

Three-particle scattering from Lattice QCD

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Bird's eye perspective: *Core goal for non-perturbative QCD (+QED)*

“Given a set of inputs I (needed to tune the theory),
reliably and precisely determine the numerical value of target output O ”

Six inputs (4 quarks, 2 couplings)...

$$M_{\pi^+}/M_{\Omega}$$

$$M_{\pi^0}/M_{\Omega}$$

$$M_{K^+}/M_{\Omega}$$

$$M_{D^+}/M_{\Omega}$$

$$\alpha_{\text{QED}}^{-1}$$

$$M_{\Omega}$$

Many potential outputs

$$m_q^{\overline{\text{MS}}}(\mu)$$

$$m_p$$

$$\alpha_{\overline{\text{MS}}}(\mu)$$

$$g_A$$

$$f_{\pi}$$

$$(g - 2)_{\mu}^{\text{HVP}}$$

QCD

$$\pi^+ \gamma \rightarrow \pi^+ \pi^0$$

$$B \rightarrow K^* \ell^+ \ell^- \quad D^0 \rightarrow \pi^+ \pi^-$$

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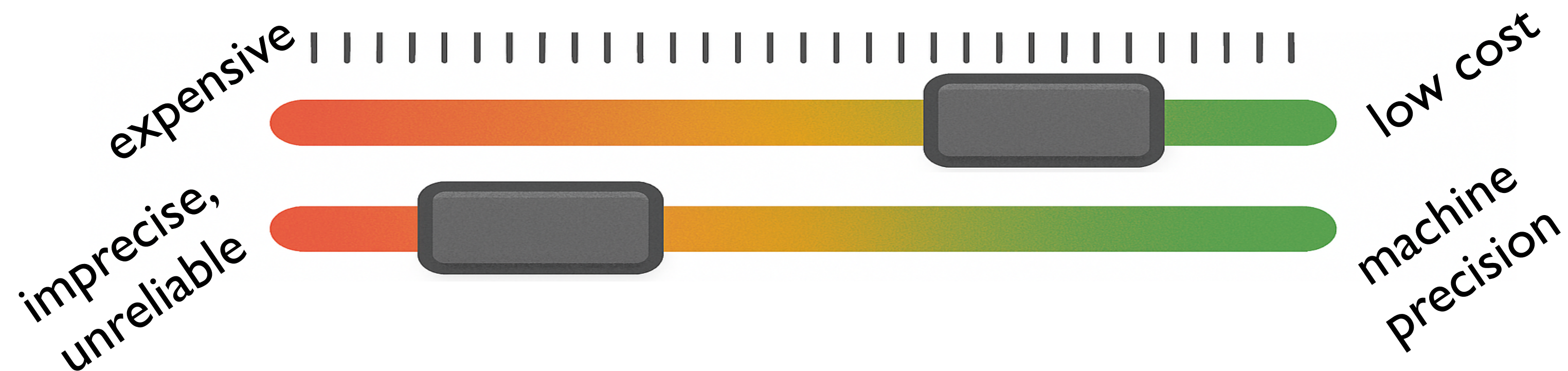
Many potential outputs

$$O = F(I)$$

- ⊗ Most ambitious dream = an analytic solution
- ⊗ Next most ambitious dream = a numerical oracle

Not all oracles are created equal

One scenario is lower cost but less reliable



Bird's eye perspective: *Core goal for non-perturbative QCD (+QED)*

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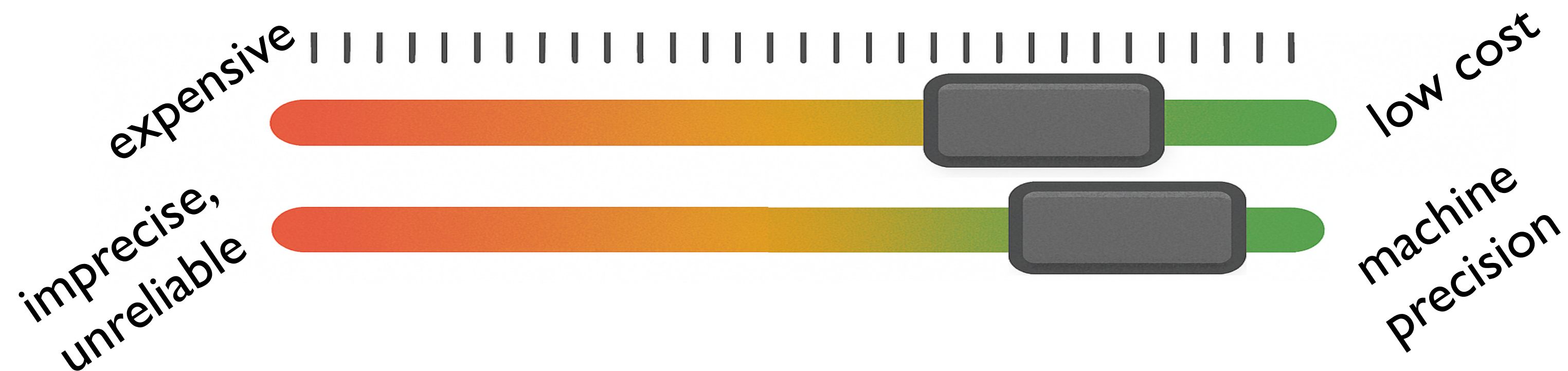
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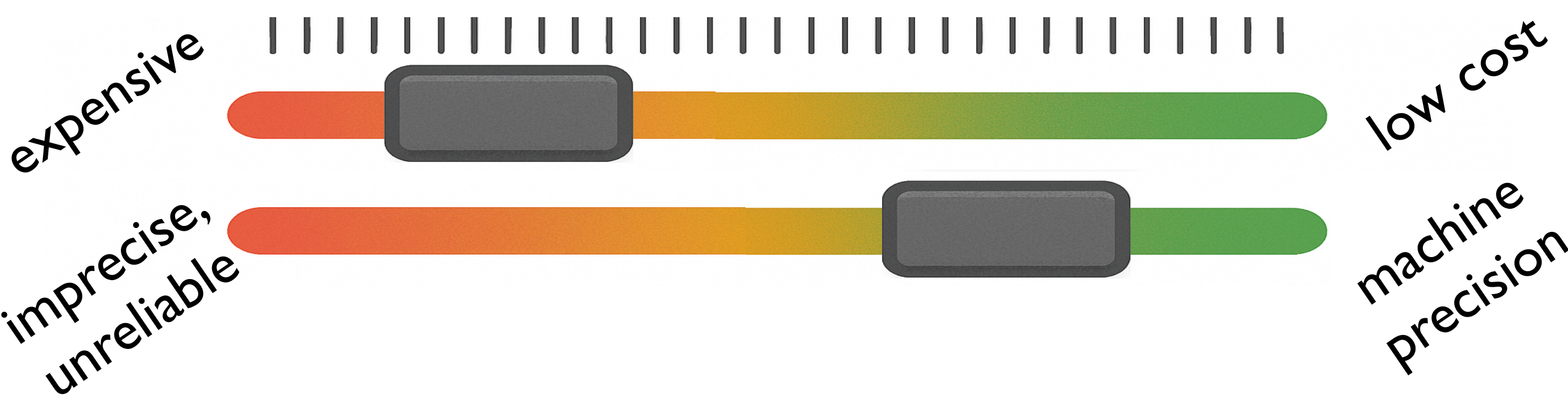
Not all oracles are created equal

Best case would be cheap and precise

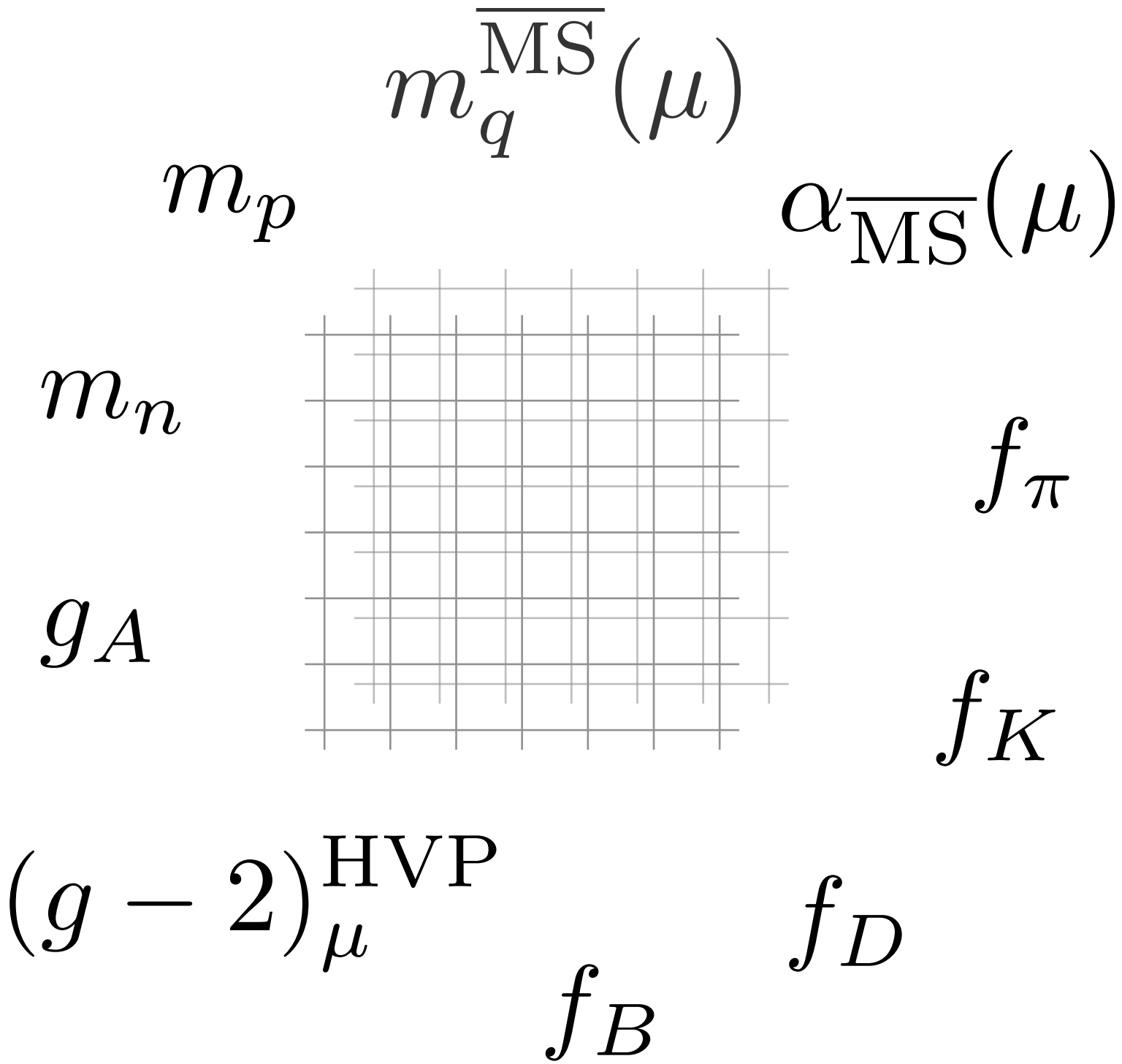


Numerical lattice QCD (+QED)

⊗ ... achieves percent-level predictions for a particular set of observables



- ⊗ Typically best with
- single-particle states
 - mesons rather than baryons
 - local quark-anti-quark currents, e.g. $\mathcal{J}_\mu = \sum_i \bar{q}_i \gamma_\mu q_i$
 - short-distance matrix elements
- (= no on-shell multi-hadron states)
 (= no strong-force phases)



Multi-hadron observables

- ⊗ Essential for **resonances** (that decay via QCD) & **shallow bound states**
 - exotics, XYZs
 - Nucleon excited states $N(1440)$
 - Deuteron

- ⊗ Relevant for electroweak physics and CP violation

CP violation in charm $D \rightarrow \pi\pi, K\bar{K}$

$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$
• LHCb (PRL, 2019) •

Various SM predictions give
a significantly lower value

$f_0(1710)$ could enhance ΔA_{CP}
• Soni (2017) •

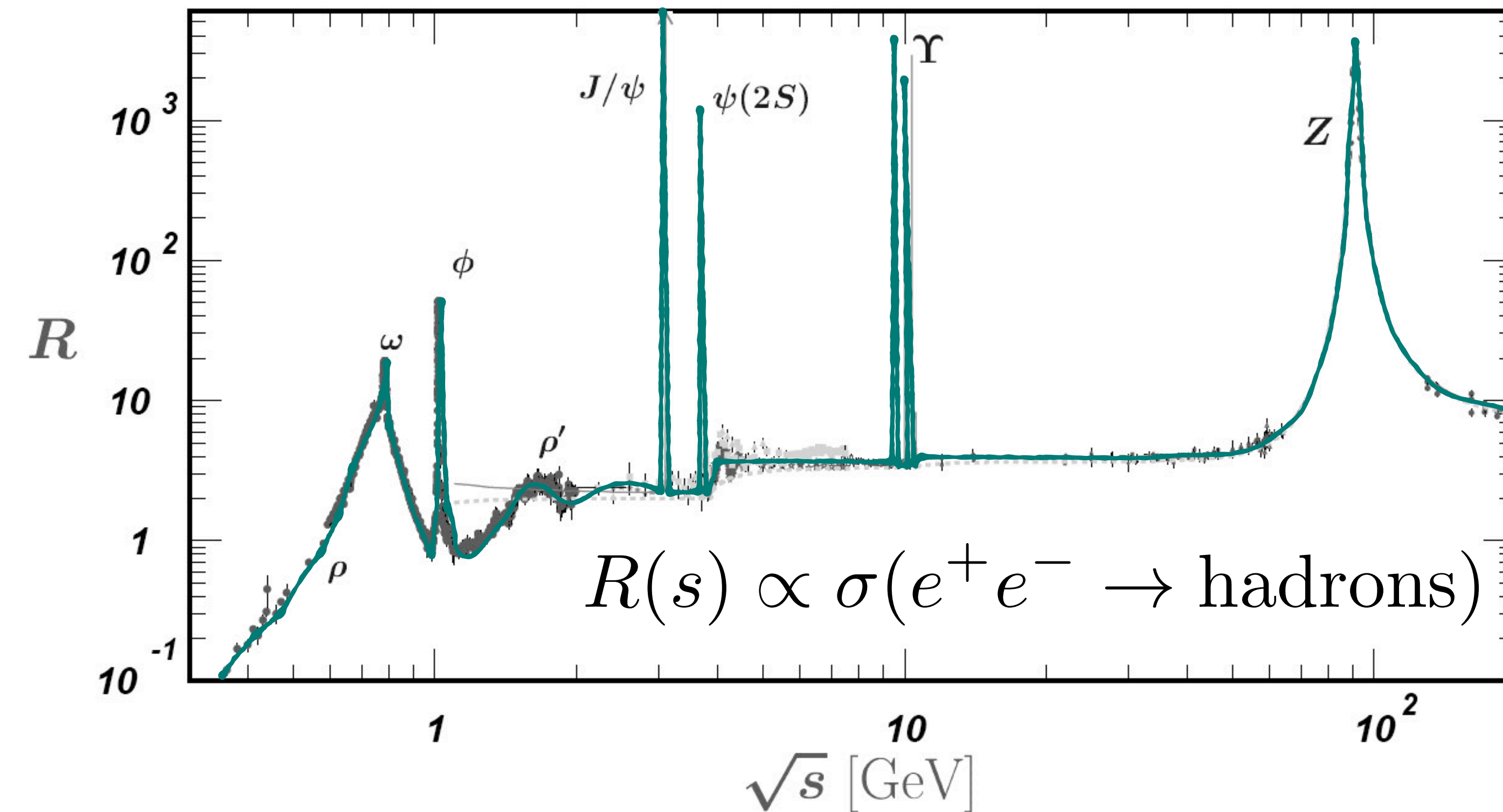
Resonant B decays $B \rightarrow K^* \ell\ell \rightarrow K\pi \ell\ell$

$|X(3872)\rangle, |K^*\rangle, |f_0(1710)\rangle$ are not QCD Hamiltonian eigenstates (not stationary since they decay)
indeed not really states at all, in the sense of the Hilbert space

A sense of the difficulty

- ⊗ An indicator of the challenge from the form of known results

electron-positron to hadrons (experiment)

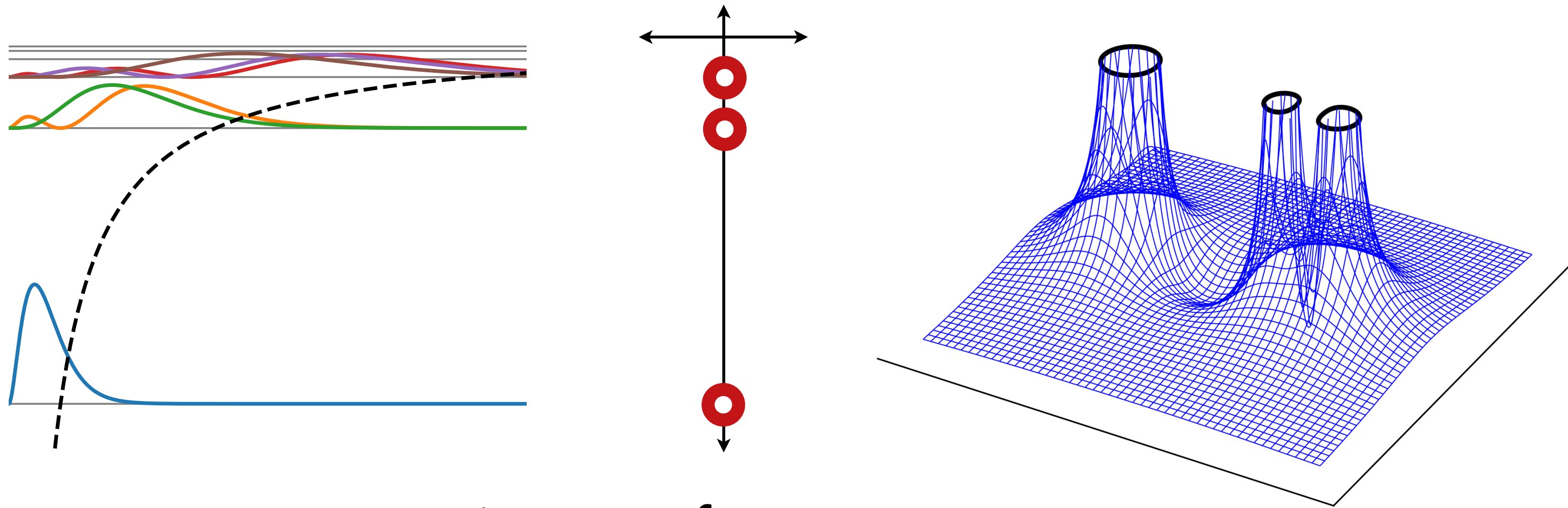


← complicated line shape

- ⊗ Complicated line shape with peaks and cuts
- ⊗ Associated with complex functions (as in $i = \sqrt{-1}$) with poles, branch points, branchcuts

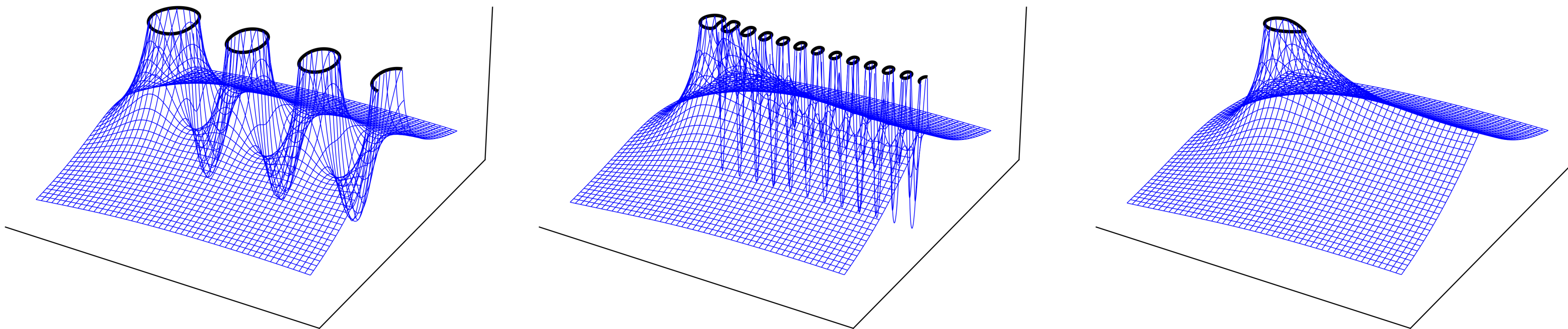
Poles and cuts

- ⊗ A trapped quantum mechanical system has discrete *quantized* energies



Leads to poles in
correlation functions

- ⊗ Scattering concerns a continuum of states



Leads to branch cuts in
correlation functions

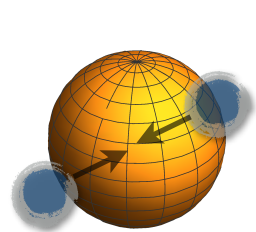
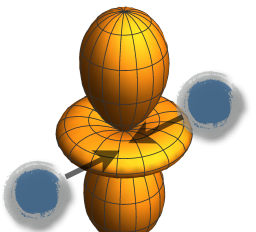
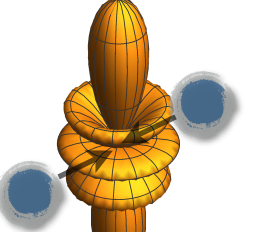
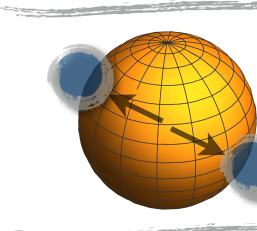
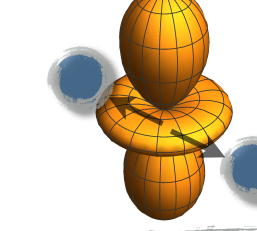
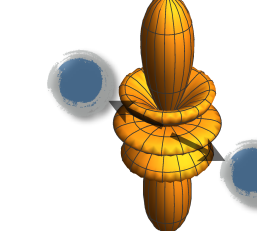
S-matrix (and unitarity)

- At low-energies, QCD = hadronic degrees of freedom
- Overlaps of multi-hadron *asymptotic states* $\rightarrow$ S matrix

$$\pi \sim \bar{u}d, \quad K \sim \bar{s}u, \quad p \sim uud$$

$$[S S^\dagger]_{\alpha\beta} = \sum_{\gamma} \langle \alpha, \text{out} | \gamma, \text{in} \rangle \langle \gamma, \text{in} | \beta, \text{out} \rangle = \mathbb{I}_{\alpha\beta}$$

$|\pi\pi, \text{in}\rangle$

			
$S(s) \equiv \langle \pi\pi, \text{out} $	 $e^{2i\delta_0(s)}$	0	0
	0	 $e^{2i\delta_1(s)}$	0
	0	0	 $e^{2i\delta_2(s)}$

depends on $s = E_{\text{cm}}^2$
and angular variables

diagonal in angular momentum

$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

- An enormous space of information

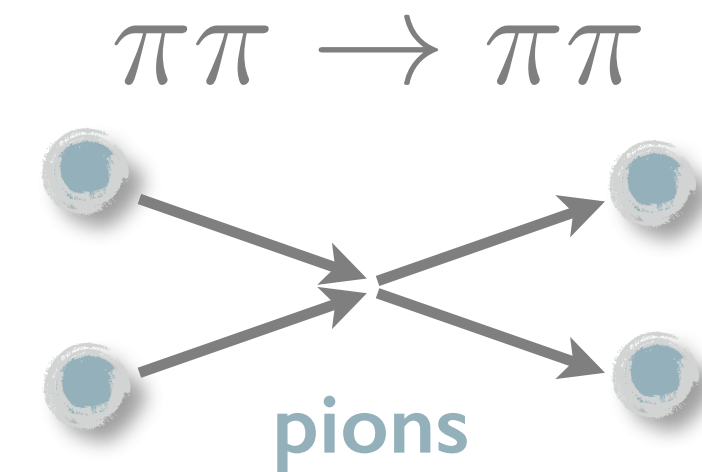
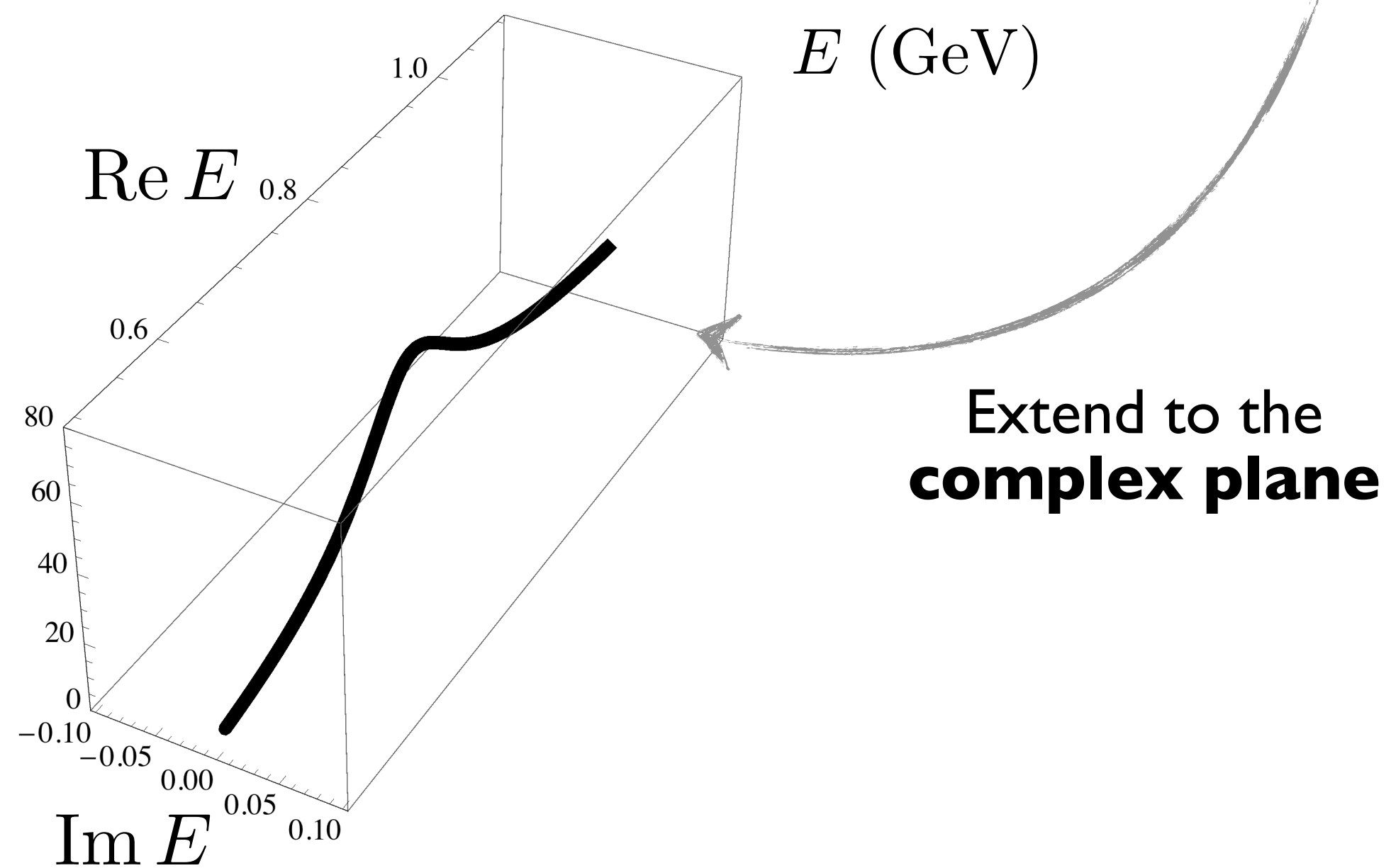
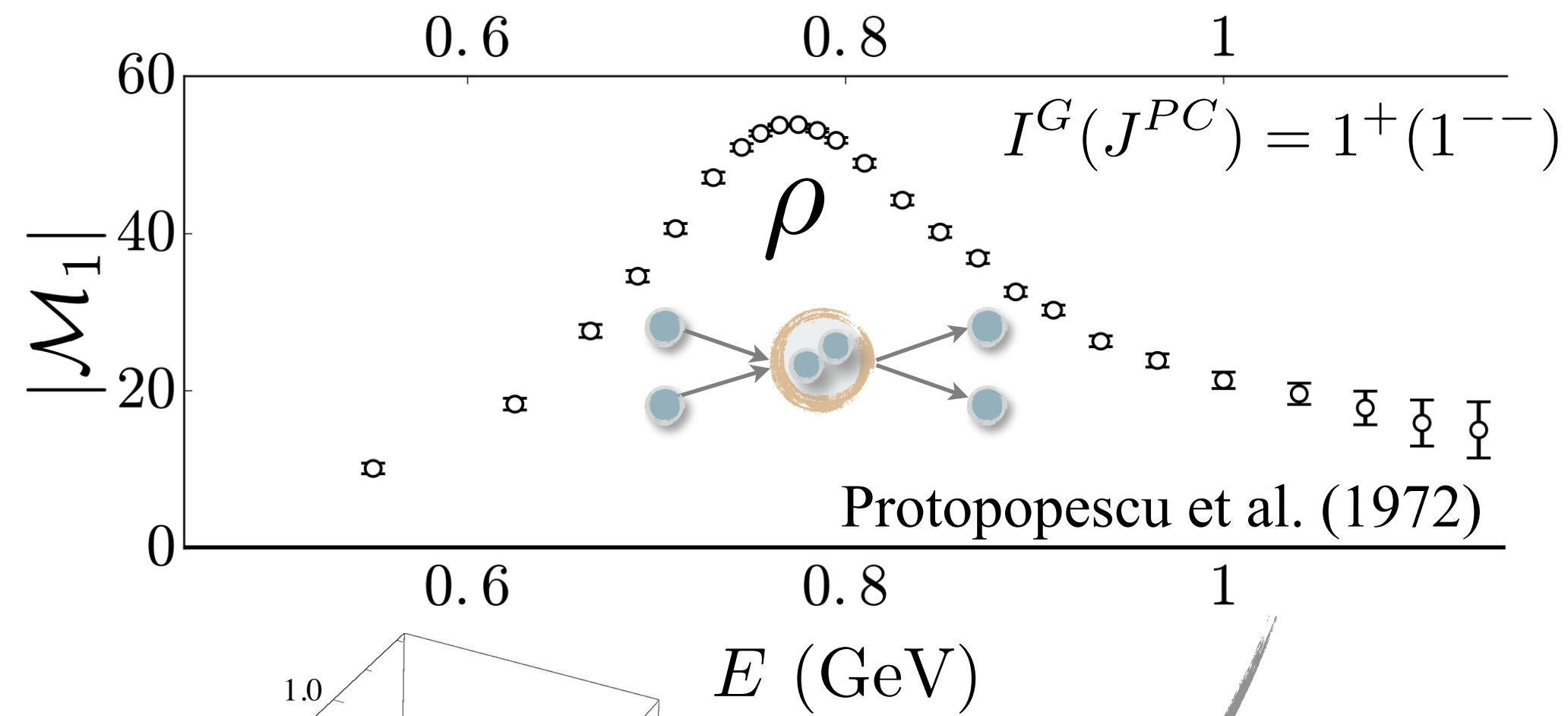
$$|\pi\pi\pi\pi, \text{in}\rangle \quad |K\bar{K}, \text{in}\rangle \quad \dots$$

Definition of a resonance

⊗ Roughly speaking, a bump in:

$$|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$$

scattering rate

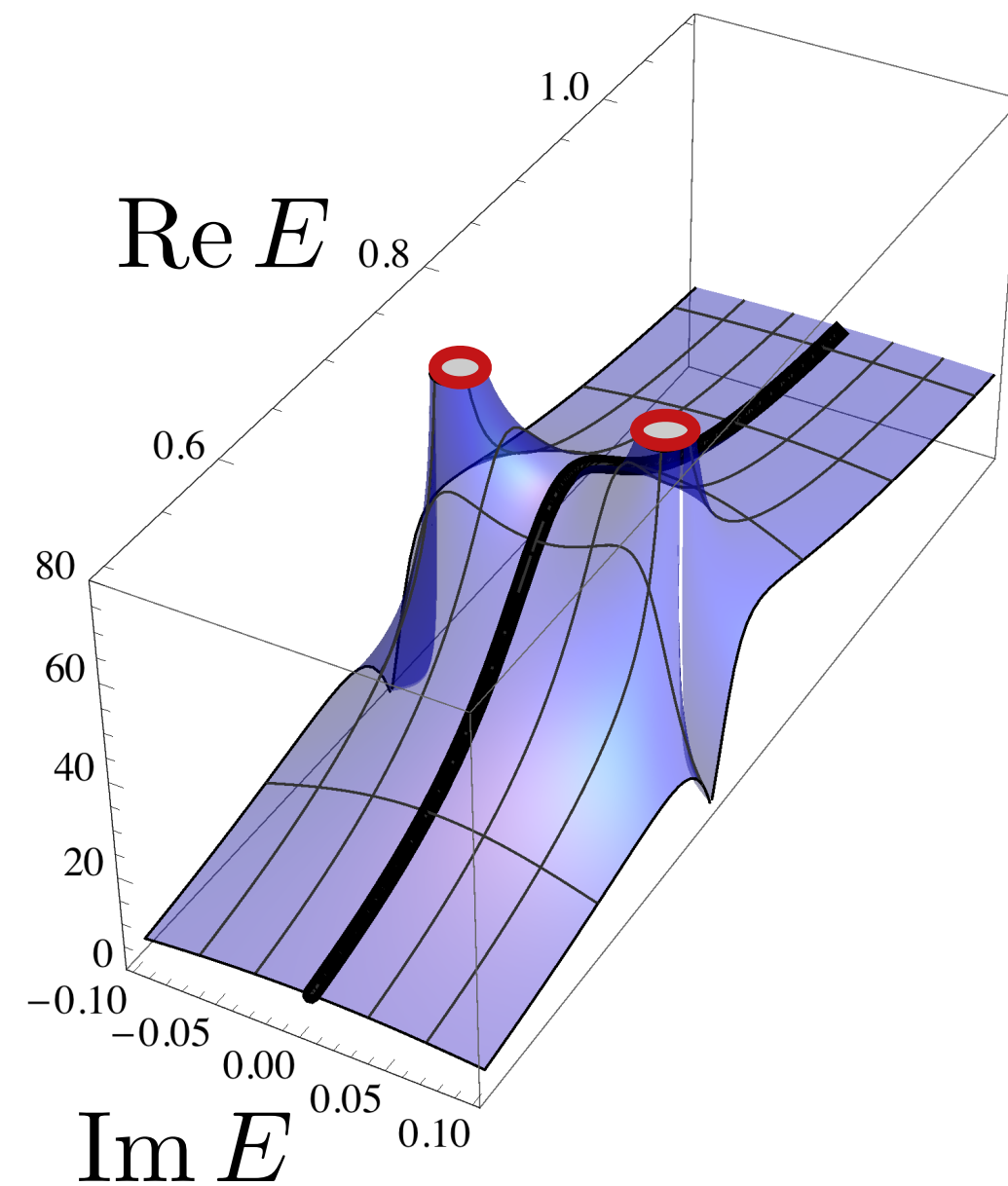
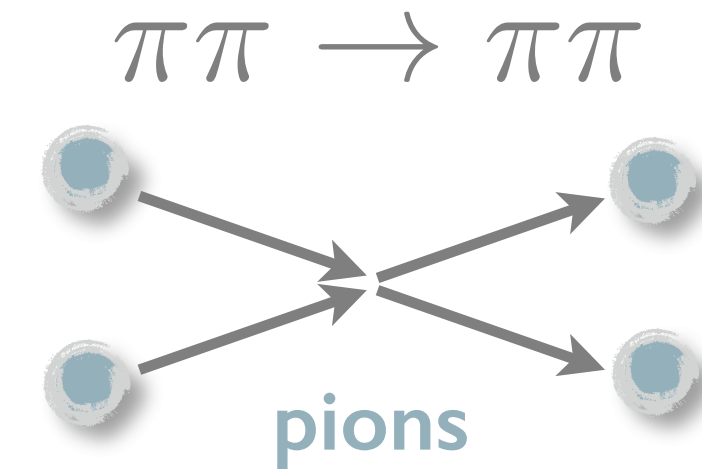
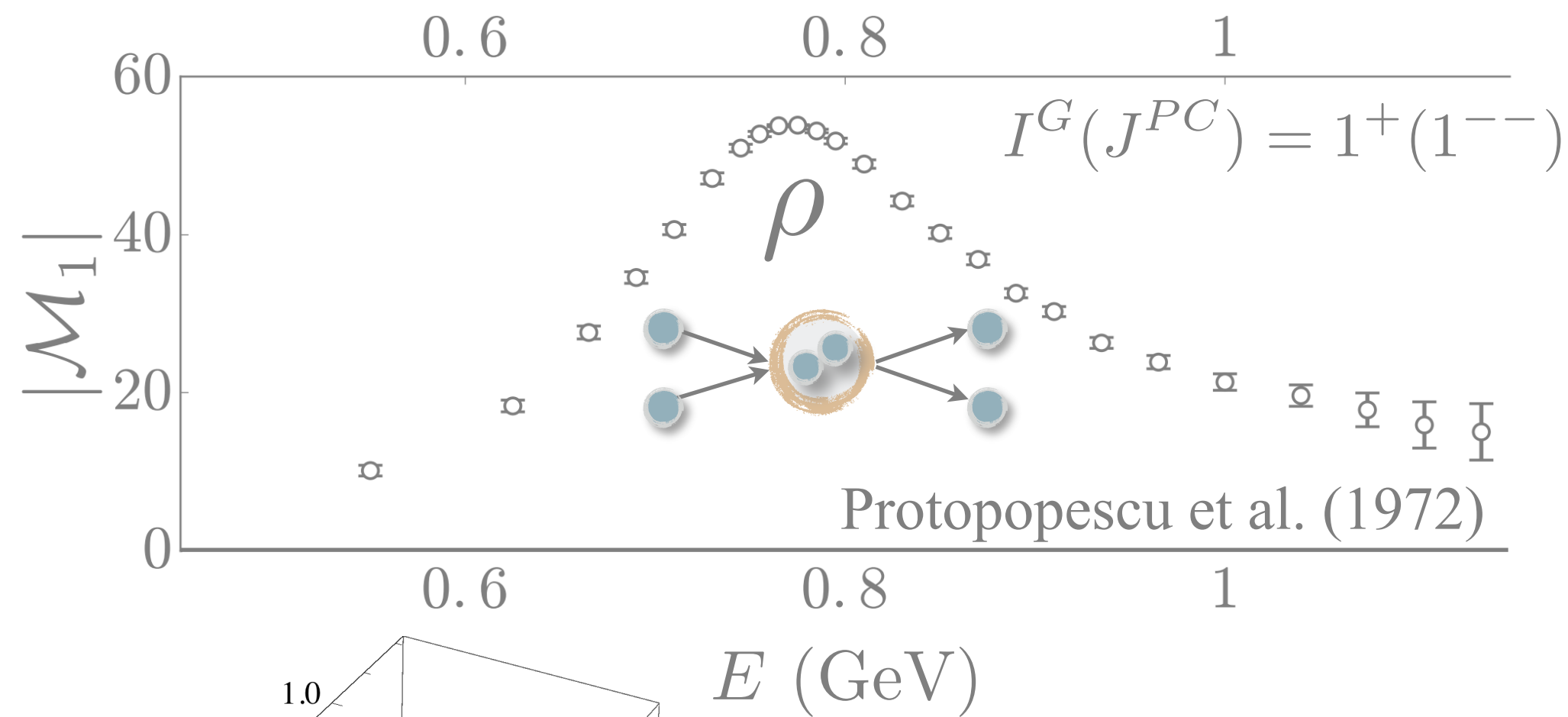


Definition of a resonance

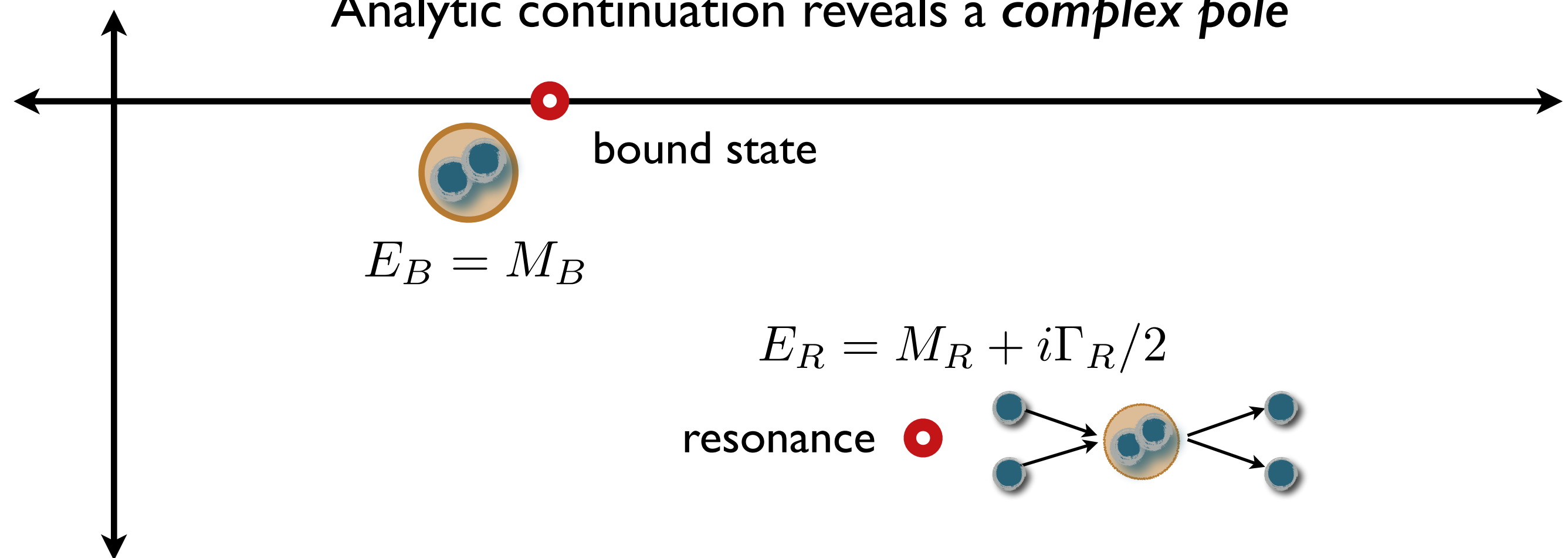
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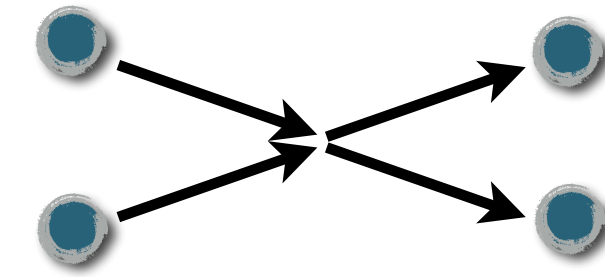
Analytic continuation reveals a **complex pole**



Analyticity

⊗ Instead of $|\mathcal{M}(s)|^2 \rightarrow$ analytically continue the *amplitude* itself

For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?



⊗ The optical theorem tells us...

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \text{Im } \mathcal{M}_\ell(s)$$

... where $\rho(s) = \frac{\sqrt{1 - 4m^2/s}}{32\pi}$ is the two-particle phase space

⊗ Unique solution is...

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

K matrix (short distance)

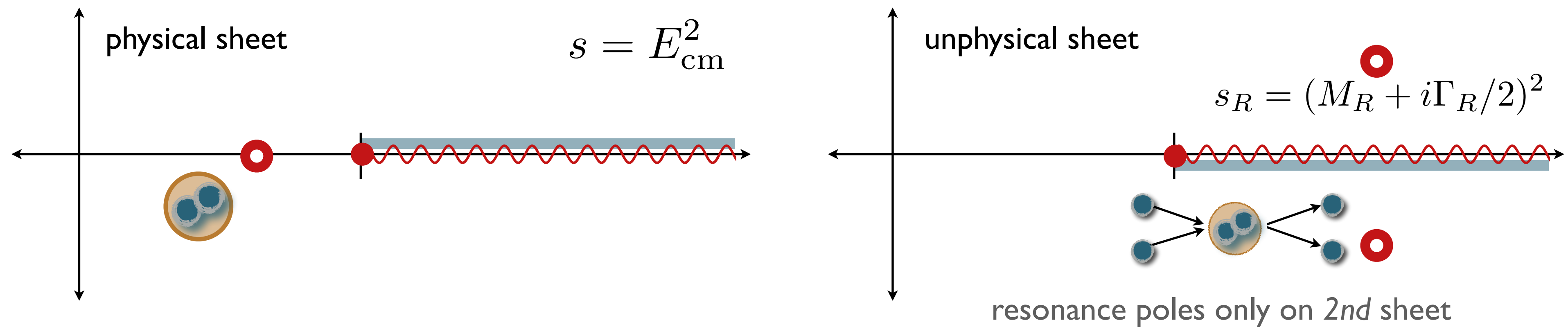
phase-space cut (long distance)

Key message: *The scattering amplitude has a square-root branch cut*

Two-to-two scattering

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)} \propto \frac{1}{p \cot \delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \quad \rho(s) \propto i\sqrt{s - (2m)^2}$$

Each channel generates a *square-root cut* $\rightarrow$ doubles the number of sheets



Important lessons:

Details of analyticity = important for quantitative understanding

Possible to separate...

(i) *long-distance kinematic singularities*

(ii) *short-distance/microscopic physics (depending on interaction details)*

Lattice QCD meets multi-hadron

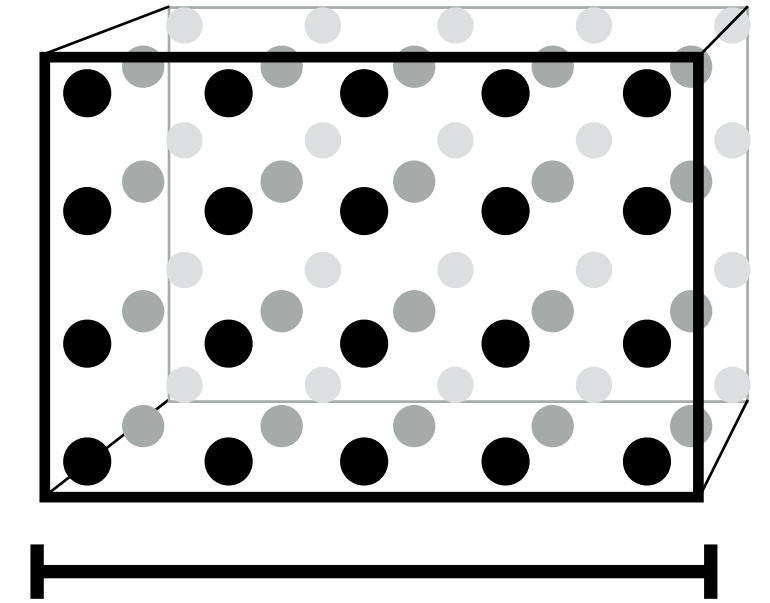
⊗ LQCD evaluates the path-integral in a finite, Euclidean space-time

⊗ The *finite volume*...

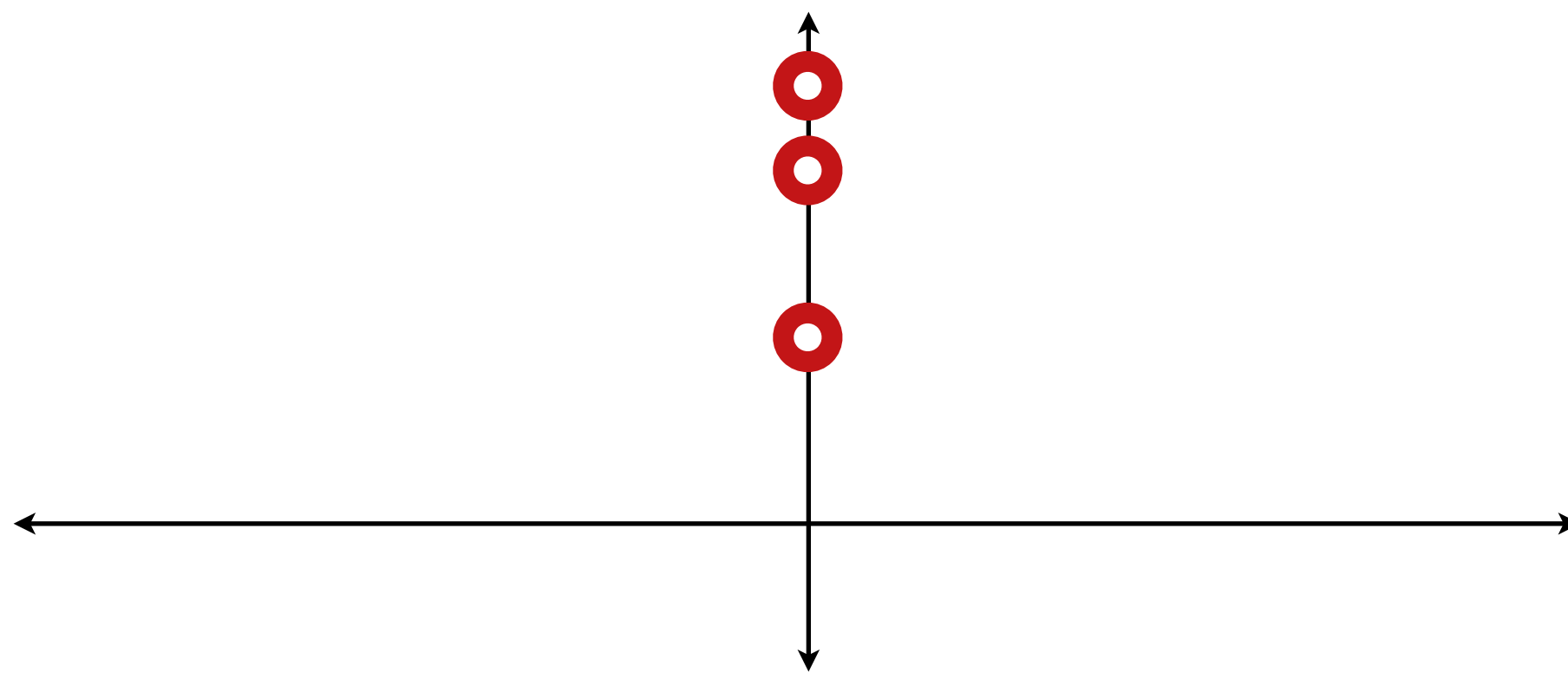
- *Discretizes* the spectrum
- *Eliminates* the branch cuts and extra sheets
- *Hides* the resonance poles

⊗ The *Euclidean signature*...

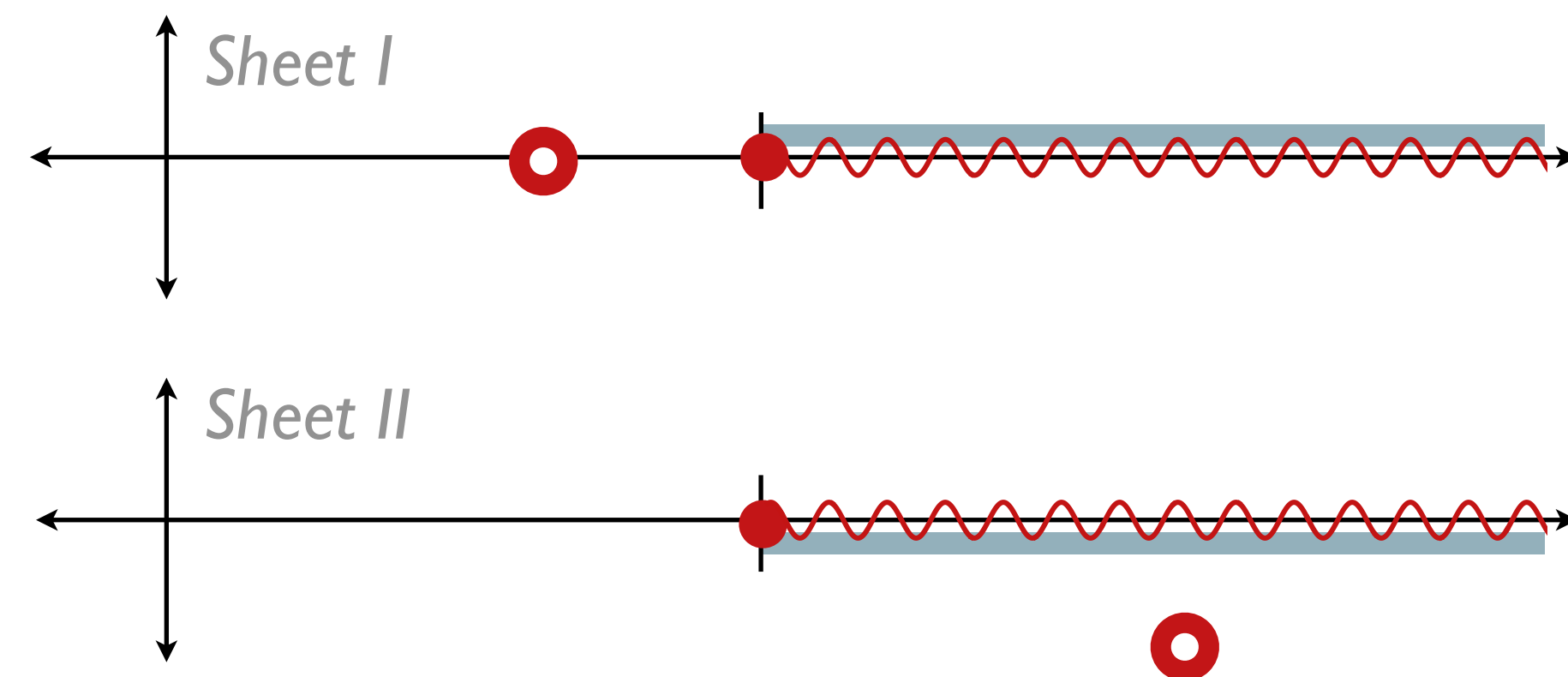
- *Prevents* simple access to timelike momenta
- *Denies* easy access to on-shell particles



Finite-volume Euclidean analytic structure



Infinite-volume Minkowski analytic structure



Two strategies

⊗ Finite-volume as a tool (Euclidean signature not “of central importance”)

- LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau) \mathcal{O}_i^\dagger(0) \rangle = \sum_n \langle 0 | \mathcal{O}_j(\tau) | E_n \rangle \langle E_n | \mathcal{O}_i^\dagger(0) | 0 \rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

- Our task is relate $E_n(L)$ and $\langle E_{m'} | \mathcal{J}(0) | E_m \rangle$ to **experimental observables**
- Applicable only in limited energy range for two- and three-hadron states

⊗ Spectral function method (finite-volume not “of central importance”)

- Formally applies for any number of particles / any energy range
- An answer to the question... “Can’t you just analytically continue?”
- Still important challenges and limitations to consider

Landscape of amplitudes

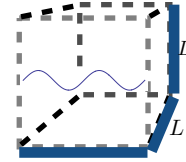
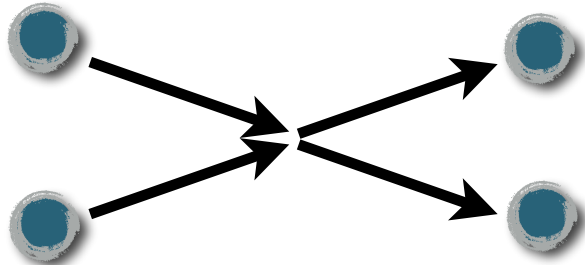
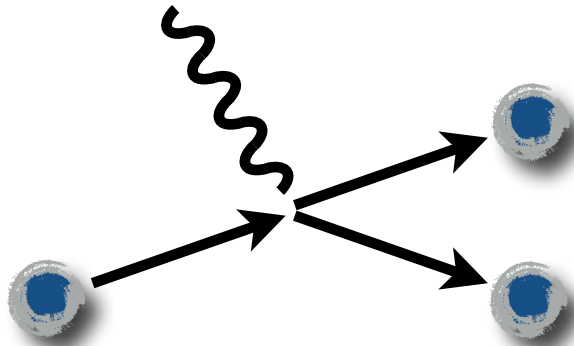
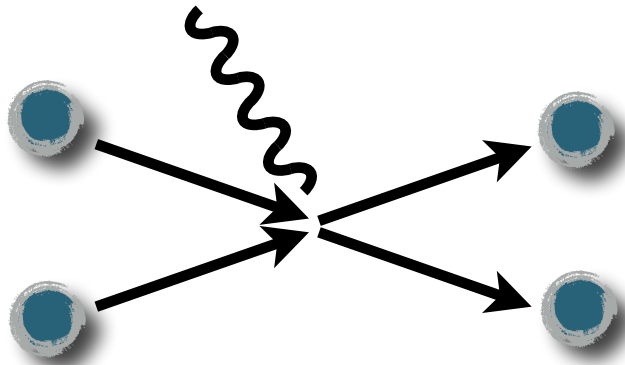
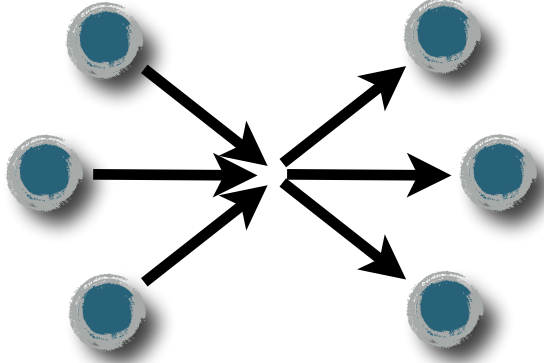
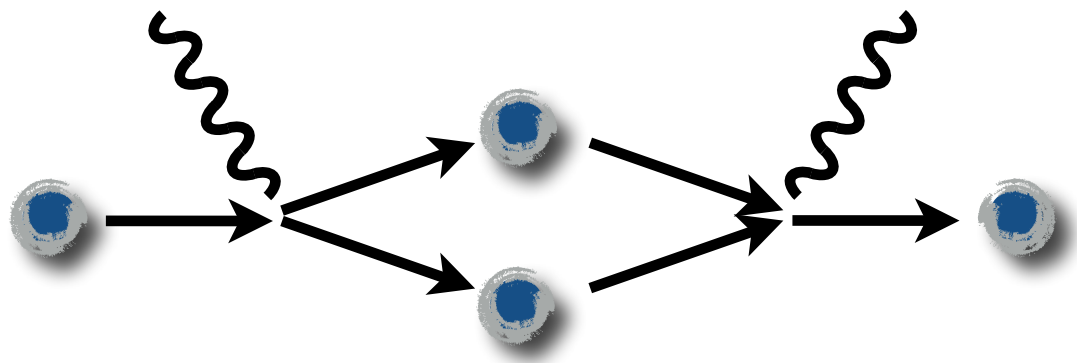
Two-to-two scattering: $2 \rightarrow 2$

Decays with an external current: $1 \xrightarrow{\mathcal{J}} 2$

Transitions with an external current: $2 \xrightarrow{\mathcal{J}} 2$

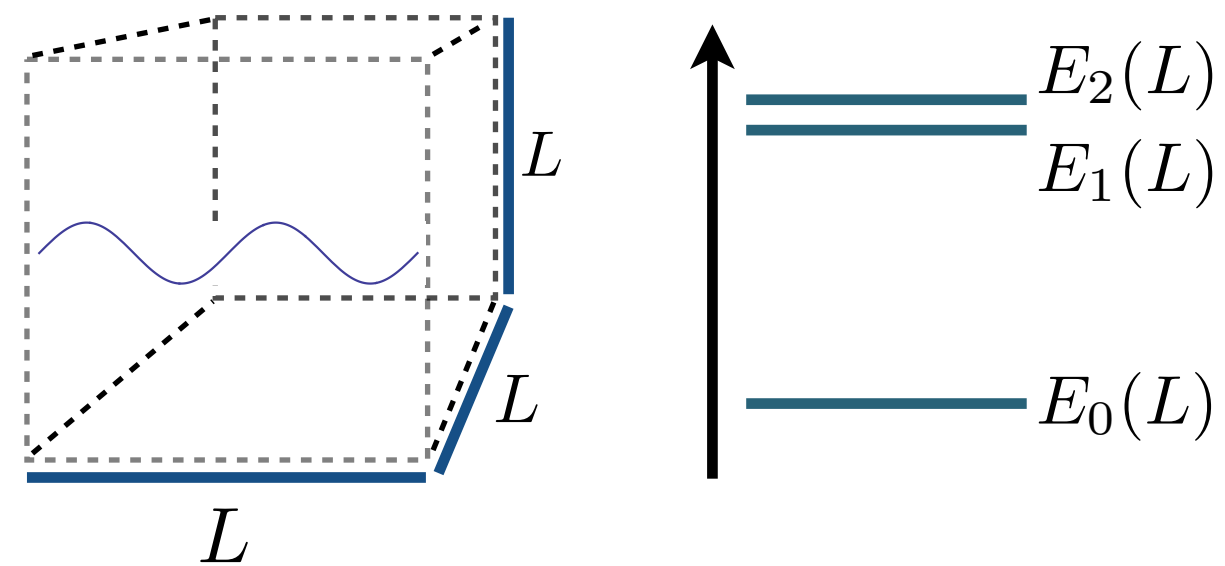
Three-to-three scattering: $3 \rightarrow 3$

Long distance matrix elements $0, 1 \xrightarrow{\mathcal{J}} X \xrightarrow{\mathcal{J}} 0, 1$

FV method 	SF method
 formalism ($E_{\text{cm}} < 4M_\pi$) numerical maturity = 🏆	formalism numerical maturity = 🤖
 formalism ($E_{\text{cm}} < 4M_\pi$) numerical maturity = 👍	formalism numerical maturity = 🤖
 formalism ($E_{\text{cm}} < 4M_\pi$) numerical maturity = 🤖	formalism numerical maturity = 🤖
 formalism ($E_{\text{cm}} < 5M_\pi$) numerical maturity = 👍	formalism numerical maturity = 🤖
 formalism ($E_{\text{cm}} < 4M_\pi$) numerical maturity = 👍	formalism 🏆 👍 🤖

The finite-volume as a tool

Finite-volume set-up

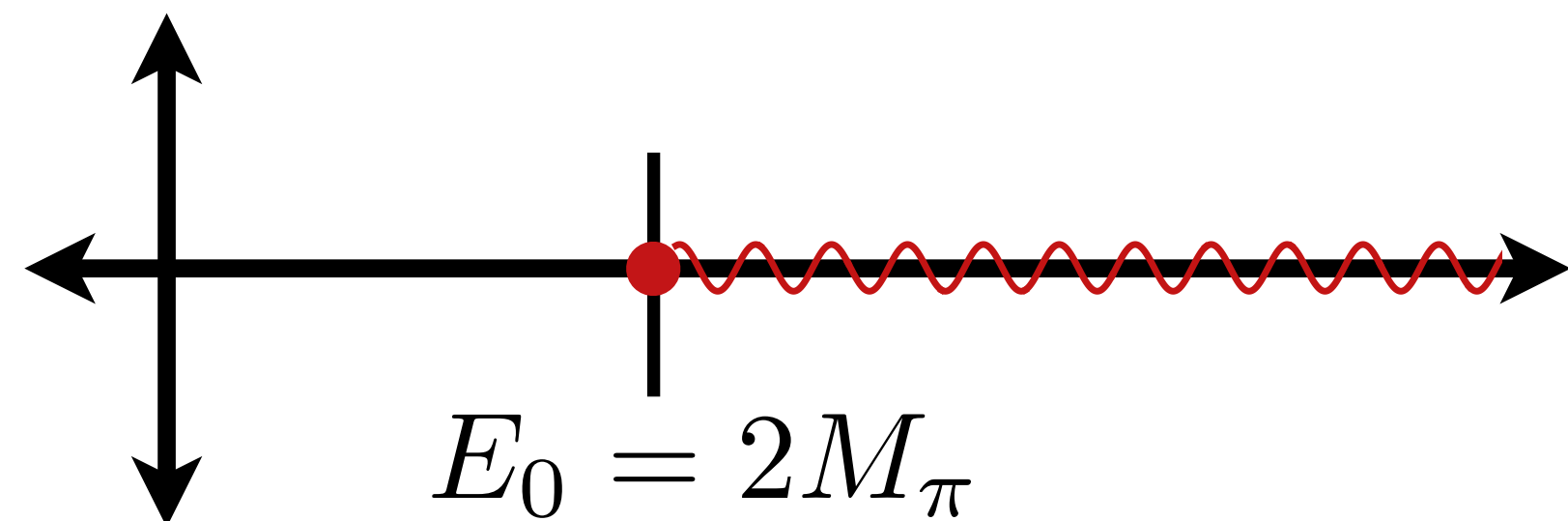


□ **cubic, periodic**, spatial volume (extent L)

$$\vec{p} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3$$

□ L is large enough to neglect $e^{-M_\pi L}$

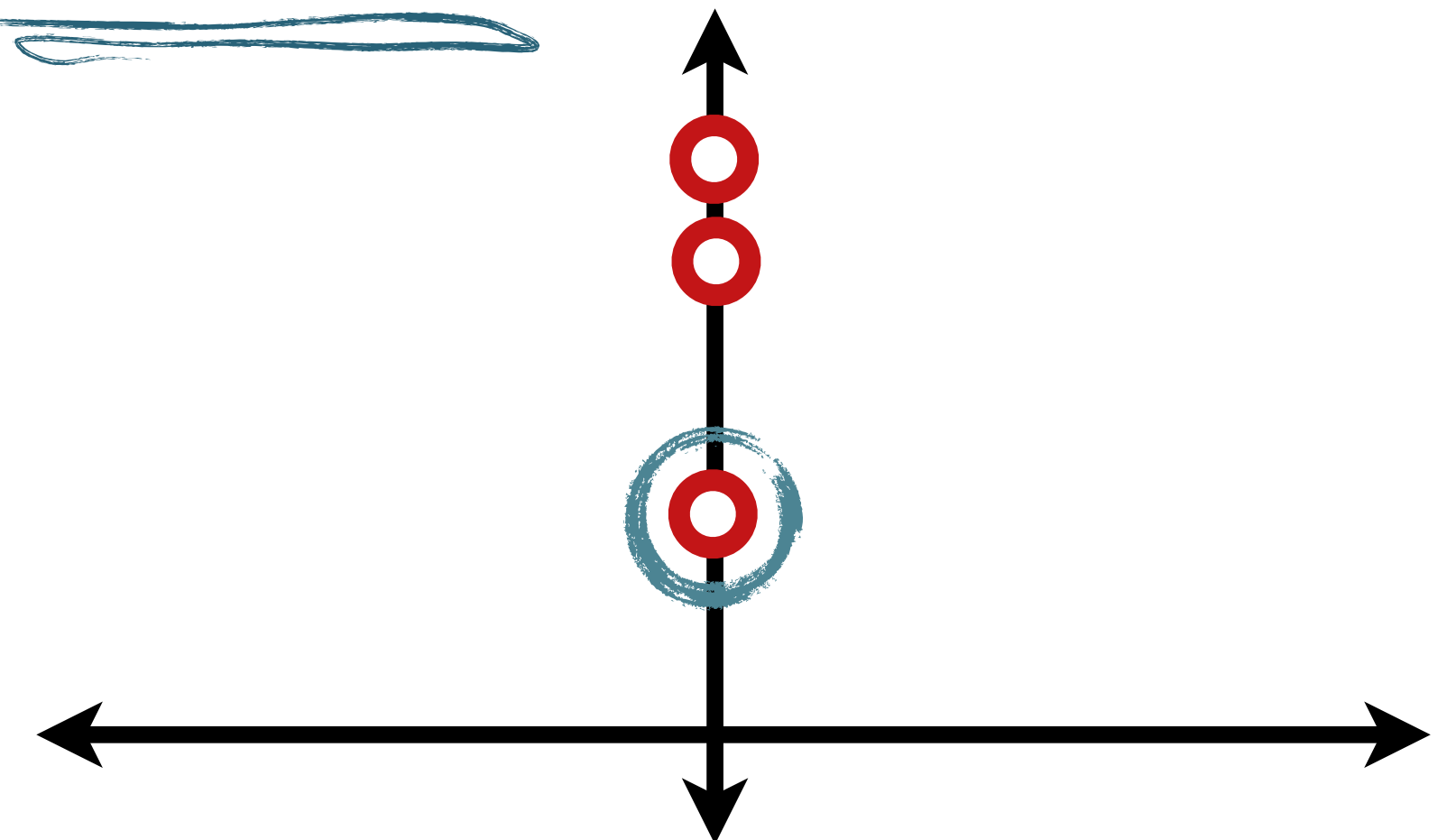
Scattering leaves an *imprint* on finite-volume quantities



Infinite-volume threshold

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a_0$$

scattering length



Finite-volume ground state

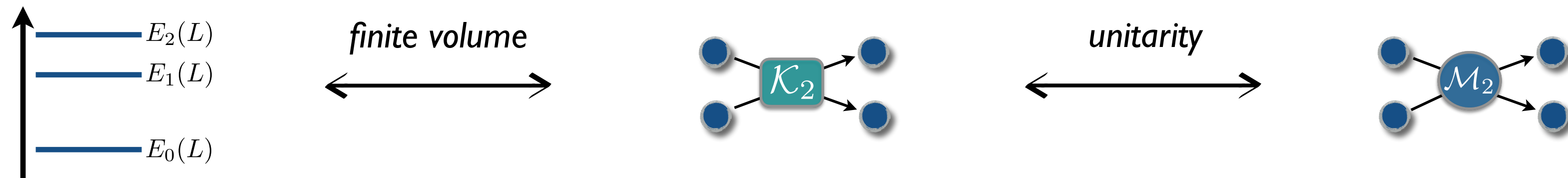
$$E_0(L) = 2M_\pi + \frac{4\pi a_0}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

General relation

$$\det[\mathcal{K}^{-1}(s) + F(P, L)] = 0$$

$$F(P, L) \equiv \text{Matrix of known geometric functions}$$



Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to *non-degenerate masses, multiple channels, spinning particles*

Encodes angular momentum mixing

Huang, Yang (1958) • Lüscher (1986, 1989) • Rummukainen, Gottlieb (1995)

Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)

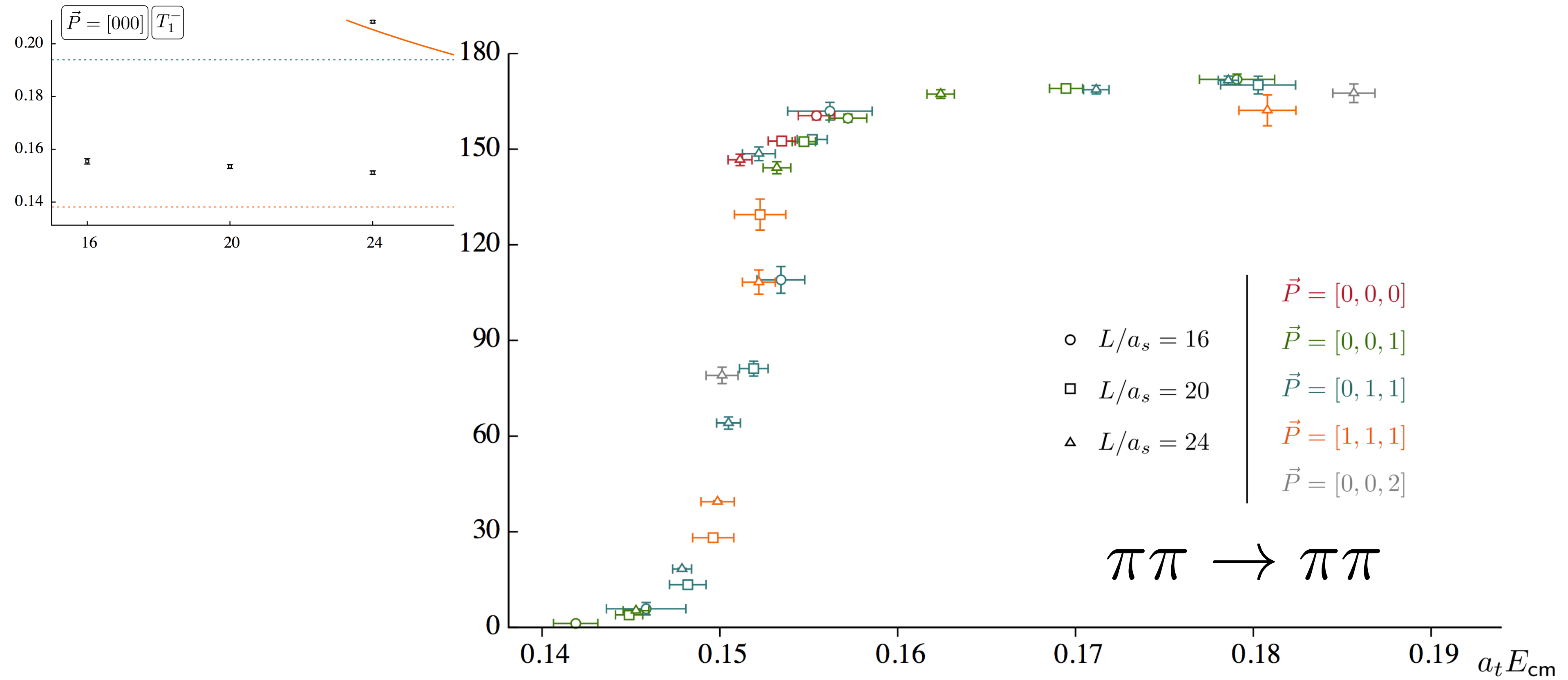
Beane, Detmold, Savage (2007) • Tan (2008) • Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012)

MTH, Sharpe (2012) • Briceño, Davoudi (2012) • Li, Liu (2013) • Briceño (2014)

Using the result

Single-channel case (*pions in a p-wave*)

$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$

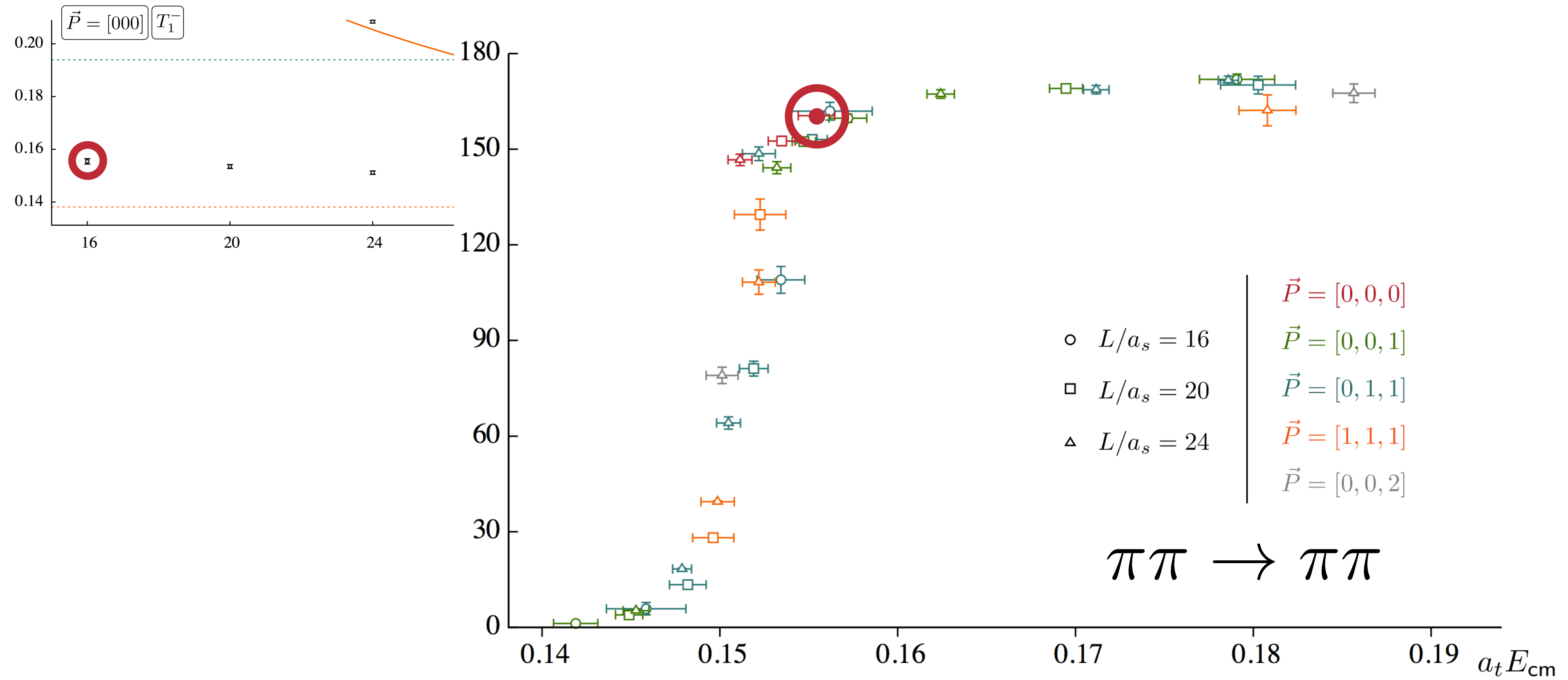


- Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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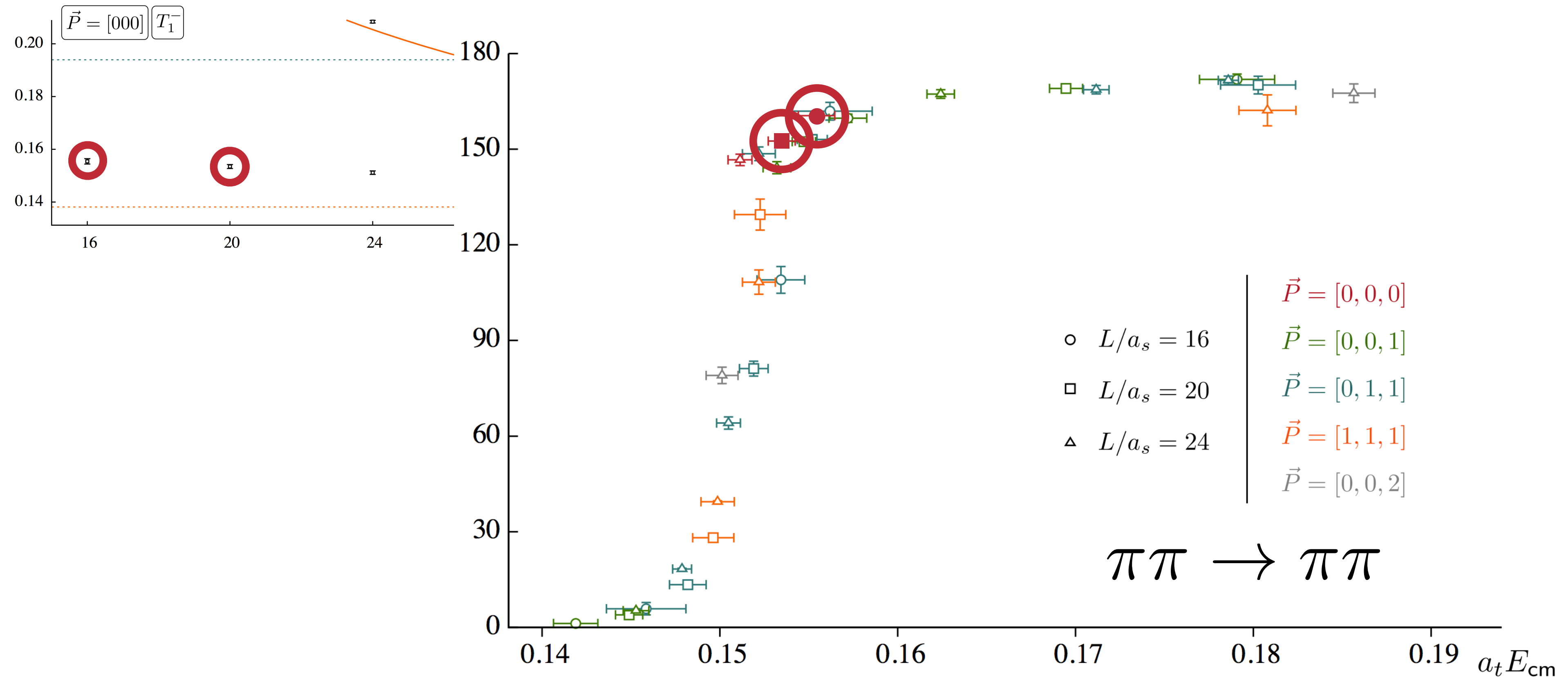


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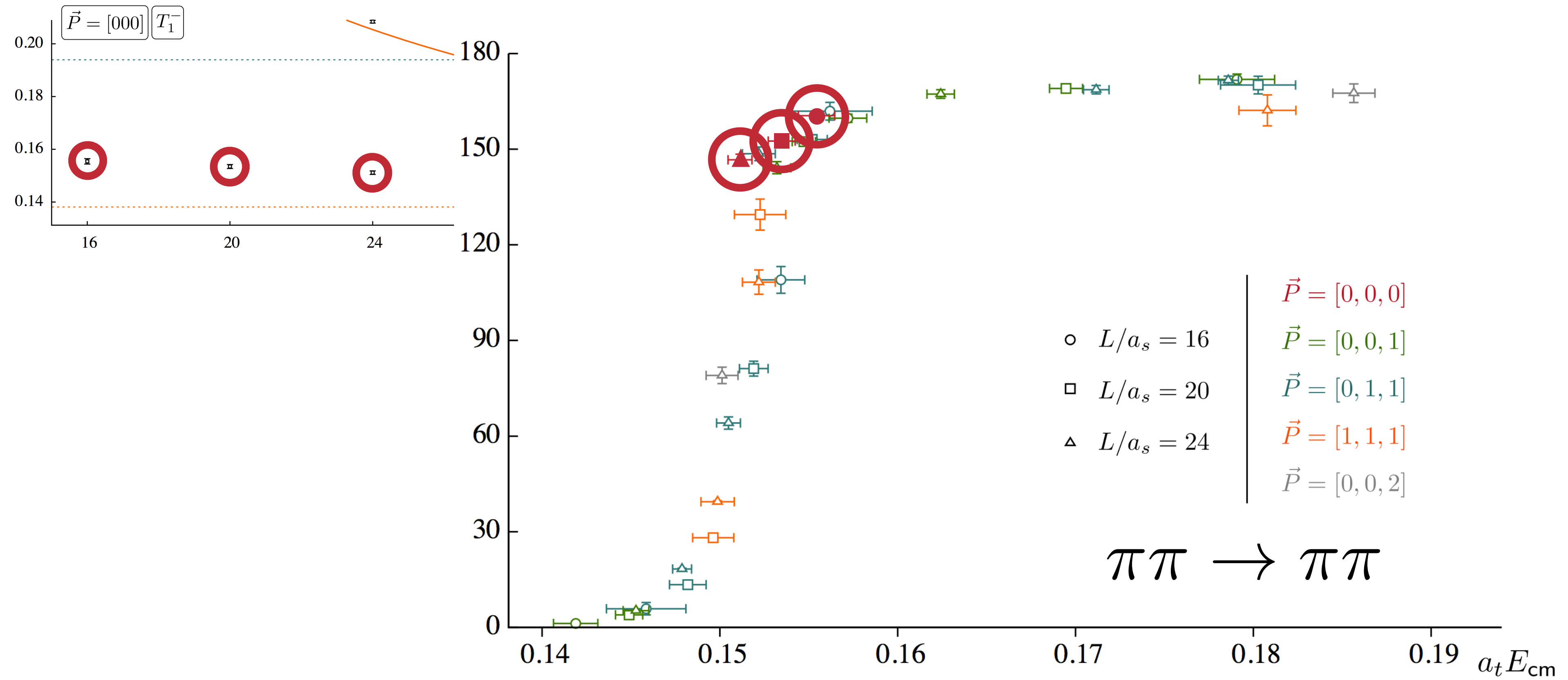


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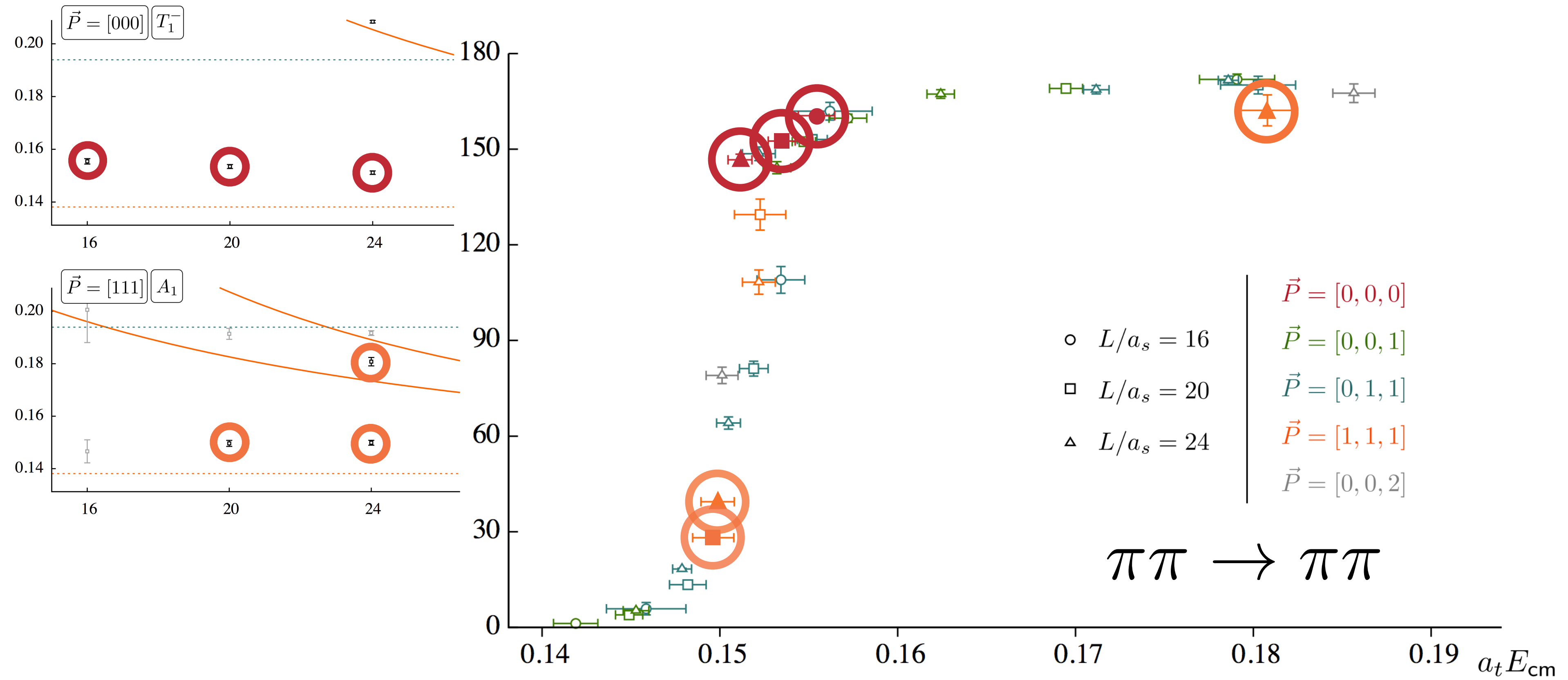


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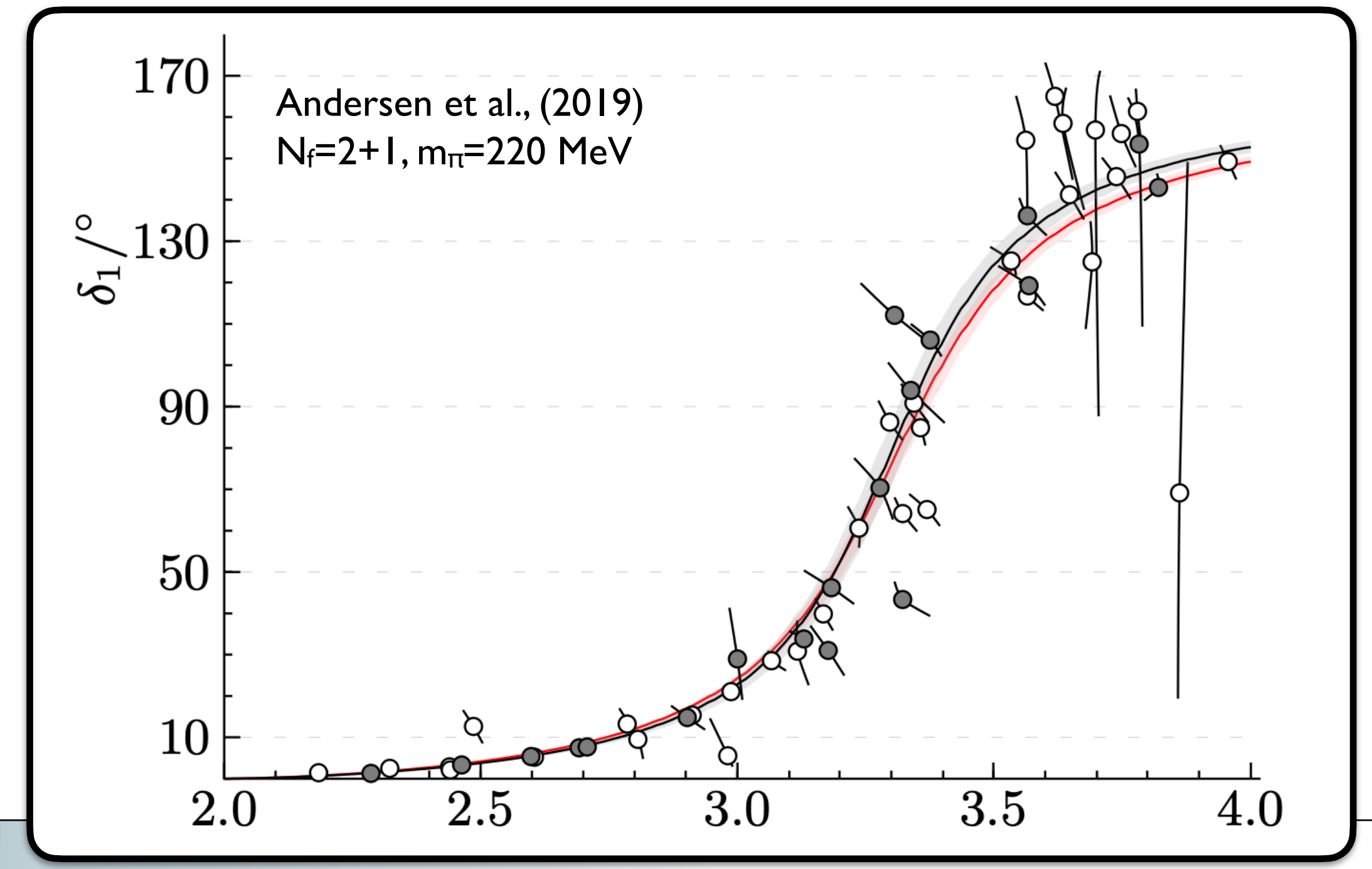
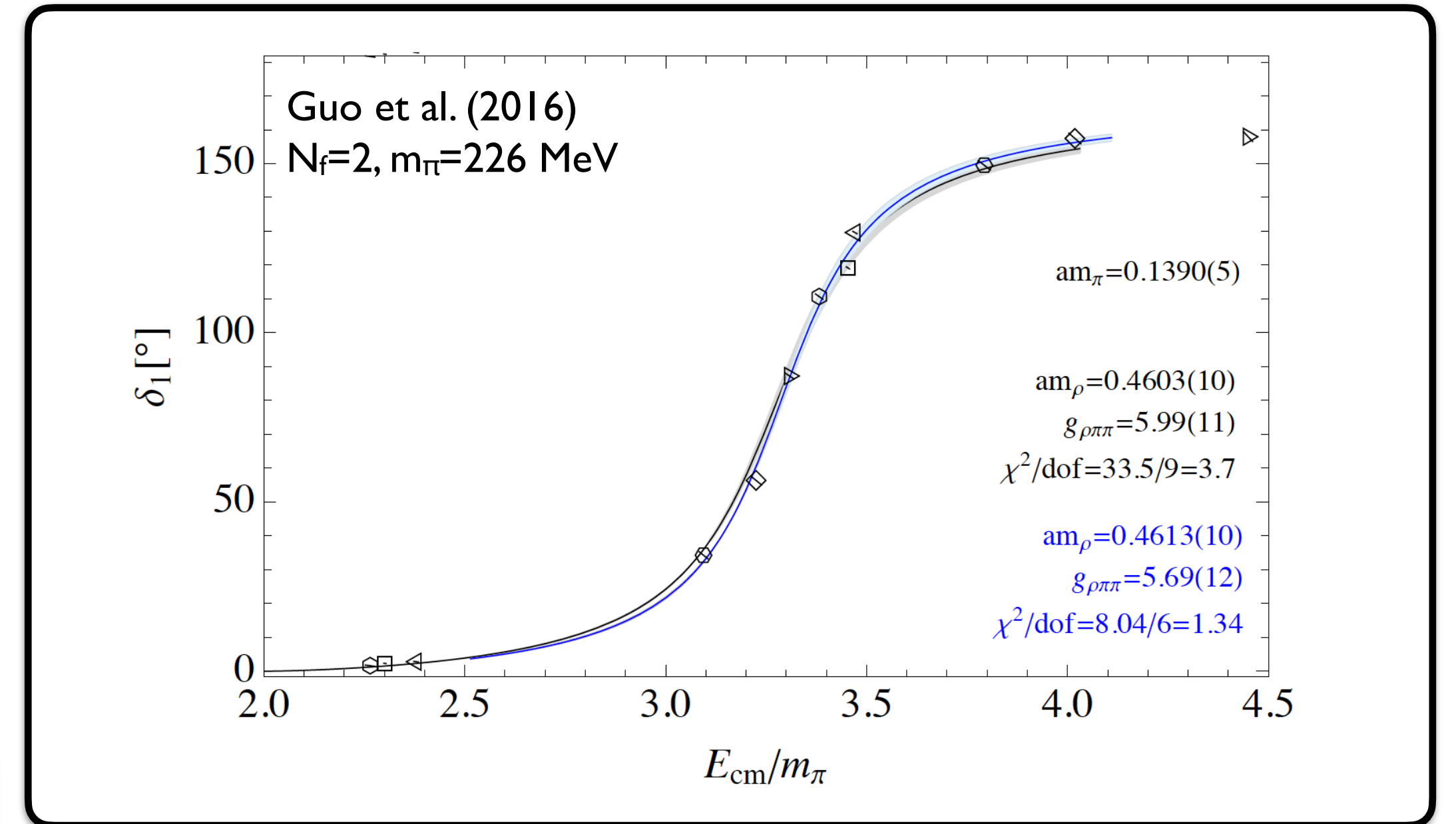
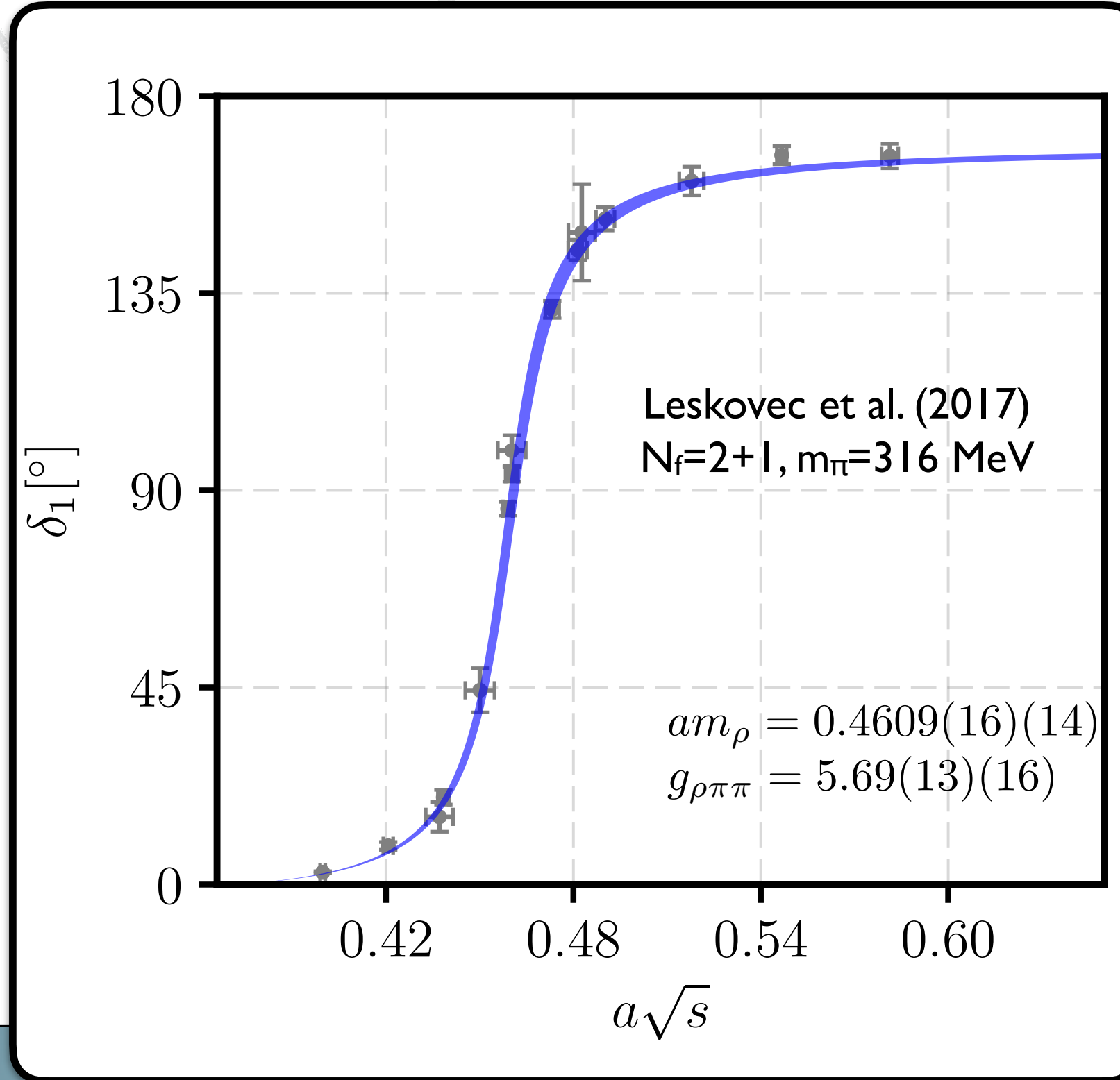
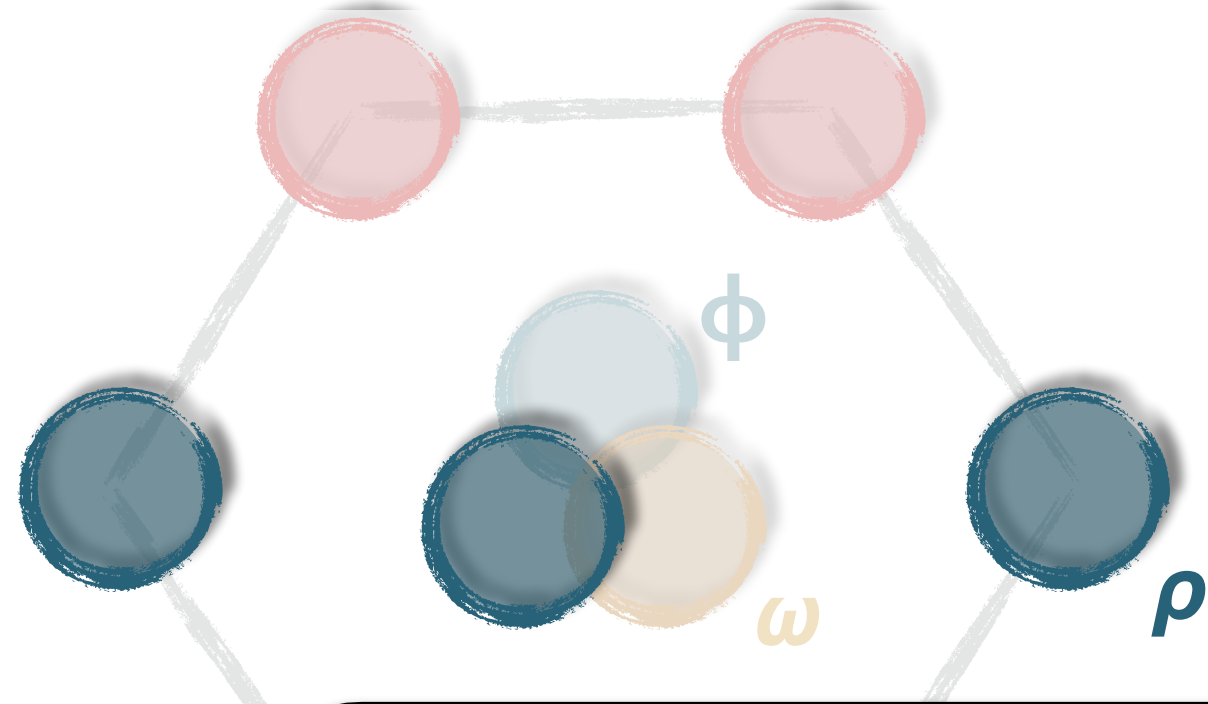
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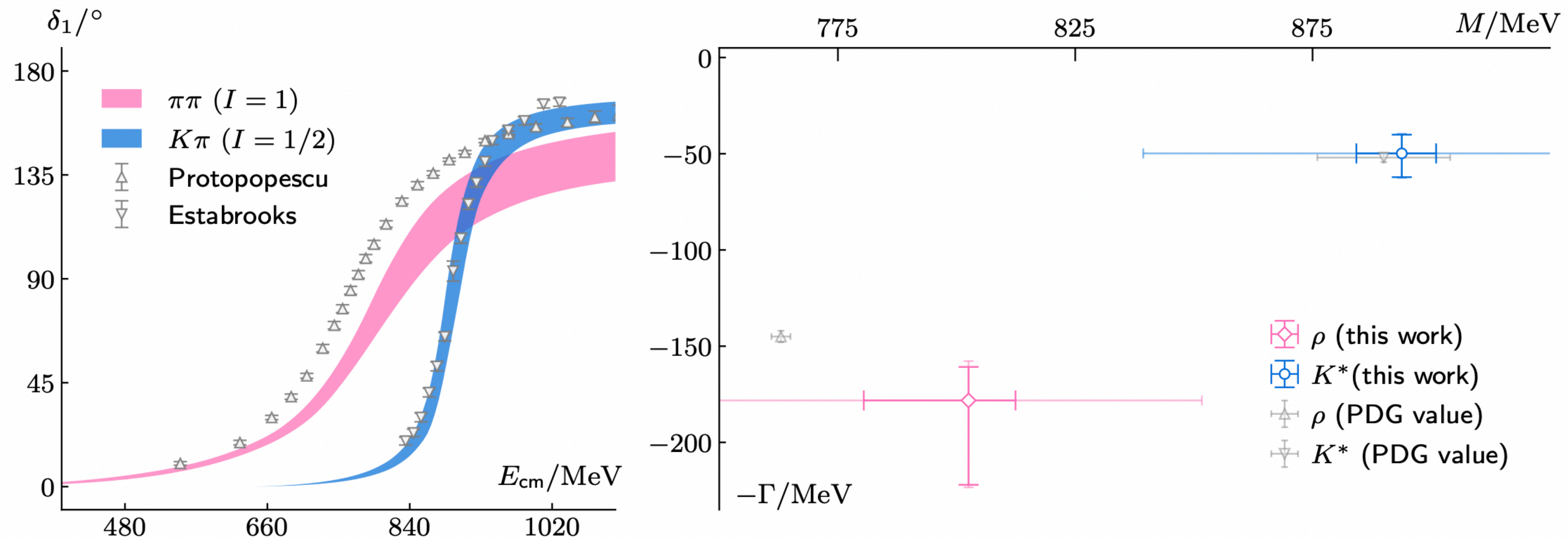
$$\rho \rightarrow \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$



One more recent result

Physical mass results (*but continuum limit still needed*)



- Boyle, Erben, Gülpers, MTH, Joswig, Marshall, **Lachini**, Portelli: *PRL* (2406.19194) and *PRD* (2406.19193) •

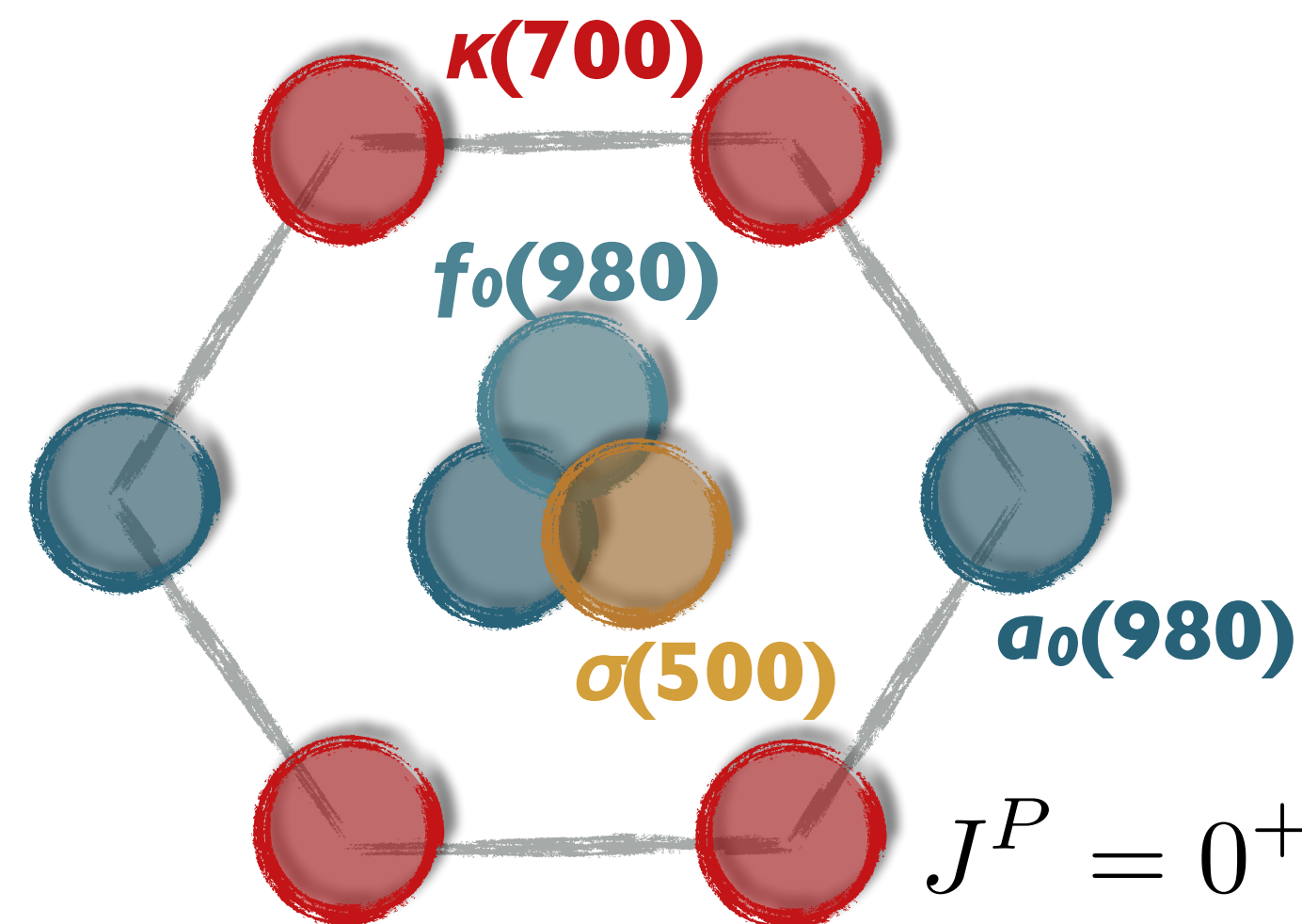
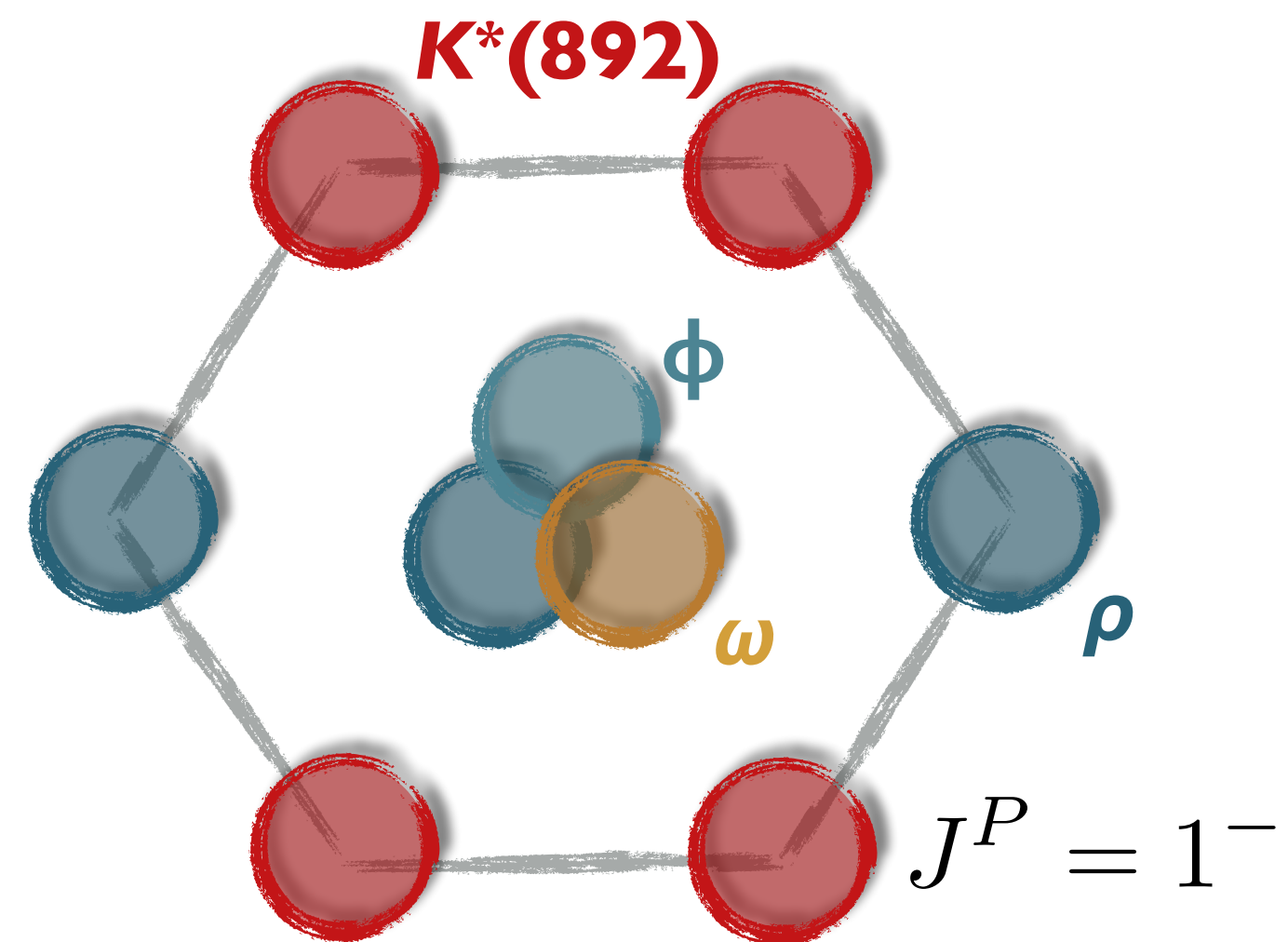
Essential step towards $B \rightarrow K^*(\rightarrow K\pi)\ell^+\ell^-$

$$\rho \rightarrow \pi\pi$$

- ☐ CP-PACS/PACS-CS 2007, 2011
- ☐ ETMC 2010
- ☐ Lang et al. 2011
- ☐ HadSpec 2012, **2016**
- ☐ Pellisier 2012
- ☐ RQCD 2015
- ☐ Guo et al. 2016
- ☐ Fu et al. 2016
- ☐ Bulava et al. 2016
- ☐ Alexandrou et al. 2017
- ☐ Andersen et al. 2018
- ☐ Fischer et al. 2020
- ☐ Erben et al. 2020

$$\sigma \rightarrow \pi\pi$$

- ☐ Prelovsek et al. 2010
- ☐ Fu 2013
- ☐ Wakayama 2015
- ☐ Howarth and Giedt 2017
- ☐ Briceño et al. 2017
- ☐ Guo et al. 2018



$$\kappa \rightarrow K\pi$$

$$K^* \rightarrow K\pi$$

- ☐ Lang et al. 2012
- ☐ Prelovsek et al. 2013
- ☐ Wilson et al. 2015
- ☐ RQCD 2015
- ☐ Brett et al. 2018
- ☐ Wilson et al. 2019
- ☐ Rendon et al. 2020

$$b_1 \rightarrow \pi\omega, \pi\phi$$

- ☐ Woss et al. 2019

$$a_0(980) \rightarrow \pi\eta, K\bar{K}$$

- ☐ Dudek et al. 2016

$$\sigma, f_0, f_2 \rightarrow \pi\pi, K\bar{K}, \eta\eta$$

- ☐ Briceño et al. 2017

See the recent review by
Briceño, Dudek and Young

Landscape of amplitudes

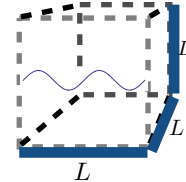
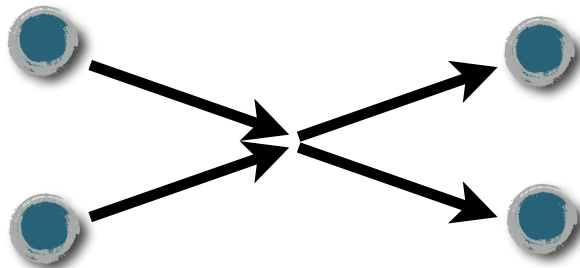
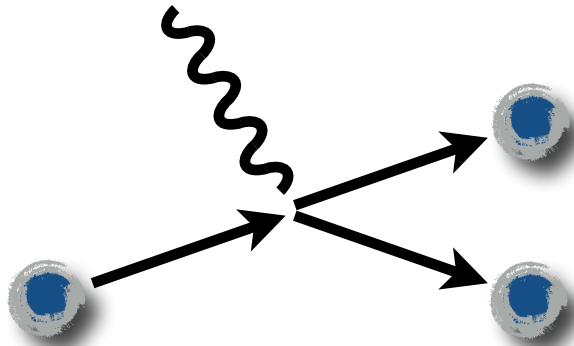
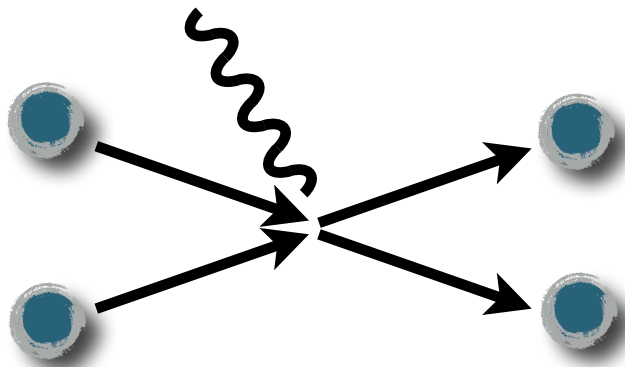
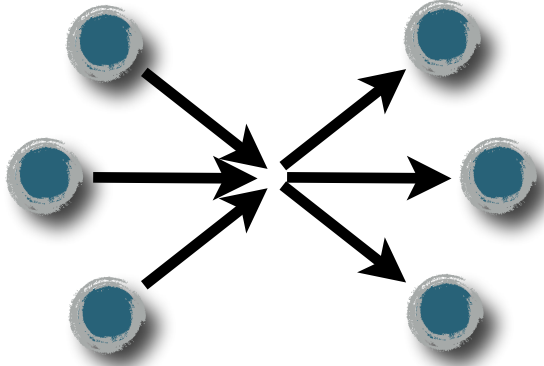
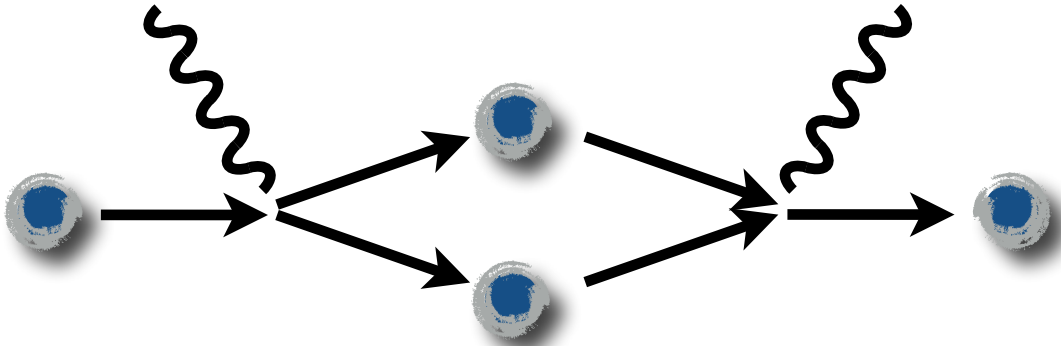
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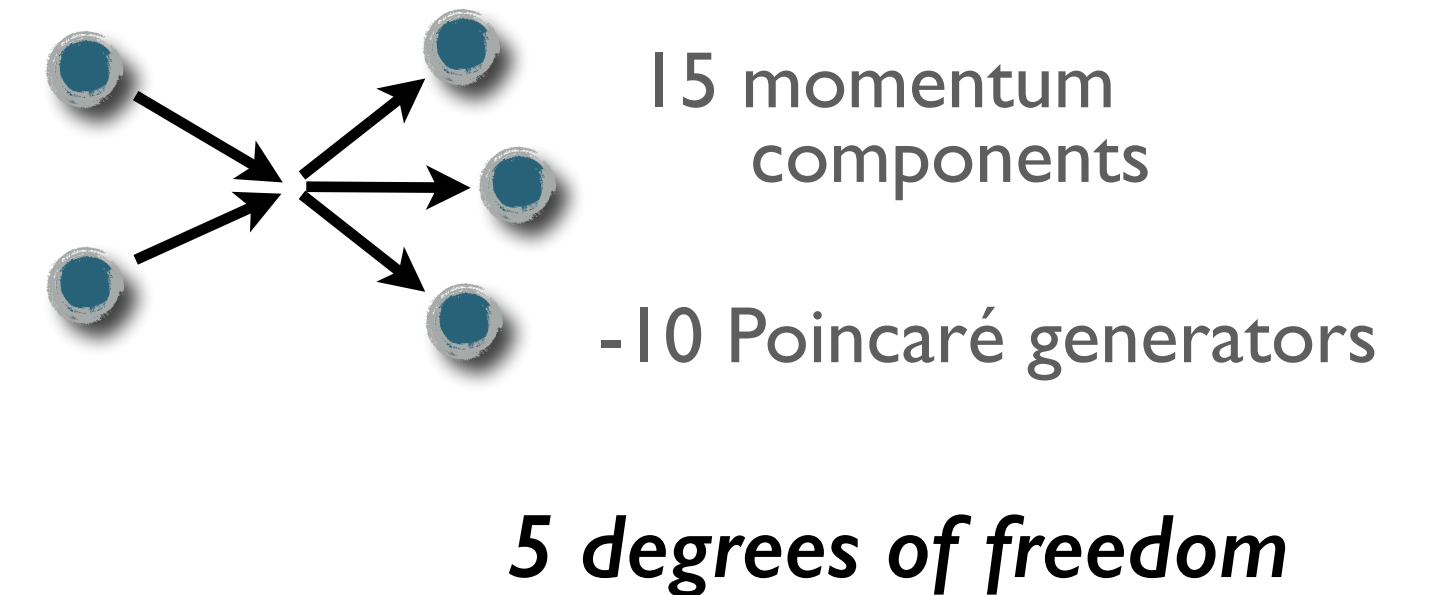
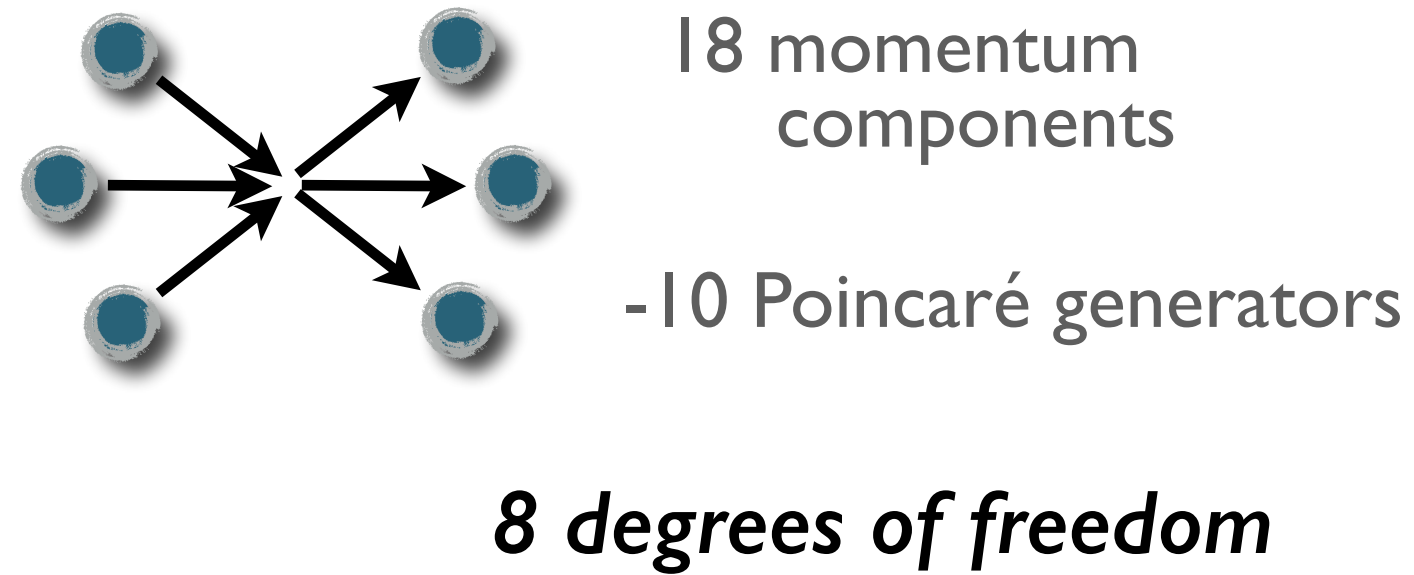
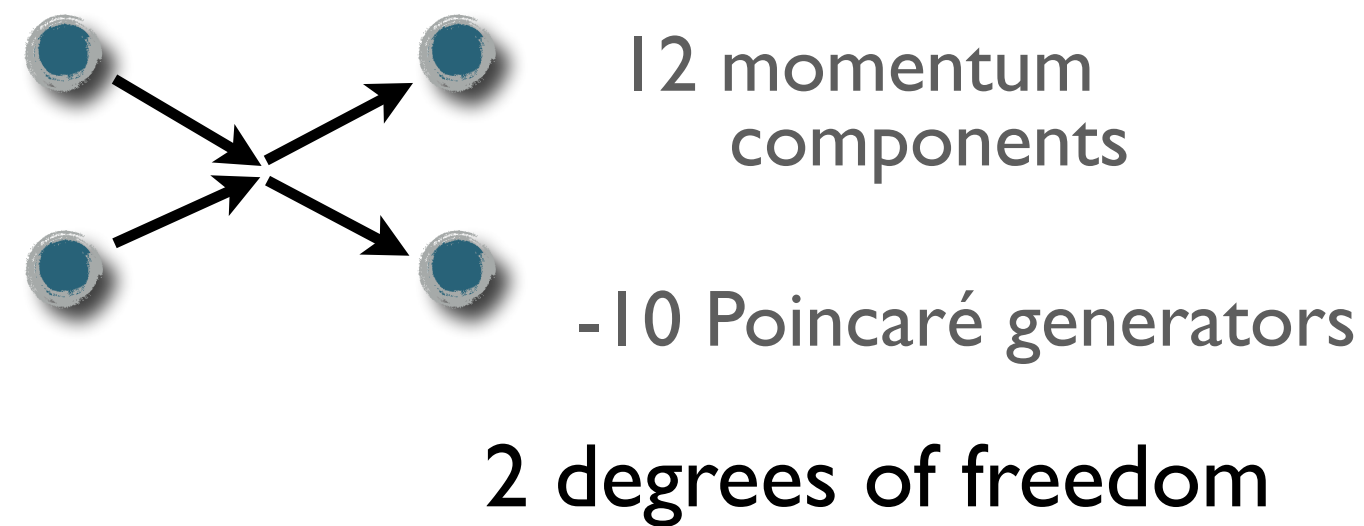
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Long distance matrix elements $0, 1 \xrightarrow{\mathcal{J}} X \xrightarrow{\mathcal{J}} 0, 1$

	FV method	SF method
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Two key complications

More degrees of freedom

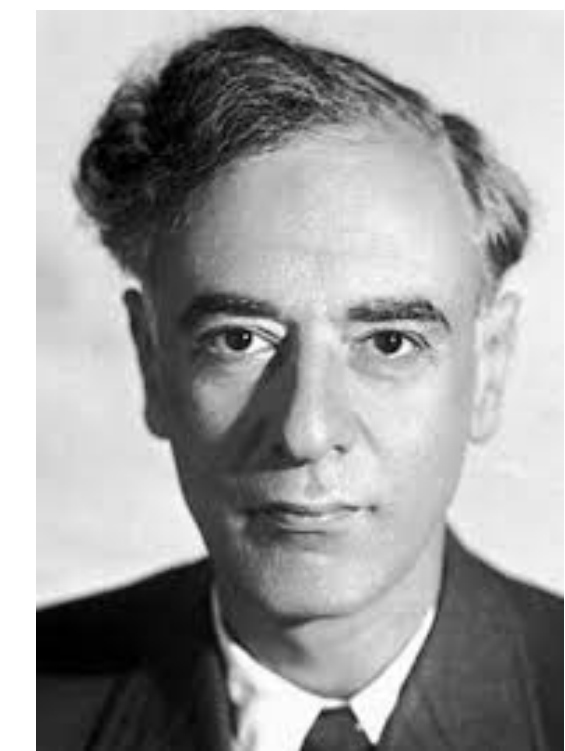


New kinds of singularities (analytic structure) (= *triangle singularities*, *anomalous thresholds*)

$$i\mathcal{M}_{3\rightarrow 3} \equiv \text{fully connected correlator} = \text{diagram 1} + \text{diagram 2} + \dots$$

complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles



Two key observations

- Can define an intermediate K matrix without these singularities

$$\mathcal{K}_{\text{df},3} \equiv \text{fully connected diagrams w/ PV pole prescription} - \text{diagram 1} + \text{diagram 2} + \dots$$

same degrees of freedom as M_3

smooth real function

relation to M_3 = known

- This quantity has a systematic low-energy expansion

$$\mathcal{K}_{\text{df},3}(p_3, p_2, p_1; k_3, k_2, k_1) = \mathcal{K}_{\text{df},3}^{\text{iso},0} + \mathcal{K}_{\text{df},3}^{\text{iso},1} \Delta + \dots \quad \Delta = \frac{s - (3m)^2}{(3m)^2}$$

smooth real function

analogous to effective range expansion

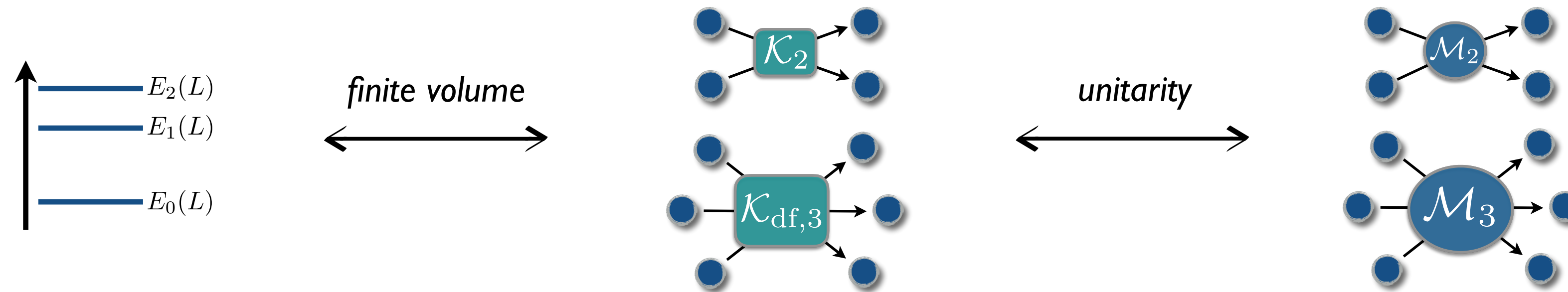
$$p \cot \delta = -\frac{1}{a} + \frac{1}{2} r p^2 + \mathcal{O}(p^4)$$

gives handle on many *degrees of freedom*
(DOFs enter order by order)

Status

⊗ Nearly general relation between *energies* and scattering *amplitudes*

Initially for 2-to-3, *no* sub-channel resonance



• MTH, Sharpe (2014, 2015) •

Subsequently extended to 2-to-3 scattering + sub-channel resonances + different isospins + non-degenerate + spin

$$\pi\pi\pi \rightarrow \rho\pi \rightarrow \omega \rightarrow \rho\pi \rightarrow \pi\pi\pi$$

MTH, Sharpe, Briceño, Romero-Lopez, Jackura, Blanton, Dawid (2014-2025)

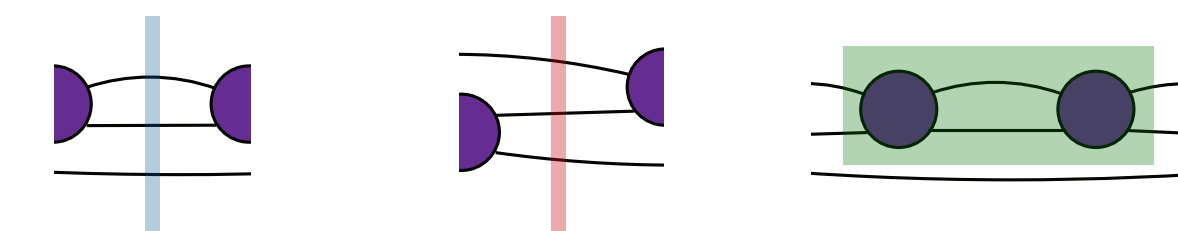
See also Döring, Mai, Hammer, Pang, Rusetsky

RFT quantization condition

RFT = relativistic field theory

$$\det [\mathbf{K}_{\text{df},3}^{-1}(E_{\text{cm}}) - \mathbf{F}_3(E, \mathbf{P}, L)] = 0$$

$\mathbf{F}_3(E, \mathbf{P}, L) =$ Matrix of functions depending on kinematics + two-particle dynamics

$$\mathbf{F}_3 = \frac{\mathbf{F}}{3} + \mathbf{F} \mathbf{K}_2 [1 - (\mathbf{F} + \mathbf{G}) \mathbf{K}_2]^{-1} \mathbf{F}$$


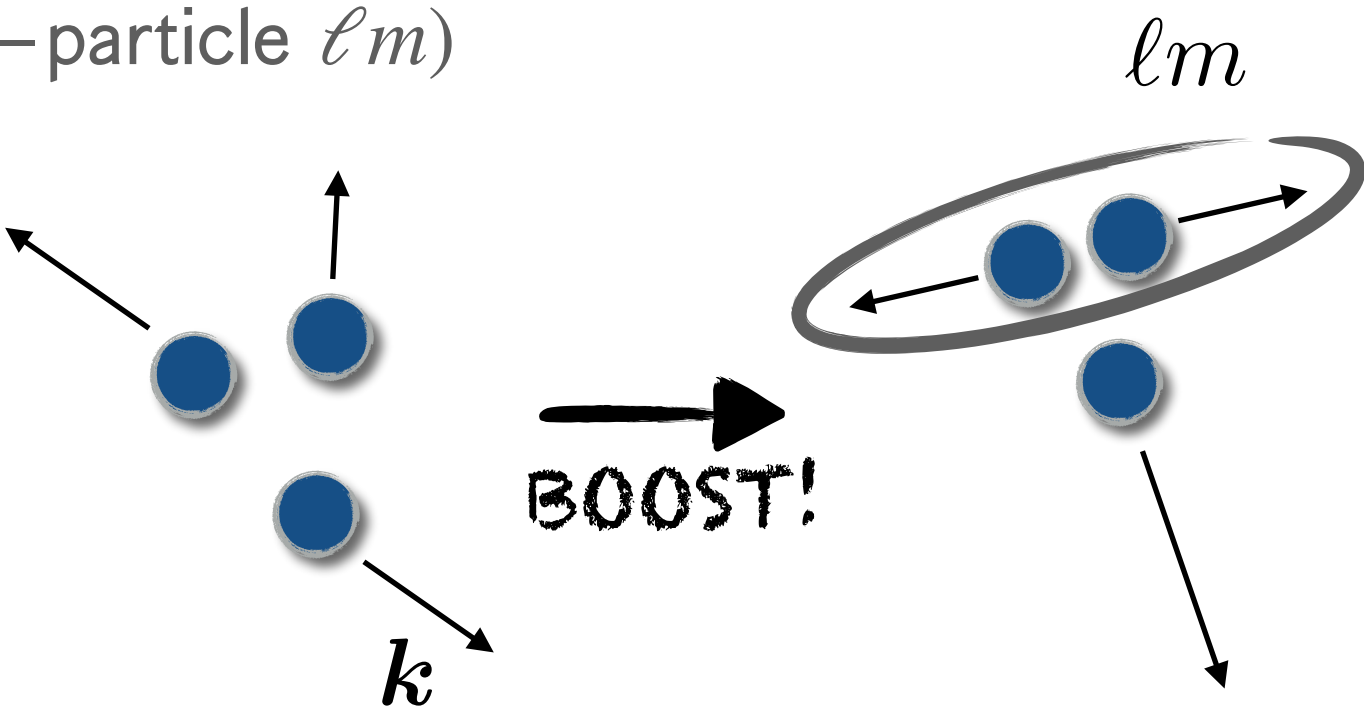
Matrices on tensor-product space: (spectator flavor space) $\otimes$ (spectator $k \in \frac{2\pi}{L} \mathbb{Z}^3$ space) $\otimes$ (two-particle ℓm)

Holds only for three-particle energies

Neglects $e^{-M_\pi L}$

Requires sub-threshold continuation of $\mathbf{K}_2$

Scheme-dependent $\mathbf{K}_{\text{df},3}$ related to physical amplitude via known on-shell integral equations



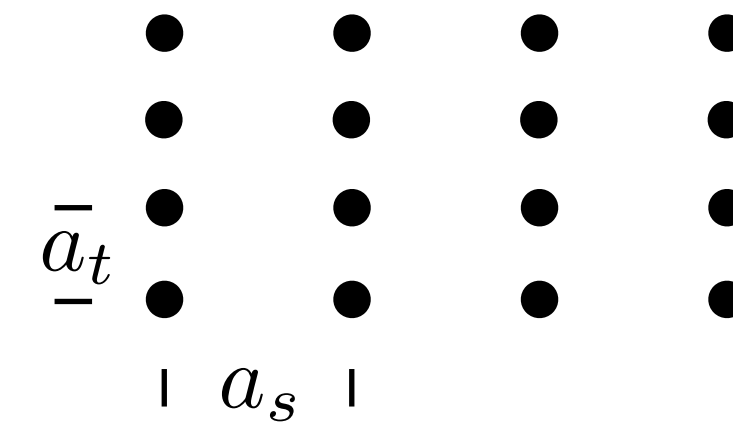
$$\pi^+ \pi^+ \pi^+ \rightarrow \pi^+ \pi^+ \pi^+$$

lattice details

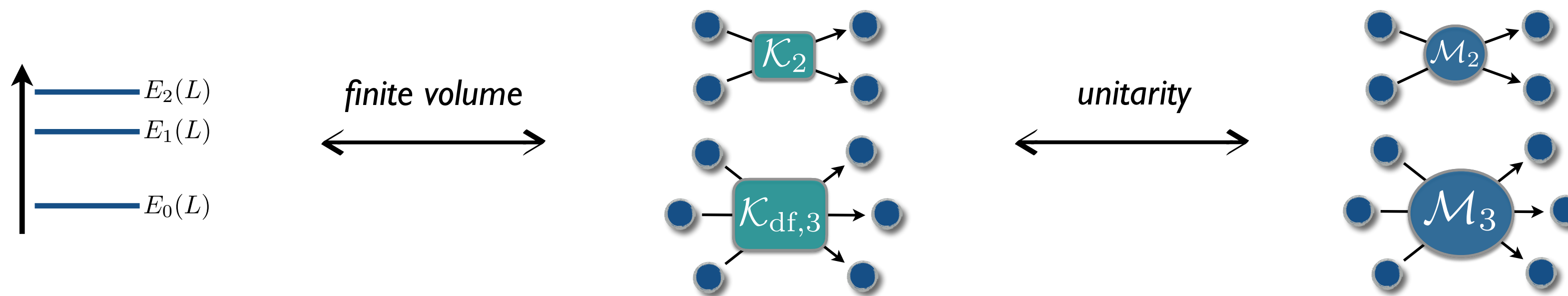
$$N_f = 2 + 1 \quad a_s/a_t = 3.444(6)$$

$$m_\pi \approx 400 \text{ MeV} \quad a_s \approx 0.12 \text{ fm}$$

$$L_s/a_s = 20, 24$$



□ Workflow outline



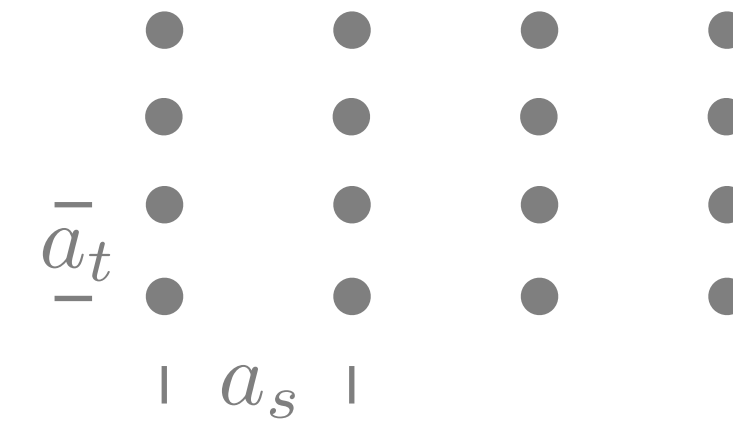
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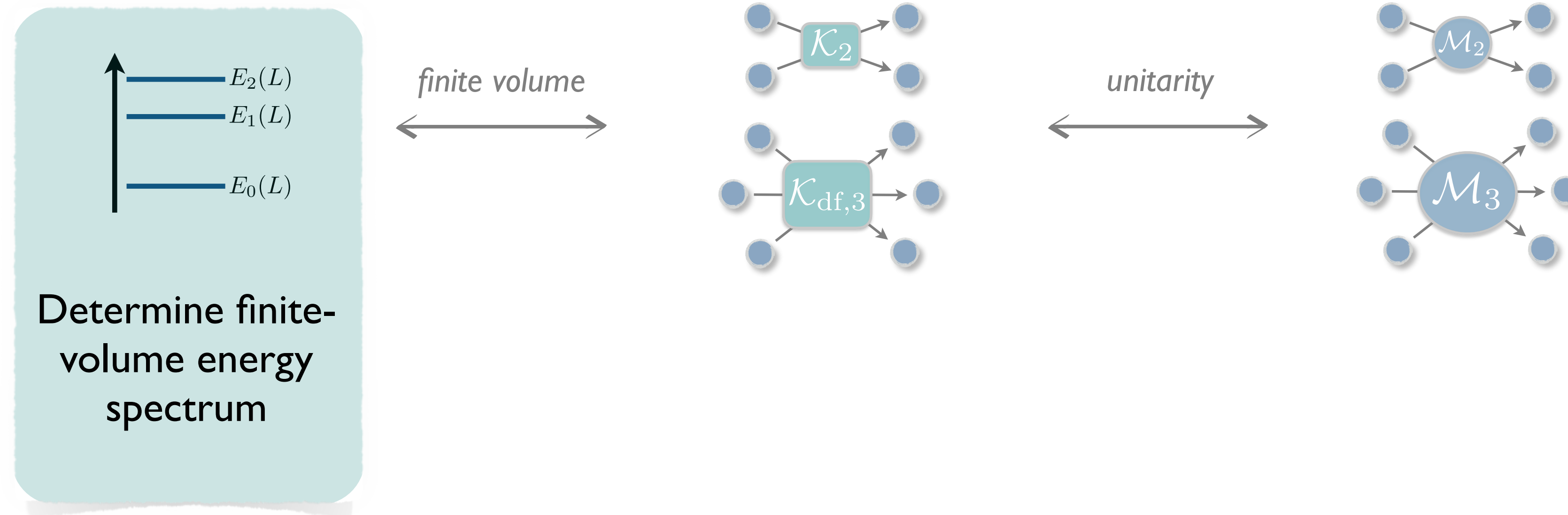
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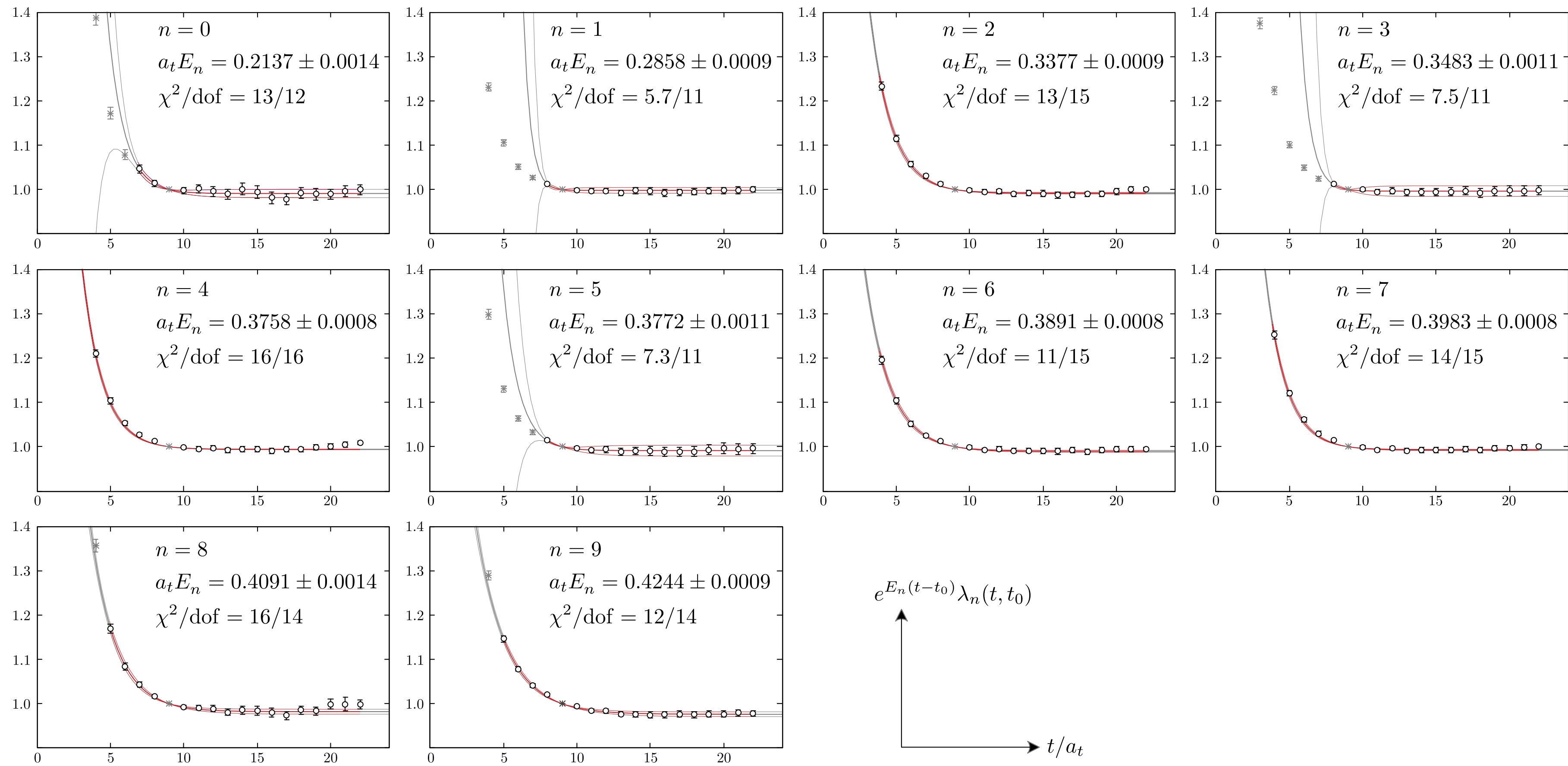
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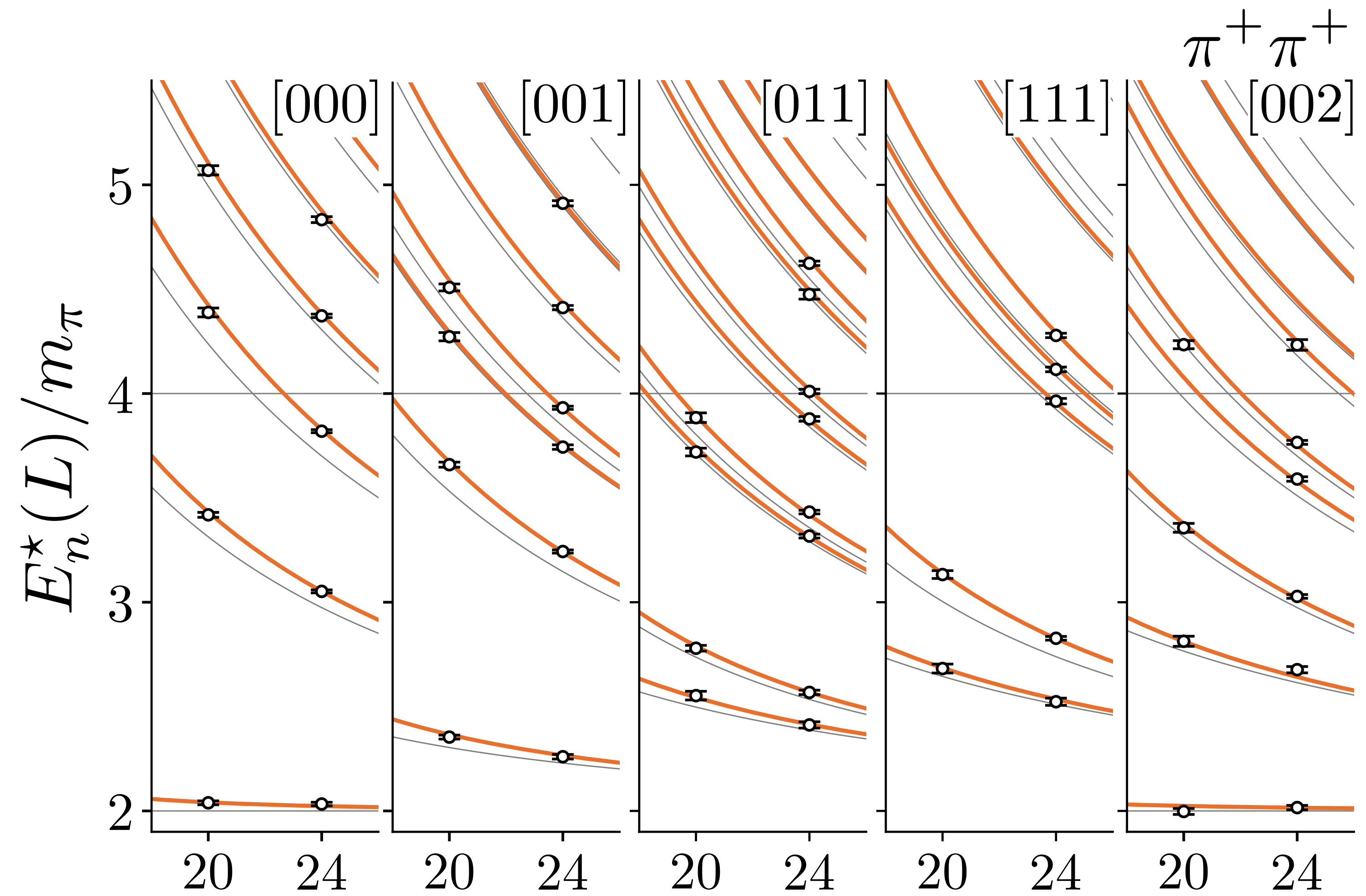
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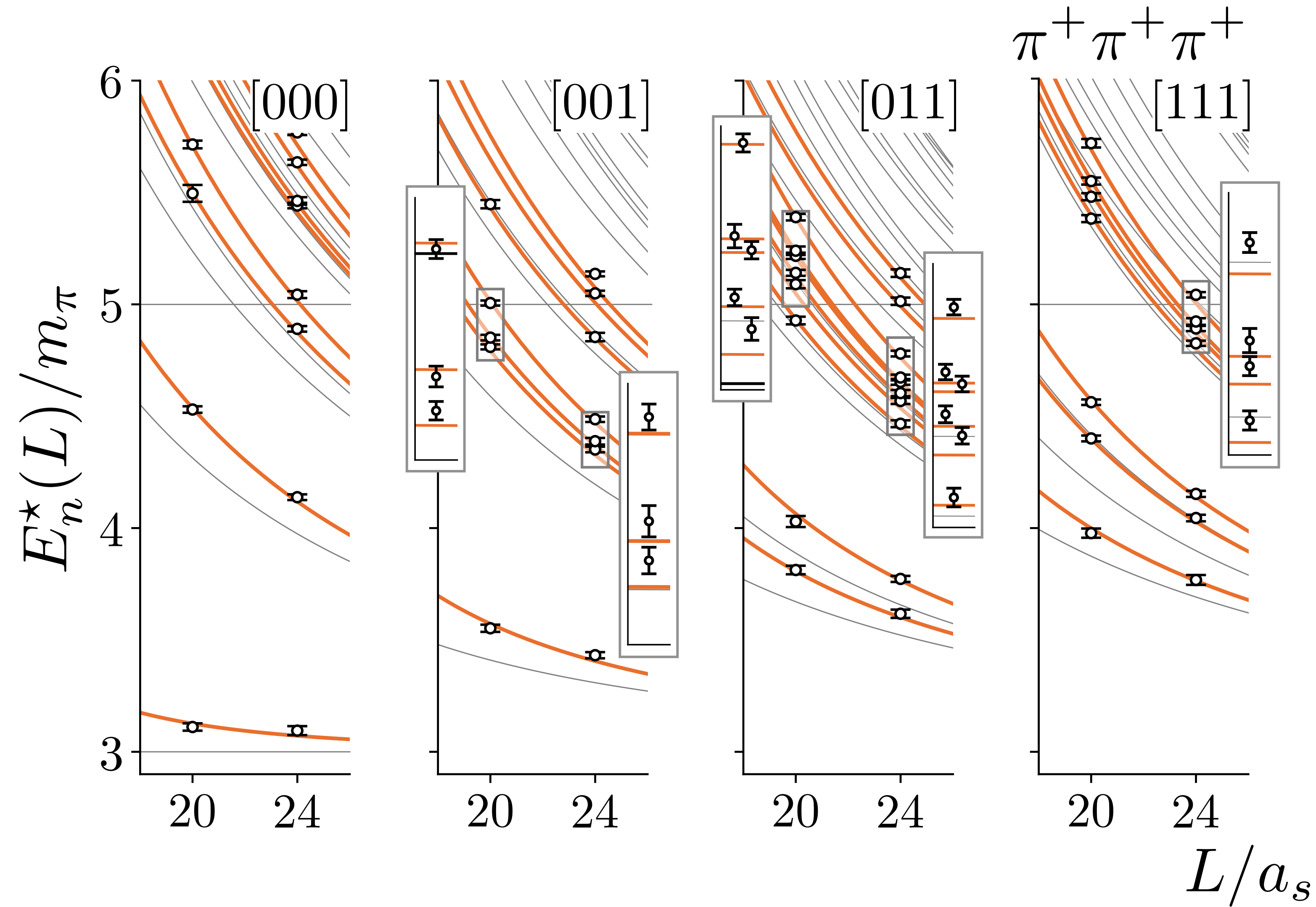
$$I = 3 (\pi^+ \pi^+ \pi^+), \quad \mathbf{P} = [000], \quad \Lambda = A_1^-, \quad L/a_s = 24$$



Two-pion energies



Three-pion energies

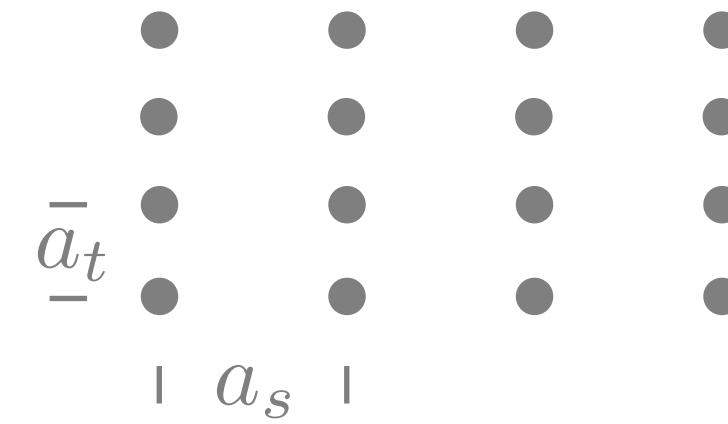


$$\pi^+ \pi^+ \pi^+ \rightarrow \pi^+ \pi^+ \pi^+$$

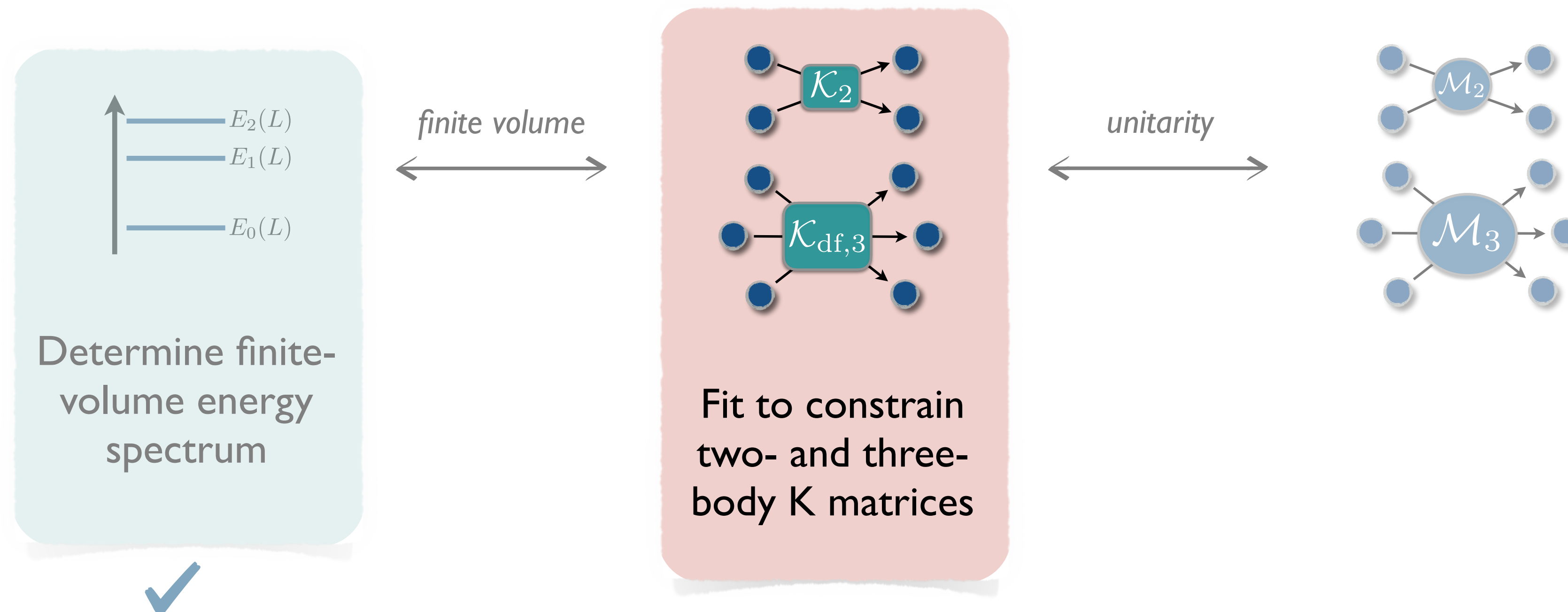
lattice details

$$N_f = 2 + 1 \quad a_s/a_t = 3.444(6) \quad L_s/a_s = 20, 24$$

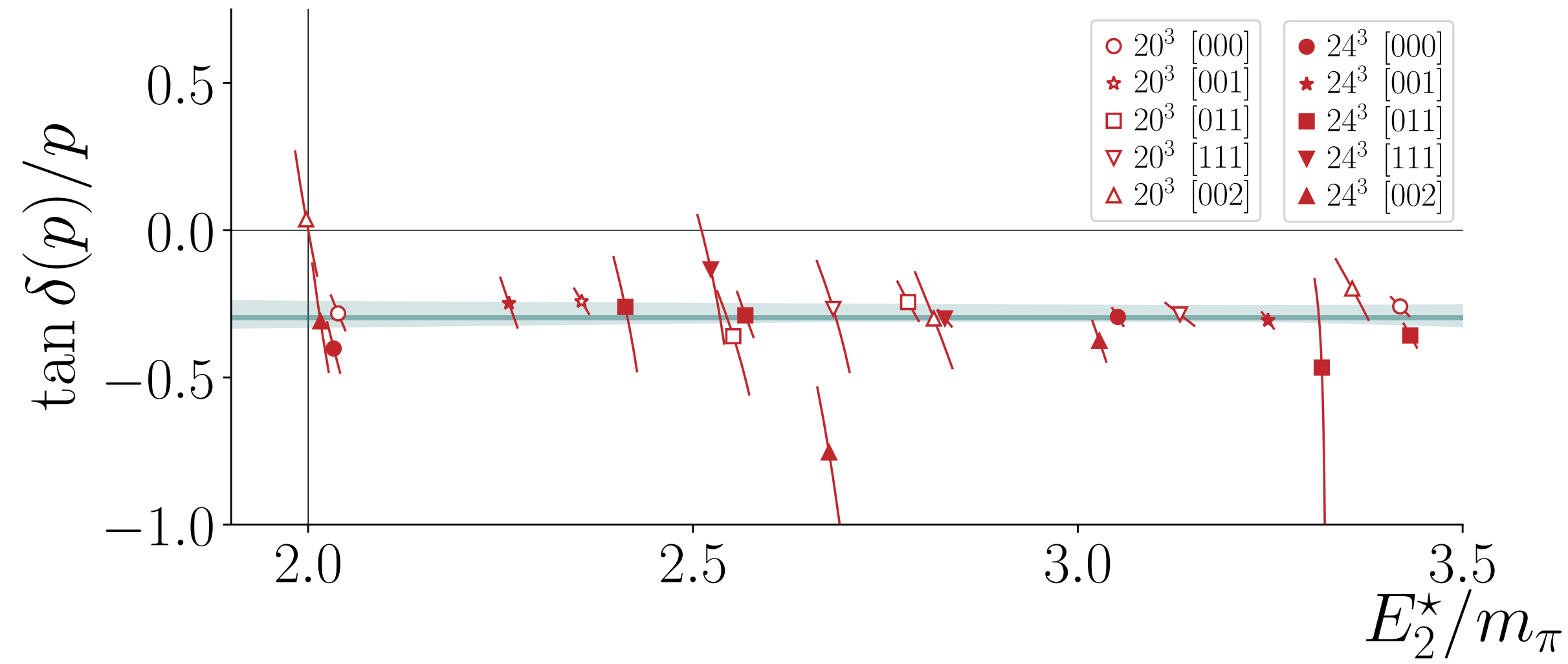
$$m_\pi \approx 400 \text{ MeV} \quad a_s \approx 0.12 \text{ fm}$$



□ Workflow outline



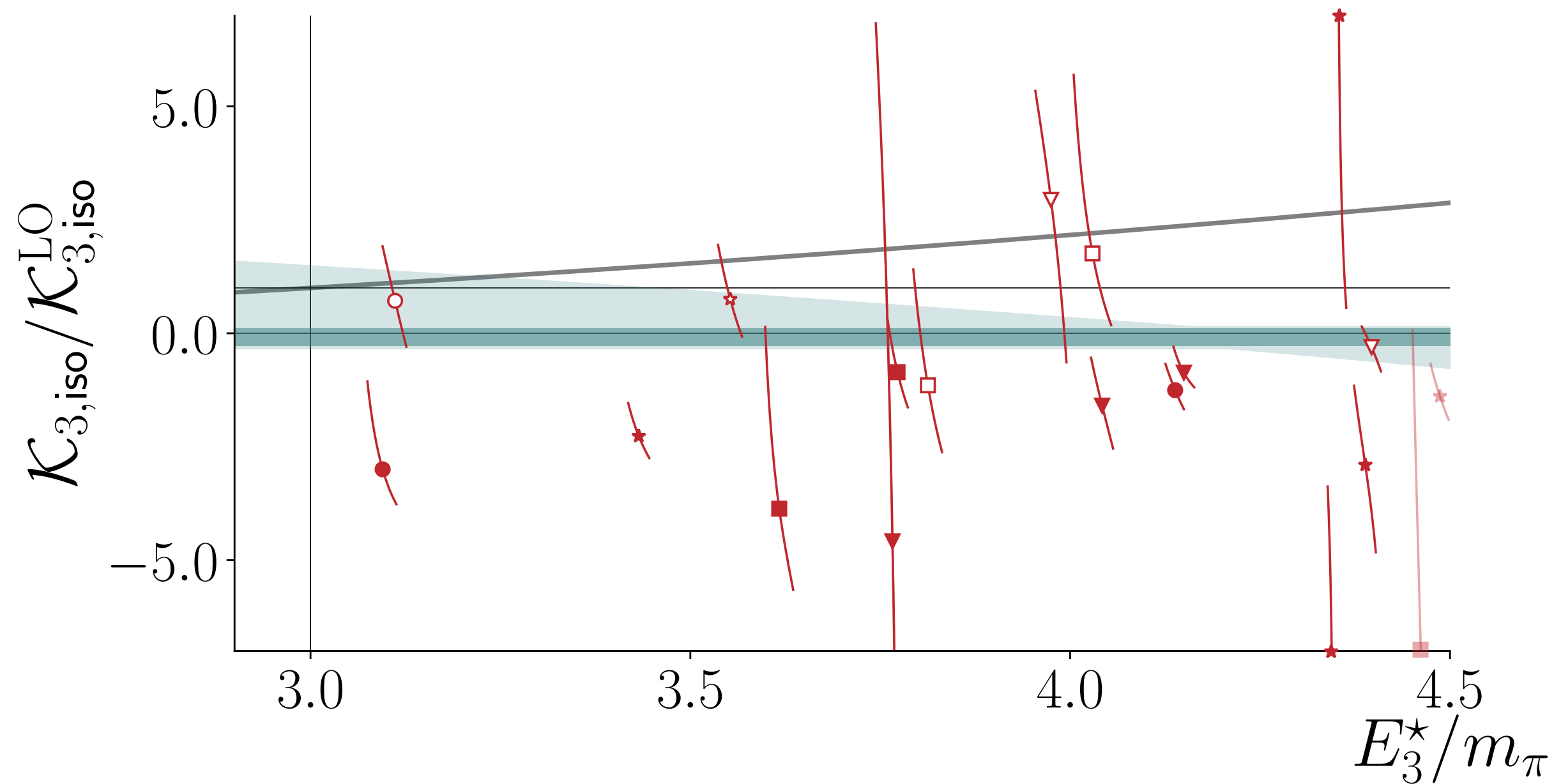
K matrix fits



Finite-volume formalism relates energies to K matrices

One-to-one for $K_{\text{df},3}$ depending only on $E_{\text{cm}} = E^*$

Fit both two and three-body K to various polynomials



Cut on the CM energy in the fits

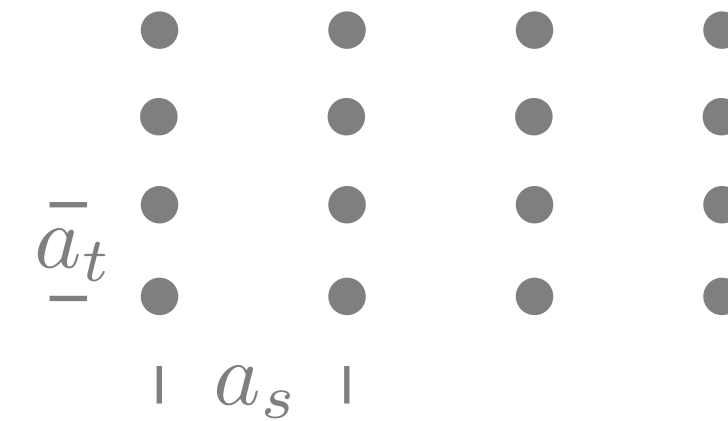
$K_{\text{df},3}$ is scheme dependent (removed upon converting to $\mathcal{M}_3$)

$$\pi^+ \pi^+ \pi^+ \rightarrow \pi^+ \pi^+ \pi^+$$

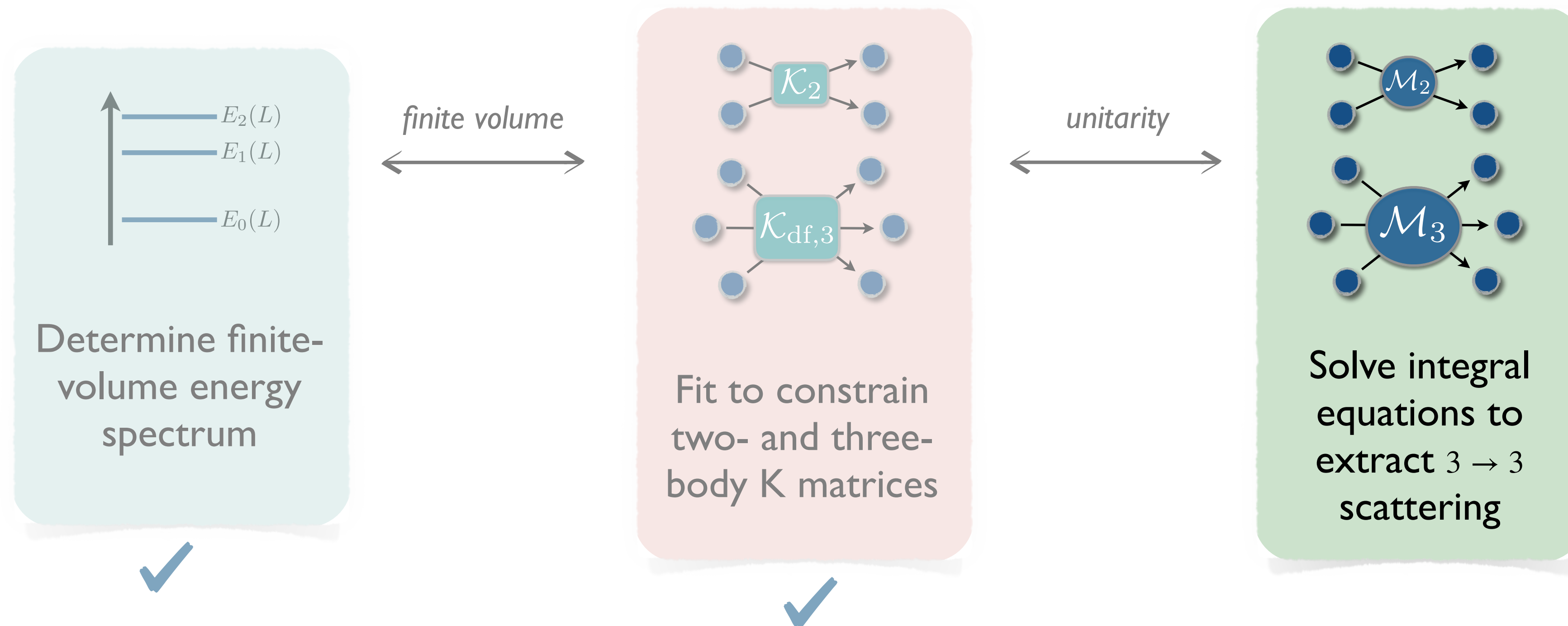
lattice details

$$N_f = 2 + 1 \quad a_s/a_t = 3.444(6) \quad L_s/a_s = 20, 24$$

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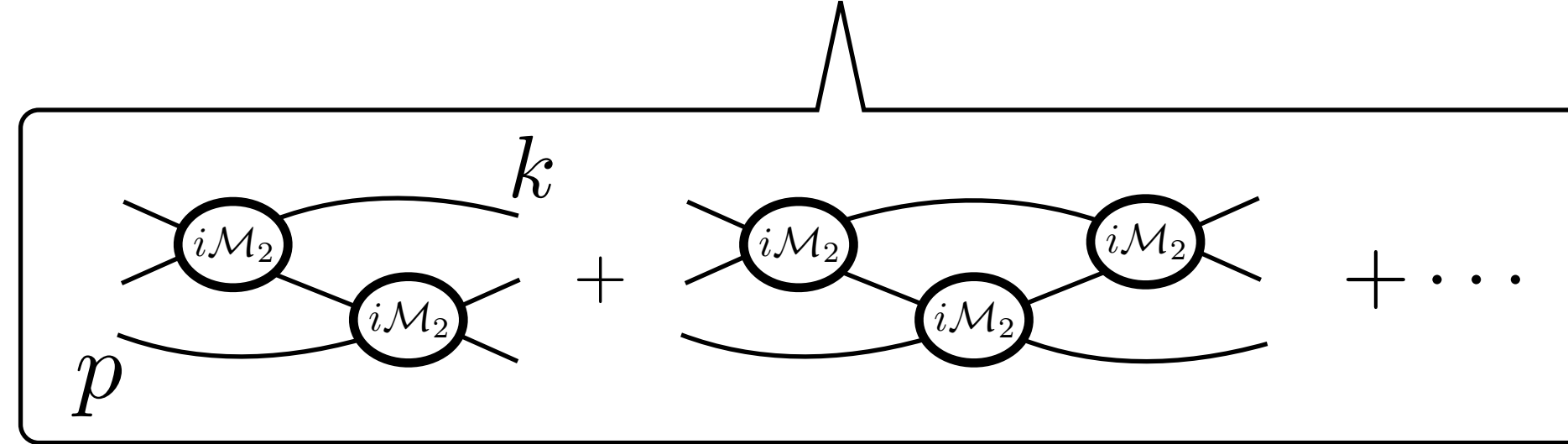


□ Workflow outline



Integral equations

$$\mathcal{M}_3^{\text{un}}(E_3^*, \mathbf{p}, \mathbf{k}) = \mathcal{D}^{\text{un}}(E_3^*, \mathbf{p}, \mathbf{k}) + \mathcal{E}^{\text{un}}(E_3^*, \mathbf{p}) \mathcal{T}(E_3^*) \mathcal{E}^{\text{un}}(E_3^*, \mathbf{k})$$



Vanishes for $K_{\text{df},3} = 0$

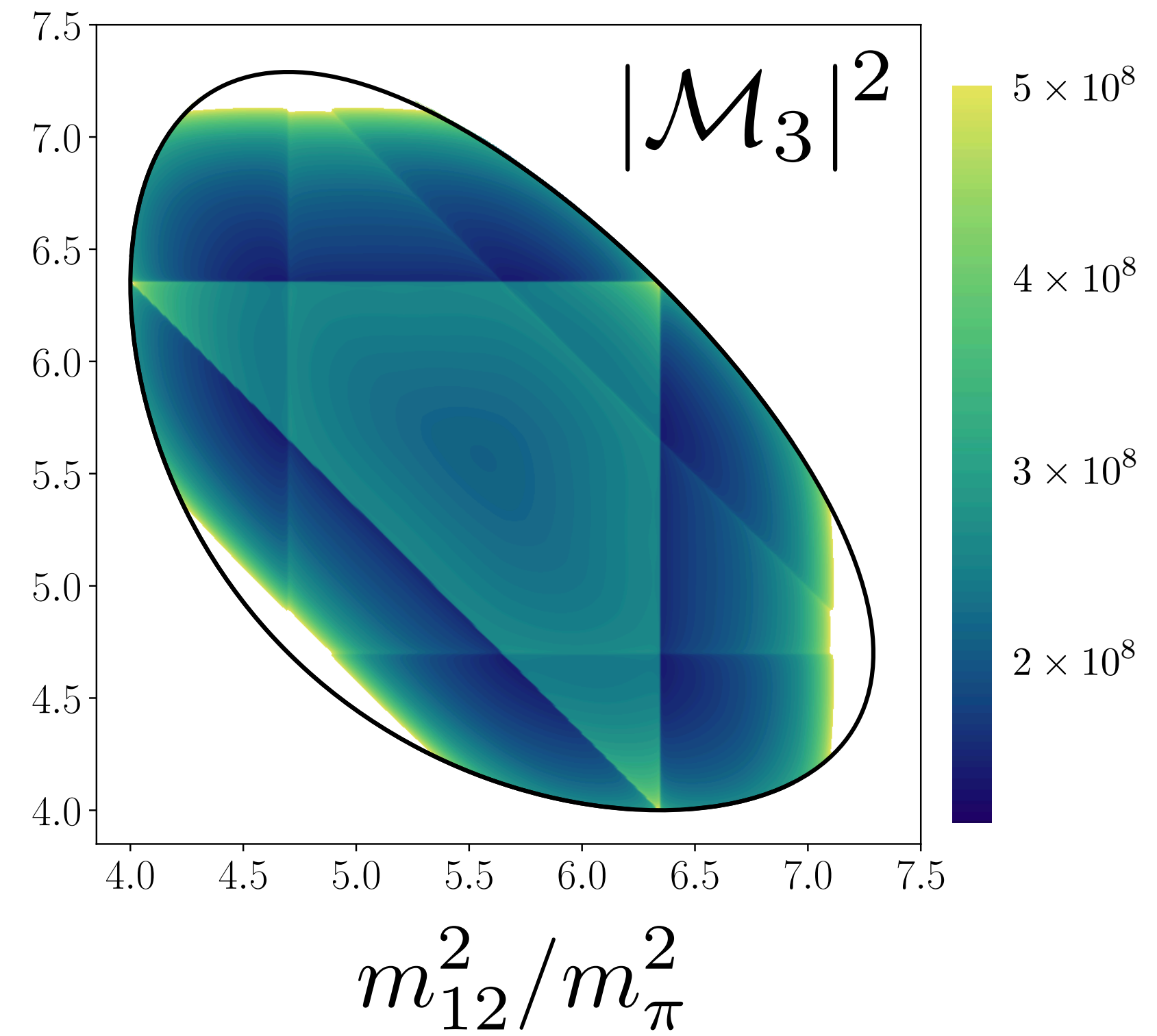
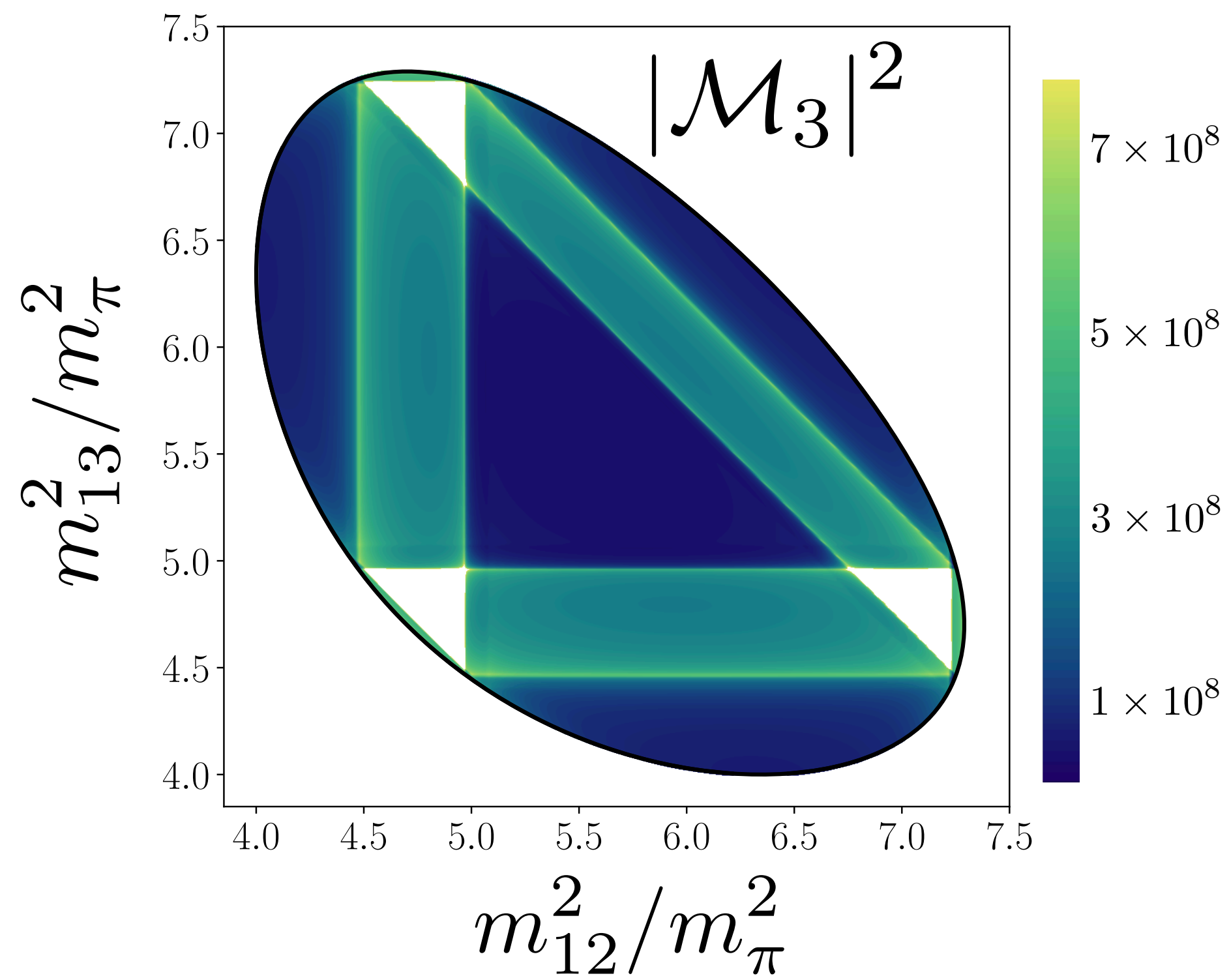
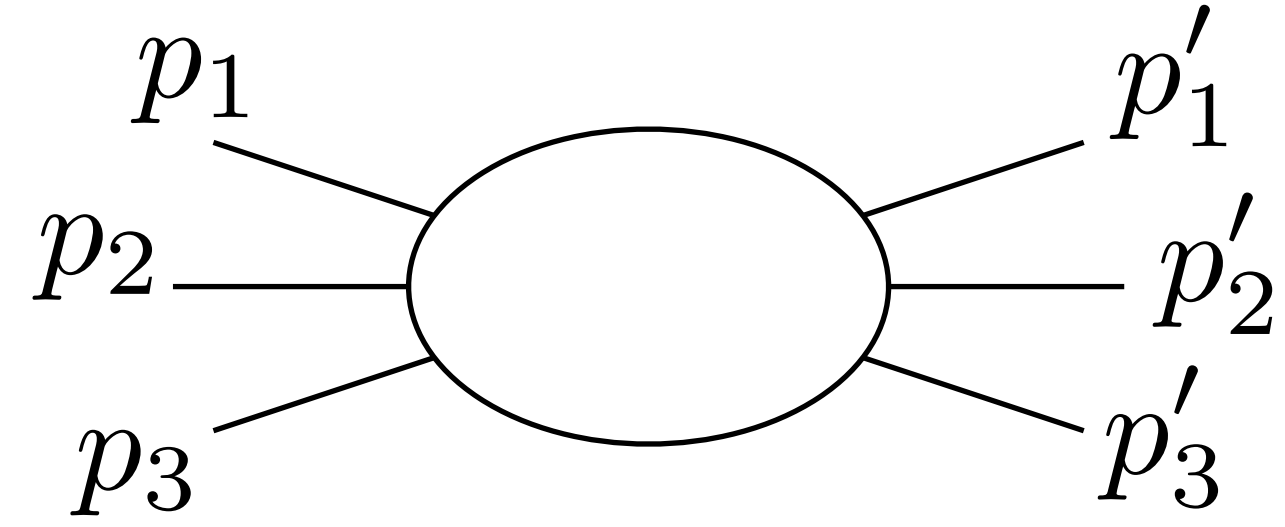
$$\mathbf{D}(N, \epsilon) = -\mathcal{M} \cdot \mathbf{G}(\epsilon) \cdot \mathcal{M} - \mathcal{M} \cdot \mathbf{G}(\epsilon) \cdot \mathbf{P} \cdot \mathbf{D}(N, \epsilon)$$

$$\mathcal{D}^{\text{un}}(E_3^*, \mathbf{p}, \mathbf{k}) = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \mathbf{D}_{pk}(N, \epsilon)$$

Solving relativistic three-body integral equations in the presence of bound states

Andrew W. Jackura,^{1,2,*} Raúl A. Briceño,^{1,2,†} Sebastian M. Dawid,^{3,4,‡} Md Habib E Islam,^{2,§} and Connor McCarty^{5,¶} *arXiv: 2010.09820*

$$\mathcal{M}_3 = \sum_{i,j \in \{1,2,3\}} \mathcal{M}_3^{\text{un}}(p'_i, p_j)$$



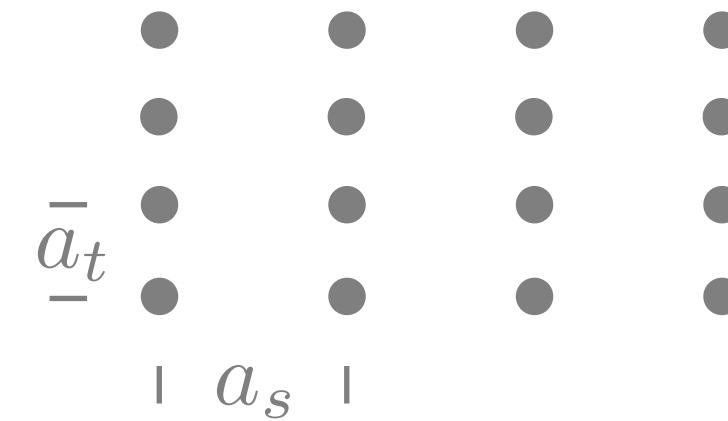
$$\pi^+ \pi^+ \pi^+ \rightarrow \pi^+ \pi^+ \pi^+$$

lattice details

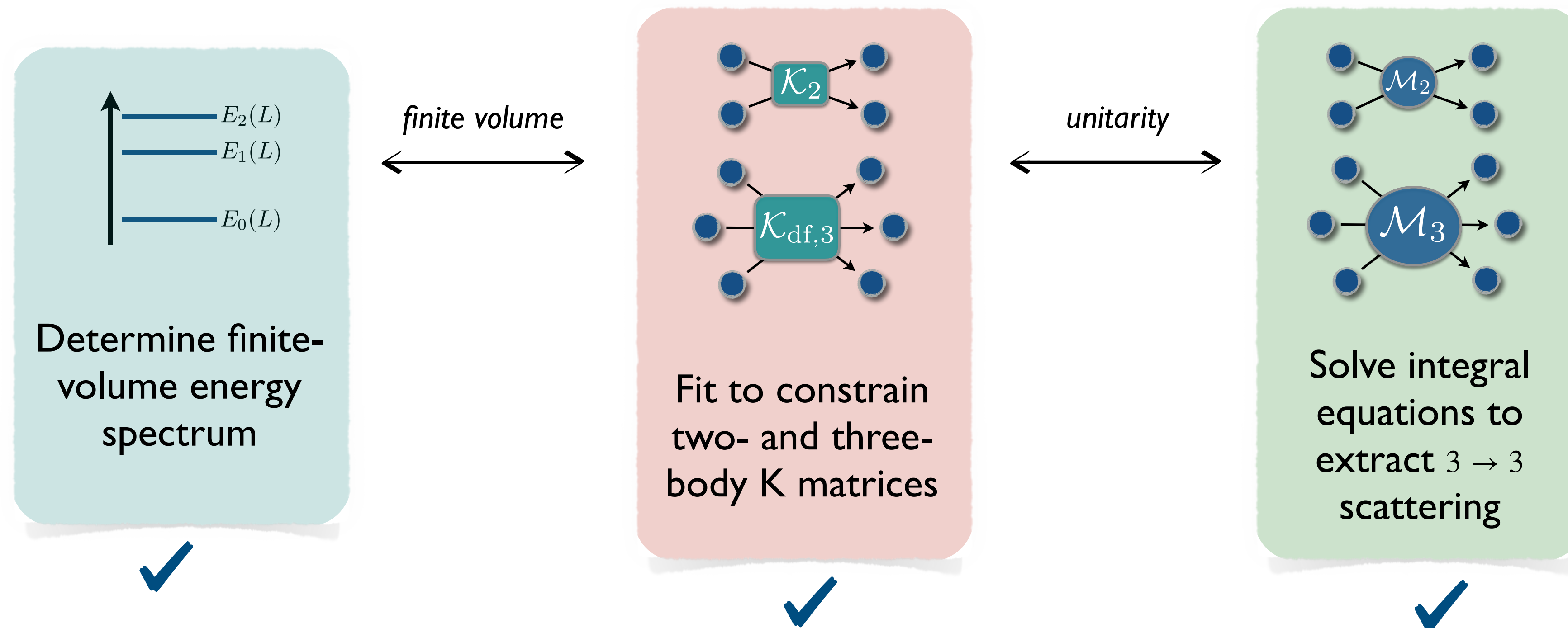
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$$L_s/a_s = 20, 24$$



□ Workflow outline



Isospin

Four possible isospin channels for three pions

$$1 \otimes 1 \otimes 1 = (0 \oplus 1 \oplus 2) \otimes 1 =$$

$$1 \oplus (0 \oplus 1 \oplus 2) \oplus (1 \oplus 2 \oplus 3)$$

$I_{\pi\pi} = 0$

$I_{\pi\pi} = 1$

$I_{\pi\pi} = 2$

Four quantization conditions

$I_{\pi\pi\pi} = 0$

(ρ)

$(\square) \quad (\rho)$

$I_{\pi\pi\pi} = 1$

$(\sigma) \quad (\rho) \quad (\pi\pi)_2$

$\begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$

$(\sigma) \quad (\rho) \quad (\pi\pi)_2$

$I_{\pi\pi\pi} = 2$

$(\rho) \quad (\pi\pi)_2$

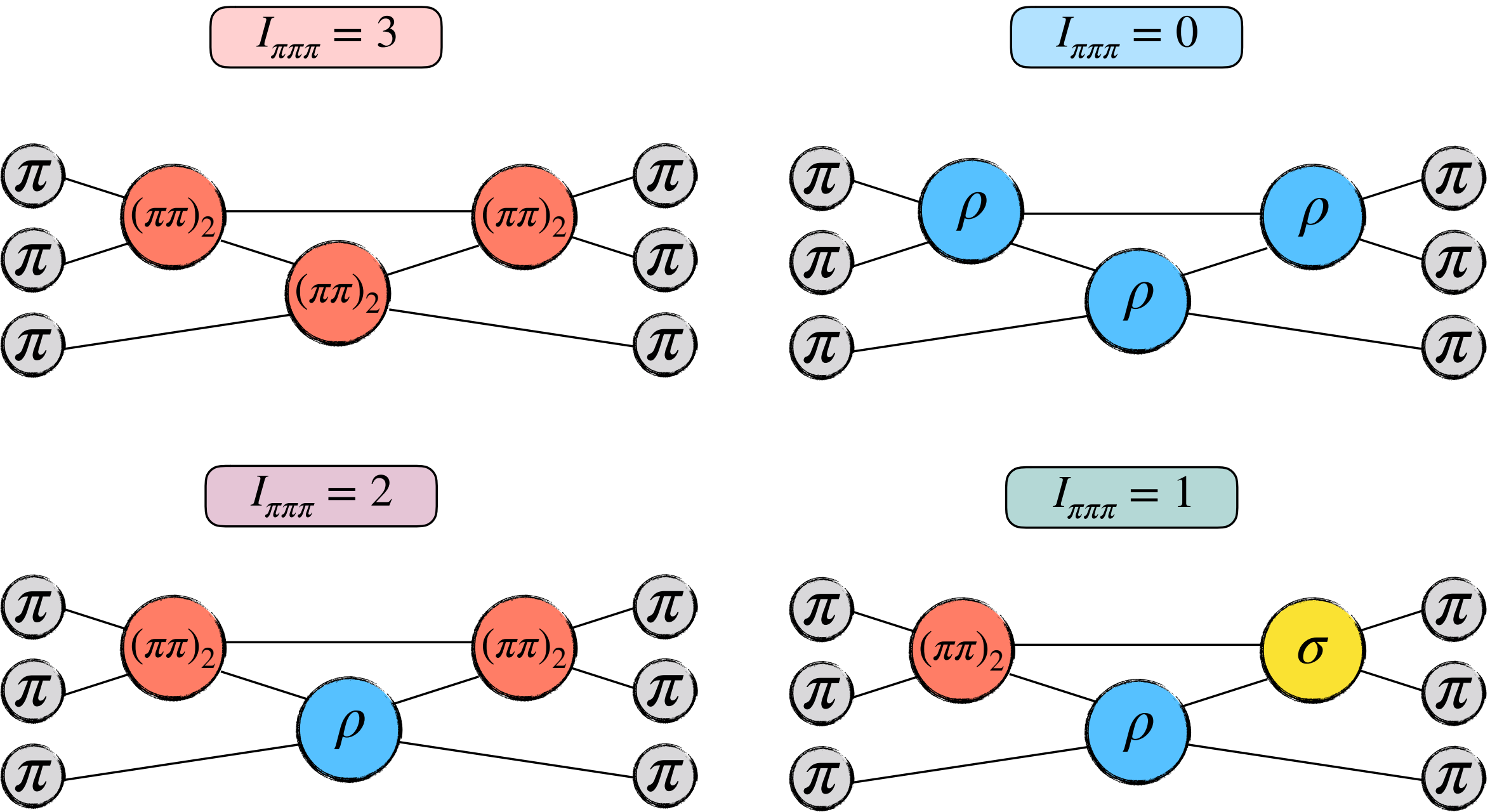
$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$

$(\rho) \quad (\pi\pi)_2$

$I_{\pi\pi\pi} = 3$

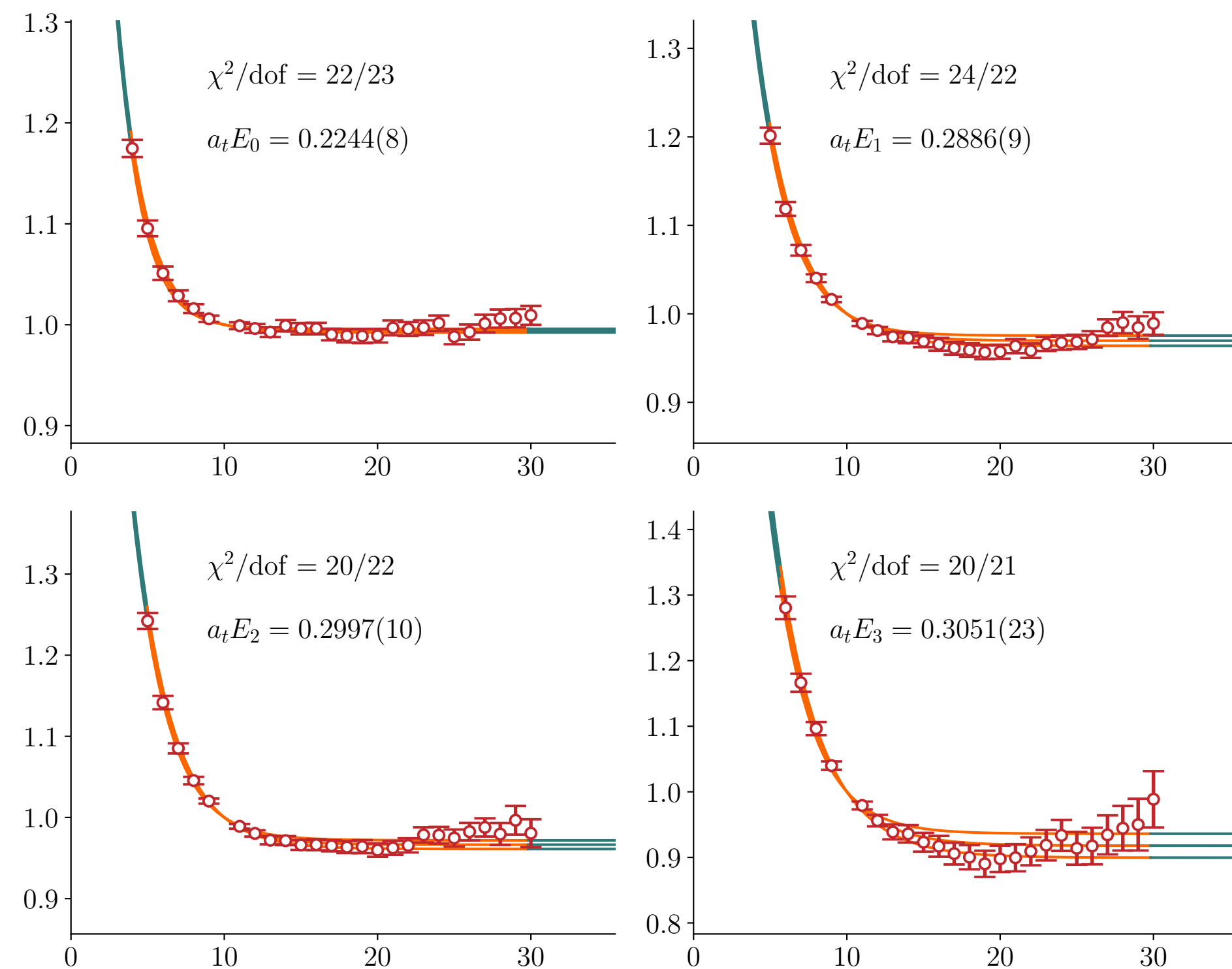
$(\pi\pi)_2$

$(\square) \quad (\pi\pi)_2$

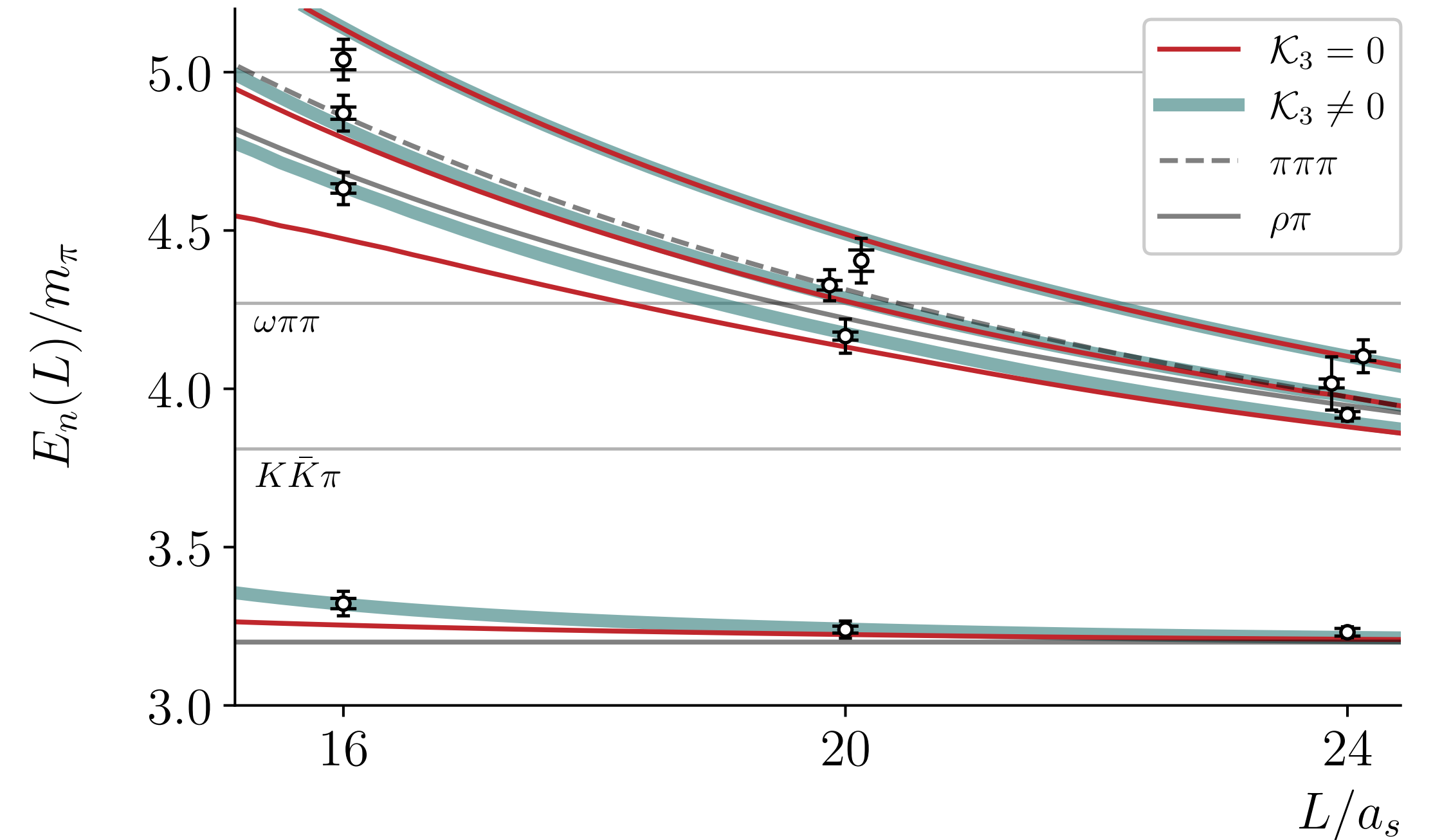


Just out on the arXiv! ... tomorrow 🙄

🔗 First lattice QCD calculation in the isotensor channel



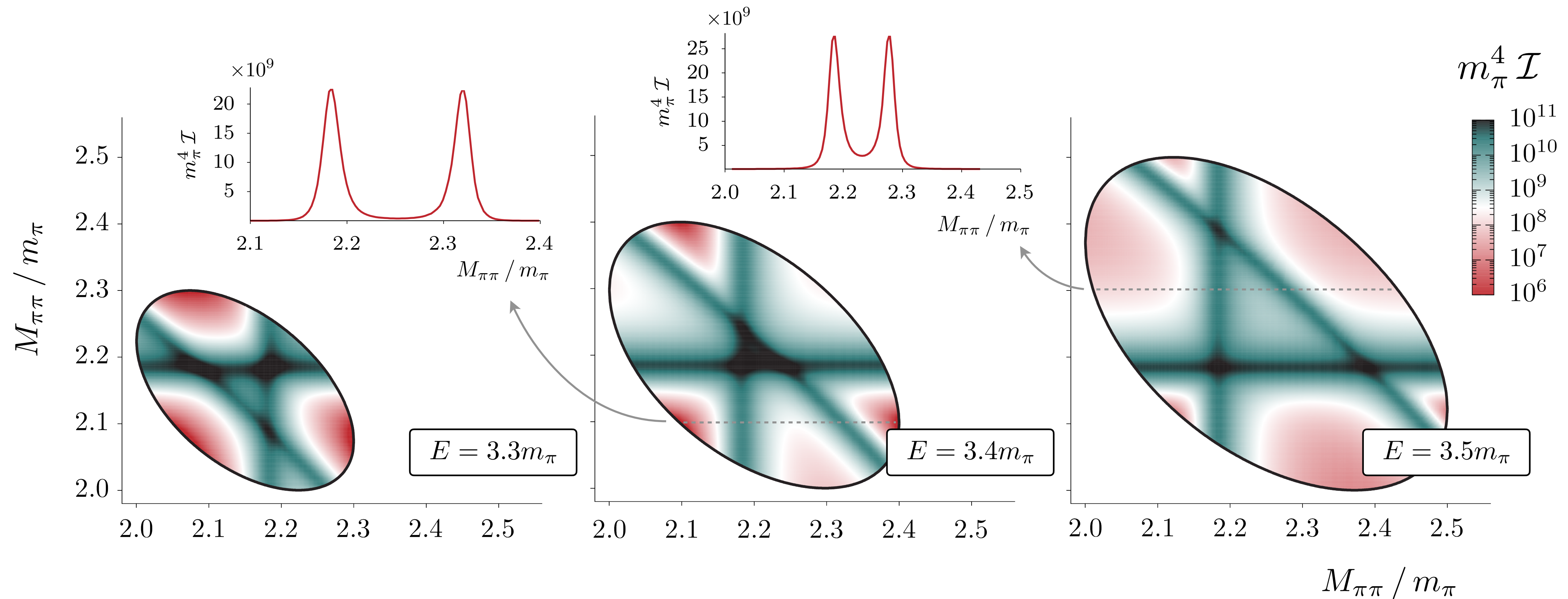
$$\lambda_n(t) e^{E_n(t-t_0)}$$



Briceño, MTH, Jackura, Edwards, Thomas (*tomorrow*) for the Hadron Spectrum Collaboration

Just out on the arXiv! ... tomorrow 🙄

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Impressive work of other groups

⊗ First lattice resonance calculations: $a_1(1260)$, $\omega(700)$, $\pi(1300)$

- Culver, Mai, Brett, Alexandru, Döring (2020) • Brett, Culver, Mai, Alexandru, Döring, Lee (2021) • Mai, Döring, Culver, Alexandru (2020)
- Mai, Alexandru, Brett, Culver, Döring, Lee, Sadasivan (2021) • Yan, Mai, Garofalo, Meißner, Liu, Liu, Urbach (2024) •
 - Yan, Mai, Garofalo, Feng, Döring, Liu, Liu, Meißner, Urbach (2025) •

⊗ Pions and kaons at the physical point

- Blanton, Romero-López, Sharpe (2020) • Fischer, Kostrzewa, Liu, Romero-López, Ueding, Urbach (2021) •
- Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, Sharpe, Skinner (2025) •

Landscape of amplitudes

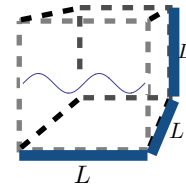
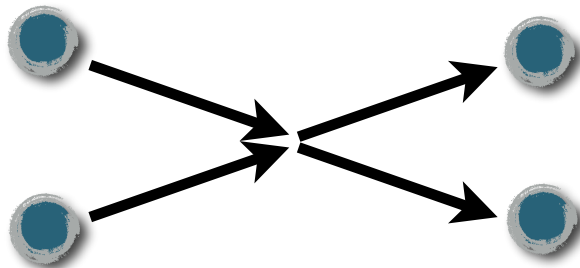
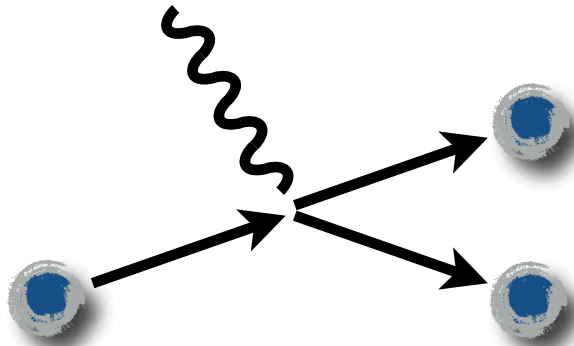
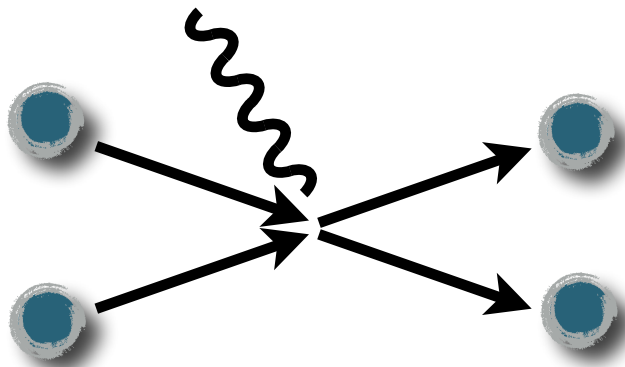
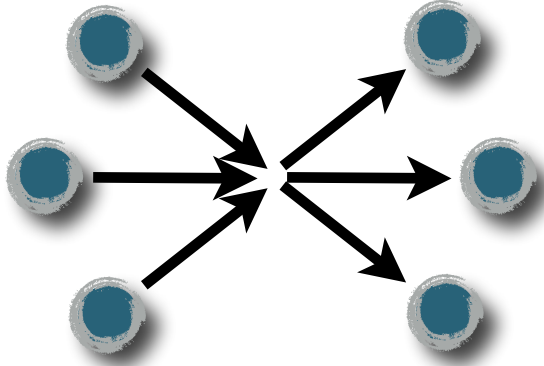
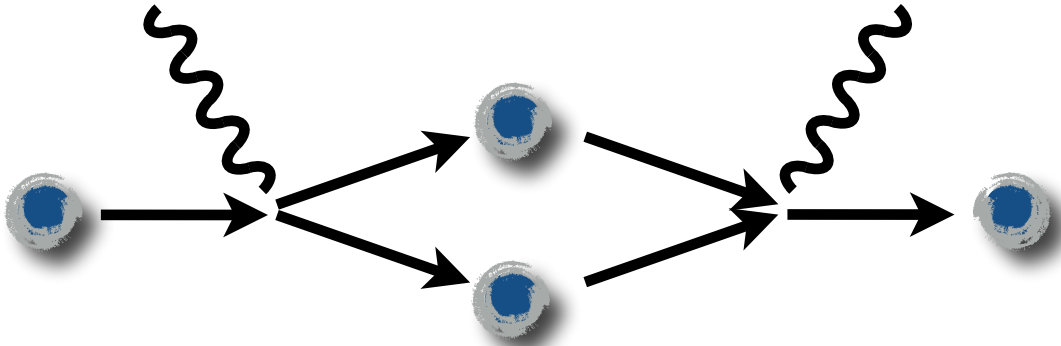
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Closing remarks

- ⊗ Lattice QCD is rich with theoretical challenges
- ⊗ Single-hadron observables are incredibly precise... (but still there's work to do!)
- ⊗ Reliable multi-hadron observables are possible!
- ⊗ The finite volume is an essential tool for multi-hadron/long-distance physics
- ⊗ Spectral reconstruction is another essential (and complimentary) tool
- ⊗ Decades worth of ambitious multi-particle physics ahead

Thanks!...

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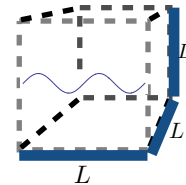
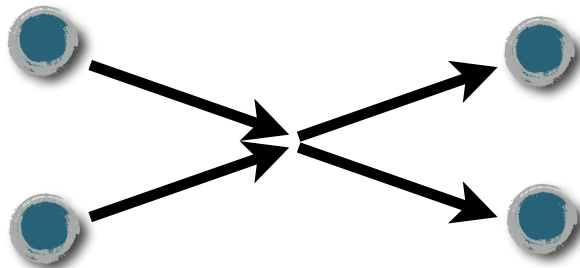
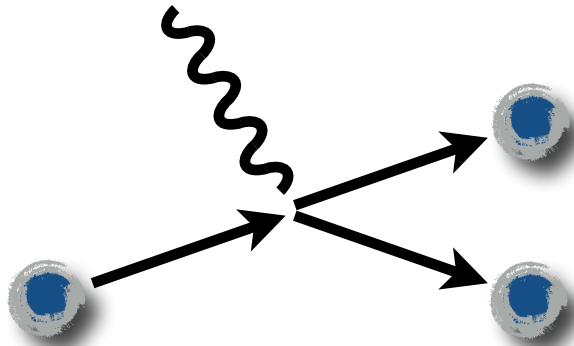
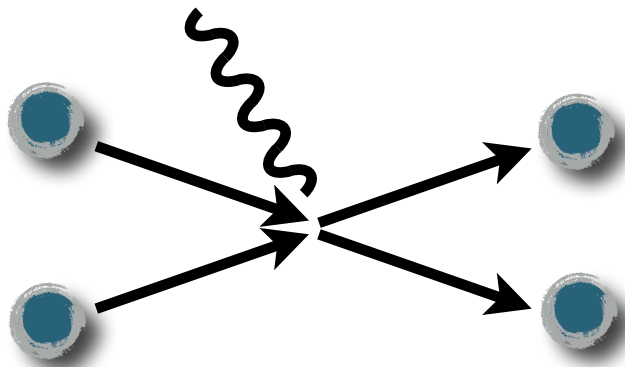
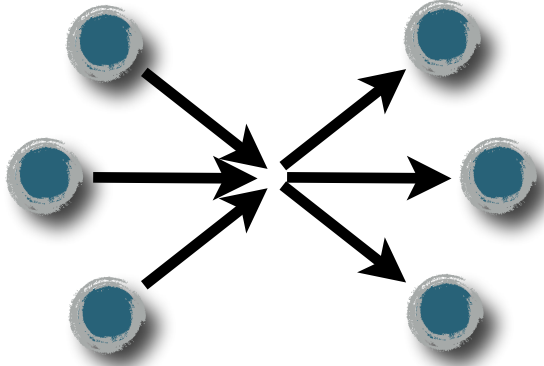
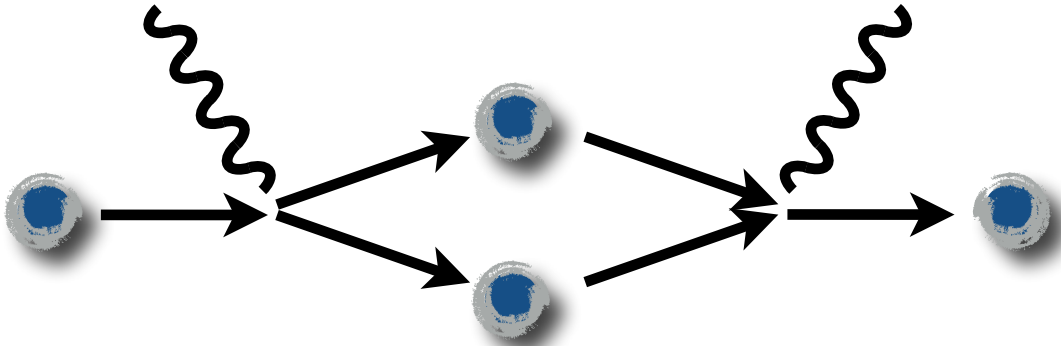
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	formalism ($E_{\text{cm}} < 4M_\pi$) numerical maturity = 	formalism   

Correlation function to observables

- ⊗ Lattice QCD gives finite-volume Euclidean correlators

$$\langle 0 | \mathcal{O}_1(0) e^{-\hat{H}\tau} \mathcal{O}_2(0) | 0 \rangle_L \quad \text{have}$$

- ⊗ Complete physical information is contained in...

$$\langle 0 | \mathcal{O}_1(0) f(\hat{H}) \mathcal{O}_2(0) | 0 \rangle_\infty \quad \text{want}$$

- ⊗ Detailed choice of $f(E)$ and operators determines the observable

R-ratio

$$\langle 0 | j_\mu(0) \delta(\hat{H} - \omega) j_\mu(0) | 0 \rangle_\infty$$

Meyer • Bailas, Hashimoto, Ishikawa (2020)
Bulava et al. (2021) • Alexandrou et al. (2022)

$\pi\pi \rightarrow \pi\pi$ amplitude

$$\langle \pi | \pi(0) \frac{1}{E - \hat{H} + i\epsilon} \pi(0) | \pi \rangle_\infty$$

Barata, Fredenhagen (1991) • Bulava, MTH (2019) • Patella, Tantalo (2025)

D-meson total lifetime

$$\langle D | \mathcal{H}_W(0) \delta(M_D - \hat{H}) \mathcal{H}_W(0) | D \rangle_\infty$$

MTH, Meyer, Robaina (2017)

long-distance and mixing amplitudes

$$\langle P | \mathcal{J}(0) \frac{1}{E - \hat{H} + i\epsilon} \mathcal{J}_\mu(0) | P' \rangle_\infty$$

Gambino, Hashimoto (2020) • Gambino et al. (2022) •
Frezzotti (2023) • ETMC (2024) • De Santis et al. (2025)

Linear reconstruction

$$\underbrace{\langle \mathcal{O}(0) e^{-\hat{H}\tau} \mathcal{O}(0) \rangle}_{\text{have}} = \int d\omega e^{-\omega\tau} \underbrace{\langle \mathcal{O}(0) \delta(\omega - \hat{H}) \mathcal{O}(0) \rangle}_{\text{want}}$$

$$\underbrace{G(\tau)}_{\text{have}} = \int d\omega e^{-\omega\tau} \underbrace{\rho(\omega)}_{\text{want}}$$

⊗ **Linear, model-independent reconstruction** (e.g. Backus-Gilbert-like, Chebyshev)

$$\begin{aligned} \sum_{\tau} \mathcal{K}(\bar{\omega}, \tau) G(\tau) &= \sum_{\tau} \mathcal{K}(\bar{\omega}, \tau) \int d\omega e^{-\omega\tau} \rho(\omega) = \int d\omega \left[\sum_{\tau} \mathcal{K}(\bar{\omega}, \tau) e^{-\omega\tau} \right] \rho(\omega) \\ &= \int d\omega \hat{\delta}_{\Delta}(\bar{\omega}, \omega) \rho(\omega) \end{aligned}$$

← δ is exactly known

⊗ **Non-linear (*not discussed here...*)**

- Maximum Entropy Method (MEM)
- Direct fits
- Neural networks

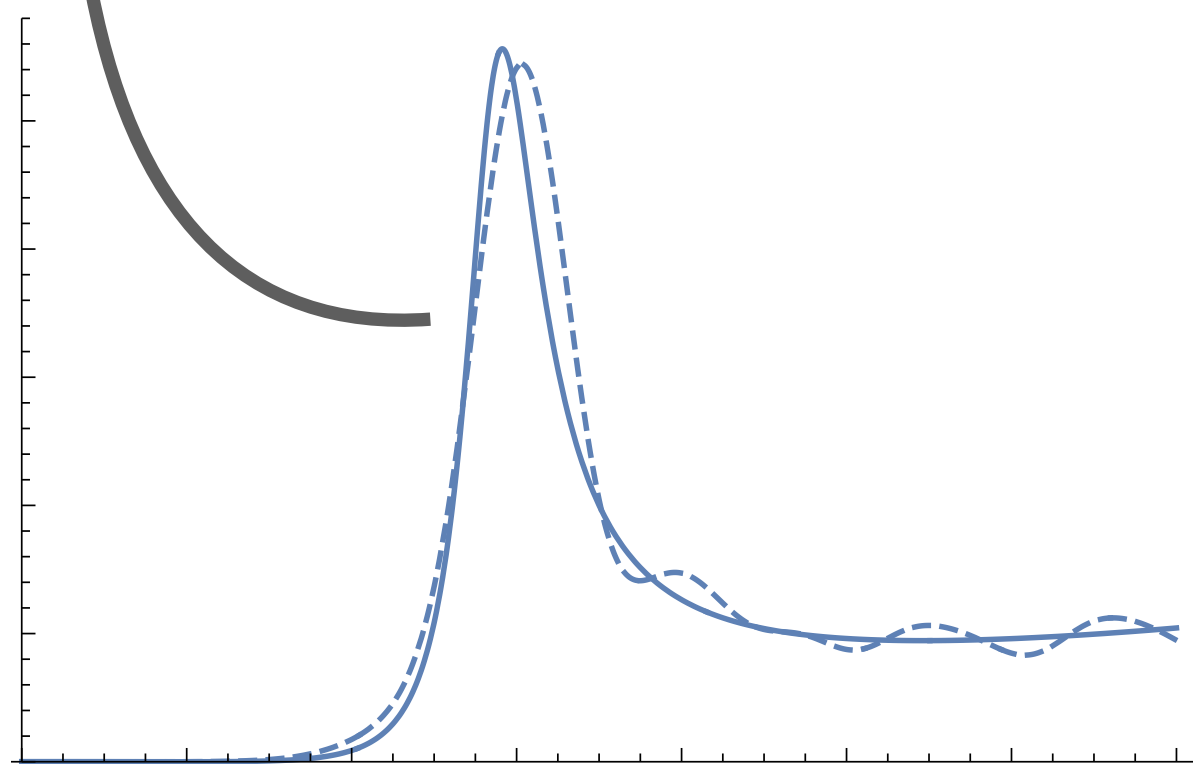
See multiple ECT and CERN workshops, work by
Aarts, Allton, Amato, Brandt, Burnier, Del Debbio, Francis,
Giudice, Hands, Harris, Hashimoto, Jäger, Karpie, Liu,
Meyer, Monahan, Orginos, Robaina, Rothkopf, Ryan, ...*

⊗ **Key idea here... we aim only to construct**

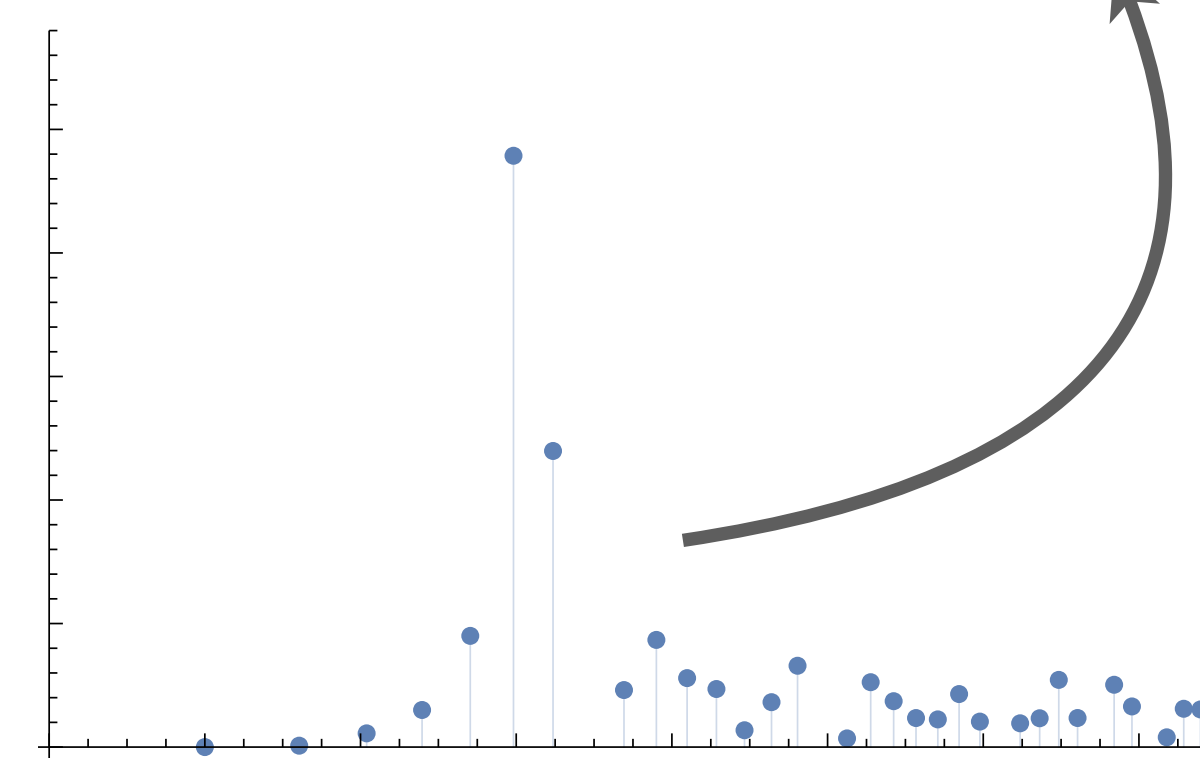
$$\hat{\rho}(\bar{\omega}) \equiv \int_{-\infty}^{\infty} d\omega \hat{\delta}_{\Delta}(\bar{\omega}, \omega) \rho(\omega)$$

Role of the finite volume

$$\hat{\rho}_L(\bar{\omega}) \equiv \int_0^\infty d\omega \hat{\delta}_\Delta(\bar{\omega}, \omega) \rho_L(\omega)$$



$$G_L(\tau) = \int d\omega e^{-\omega\tau} \rho_L(\omega)$$



⊗ Any reconstructed spectral function ($\neq$ forest of deltas) must be smeared (or else $L \rightarrow \infty$)

We require...

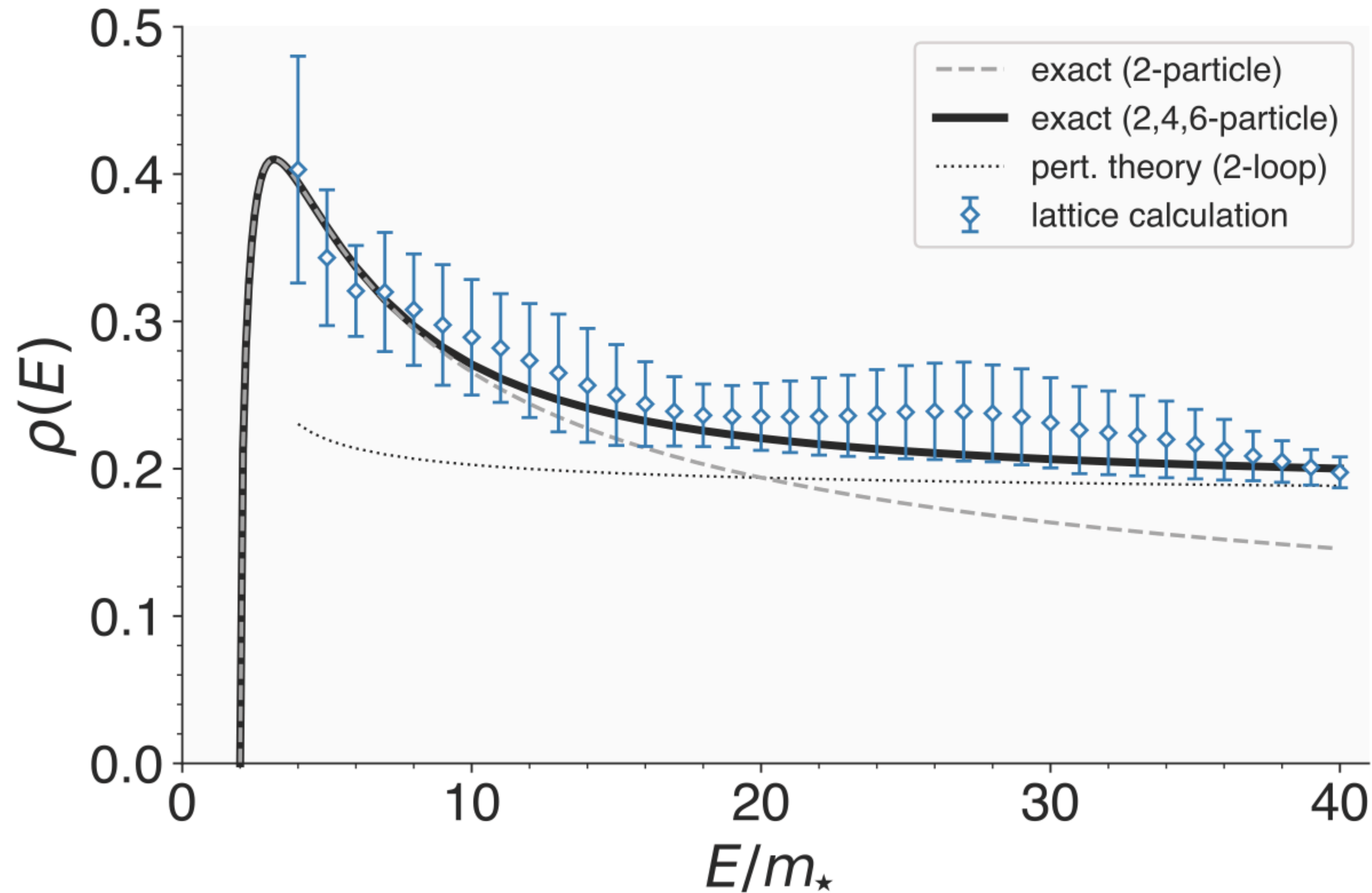
$$1/L \ll \Delta \ll \mu_{\text{physical}}$$

smearing function covers
many delta peaks

smearing does not overly
distort observable

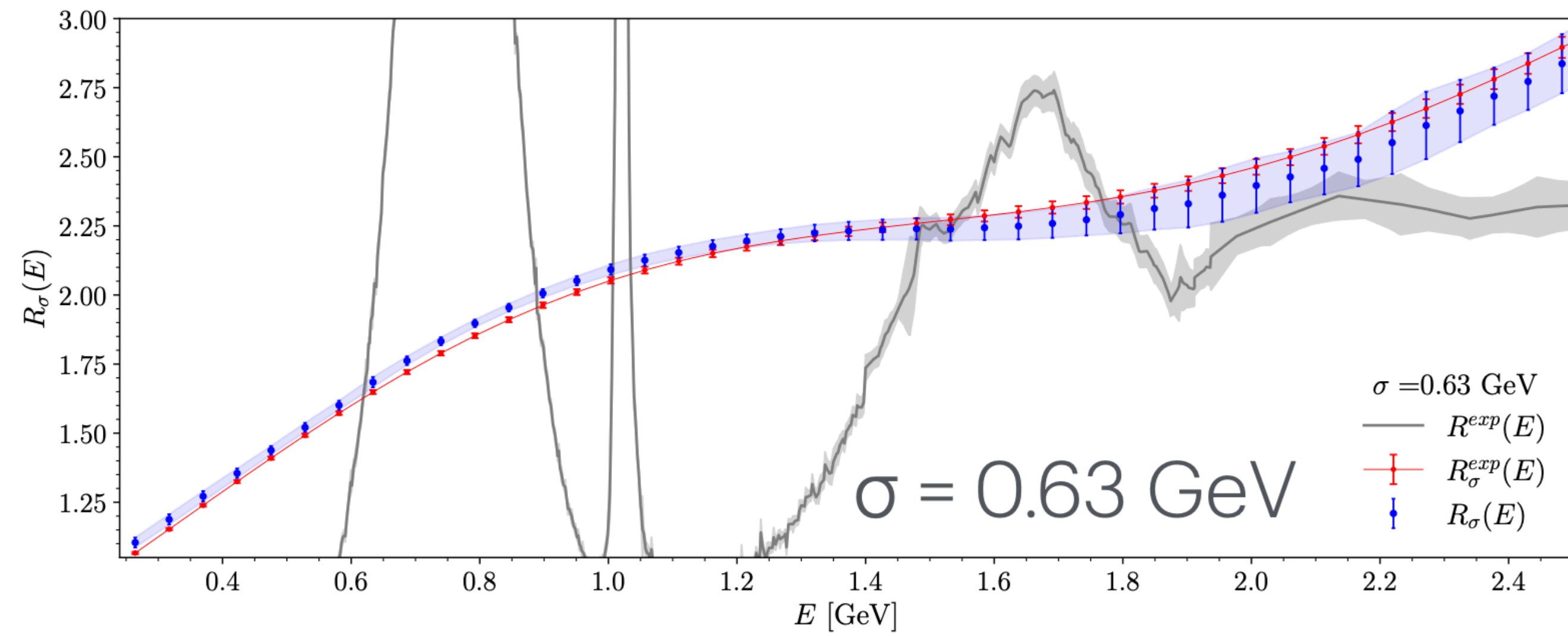
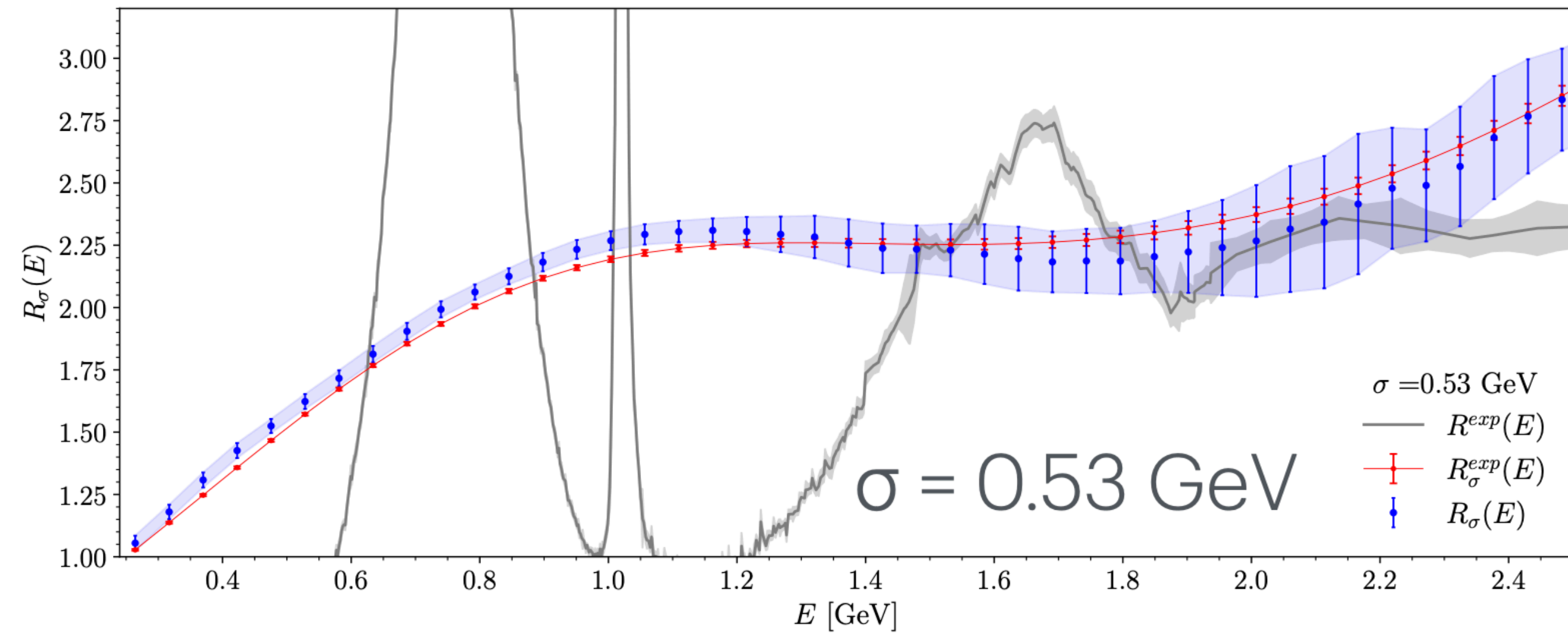
MTH, Meyer, Robaina (2017)

Monte Carlo test: 1+1d non-linear O(3) sigma model



Bulava, MTH, Hansen, Patella, Tantalo (2021) *using method of* Hansen, Lupo, Tantalo (2019)

Smeared R ratio



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