# Moments of PDFs of the pion from Lattice QCD using gradient flow

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## Introduction

Despite the **parton distribution functions** (PDFs) having a long history among hadron structure functions, we are now in an era in which the proton PDFs are a dominant source of uncertainty at several collider experiments. Moreover, the PDFs of the pion remain significantly less constrained from experimental data as they cannot be probed as fixed targets, and their large Bjorken $x$ behavior has been the topic of debates motivating future measurements at JLab and the EIC. Thus, there is increasing urgency for improved theoretical input to PDFs from **lattice quantum chromodynamics** (QCD).

In the **Euclidean space** where lattice QCD is formulated, the non-local operators defining PDFs collapse to a single point, making their direct computation impossible. One can instead consider their **Mellin moments**, which, for quarks, are defined as

$$\langle x^{n-1}\rangle_q(\mu) = \int_0^1 dx\, x^{n-1}\left(q(x,\mu) + (-1)^n \bar{q}(x,\mu)\right), \qquad (1)$$

where $q(x,\mu)$ is a PDF at renormalization scale $\mu$. Moments are defined from matrix elements of local twist-2 operators,

$$\widehat{O}_n^{rs}(x) \equiv i^{n-1}\bar{\psi}_r\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\dots\overleftrightarrow{D}_{\mu_n\}}\psi_s(x) - \text{traces}, \qquad (2)$$

where $\psi_s/\bar{\psi}_r$ is a quark/antiquark field of flavor $s/r$, $\overleftrightarrow{D}_\mu$ is the symmetric forward–backward covariant derivative, and $\{\cdots\}$ denotes normalized symmetrization over Lorentz indices.

However, the $O(4)$ group symmetry of the theory reduces to the **hypercubic group** $H(4)$ on the lattice. As $n > 2$ and the dimension of the operator increases, there is induced **power divergent mixing** with respect to the lattice spacing with lower dimensional operators transforming under the same irreducible representations (irreps). That **can be avoided for $n = 3$ and $4$** by choosing particular irreps which however **require boosting** the state to 2 and 3 different directions respectively, **significantly degrading the signal**, while **no safe irrep exists for $n > 4$**.

Here, we explore a **novel method** to avoid this mixing and obtain precise moments of PDFs of arbitrary order using **gradient flow** [1]. We demonstrate the feasibility of this approach, and compute ratios of moments **up to $\langle x^5\rangle$** of flavor non-singlet PDFs of the **pion** using ensembles tuned at an unphysically heavy pion mass of **411 MeV** [2, 3].

## Lattice calculation

We compute zero-momentum projected connected **three-point functions**

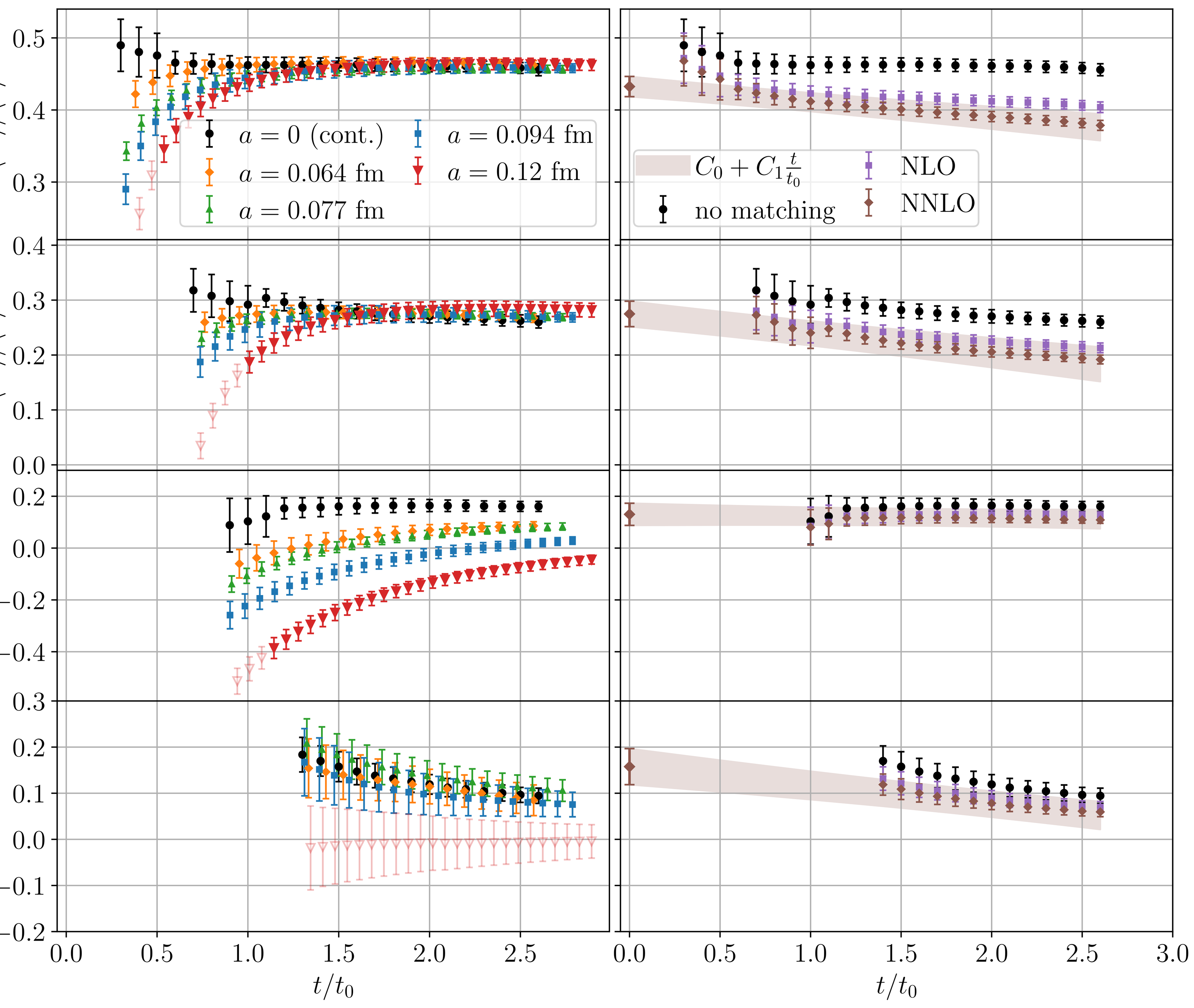
$$C_n^{\text{3-pt}}(\tau_s,\tau_{\mathcal{O}},t) = a^6\sum_{\mathbf{x},\mathbf{y}}\langle\bar{\psi}_d(\tau_s,\mathbf{x})\gamma_5\psi_u(\tau_s,\mathbf{x})\widehat{O}_n^{ud}(\mathbf{y},\tau_{\mathcal{O}},t)\bar{\psi}_u(0)\gamma_5\psi_d(0)\rangle_c \qquad (3)$$

for $n$=2-6 using **four $\mathrm{SU}(3)_f$ Stabilized Wilson Fermion ensembles** with $m_\pi \simeq$ 411 MeV generated by the **OpenLat initiative**, with lattice spacings **$a \simeq$[0.12, 0.094, 0.077, 0.064] fm**. We consider operators with **only temporal Euclidean components**, $\widehat{O}_n = \widehat{O}_{n,4\dots4}$, made traceless by subtracting the appropriate spatial or mixed components, such that **no boosting is required**, and the flowed moment ratios are directly proportional to ground-state matrix elements extracted from three-point functions:

$$\frac{C_n^{\text{3-pt}}(\tau_s,\tau_{\mathcal{O}},t)}{C_2^{\text{3-pt}}(\tau_s,\tau_{\mathcal{O}},t)} = \frac{\langle\pi^+(\mathbf{0})|\widehat{O}_{n,4\dots4}(t)|\pi^+(\mathbf{0})\rangle}{\langle\pi^+(\mathbf{0})|\widehat{O}_{44}(t)|\pi^+(\mathbf{0})\rangle} + \dots \propto \frac{\langle x^{n-1}\rangle(t)}{\langle x\rangle(t)}. \qquad (4)$$

The denominator is set to $n = 2$, since no advantage is found by considering other combinations.

After determining the ratios of flowed moments by investigating the plateau behavior of three-point function ratios, we take the continuum $a \to 0$ limit at each fixed flow time $t/t_0$, and perform the perturbative SFTX matching. The matched ratios demonstrate linear dependence in $t$, which is fit to obtain the final results for $\langle x^{n-1}\rangle/\langle x\rangle$ at $\overline{\text{MS}}$ and $\mu = 2$ GeV, as shown below.




## Flowed moments

To circumvent **power divergent** mixing for twist-2 operators on the lattice, Ref. [1] proposed to use **gradient flow** as a UV regulator: flowed operator matrix elements are finite in the continuum limit if the operator is purely gluonic, and renormalize multiplicatively if fermion fields are present. The **multiplicative renormalization can be avoided by considering ratios of matrix elements with identical fermion content**, which are finite and have a well-defined continuum limit. Thus, ratios of PDF moments in the $\overline{\text{MS}}$ scheme can be obtained as

$$\frac{\langle x^{n-1}\rangle^{\overline{\text{MS}}}(\mu)}{\langle x^{m-1}\rangle^{\overline{\text{MS}}}(\mu)} = \frac{\zeta_m(t,\mu)}{\zeta_n(t,\mu)}\frac{\langle x^{n-1}\rangle(t)}{\langle x^{m-1}\rangle(t)} + O(t), \qquad (5)$$

where the matching coefficients $\zeta_n(t,\mu)$ for the operator of Eq. (2) were calculated at next-to-leading order (NLO) for arbitrary $n$ in Ref. [1] and have been extended to NNLO up to $n = 6$ in Ref. [2], while the $O(t)$ terms are contributions from higher-dimensional operators in the short flow-time expansion (SFTX).

The strategy for the calculation can be summarized as follows:

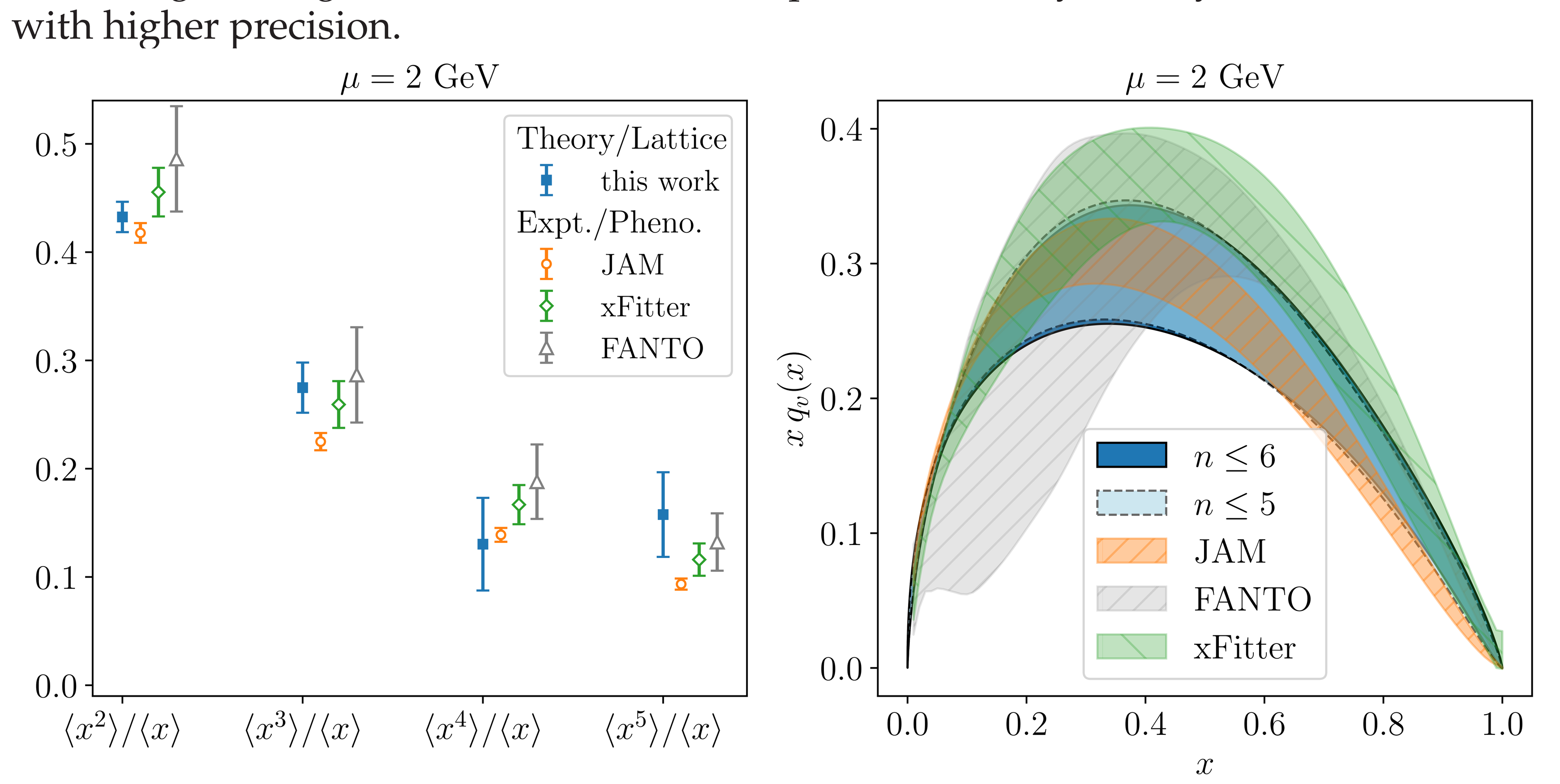
- **determine the ratios of flowed moments that appear on the RHS of Eq.** (5),
- **take the continuum limit at fixed flow time** $t$,
- **match to the** $\overline{\text{MS}}$ **scheme at scale** $\mu$ **using the coefficients** $\zeta_n(t,\mu)$ **in Eq.** (5), and
- **fit any residual $t$-dependence of the results to obtain the physical result at** $t \to 0$.

## Results and comparisons

| | $a$ (fm) | $t_0/a^2$ | $\beta$ | $\kappa_{ud} = \kappa_s$ | $T/a \times (L/a)^3$ | $\mathrm{N}_{\text{cfg}}$ | $\tau_s/a$ |
|---|---|---|---|---|---|---|---|
| `a12m412_mL6.0` | 0.12 | 1.4868(04) | 3.685 | 0.1394305 | $96 \times 24^3$ | 838 | 35,40 |
| `a094m412_mL6.2` | 0.094 | 2.4400(01) | 3.8 | 0.138963 | $96 \times 32^3$ | 700 | 35,40 |
| `a077m412_mL7.7` | 0.077 | 3.6239(12) | 3.9 | 0.138603 | $96 \times 48^3$ | 400 | 40,42 |
| `a064m412_mL6.4` | 0.064 | 5.2471(26) | 4.0 | 0.138272 | $96 \times 48^3$ | 500 | 40,42 |

The specifics of the ensembles used in this work, as well as the statistics of the measurements, are summarized in the table above. **On the left panel of the figure below, we compare the moment ratios against four different phenomenological extractions** of valence PDFs from the past few years. As a reminder, our results are not at the physical point but at $m_\pi \simeq 411$ MeV.

**On the right panel, we present a parametric reconstruction of the PDF using the canonical ansatz** $q_v(x) \propto x^\alpha(1-x)^\beta$. The fitting procedure is not trivial; while Gaussian priors help stabilize the fit, the results remain highly sensitive to the prior width of $\alpha$. The resulting PDF is in reasonable agreement with phenomenological extractions, but, for these reasons, should be regarded as illustrative rather than a robust reconstruction. Despite that, it is worth noting that **the preferred values for the large-$x$ behavior parameter $\beta$**, which are stable across fitting procedures, **are $\beta \sim 1$**. We find that excluding the highest-order moment ($n = 6$) has little impact on the extracted values of $\alpha$ and $\beta$, indicating that higher moments affect these parameters only weakly unless determined with higher precision.




## Conclusion and remarks

These results [2, 3] constitute the first direct lattice QCD extraction of flavor non-singlet pion PDF moment ratios using a novel method that employs the gradient flow. With a modest statistical sample, ratios up to $\langle x^5\rangle/\langle x\rangle$ show **good agreement and competitive precision with phenomenological extractions**. This opens the possibility of **incorporating lattice QCD results from this method into global PDF fits**, either directly in the form of ratios as presented here or as individual moments, after the determination of the normalization $\langle x\rangle$ from standard methods. Additionally, this method is **complementary to established approaches for the $x$-dependence of PDFs**, such as pseudo- and quasi-PDFs, and combining results from both should substantially enhance the impact of lattice QCD in this area.

Future investigations will include: 1) a chiral extrapolation of the results, 2) a better understanding of cutoff effects and their possible reduction by exploiting different discretizations of the twist-2 operators based on the subduced hypercubic irreps, 3) the impact of higher-order corrections in the perturbative matching, and 4) alternative methods like Gaussian Processes for the PDF reconstruction.

Finally, this work opens the way for applying this method to other quantities of interest, including **the proton PDFs, the flavor-singlet and gluon PDFs, and off-forward structure functions such as the Generalized Parton Distributions**.

## References

[1] Andrea Shindler. Moments of parton distribution functions of any order from lattice QCD. *Phys. Rev. D*, 110(5):L051503, 2024.

[2] Anthony Francis, Patrick Fritzsch, Robert V. Harlander, Rohith Karur, Jangho Kim, Jonas T. Kohnen, Giovanni Pederiva, Dimitra A. Pefkou, Antonio Rago, Andrea Shindler, André Walker-Loud, and Savvas Zafeiropoulos. Gradient flow for parton distribution functions: first application to the pion. arXiv:2509.02472.

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