Workshop QCD@Work – Theory and Experiment

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Gauge invariant quarkonium bound state equation

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Gauge invariant Green's functions along polygonal lines

Objective: Study quark Green's function properties by means of a gauge invariant formalism to improve predictivity.

Necessity of using gluon field path-ordered phase factors.

Green's functions with paths along polygonal lines are of particular interest, since they can be decomposed into a succession of straight line segments. The latter have Lorentz invariant forms and the same is true for the polygonal lines.

A phase factor with a single straight line segment going from x to y:

$$U(y,x) = Pe^{-ig\int_x^y dz^\mu A_\mu(z)}.$$

For a rigid straight line segment a displacement of one end of the segment generates also displacements of the interior points of the segment with appropriate weights.

$$\frac{\partial U(y,x)}{\partial y^\alpha} = -igA_\alpha(y)U(y,x)+ig(y-x)^\beta \int_0^1 d\lambda\, \lambda\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0),$$

$$\frac{\partial U(y,x)}{\partial x^\alpha} = +igU(y,x)A_\alpha(x)+ig(y-x)^\beta \int_0^1 d\lambda\, (1-\lambda)\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0).$$

Conventions to represent the contributions of the integrals:

$$\frac{\bar\delta U(y,x)}{\bar\delta y^{\alpha+}} \equiv ig(y-x)^\beta \int_0^1 d\lambda\, \lambda\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0),$$

$$\frac{\bar\delta U(y,x)}{\bar\delta x^{\alpha-}} \equiv ig(y-x)^\beta \int_0^1 d\lambda\, (1-\lambda)\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0).$$

$\Longleftrightarrow$ Rigid path derivatives.

Gauge invariant Green's functions with polygonal lines can be classified according to the number of segments they contain.

The gauge invariant two-point quark Green's function with a polygonal line with n segments and $n-1$ junction points $y_1, y_2, \ldots, y_{n-1}$ between the segments is defined as

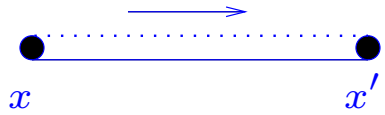
$$S_{(n)}(x, x'; y_{n-1}, \ldots, y_1) = -\frac{1}{N_c} \langle \overline{\psi}(x') U(x', y_{n-1}) \ldots U(y_1, x) \psi(x) \rangle,$$

where each U is along a straight line segment.

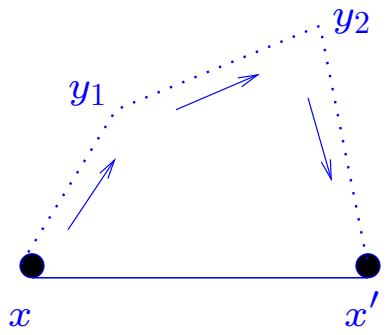
For one straight line, one has:

$$S_{(1)}(x, x') \equiv S(x, x') = -\frac{1}{N_c} \langle \overline{\psi}(x')\, U(x', x)\, \psi(x) \rangle.$$

Pictorially:



$$S(x,x') \equiv S_{(1)}(x,x') = -\tfrac{1}{N_c} < \overline{\psi}(x')\, U(x',x)\, \psi(x) >$$



$$S_{(3)}(x,x';y_2,y_1) = -\tfrac{1}{N_c} < \overline{\psi}(x')\, U(x',y_2) U(y_2,y_1) U(y_1,x)\, \psi(x) >$$

Wilson loop

$$\Phi(C) = \frac{1}{N_c}\mathrm{tr}\, P e^{-ig\oint_C dx^\mu A_\mu(x)}.$$

Vacuum expectation value:

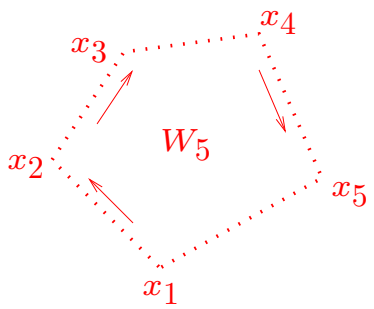
$$W(C) = \langle \Phi(C) \rangle.$$

Functional representation:

$$W(C) = e^{F(C)}.$$

If the contour C is a polygon C_n with n sides and n successive junction points $x_1,\ x_2,\ \ldots,\ x_n$, then we write:

$$W(x_n, x_{n-1}, \ldots, x_1) = W_n = e^{F_n(x_n, x_{n-1}, \ldots, x_1)} = e^{F_n}.$$



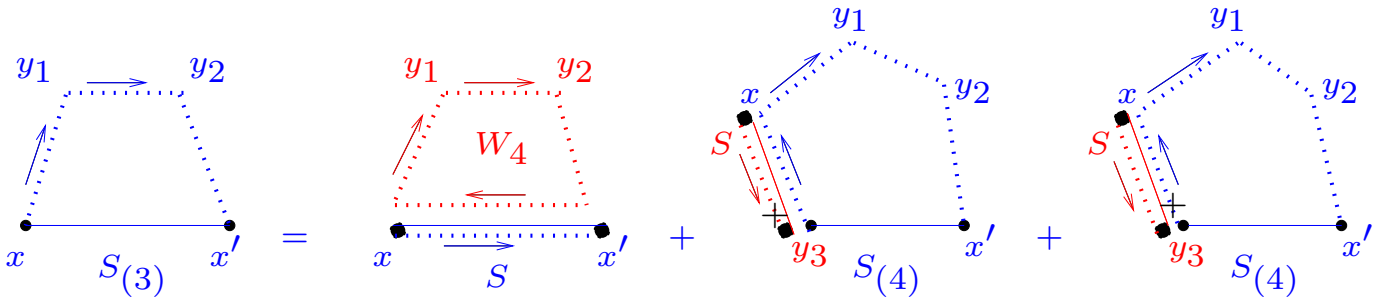
$$W_5(x_5, x_4, \ldots, x_1) = e^{F_5(x_5, \ldots, x_1)}$$

Functional relations for Green's functions

Use of equations of motion of quark fields and integrations yield functional relations between the various Green's functions with polygonal lines.

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = S(x,x')\, e^{F_{n+1}(x',y_{n-1},\ldots,y_1,x)}$$

$$+\left(\frac{\bar\delta S(x,y_n)}{\bar\delta y_n^{\alpha+}} + S(x,y_n)\frac{\bar\delta}{\bar\delta y_n^{\alpha-}}\right)\gamma^\alpha\, S_{(n+1)}(y_n,x';y_{n-1},\ldots,y_1,x).$$



Integrodifferential equation for the two-point function

S satisfies the following equation of motion:

$$(i\gamma.\partial_{(x)} - m)S(x,x') = i\delta^4(x-x') + i\gamma^\mu \frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}}.$$

The rigid path derivative $\bar{\delta}S(x,x')/\bar{\delta}x^{\mu-}$ is calculated using the functional relations between Green's functions. One establishes the following integrodifferential equation for the Green's function $S(x,x')$:

$$(i\gamma.\partial_{(x)} - m)S(x,x') = i\delta^4(x-x') + i\gamma^\mu \Big\{ K_{2\mu}(x',x,y_1)\, S_{(2)}(y_1,x';x) + \sum_{n=3}^{\infty} K_{n\mu}(x',x,y_1,\ldots,y_{n-1})\, S_{(n)}(y_{n-1},x';x,y_1,\ldots,y_{n-2}) \Big\}.$$

The kernel K_n contains globally n derivatives of Wilson loops with a $(n+1)$-sided polygonal contour and also the Green's function S and its derivative.

The Green's functions $S_{(n)}$ themselves are related to the simplest Green's function S with functional relations.

The dominant parts of the kernel are expected to be those containing the least number of derivatives of Wilson loops.

Thus the leading term is the second-order term (the first-order one being zero for symmetry reasons).

$$\frac{\bar{\delta} S(x,x')}{\bar{\delta} x^{\mu -}} \simeq - \int d^4 y_1 \, \frac{\bar{\delta}^2 F_3(x',x,y_1)}{\bar{\delta} x^{\mu -} \bar{\delta} y_1^{\alpha_1 +}} \, e^{F_3(x',x,y_1)} \, S(x,y_1) \, \gamma^{\alpha_1} \, S(y_1,x').$$

Four-point Green's functions

Two quark fields ψ_1 and ψ_2 with masses m_1 and m_2, respectively.

Four-point function with one straight line between the quark and the antiquark:

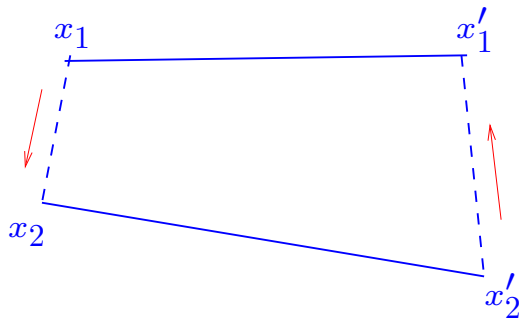
$$G_{(1)}(x_1, x_2; x_2', x_1') = -\frac{1}{N_c} \langle \overline{\psi}_2(x_2) U(x_2, x_1) \psi_1(x_1)\, \overline{\psi}_1(x_1') U(x_1', x_2') \psi_2(x_2') \rangle .$$

Four-point function with a polygonal line of n sides between the quark and the antiquark:

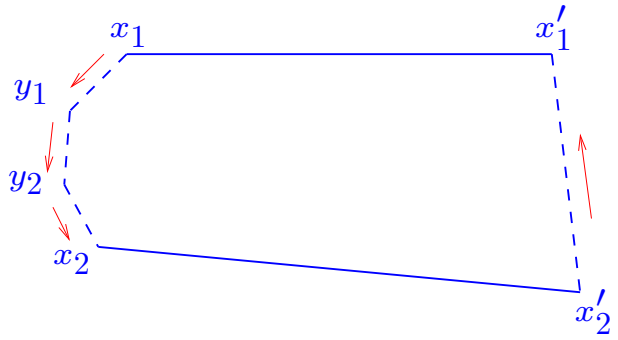
$$G_{(n)}(x_1, x_2; x_2', x_1'; y_{n-1}, \ldots, y_2, y_1) = -\frac{1}{N_c} \langle \overline{\psi}_2(x_2) U(x_2, y_{n-1}) U(y_{n-1}, y_{n-2})$$

$$\times \ldots U(y_2, y_1) U(y_1, x_1) \psi_1(x_1)\, \overline{\psi}_1(x_1') U(x_1', x_2') \psi_2(x_2') \rangle .$$

$G_{(1)} =$



$G_{(3)} =$



Bound state wave functions

Bound state with total momentum P. Wave functions:

$$\Phi_{(1)}(P, x_1, x_2) = \frac{-1}{\sqrt{N_c}} \langle 0|T\,\overline{\psi}_2(x_2)U(x_2, x_1)\psi_1(x_1)|P\rangle,$$

$$\Phi_{(n)}(P, x_1, x_2; y_{n-1}, \ldots, y_2, y_1) = \frac{-1}{\sqrt{N_c}} \langle 0|T\overline{\psi}_2(x_2)U(x_2, y_{n-1})U(y_{n-1}, y_{n-2})$$

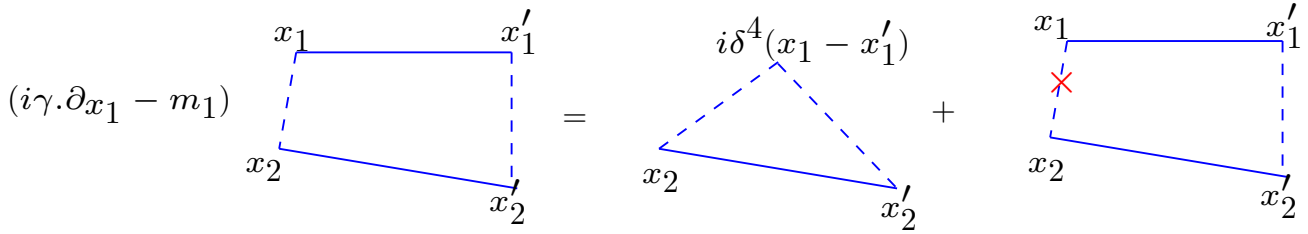
$$\times \ldots U(y_2, y_1)U(y_1, x_1)\psi_1(x_1)|P\rangle,$$

Functional relations are established between $\Phi_{(n)}$, $\Phi_{(1)}$ and Wilson loops, similar to those of the Green's functions.

Wave equations

The four-point Green's functions satisfy equations of motion:

$$(i\gamma.\partial_{x_1} - m_1)G_{(1)}(x_1, x_2; x_1', x_2') = i\delta^4(x_1 - x_1')S_{2(2)}(x_2', x_2; x_1) + i\gamma^\mu \frac{\overline{\delta} G_{(1)}}{\overline{\delta} x_1^{\mu -}}\bigg|_{x_1 x_2}.$$



$$G_{(1)}(x_1, x_2; x_1', x_2')(-i\gamma.\overleftarrow{\partial}_{x_2} - m_2) = i\delta^4(x_2 - x_2')S_{1(2)}(x_1, x_1'; x_2) - i\frac{\overline{\delta} G_{(1)}}{\overline{\delta} x_2^{\mu +}}\bigg|_{x_2 x_1}\gamma^\mu$$

Taking large timelike separations between the sets of points (x_1, x_2) and (x_1', x_2'), one may select the bound states with appropriate projection operators.

$$(i\gamma.\partial_{x_1} - m_1)\Phi_{(1)}(P, x_1, x_2) = i\gamma^\mu \frac{\overline{\delta}\Phi_{(1)}}{\overline{\delta} x_1^{\mu -}}\Big|_{x_1 x_2}.$$

$$\Phi_{(1)}(P, x_1, x_2)(-i\gamma.\overleftarrow{\partial}_{x_2} - m_2) = -i\frac{\overline{\delta}\Phi_{(1)}}{\overline{\delta} x_2^{\mu +}}\Big|_{x_2 x_1}\gamma^\mu.$$

The action of the operators $\frac{\overline{\delta}}{\overline{\delta} x_1^{\mu -}}$ and $\frac{\overline{\delta}}{\overline{\delta} x_2^{\mu +}}$ can be expressed through their action on Wilson loops by using the functional relations between the various wave functions defined with polygonal lines. One ends up with wave equations involving as kernels the Wilson loops with polygonal contours and their functional derivatives, the wave functions $\Phi_{(n)}$ and the two-point Green's functions $S_{1(n)}$ and $S_{2(n)}$.

$$(i\gamma.\partial_{x_1} - m_1)\Phi_{(1)}(P, x_1, x_2) = i\gamma^{\mu}\Big\{\sum_{n=1}^{\infty} K_{1,n\mu} * \Phi_{(n)}\Big\}.$$

$$\Phi_{(1)}(P, x_1, x_2)(-i\gamma.\overleftarrow{\partial}_{x_2} - m_2) = -i\Big\{\sum_{n=1}^{\infty} K_{2,n\nu} * \Phi_{(n)}\Big\}\gamma^{\nu}.$$

These two bound state equations are independent but compatible among themselves.

The leading term in the kernel is represented by the Wilson loop along a triangular contour with two derivatives. Higher-order terms involve Wilson loops along polygonal contours with an increasing number of segments and derivatives.

Chiral symmetry breaking

A criterion for spontaneous chiral symmetry breaking can be obtained, as the one obtained by Baker, Johnson and Lee (1964) with the self-energy part of the fermion Green's function in QED.

Consider the equation of the gauge invariant two-point Green's function $S_{(1)}$ and extract from it the equation satisfied by the anticommutator $\{\gamma_5, S_{(1)}\}$, which selects its scalar part.

Compare the latter equation in the limit of massless quarks ($m_1 = m_2 = 0$) to the bound state equation satisfied by $\Phi_{(1)}$. One finds that the two equations are identical if $P = 0$.

This means that if the equation of $S_{(1)}$ contains a nontrivial (dynamical) solution for its scalar part, then the latter is also a solution of the bound state equation with zero total momentum, corresponding to a massless pseudoscalar bound state, which is then the Goldstone boson of the spontaneous chiral symmetry breaking.