end{cases}$$

$$x_{i,g}^{\pm} = g(x_i^{\pm}) \qquad x_i^{\pm} = x_i^{\pm}(u_1,u_2,u_3,u_4,u_5,u_6)$$

$$\mathcal{L}_3(x^+,x^-) = \frac{1}{18}\left(\ell_1(x^+) - \ell_1(x^-)\right)^3 + L_3(x^+,x^-)$$

$$\Delta_9 = (1 - u_1 - u_2 - u_3 + u_4 u_1 u_2 + u_5 u_2 u_3 + u_6 u_3 u_1 - u_1 u_2 u_3 u_4 u_5 u_6)^2 - 4 u_1 u_2 u_3 (1-u_4)(1-u_5)(1-u_6)$$

reduces to Δ in the massless limit

8-edged Wilson loop in AdS_3

- at strong coupling, Alday & Maldacena have considered $2n$-sided polygons embedded into the boundary of AdS_3
- $2n$-sided remainder function depends on $2(n-3)$ variables

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$$R_{8,WL}^{strong} = -\frac{1}{2}\ln\left(1+\chi^-\right)\ln\left(1+\frac{1}{\chi^+}\right)+\frac{7\pi}{6} + \int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{|m|\,\sinh t}{\tanh(2t+2i\phi)}\,\ln\left(1+e^{-2\pi|m|\cosh t}\right)$$

Alday Maldacena 09

where $\chi^+ = e^{2\pi \mathrm{Im}\, m}$ $\chi^- = e^{-2\pi \mathrm{Re}\, m}$ $m = |m|e^{i\phi}$

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- at weak coupling, the 2-loop octagon remainder function is

$$R^{(2)}_{8,WL}(\chi^+,\chi^-) = -\frac{\pi^4}{18} - \frac{1}{2}\ln\left(1+\chi^+\right)\ln\left(1+\frac{1}{\chi^+}\right)\ln\left(1+\chi^-\right)\ln\left(1+\frac{1}{\chi^-}\right)$$

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- 2-loop 2*n*-sided polygon *R* conjectured through collinear limits Heslop Khoze 10
proven through OPE Gaiotto Maldacena Sever Vieira 10

Amplitudes in twistor space

- twistors live in the fundamental irrep of *SO*(2,4)
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  $x_a \leftrightarrow (Z_a, Z_{a+1})$

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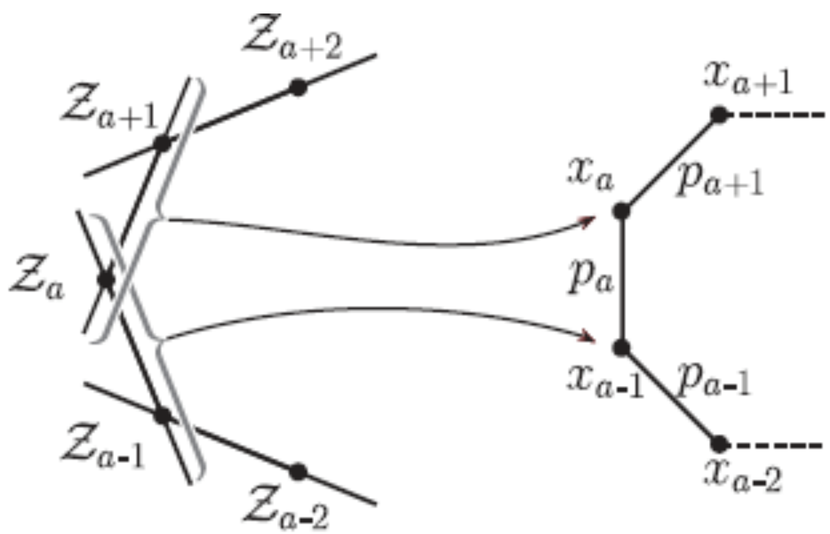

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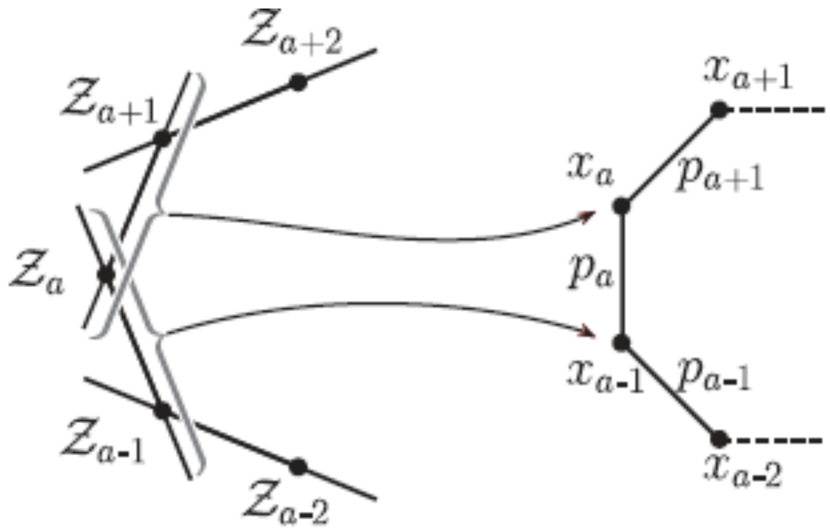
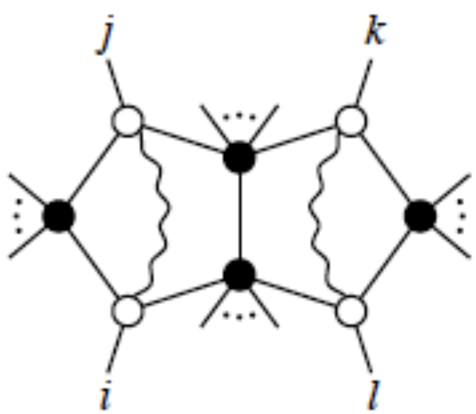


2-loop *n*-pt MHV amplitudes can be written as sum of pentaboxes in twistor space

$$m_n^{(2)} = \frac{1}{2} \sum_{i<j<k<l<i}$$



Arkani-Hamed Bourjaily Cachazo Trnka10