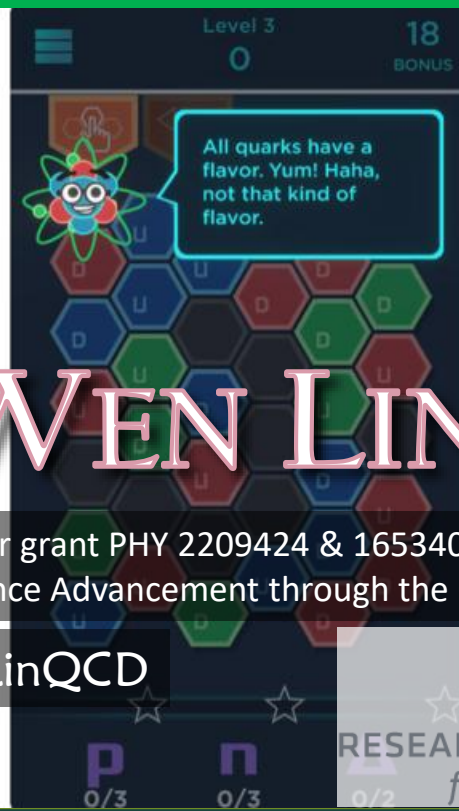


Recent Progress on x -dependent GPDs from the Lattice



HUEY-WEN LIN

This work of HL is supported by the NSF under grant PHY 2209424 & 1653405, DOE under DE-SC0024053 and the Research Corporation for Science Advancement through the Cottrell Scholar Award



@LinQCD

RESEARCH CORPORATION
for SCIENCE ADVANCEMENT

Outline

§ Backgrounds

∞ LaMET Method and Progress on PDFs

§ Selected GPDs Results

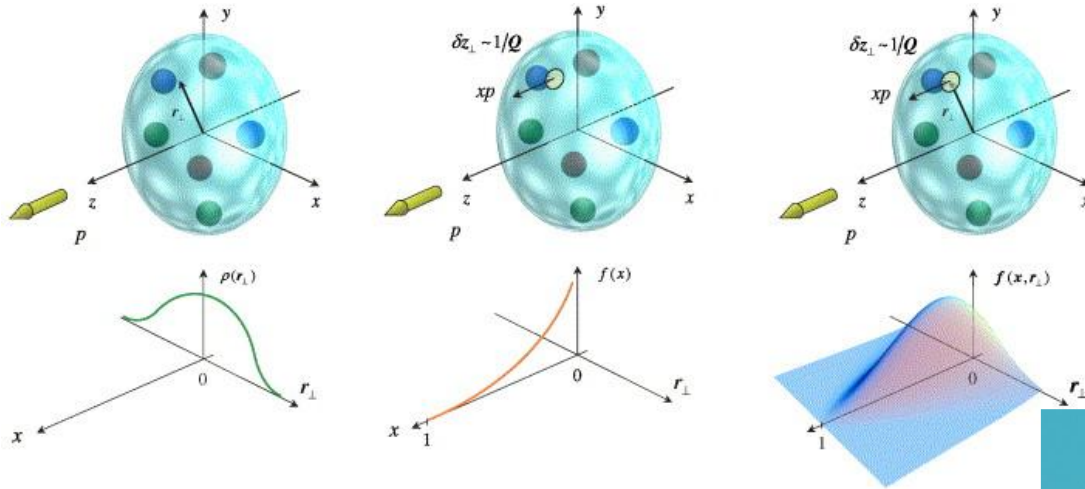
∞ Progress on the zero-skewness GPDs

∞ zero-skewness GPDs



3D Imaging

§ GPDs encode information about the spatial structure & the partonic distribution of spin and orbital angular momenta



Picture from A. Belitskya and A Radyushkin, Physics Report, 416 (2015)

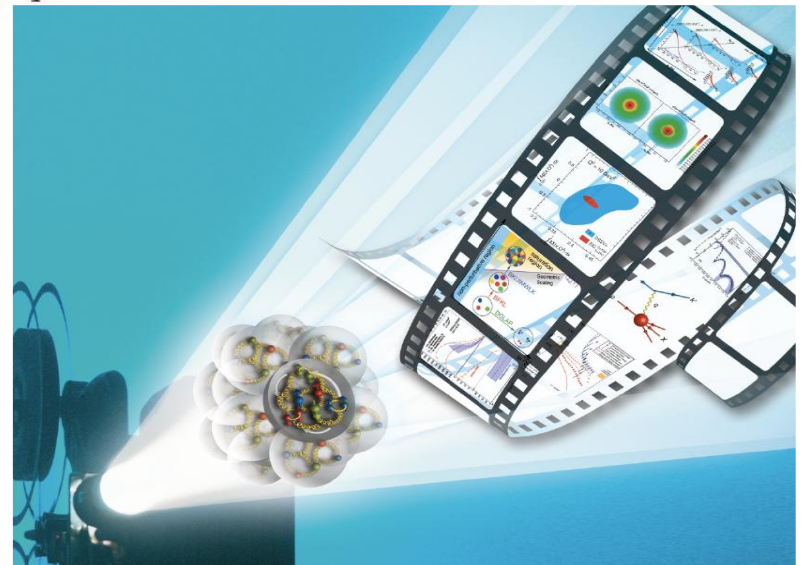
**Electron Ion Collider:
The Next QCD Frontier**

Imaging of the proton

EIC White Paper, 1212.1701

Yellow Report, 2103.05419

Christine Aidala & Marco Radici on Mon



What is Lattice QCD?

§ Lattice QCD is an ideal theoretical tool for investigating the strong-coupling regime of quantum field theories

§ Physical observables are calculated from the path integral

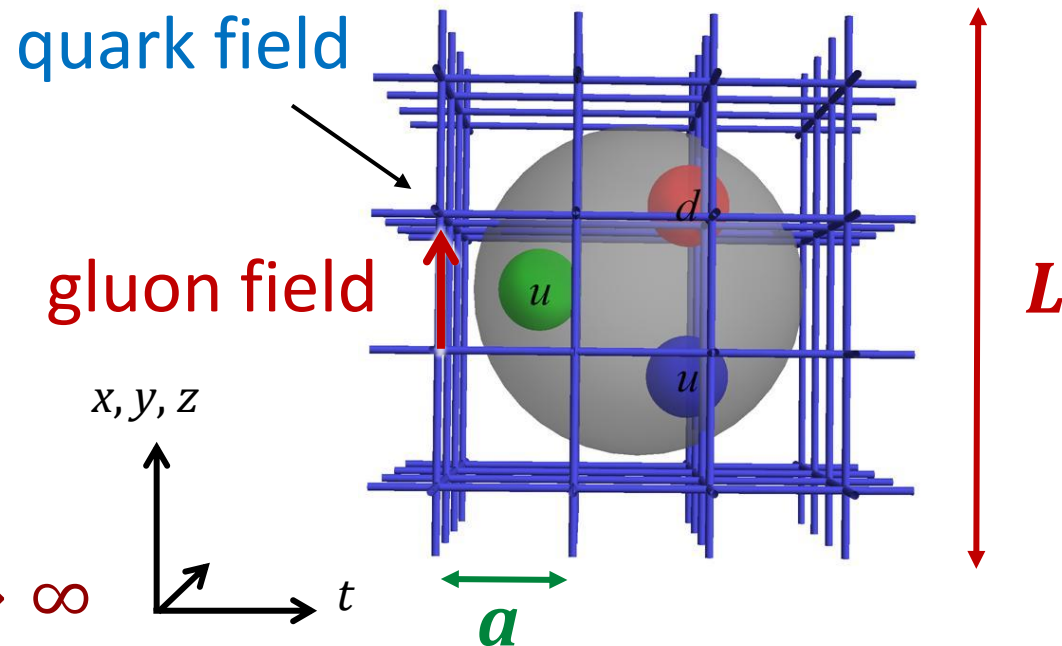
$$\langle 0 | O(\bar{\psi}, \psi, A) | 0 \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS(\bar{\psi}, \psi, A)} O(\bar{\psi}, \psi, A)$$

in **Euclidean** space

- ∞ Quark mass parameter (described by m_π)
- ∞ Impose a UV cutoff
discretize spacetime
- ∞ Impose an infrared cutoff
finite volume

§ Recover physical limit

$$m_\pi \rightarrow m_\pi^{\text{phys}}, a \rightarrow 0, L \rightarrow \infty$$

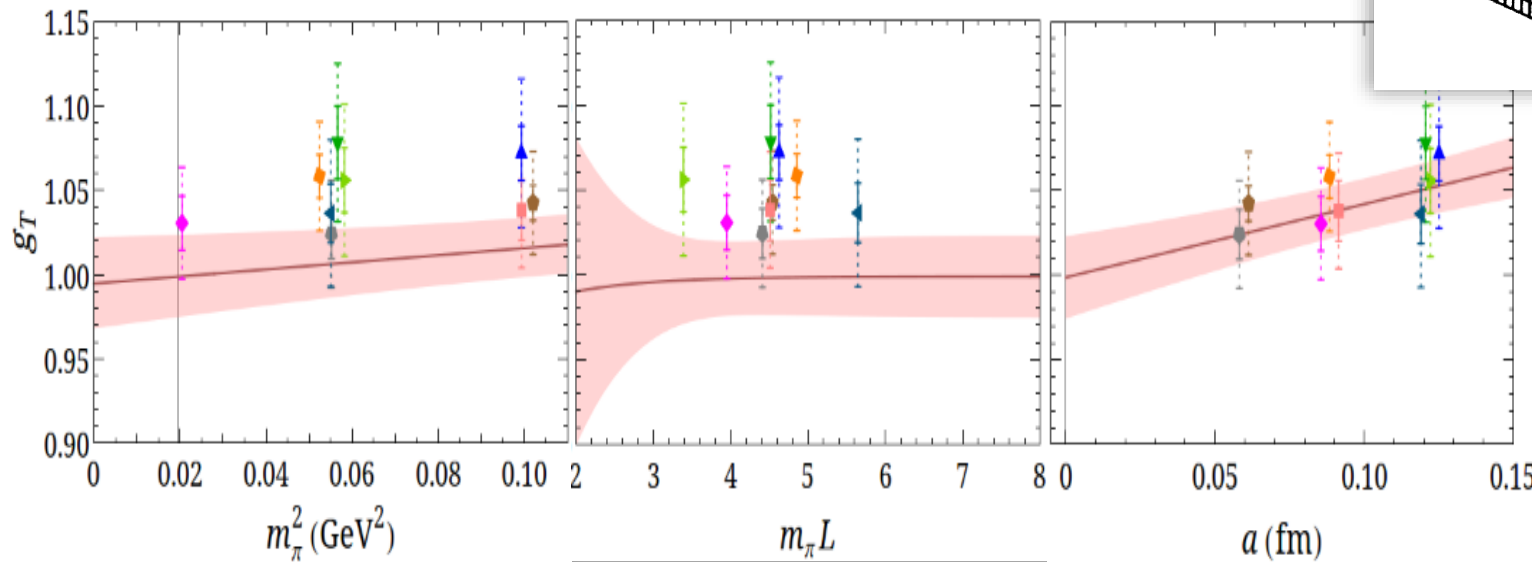
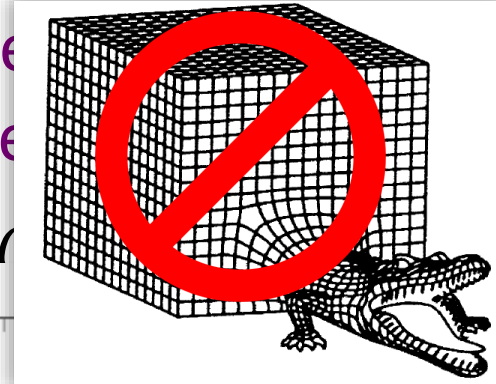


Lattice QCD 101

§ Lattice QCD is an ideal theoretical tool for investigating the strong-coupling regime of quantum field theory

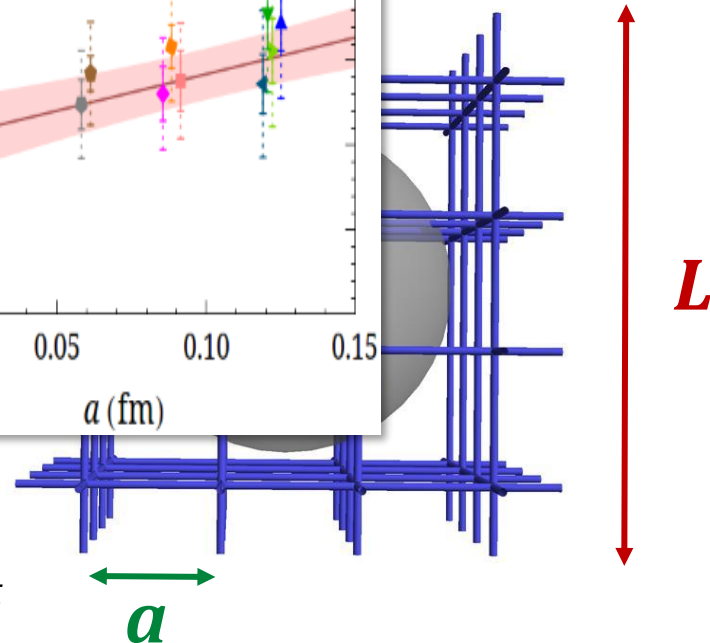
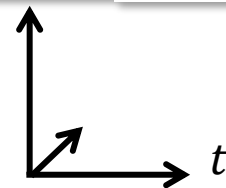
§ Physical observables are calculated from the

$$\langle 0 | O(\bar{\psi}, \psi, A) | 0 \rangle = \frac{1}{\mathcal{N}} \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS(\bar{\psi}, \psi, A)}$$



§ Recover physical limit

$$m_\pi \rightarrow m_\pi^{\text{phys}}, \quad a \rightarrow 0, \quad L \rightarrow \infty$$



Moments of PDFs

§ PDG-like rating system or average

§ LatticePDF Workshop

↻ Lattice representatives came together and devised a rating system

§ Lattice QCD/global fit status

$$\langle x^{n-1} \rangle_{\delta q} = \int_{-1}^1 dx x^{n-1} \delta q(x)$$



LatticePDF Report, 1711.07916, 2006.08636

Moment	Collaboration	Reference	N_f	DE	CE	FV	RE	ES	Value	Global Fit
g_T	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	** 0.926(32)	0.10 — 1.1
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	* 0.989(32)(10)	
	χ QCD 20	(Horkel <i>et al.</i> , 2020)	2+1	■	★	○	★	★	† 1.096(30)	
	LHPC 19	(Hasan <i>et al.</i> , 2019)	2+1	○	★	○	★	★	* 0.972(41)	
	Mainz 19	(Harris <i>et al.</i> , 2019)	2+1	★	○	★	★	★	0.965(38)(⁺¹³ ₋₄₁)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	1.08(3)(3)(9)	
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2	■	★	○	★	★	** 0.974(33)	
	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	1.004(21)(02)(19)	
$\langle 1 \rangle_{\delta u^-}$	RQCD 14	(Bali <i>et al.</i> , 2015)	2	○	★	★	★	■	1.005(17)(29)	-0.14 — 0.91
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	** 0.716(28)	
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	* 0.784(28)(10)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	0.85(3)(2)(7)	
$\langle 1 \rangle_{\delta d^-}$	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	0.782(16)(2)(13)	-0.97 — 0.47
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	** -0.210(11)	
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	* -0.204(11)(10)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	-0.24(2)(0)(2)	
$\langle 1 \rangle_{\delta s^-}$	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	-0.219(10)(2)(13)	N/A
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	** -0.0027(58)	
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	* -0.0027(16)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	-0.012(16)(8)	
	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	-0.00319(69)(2)(22)	

Moments of PDFs

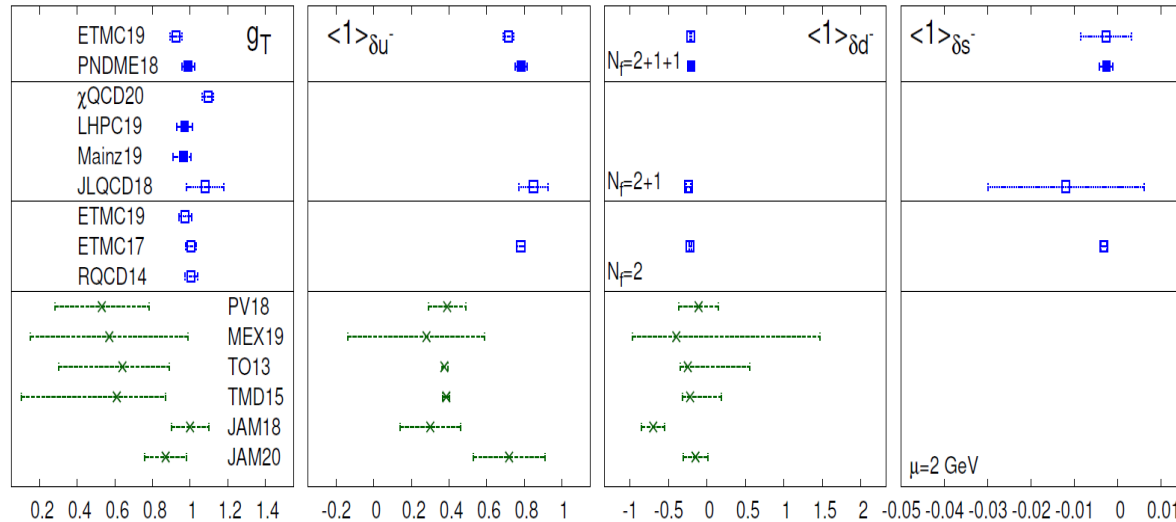
§ PDG-like rating system or average

§ LatticePDF Workshop

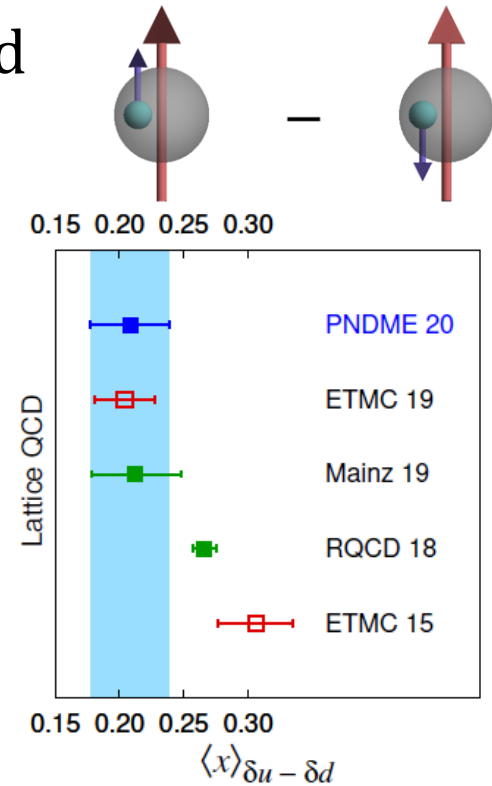
∞ Lattice representatives came together and devised a rating system

§ Recent lattice QCD/global fit status

LatticePDF Reports, 1711.07916, 2006.08636

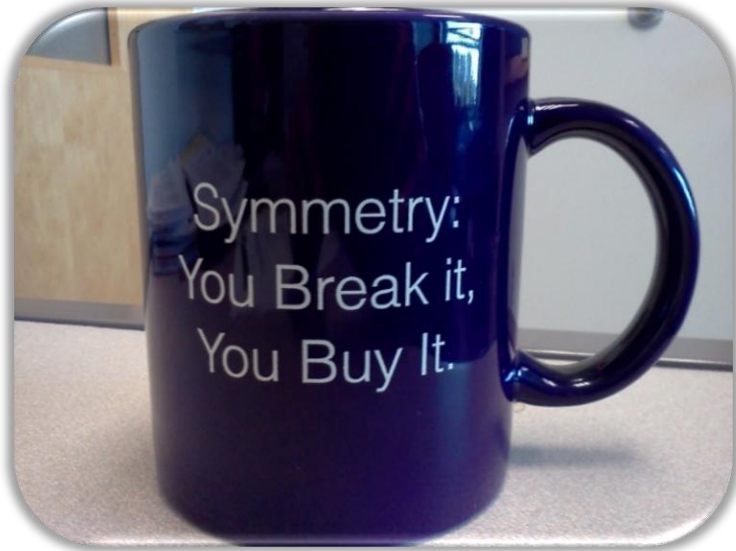


$$\langle x^{n-1} \rangle_{\delta q} = \int_{-1}^1 dx x^{n-1} \delta q(x)$$



PNDME, 2005.13779

Lattice Structure Limitation



§ Lattice calculations rely on operator product expansion, only provide moments

$$\langle x^{n-1} \rangle_q = \int_{-1}^1 dx x^{n-1} q(x)$$

§ Longstanding obstacle!

⌘ Holy grail of structure calculations

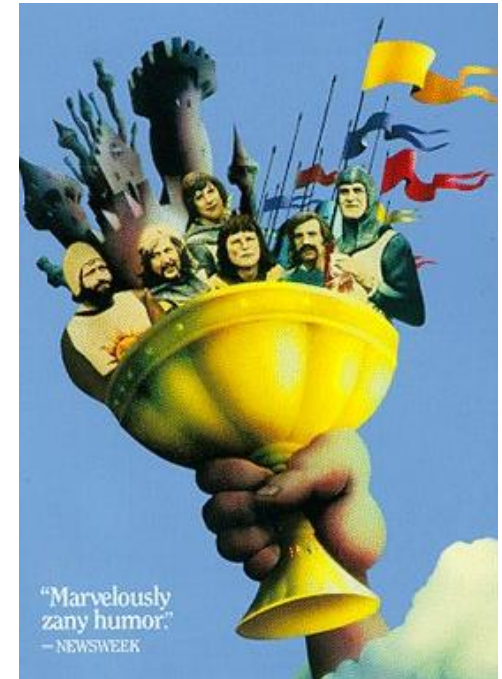
§ Applies to many structure quantities:

⌘ Parton distribution functions (PDF)

⌘ Generalized parton distributions (GPD)

⌘ Transverse-momentum distributions (TMD)

⋮



A NEW HOPE

It is a period of war and economic uncertainty.

Turmoil has engulfed the galactic republics.

Basic truths at foundation of the human civilization are disputed by the dark forces of the evil empire.

A small group of QCD Knights from United Federation of Physicists has gathered in a remote location on the third planet of a star called Sol on the inner edge of the Orion-Cygnus arm of the galaxy.

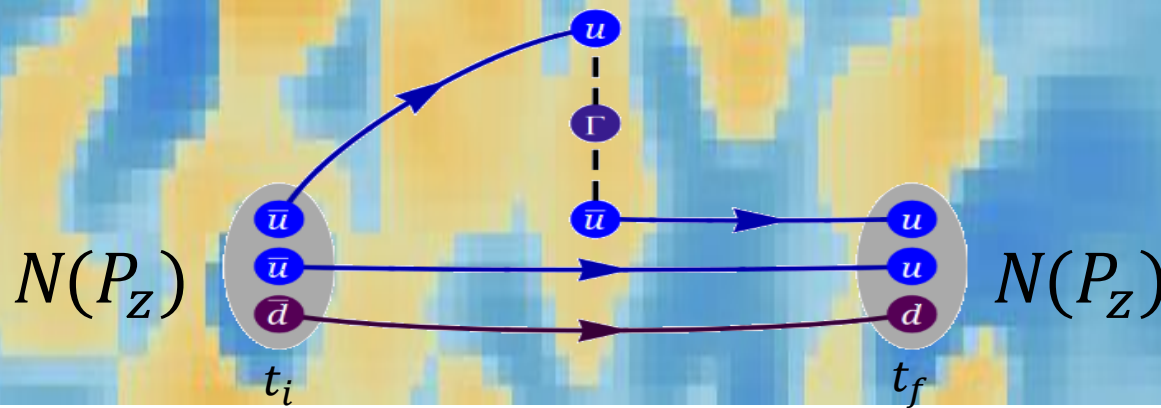
The QCD Knights are the only ones who can tame the power of the Strong Force, responsible for holding atomic nuclei together, for giving mass and shape to matter in the Universe.

They carry secret plans to build the most powerful

Lattice Parton Method

§ Large-momentum effective theory (LaMET)/quasi-PDF

(X. Ji, 2013; See 2004.03543 for review)



§ Compute quasi-distribution via

$$\tilde{q}(x, \mu, P_z) = \int \frac{dz}{4\pi} e^{-izk_z} \left\langle P \left| \bar{\psi}(z) \Gamma \exp \left(-ig \int_0^z dz' A_z(z') \right) \psi(0) \right| P \right\rangle$$

§ Recover true distribution (take $P_z \rightarrow \infty$ limit)

$$\tilde{q}(x, \mu, P_z) = \int_{-\infty}^{\infty} \frac{dy}{|y|} C \left(\frac{x}{y}, \frac{\mu}{P_z} \right) \mathbf{q}(\mathbf{y}, \mu) + \mathcal{O} \left(\frac{M_N^2}{P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(x P_z)^2}, \frac{\Lambda_{\text{QCD}}^2}{((1-x) P_z)^2} \right)$$

X. Xiong e.a., 1310.7471; J.-W. Chen e.a., 1603.06664

Lattice Parton Method

§ Large-momentum effective theory (LaMET)/quasi-PDF

(X. Ji, 2013; See 2004.03543 for review)

Additional source of systematics: P_z

Smaller P_z gives better signal but larger systematics
(like how heavier pion mass gives better precision)

New parameters in x-dependent methods to
pay attention to

§ Compute quasi-distribution via

$$\tilde{q}(x, \mu, P_z) = \int \frac{dz}{4\pi} e^{-izk_z} \left\langle P \left| \bar{\psi}(z) \Gamma \exp \left(-ig \int_0^z dz' A_z(z') \right) \psi(0) \right| P \right\rangle$$

§ Recover true distribution (take $P_z \rightarrow \infty$ limit)

$$\tilde{q}(x, \mu, P_z) = \int_{-\infty}^{\infty} \frac{dy}{|y|} C \left(\frac{x}{y}, \frac{\mu}{P_z} \right) q(y, \mu) + \mathcal{O} \left(\frac{M_N^2}{P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(x P_z)^2}, \frac{\Lambda_{\text{QCD}}^2}{((1-x) P_z)^2} \right)$$

X. Xiong e.a., 1310.7471; J.-W. Chen e.a., 1603.06664

Direct x -Dependent Structure

§ Longstanding obstacle to lattice calculations!



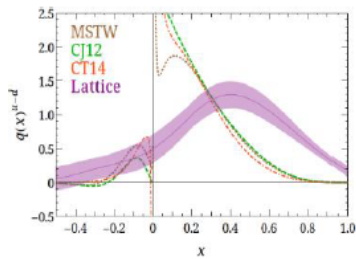
- ✧ **Quasi-PDF**/large-momentum effective theory (LaMET)
(X. Ji, 2013; See 2004.03543 for review)
- ✧ **Pseudo-PDF** method: differs in FT (A. Radyushkin, 2017)
- ✧ Lattice cross-section method (**LCS**) (Y Ma and J. Qiu, 2014, 2017)
- ✧ Compton amplitude method (A.J. Chambers et al., 1703.01153)
- ✧ Hadronic tensor currents (Liu et al., hep-ph/9306299, ... 1603.07352)
- ✧ Euclidean correlation functions (RQCD, 1709.04325)

Lattice Parton Calculations

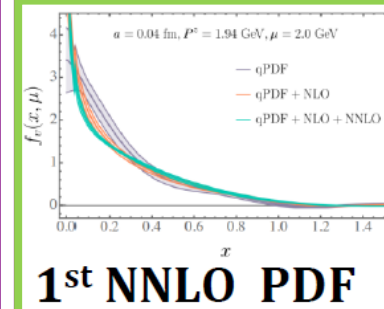
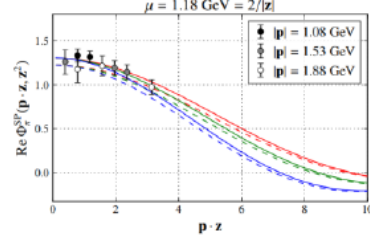
§ Rapid developments!

HL, Prog.Part.Nucl.Phys. 144 (2025)

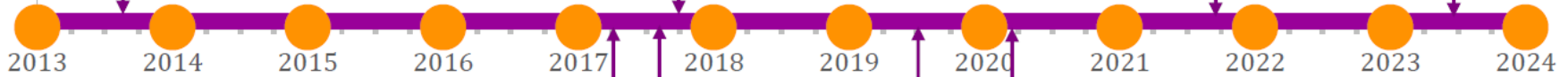
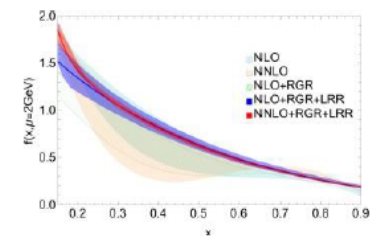
LaMET/quasi-PDF lattice calculation



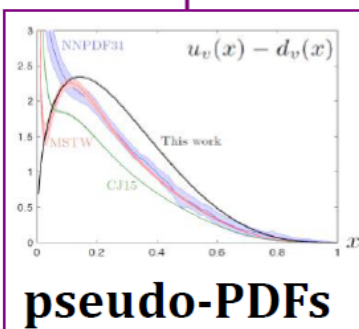
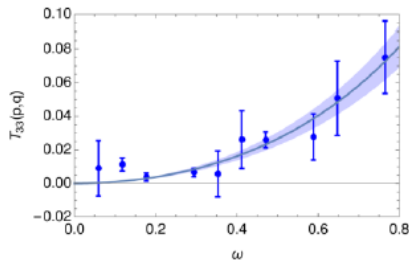
Euclidean correlation functions



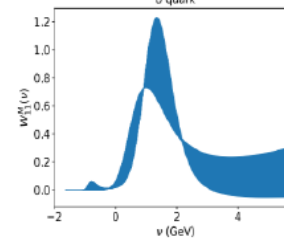
1stPDF w/ LRR+RGR



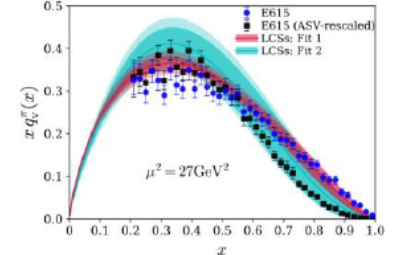
Compton amplitude



Hadronic tensor



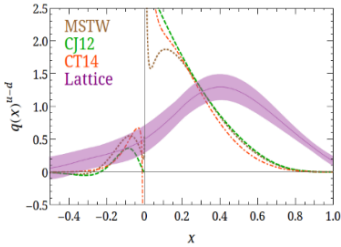
LCS



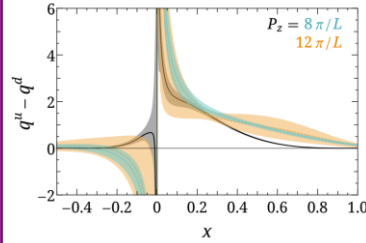
Lattice Parton Calculations

§ Physics-quantity milestones

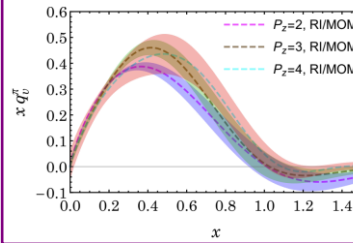
First unpol. lattice PDF



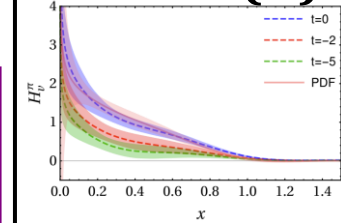
First PDFs at M_π^{phys}



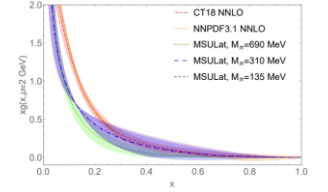
Pion v-PDF



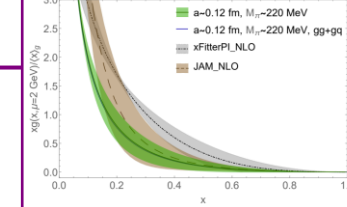
1st GPD (π)



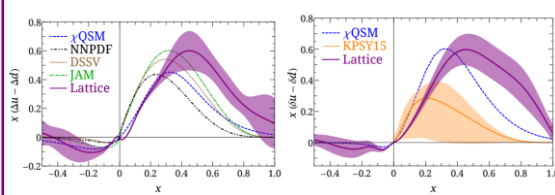
N g-PDF



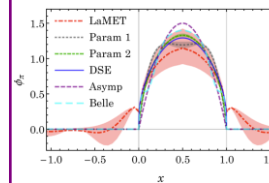
π g-PDF



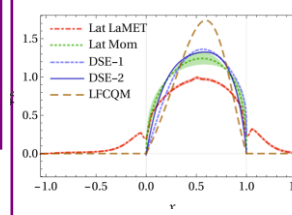
Pol. PDFs and mass corrections



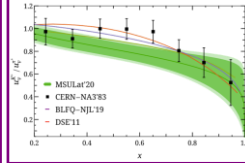
Pion DA



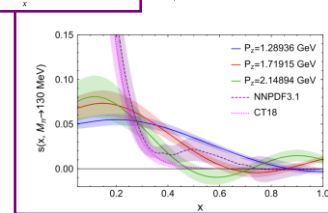
Kaon DA



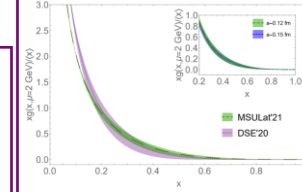
K PDF



s, c PDF



Kaon g-PDF

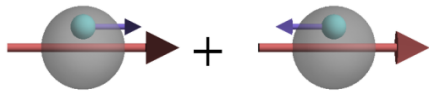


HL, Prog.Part.Nucl.Phys. 144 (2025) 104177

Lattice Example Results

§ Summary of PDF results at physical pion mass

unpolarized



$$u(x) - d(x)$$

longitudinally polarized

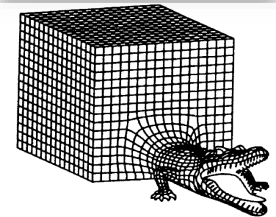
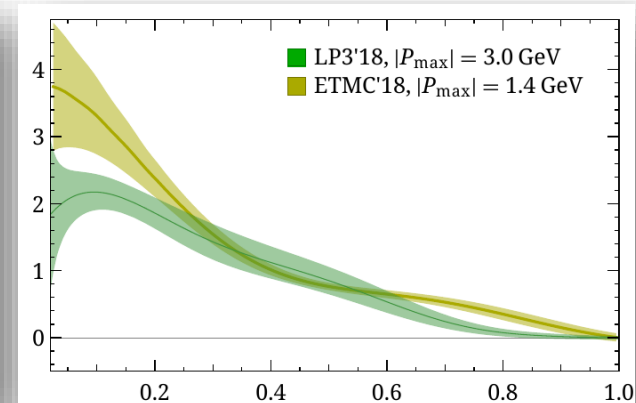
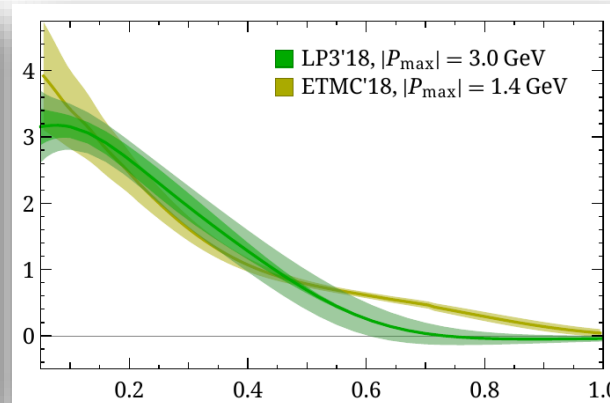
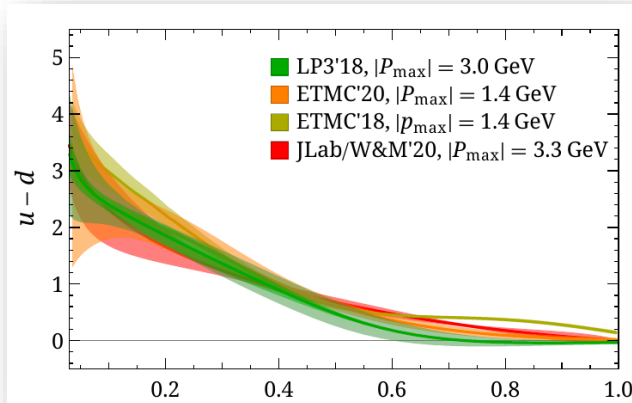


$$\Delta u(x) - \Delta d(x)$$

transversely polarized



$$\delta u(x) - \delta d(x)$$



Finite volume,
Discretization,
...

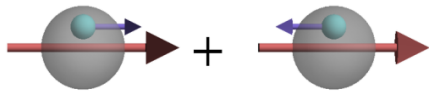


2006.08636 (PDFLattice2019)

Lattice Example Results

§ Summary of PDF results at physical pion mass

unpolarized



$$u(x) - d(x)$$

longitudinally polarized

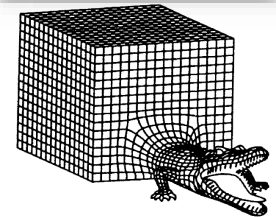
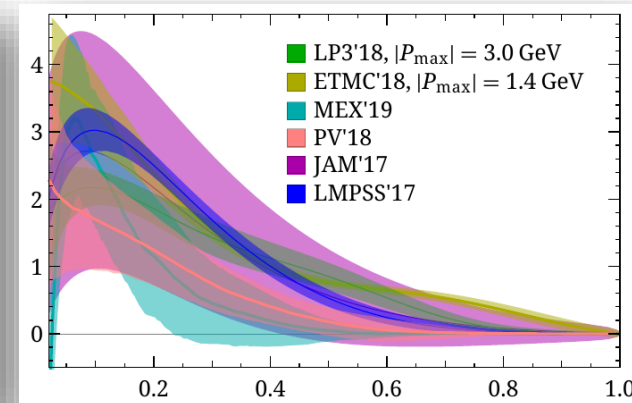
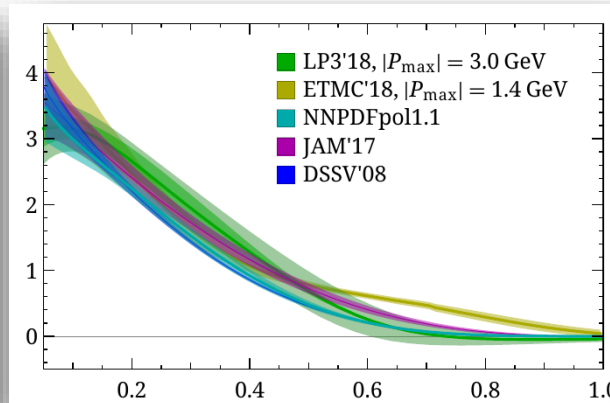
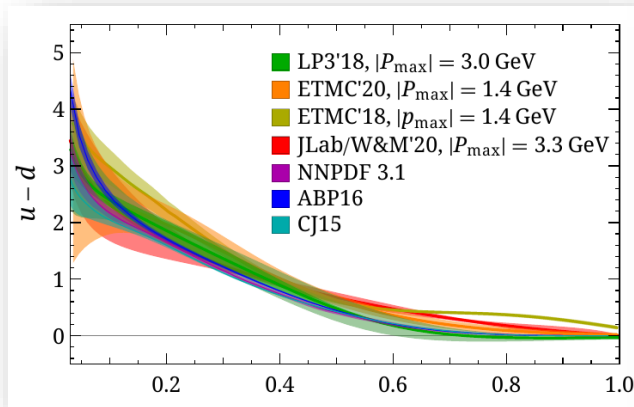


$$\Delta u(x) - \Delta d(x)$$

transversely polarized



$$\delta u(x) - \delta d(x)$$



Finite volume,
Discretization,
...

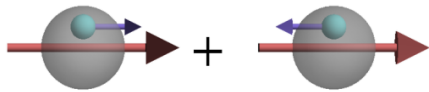


2006.08636 (PDFLattice2019)

Lattice Example Results

§ Summary of PDF results at physical pion mass

unpolarized



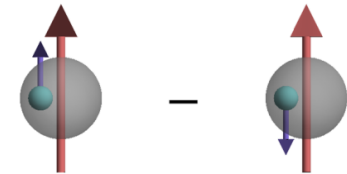
$$u(x) - d(x)$$

longitudinally polarized



$$\Delta u(x) - \Delta d(x)$$

transversely polarized



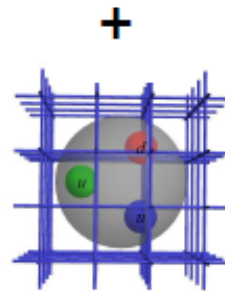
$$\delta u(x) - \delta d(x)$$

§ Complementary lattice inputs for best PDFs

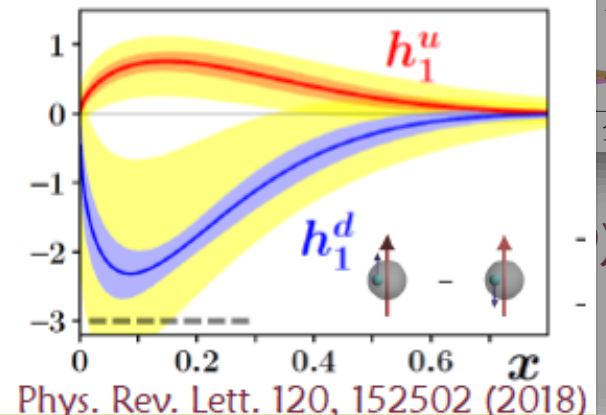
Theory
Input

Exp't
Input

Global Analysis
of PDFs



Work has been made in
this direction



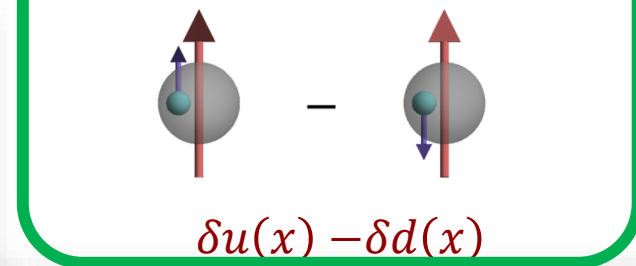
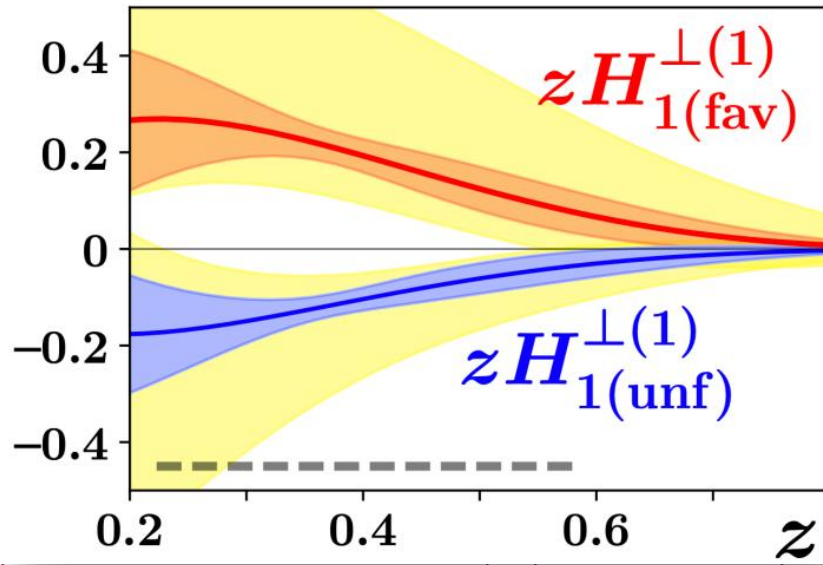
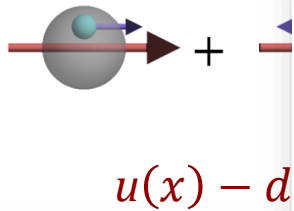
Lattice Example Results

§ Summary of PDF results at physical pion mass

unpolarized

longitudinally polarized

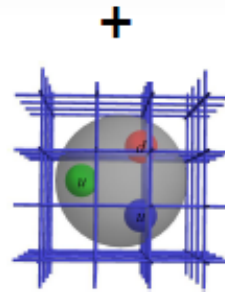
transversely polarized



§ Complement

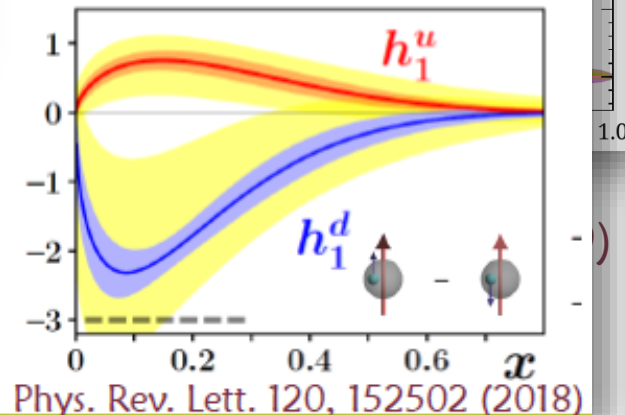
Theory
Input

Global Analysis
of PDFs



at PDFs

Work has been made in
this direction



Isvector PDFs Update

§ Nucleon isovector PDF calculated directly at **physical pion mass**

∞ NNLO matching & treat leading-renormalon effects



∞ Leading-renormalon resummation (LRR)

R. Zhang, et. al.

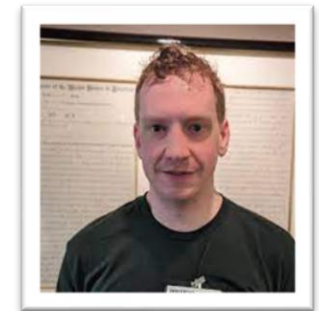
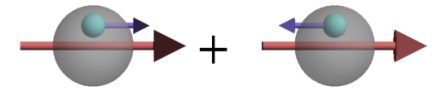
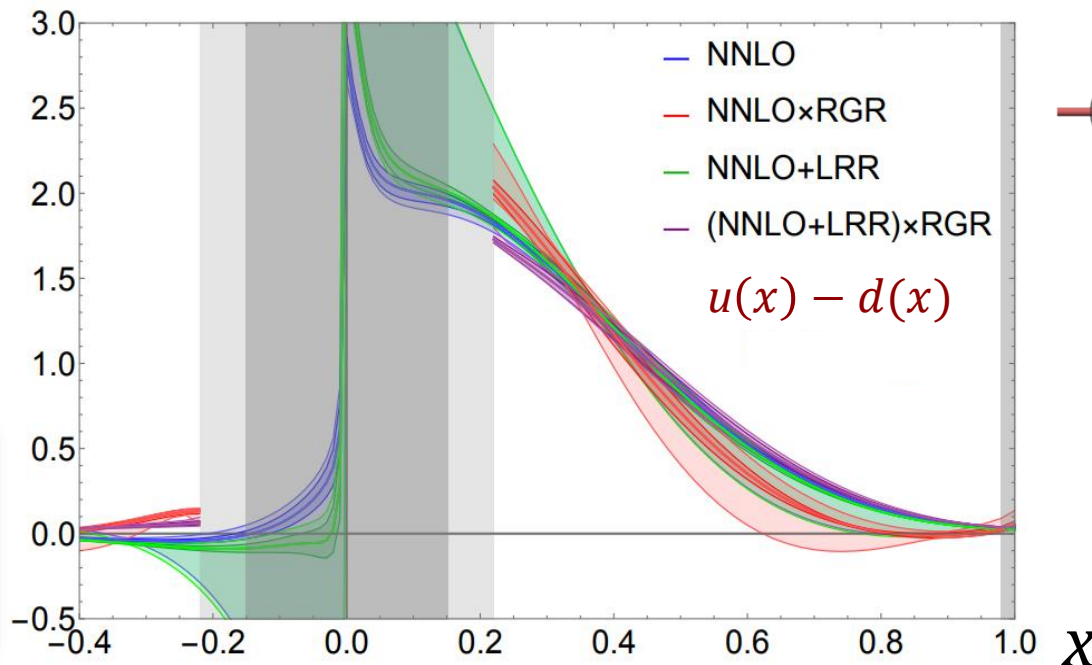
∞ Renormalization-group resummation (RGR)

PLB 844, 138081 (2023)

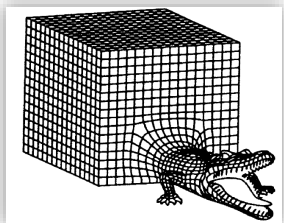
∞ $N_f = 2+1+1$ clover/HISQ, $a \approx 0.09$ fm, $P_z \approx 2$ GeV

J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

Wanted
PDFs,
GPDs,
etc...



P: Jack Holligan



Isvector PDFs Update

§ Nucleon isovector PDF calculated directly at **physical pion mass**

∞ NNLO matching & treat leading-renormalon effects



∞ Leading-renormalon resummation (LRR)

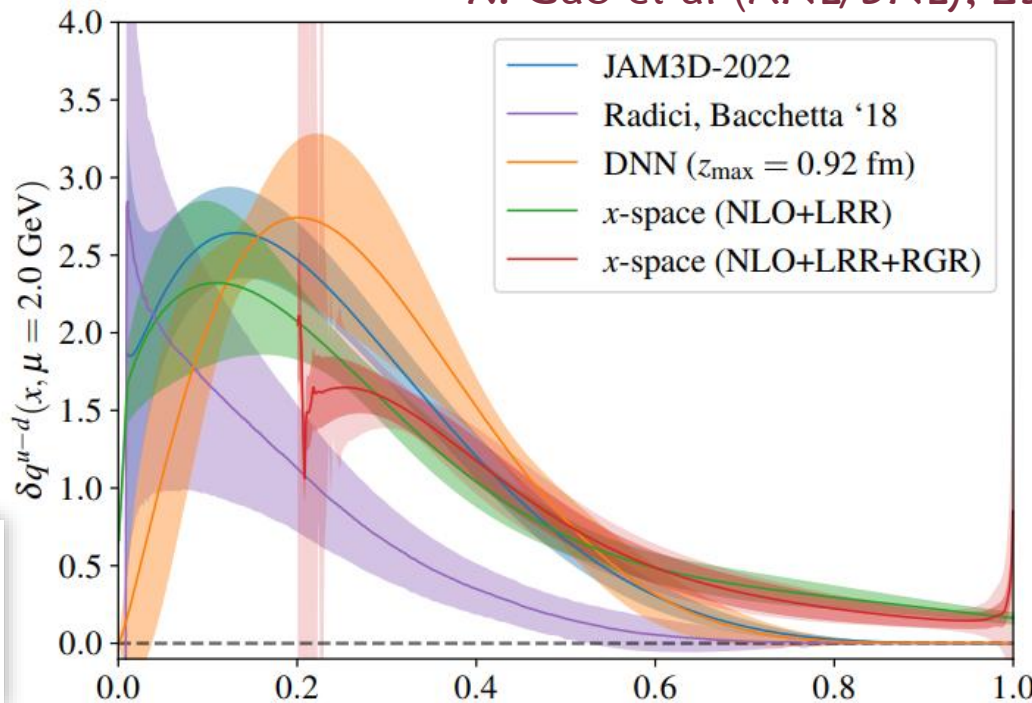
R. Zhang, et. al.

∞ Renormalization-group resummation (RGR)

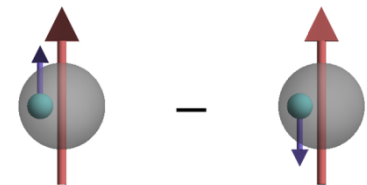
PLB 844, 138081 (2023)

∞ $N_f = 2+1$ clover/HISQ, $a \approx 0.076$ fm, $P_z \approx 1.5$ GeV

X. Gao et al (ANL/BNL), 2310.19047 [hep-lat]

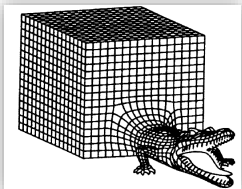


Transversity



$$\delta u(x) - \delta d(x)$$

Wanted
PDFs,
GPDs,
etc...



Continuum PDF

§ Nucleon PDFs using quasi-PDFs in the continuum limit

∞ Lattice details: clover/2+1 clover (LPC)

$a \approx \{0.49, 0.64, 0.85, 0.98\}$ fm,

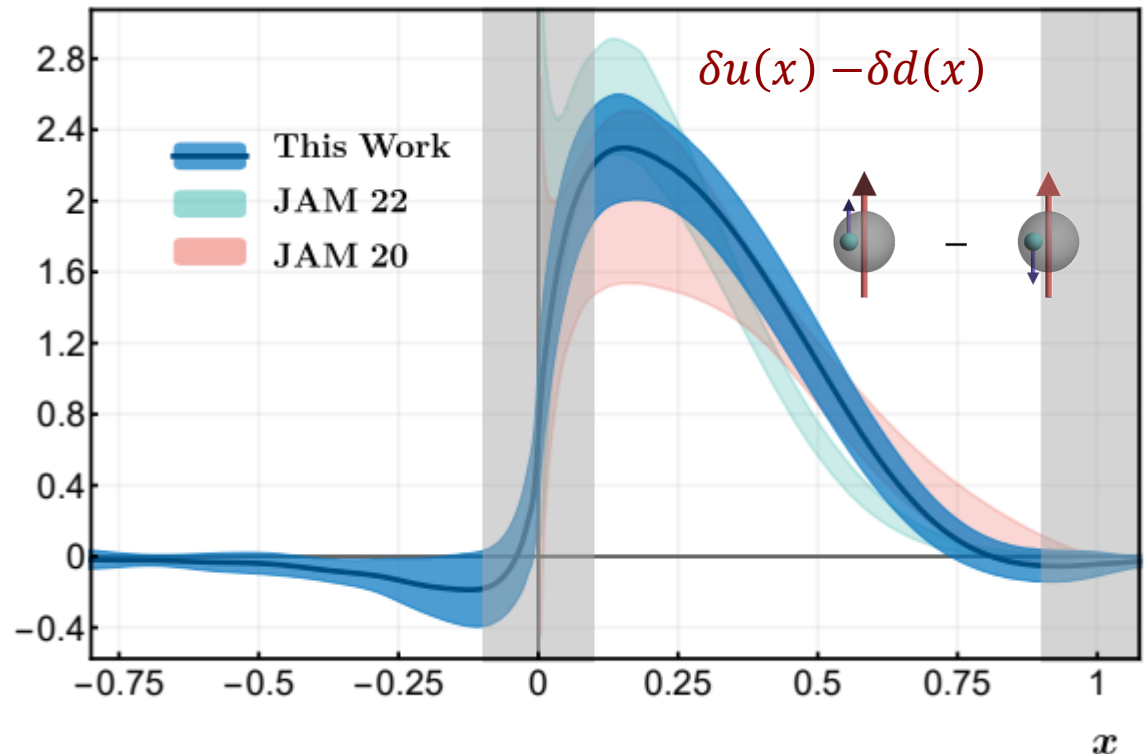
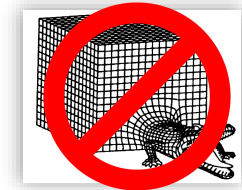
$M_\pi \in [222, 354]$ -MeV pion,

$M_\pi L \in [3.9, 8.1]$

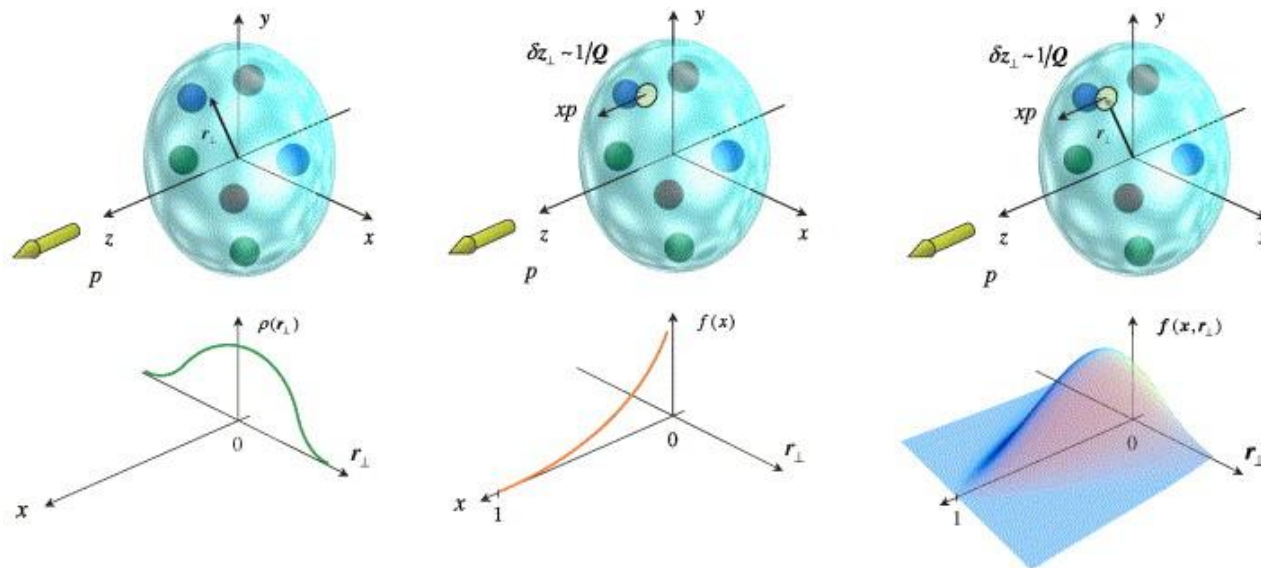
$P_z \in [1.8, 2.8]$

Fei Yao et al (LPC),
2208.08008

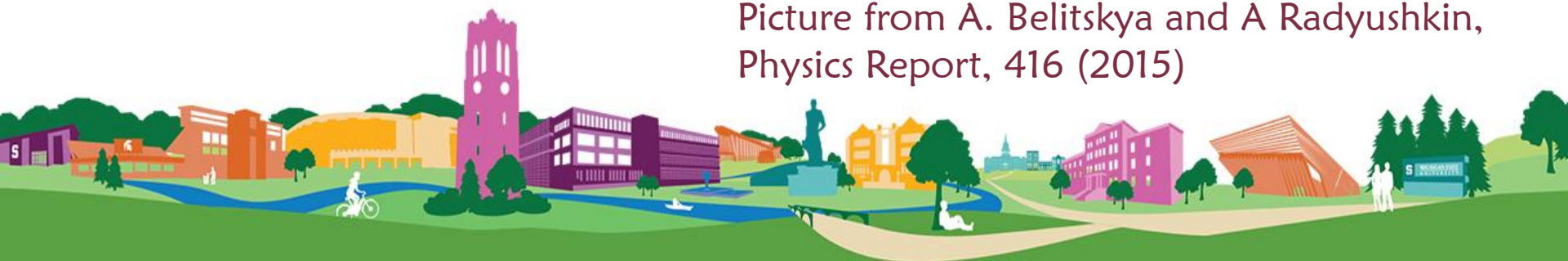
Wanted
PDFs, GPDs,
etc...



Bjorken- x Dependent GPDs

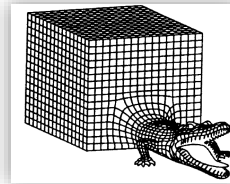


Picture from A. Belitskya and A Radyushkin,
Physics Report, 416 (2015)



Bjorken- x Dependent GPDs

Single-ensemble result



finite-volume,
discretization,
heavier quark mass,
Smaller boosted P_z ,

...



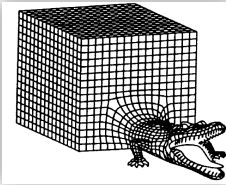
First Lattice GPDs

§ First glimpse into pion GPD using Quasi-PDF/LaMET

∞ Lattice details: clover/HISQ, **0.12fm**, **310-MeV** pion mass

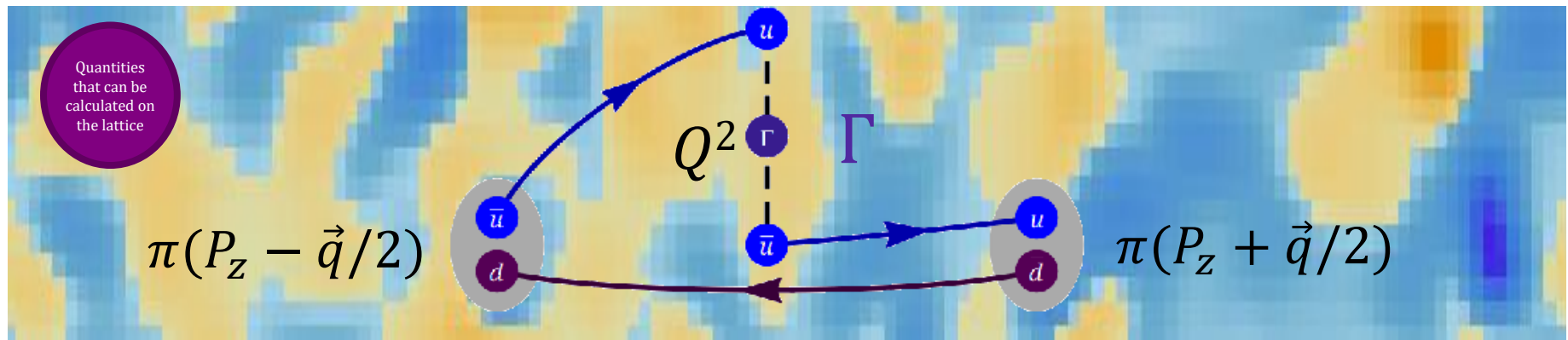
$$P_z \approx 1.6 \text{ GeV}$$

MILC, Phys. Rev. D, 82 (2010), 074501;
Phys. Rev. D, 87 (2013), 0545056



J. Chen, HL, J. Zhang, 1904.1237;

$$H_q^\pi(x, \xi, t, \mu) = \int \frac{d\eta^-}{4\pi} e^{-ix\eta^- P^+} \left\langle \pi(P + \Delta/2) \left| \bar{q} \left(\frac{\eta^-}{2} \right) \gamma^+ \Gamma \left(\frac{\eta^-}{2}, -\frac{\eta^-}{2} \right) q \left(-\frac{\eta^-}{2} \right) \right| \pi(P - \Delta/2) \right\rangle$$



First Lattice GPDs

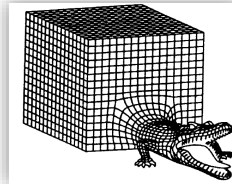
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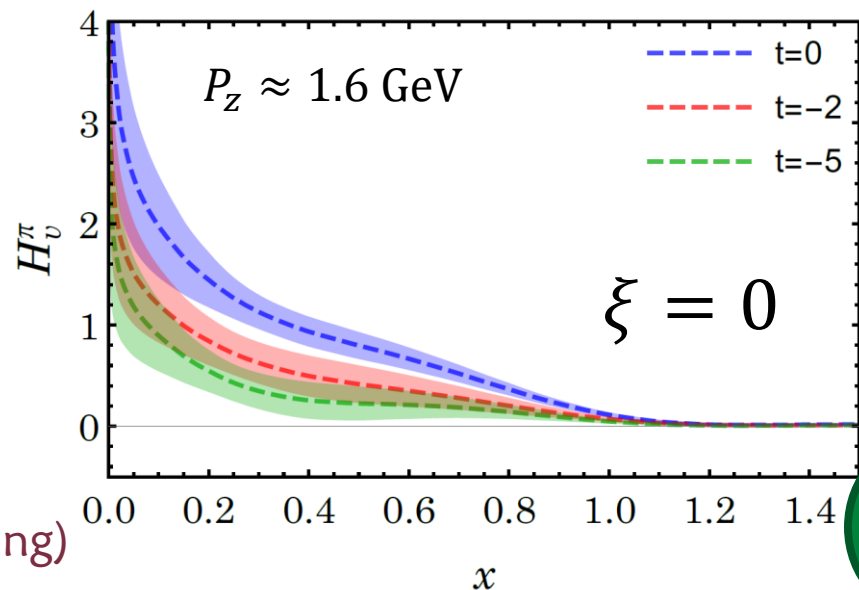
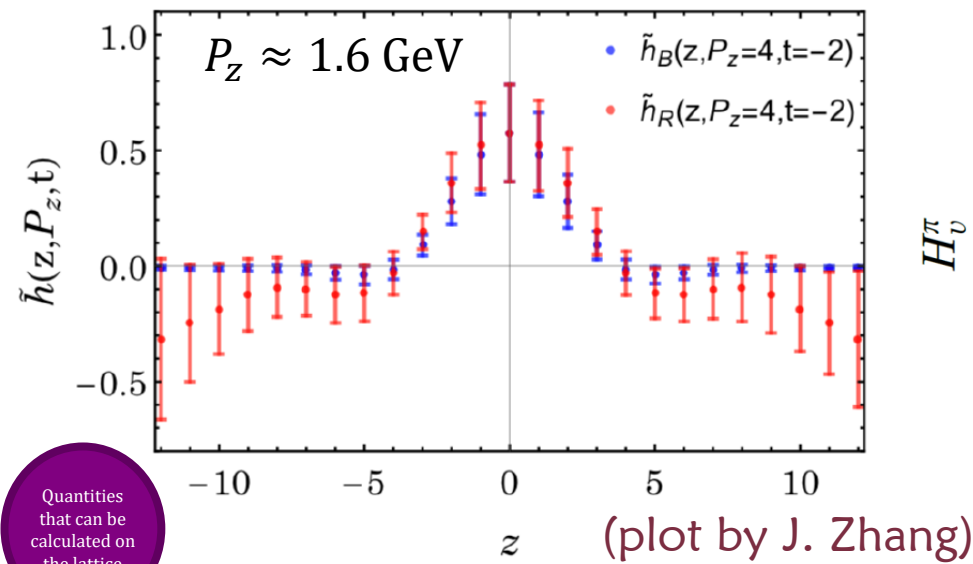
MILC, Phys. Rev. D, 82 (2010), 074501;

Phys. Rev. D, 87 (2013), 0545056



J. Chen, HL, J. Zhang, 1904.1237;

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Quantities
that can be
calculated on
the lattice

Wanted
PDFs,
GPDs,
etc...

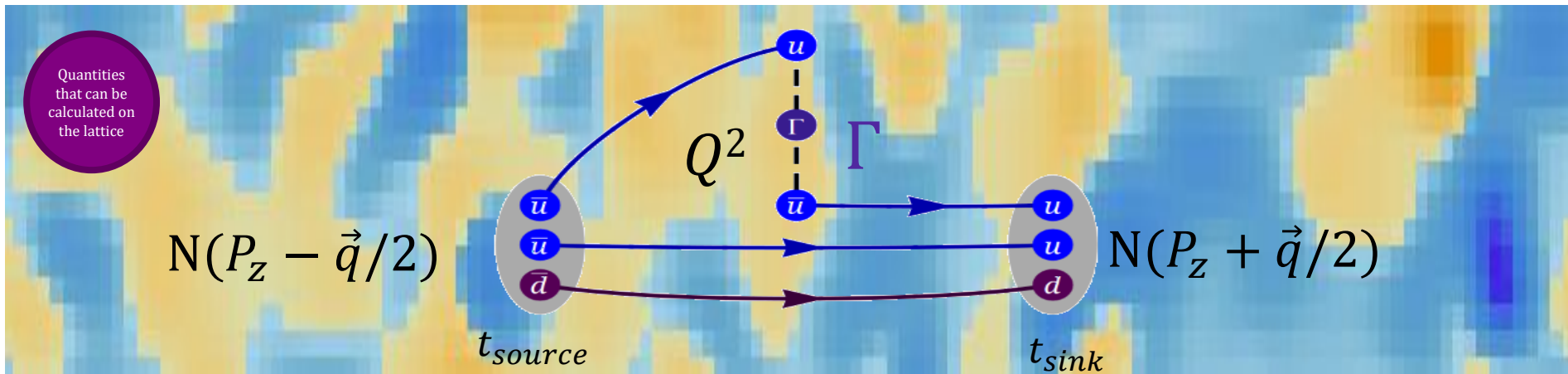
Isvector Nucleon GPDs

§ Nucleon GPD using quasi-PDFs at physical pion mass

⌘ MSULat: clover/2+1+1 HISQ

0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV

⌘ $\xi = 0$ isovector nucleon GPD results



$$\tilde{F}(x, \xi, t, \bar{P}_Z) = \frac{\bar{P}_Z}{\bar{P}_0} \int \frac{dz}{4\pi} e^{ixz\bar{P}_Z} \langle P' | \tilde{O}_{\gamma_0}(z) | P \rangle = \frac{\bar{u}(P')}{2\bar{P}^0} \left(H(x, \xi, t, \bar{P}_Z) \gamma^0 + E(x, \xi, t, \bar{P}_Z) \frac{i\sigma^{0\mu} \Delta_\mu}{2M} \right) u(P'')$$

$$p^\mu = \frac{p''^\mu + p'^\mu}{2}, \quad \Delta^\mu = p''^\mu - p'^\mu, \quad t = \Delta^2, \quad \xi = \frac{p''^+ - p'^+}{p''^+ + p'^+}$$

HL, Phys.Rev.Lett. 127 (2021) 18, 182001

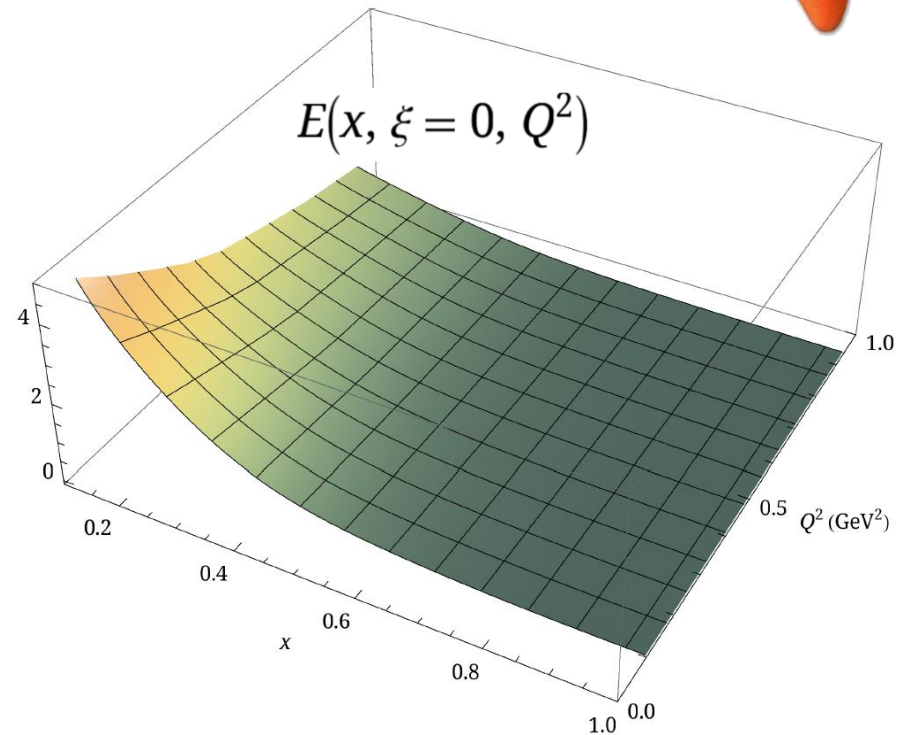
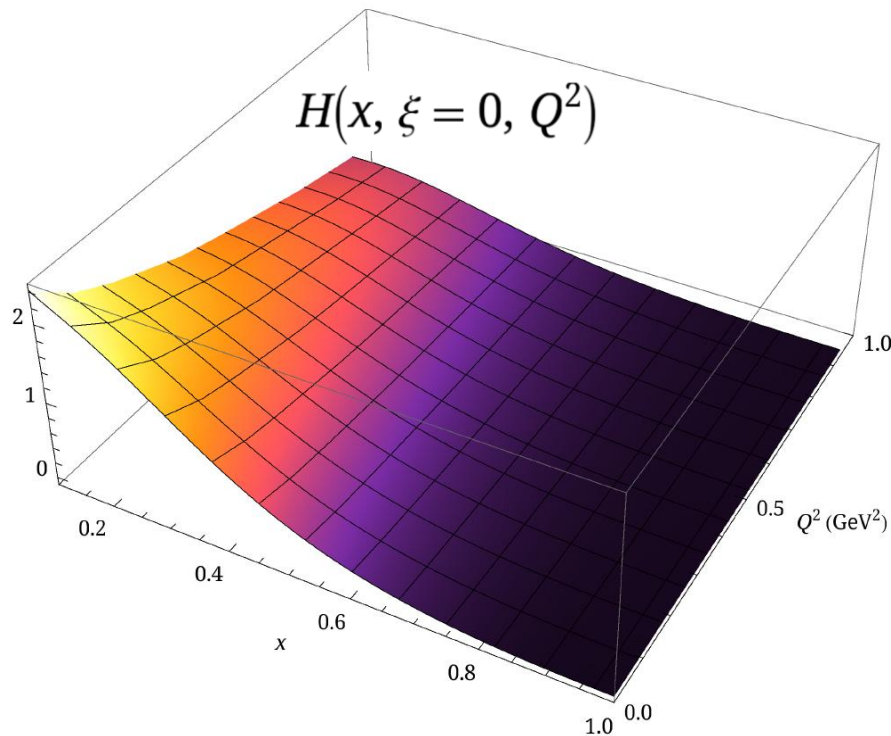
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HL, Phys.Rev.Lett. 127 (2021) 18, 182001

Isvector Nucleon GPDs

§ Nucleon GPD using quasi-PDFs at physical pion mass

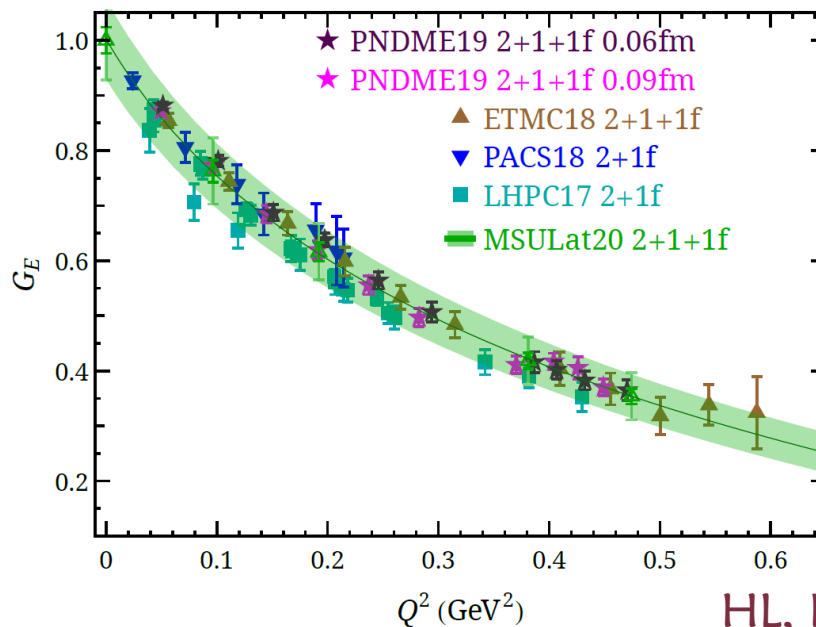
⌘ Lattice details: clover/2+1+1 HISQ (MSULat)

0.09 fm, **135-MeV** pion mass, $P_z \approx 2$ GeV

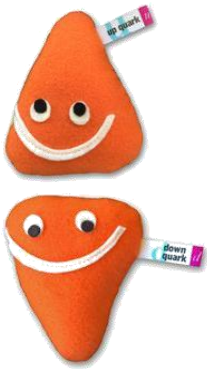
⌘ $\xi = 0$ isovector nucleon GPD results

$$\int_{-1}^{+1} dx x^{n-1} = \sum_{i=0, \text{even}}^{n-1} (-2\xi)^i A_{ni}^q(t) + (-2\xi)^n C_{n0}^q(t) \Big|_{n \text{ even}}$$

$n = 1$



**Checked for the
very first time on
the lattice!**



HL, Phys.Rev.Lett. 127 (2021) 18, 182001

Isvector Nucleon GPDs

§ Nucleon GPD using quasi-PDFs at physical pion mass

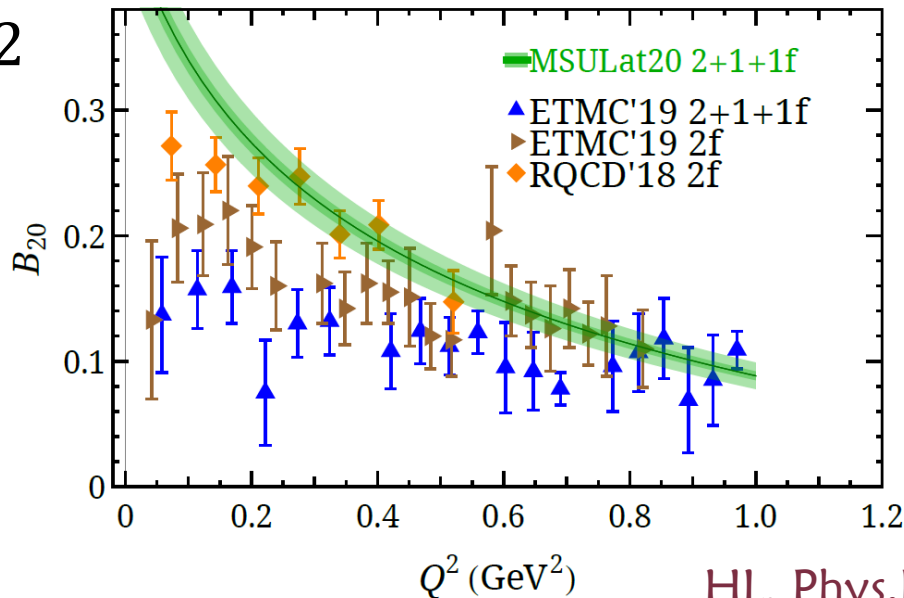
⌘ Lattice details: clover/2+1+1 HISQ (MSULat)

0.09 fm, **135-MeV** pion mass, $P_z \approx 2$ GeV

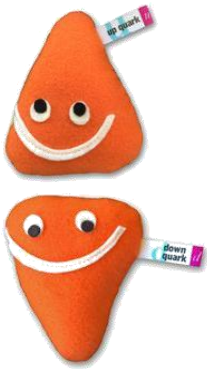
⌘ $\xi = 0$ isovector nucleon GPD results

$$\int_{-1}^{+1} dx x^{n-1} = \sum_{i=0, \text{even}}^{n-1} (-2\xi)^i B_{ni}^q(t) - (-2\xi)^n C_{n0}^q(t) \Big|_{n \text{ even}}$$

$n = 2$



**Checked for the
very first time on
the lattice!**

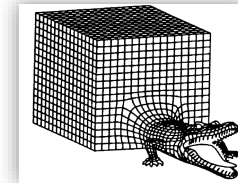


HL, Phys.Rev.Lett. 127 (2021) 18, 182001

First $\mathcal{LQCD}$ Tomography

§ Nucleon GPD using quasi-PDFs at physical pion mass

- ∞ Lattice details: clover/2+1+1 HISQ
0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV
- ∞ $\xi = 0$ isovector nucleon GPD results

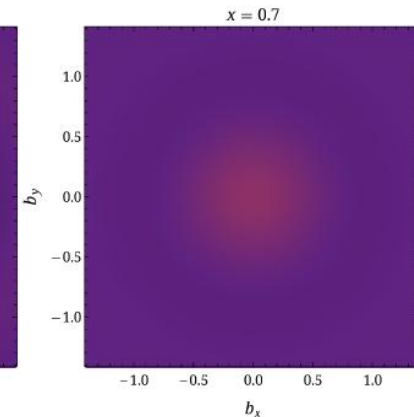
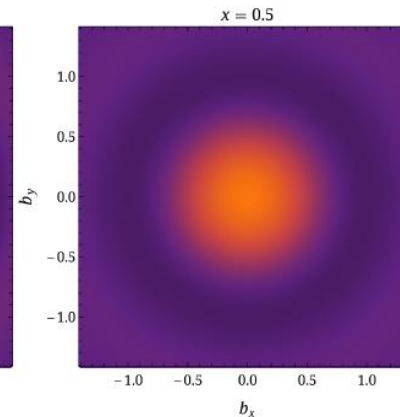
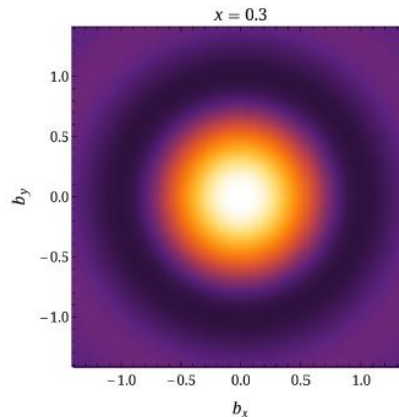
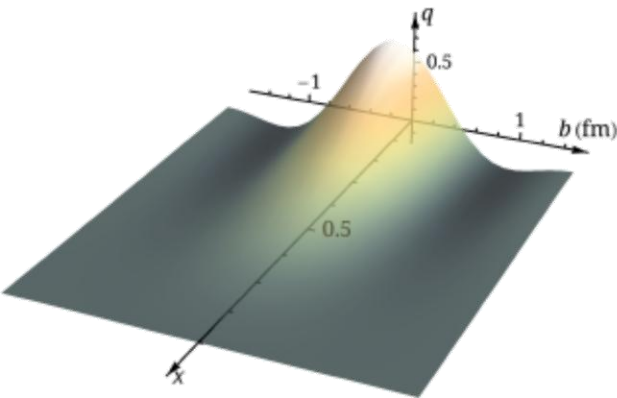


finite-volume,
discretization,

$$q(x, b) = \int \frac{d\vec{q}}{(2\pi)^2} H(x, \xi = 0, t = -\vec{q}^2) e^{i\vec{q} \cdot \vec{b}}$$



HL, Phys. Rev. Lett. 127 (2021) 18, 182001



§ Nucleon helicity GPD ($\tilde{H}$) and pion GPD (H^π) using quasi-PDFs at physical pion mass

HL (MSULat), Phys.Lett.B 824 (2022) 136821;
Phys. Lett. B 846 (2023) 138181

Tomography @ Physical Pion Mass

§ Lots of progress on tomography by many collaborations

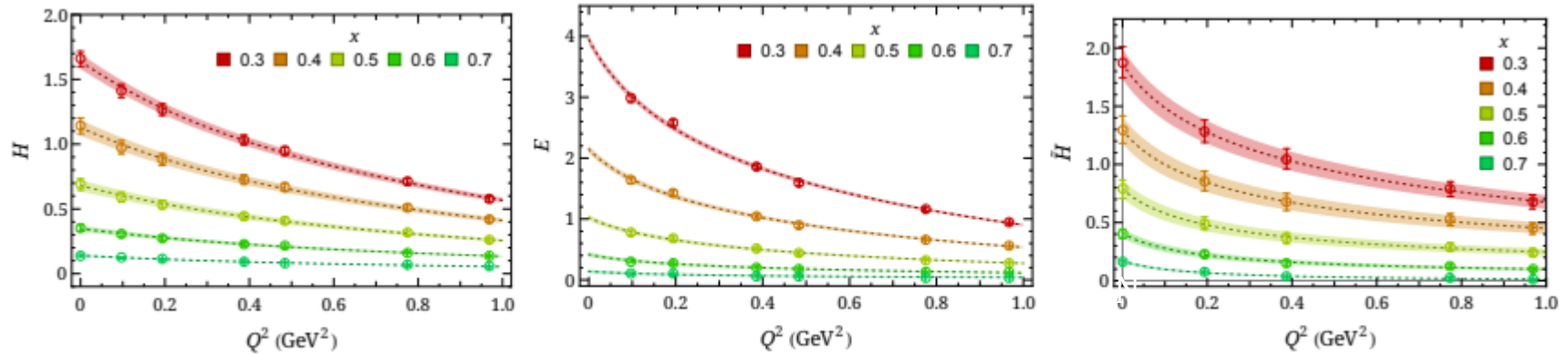


Figure 4.1: Nucleon isovector H (left), E (middle) and $\tilde{H}$ (right) GPDs at $\xi = 0$ with z -expansion to Q^2 at selected x values.

HL, Prog.Part.Nucl.Phys. 144 (2025) 104177

Tomography @ Physical Pion Mass

§ Lots of

borrowings

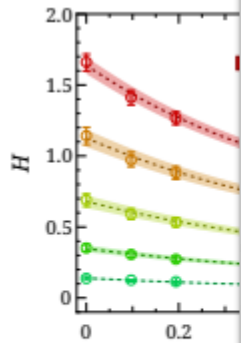
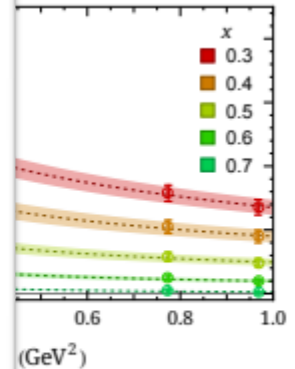
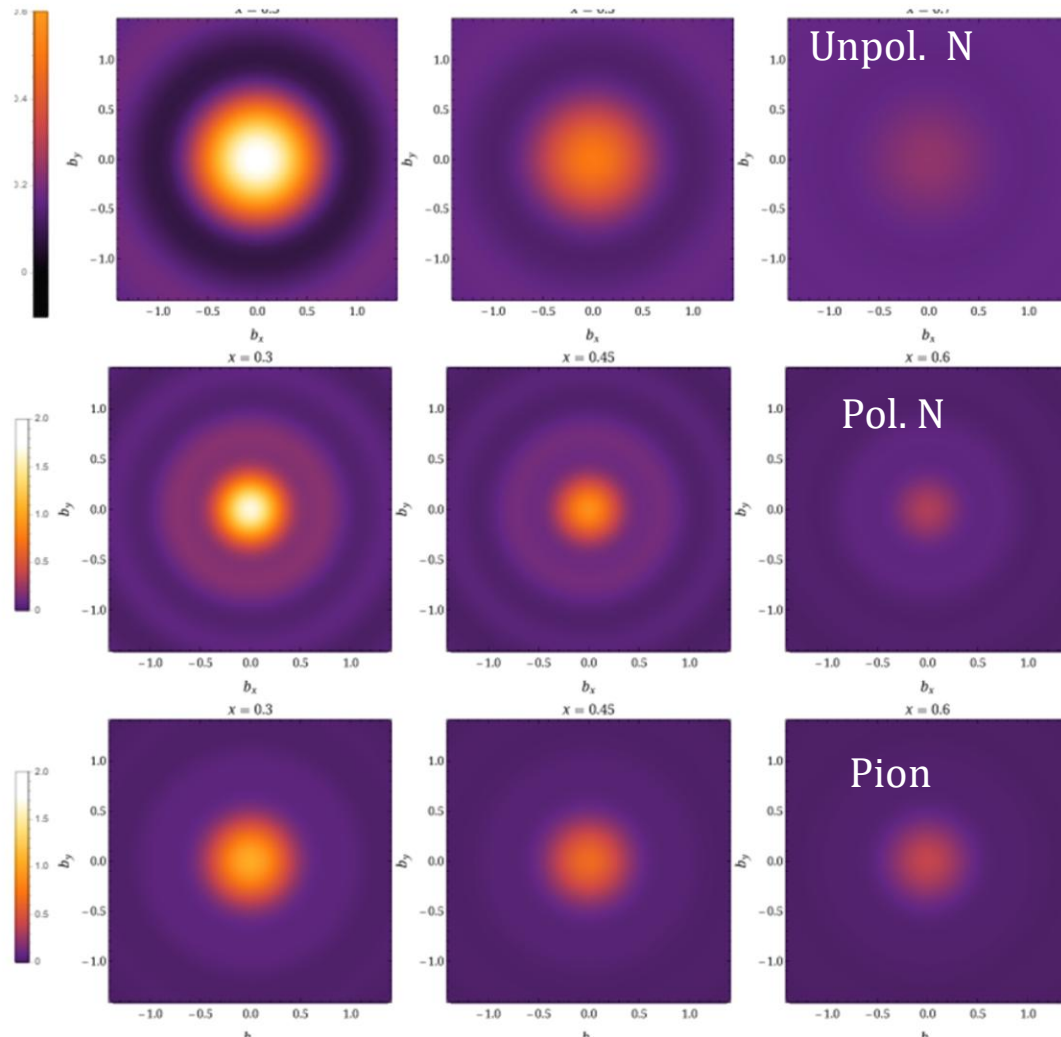


Figure 4.1:



selected x values.

HL, Prog.Part.Nucl.Phys. 144 (2025) 104177

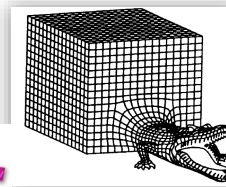
Asymmetric-Frame GPD

§ New calculations by ANL/BNL/ETMC using asymmetric frame

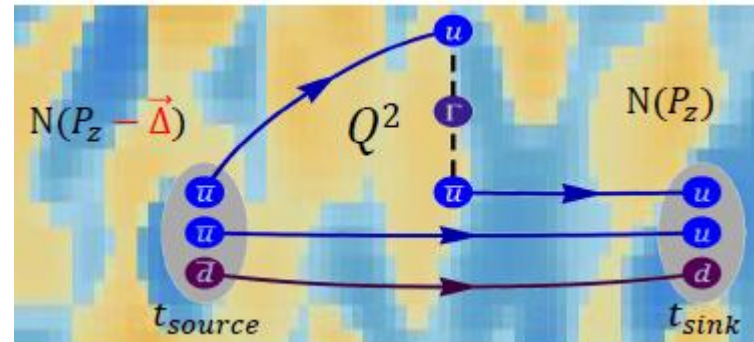
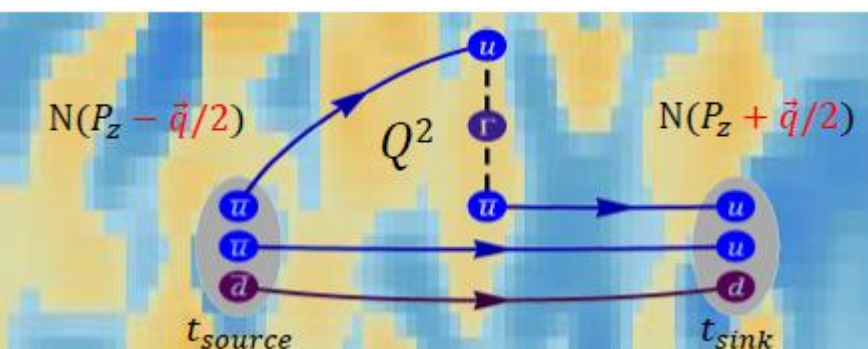
2+1+1 twisted-Wilson, 0.09 fm, $P_z \approx 1.3$ GeV

260-MeV pion, one source-sink used

$\xi = 0$ isovector nucleon GPD results



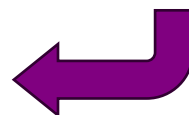
finite-volume,
discretization,
heavy quark,
excited-state, ...



$$\tilde{F}(x, \xi, t, \bar{P}_Z) = \frac{\bar{u}(P')}{2\bar{P}^0} \left(\mathbf{H}(x, \xi, t, \bar{P}_Z) \gamma^0 + \mathbf{E}(x, \xi, t, \bar{P}_Z) \frac{i\sigma^{0\mu} \Delta_\mu}{2M} \right) u(P'')$$

$$F^\mu(z, P, \Delta) = \bar{u}(p_f, \lambda') \left[\frac{P^\mu}{m} \mathbf{A}_1 + m z^\mu \mathbf{A}_2 + \frac{\Delta^\mu}{m} \mathbf{A}_3 + i m \sigma^{\mu z} \mathbf{A}_4 + \frac{i \sigma^{\mu \Delta}}{m} \mathbf{A}_5 + \frac{P^\mu i \sigma^{z \Delta}}{m} \mathbf{A}_6 + m z^\mu i \sigma^{z \Delta} \mathbf{A}_7 + \frac{\Delta^\mu i \sigma^{z \Delta}}{m} \mathbf{A}_8 \right] u(p_i, \lambda)$$

$$H(z \cdot P^{s/a}, z \cdot \Delta^{s/a}, (\Delta^{s/a})^2) = A_1 + \frac{\Delta^{s/a} \cdot z}{P^{s/a} \cdot z} A_3$$



ANL/BNL/ETMC, [2209.05373](#), [2310.13114](#)

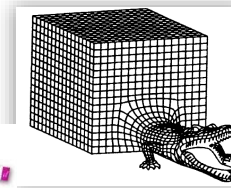
Asymmetric-Frame GPD

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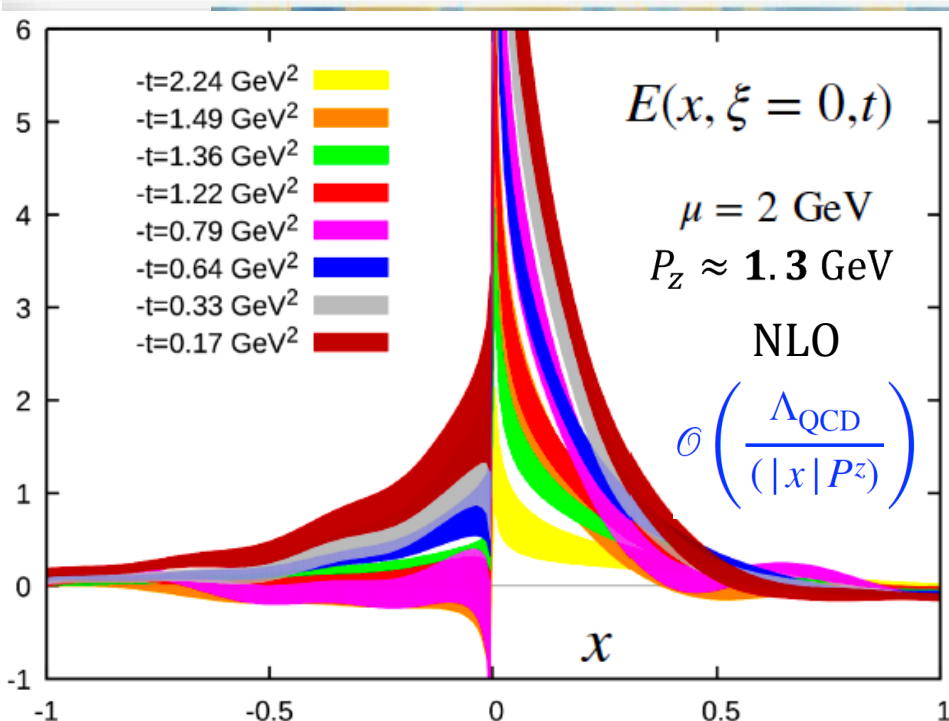
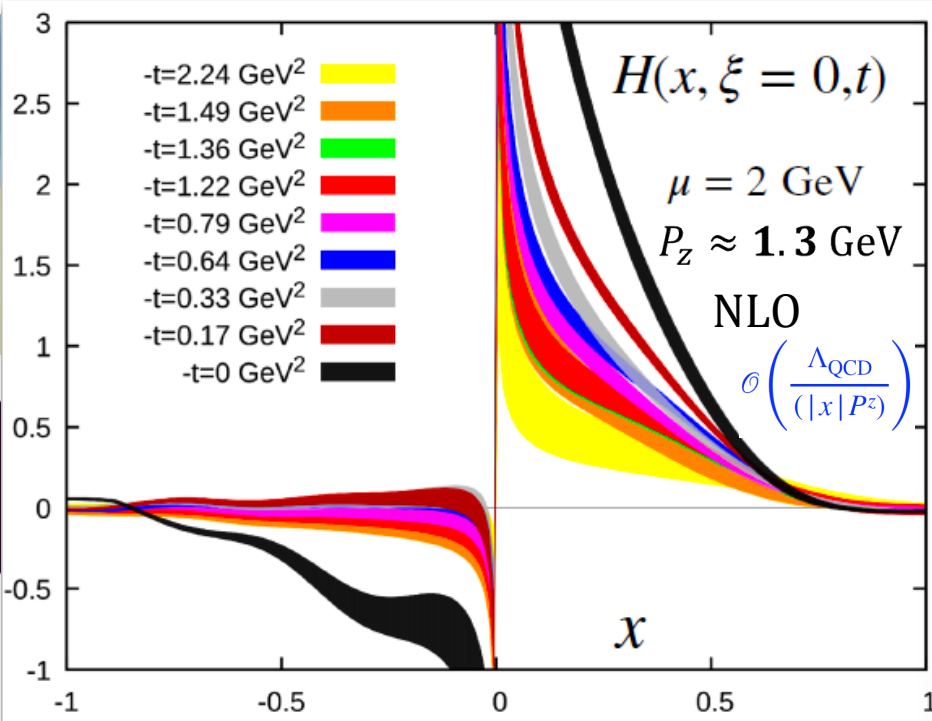
⌘ 2+1+1 twisted-Wilson, 0.09 fm, $P_z \approx 1.3$ GeV

260-MeV pion, one source-sink used

⌘ $\xi = 0$ isovector nucleon GPD results



finite-volume,
discretization,
heavy quark,
excited-state, ...



See Joshua Miller talk next

ANL/BNL/ETMC, [2209.05373](#), [2310.13114](#)

GPD Systematic Update

§ Nucleon isovector GPDs calculated directly at physical pion mass

∞ NNLO matching & treat leading-renormalon effects



∞ Leading-renormalon resummation (LRR)

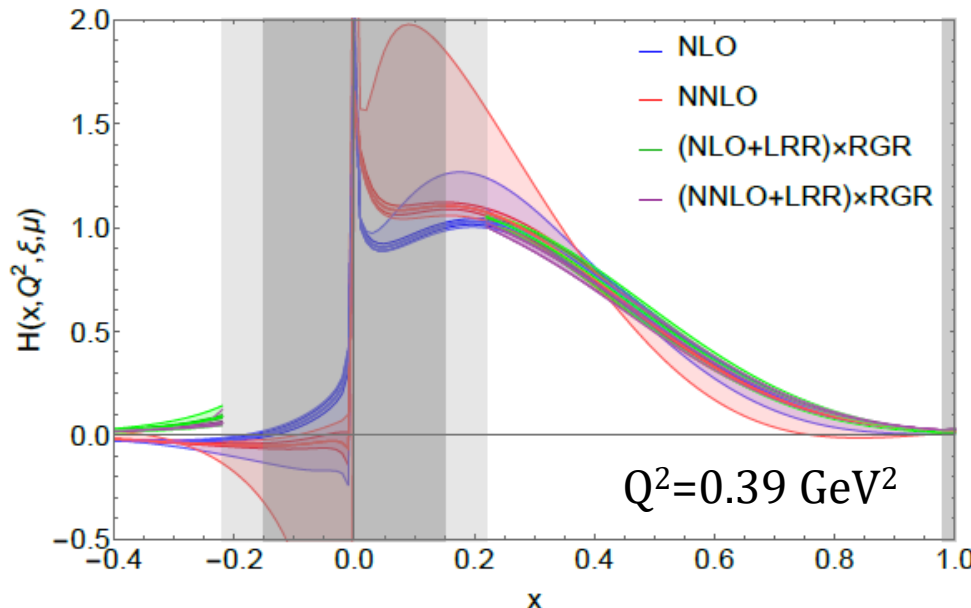
R. Zhang, et. al.

∞ Renormalization-group resummation (RGR)

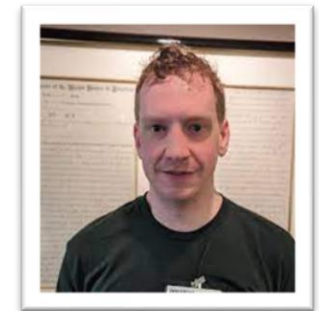
PLB 844, 138081 (2023)

∞ $N_f = 2+1+1$ clover/HISQ, $a \approx 0.09$ fm, 135-MeV pion, $P_z \approx 2$ GeV
J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

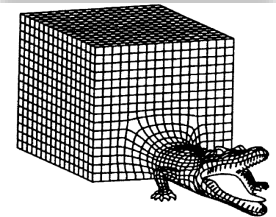
Wanted
PDFs,
GPDs,
etc...



$$\mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{(xP_z)^2}, \frac{\Lambda_{\text{QCD}}^2}{((1-x)P_z)^2}\right)$$



P: Jack Holligan



GPD Systematic Update

§ Nucleon isovector GPDs & pion valence-quark GPDs

⌘ NNLO matching & treat leading-renormalon effects



⌘ Leading-renormalon resummation (LRR)

R. Zhang, et. al.

⌘ Renormalization-group resummation (RGR)

PLB 844, 138081 (2023)

⌘ $N_f = 2+1+1$ clover/HISQ, $a \approx 0.09$ fm,
135-MeV pion, $P_z \approx 2$ GeV

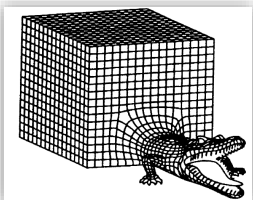
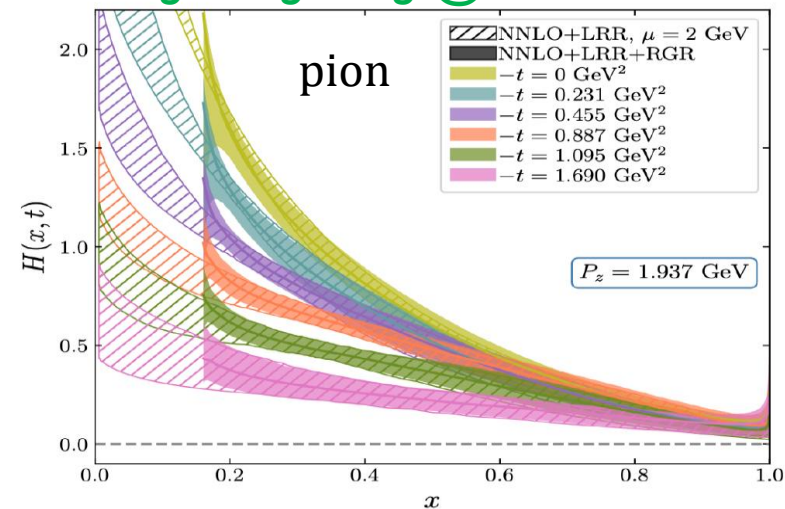
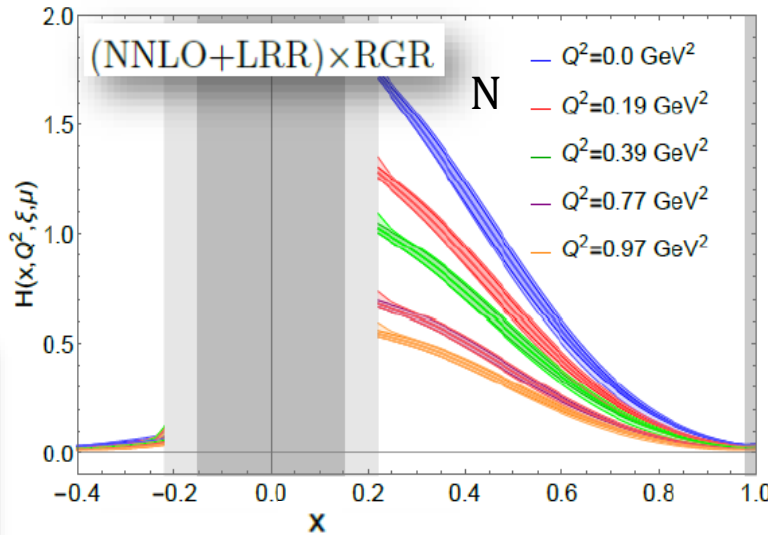
J. Holligan, HL (MSULat),
2312.10829 [hep-lat]

⌘ $N_f = 2+1$ clover/HISQ, $a \approx 0.09$ fm,
300-MeV pion, $P_z \approx 2$ GeV

H. Ding et al (ANL/BNL/Wuhan),
2407.03516 [hep-lat]

Heng-Tong Ding @ Tue. 9:30 AM

Wanted
PDFs,
GPDs,
etc...



GPD Systematic Update

§ Nucleon isovector GPDs & pion valence-quark GPDs

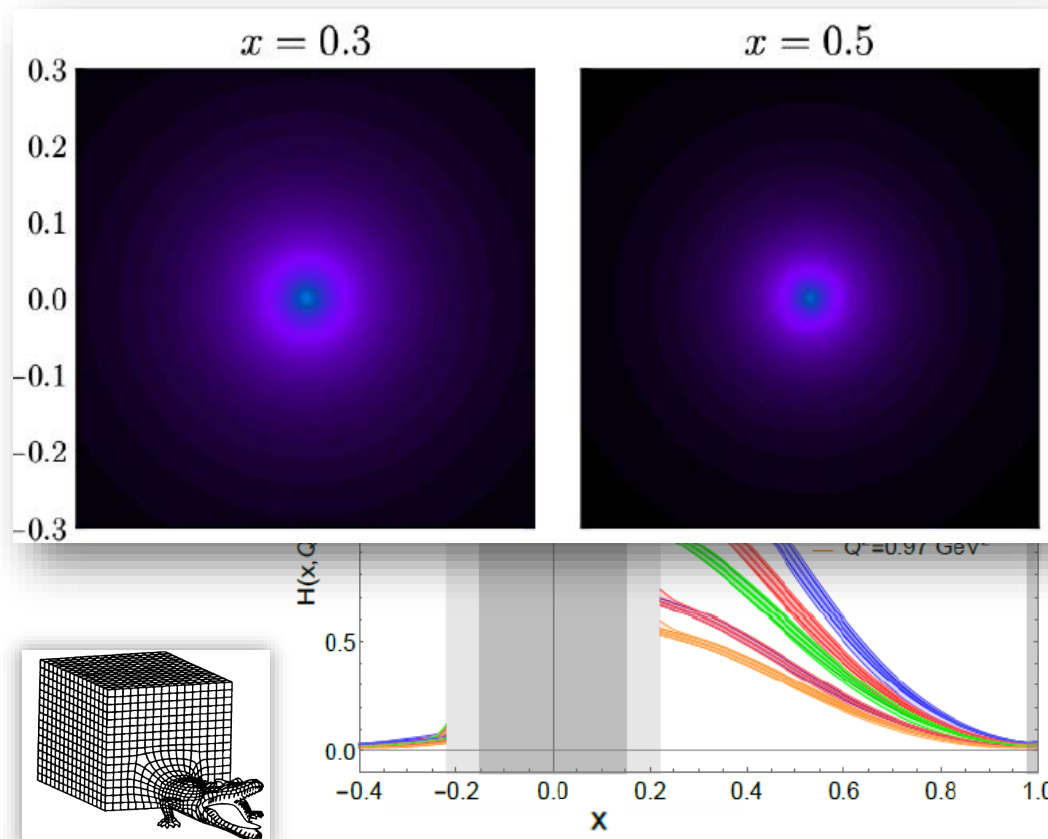
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∞ Leading-renormalon resummation (LRR)

∞ Renormalization-group resummation (RGR)

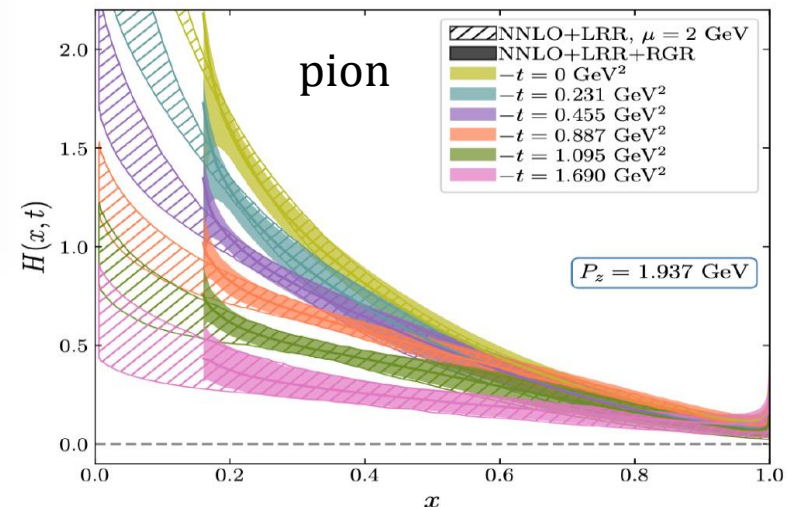
R. Zhang, et. al.

PLB 844, 138081 (2023)



• $N_f = 2+1$ clover/HISQ, $a \approx 0.09$ fm,
300-MeV pion, $P_z \approx 2$ GeV

H. Ding et al (ANL/BNL/Wuhan),
2407.03516 [hep-lat]



$\xi \neq 0$ GPDs

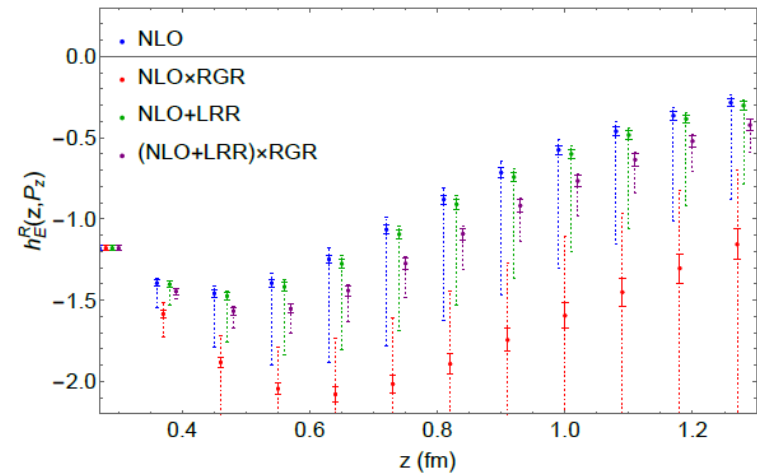
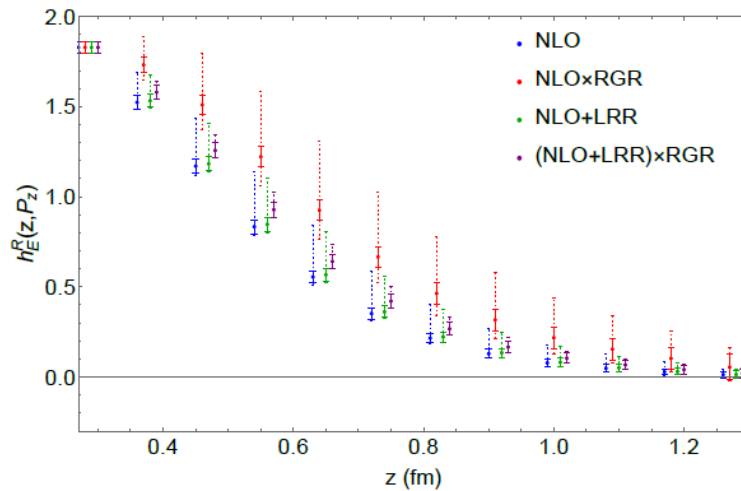
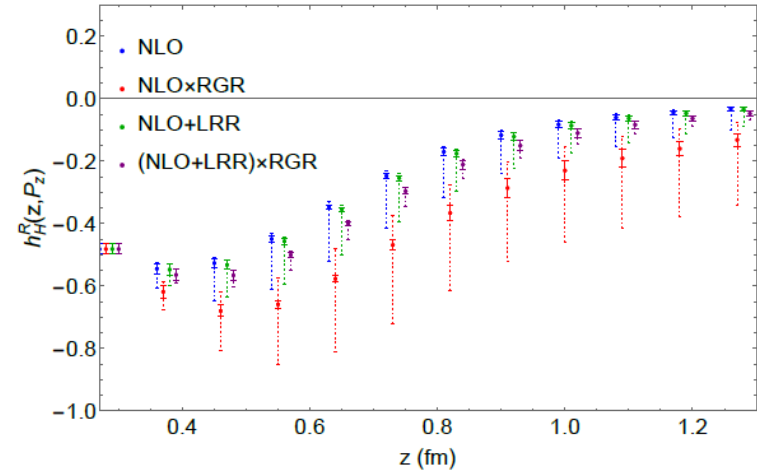
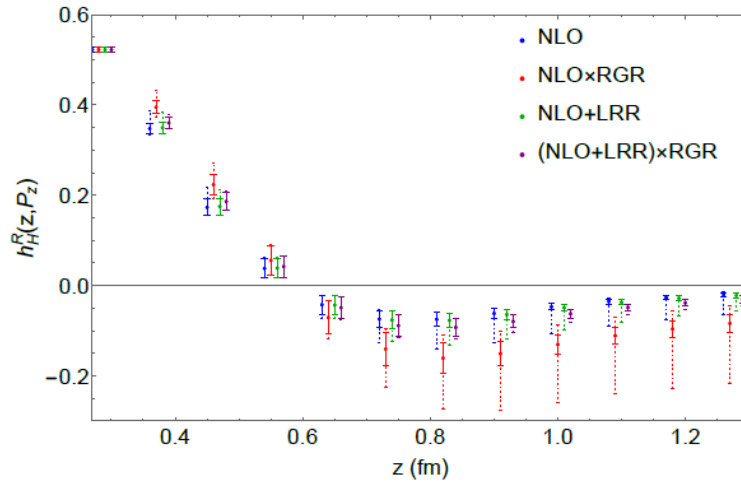
§ Only the NLO matching kernel is available

$$\begin{aligned} & \mathcal{K}(x, y, \mu, \xi, P_z) \\ &= \delta(x - y) \\ &+ \frac{\alpha_s C_F}{4\pi} \left[\left(\frac{|\xi + x|}{2\xi(\xi + y)} + \frac{|\xi + x|}{(\xi + y)(y - x)} \right) \left(\ln \left(\frac{4y^2(\xi + x)^2 P_z^2}{\mu^2} \right) - 1 \right) \right. \\ &+ \left(\frac{|\xi - x|}{2\xi(\xi - y)} + \frac{|\xi - x|}{(\xi - y)(x - y)} \right) \left(\ln \left(\frac{4y^2(\xi - x)^2 P_z^2}{\mu^2} \right) - 1 \right) \\ &\left. + \left(\left(\frac{\xi + x}{\xi + y} + \frac{\xi - x}{\xi - y} \right) \frac{1}{|x - y|} - \frac{|x - y|}{\xi^2 - y^2} \right) \left(\ln \left(\frac{4y^2(x - y)^2 P_z^2}{\mu^2} \right) - 1 \right) \right] \end{aligned}$$

Fei Yao, Yao Ji & Jian-Hui Zhang, JHEP 11(2023) 021

$\xi \neq 0$ GPDs

§ Physical pion mass; NLO $P_z \approx 2$ GeV, $\xi = 0.1$, $Q^2 = 0.23$ GeV²

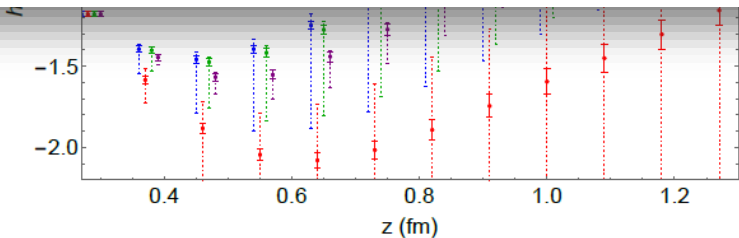
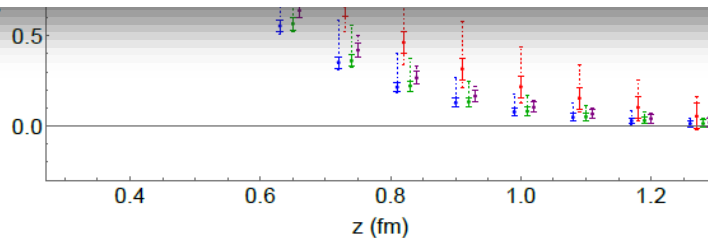
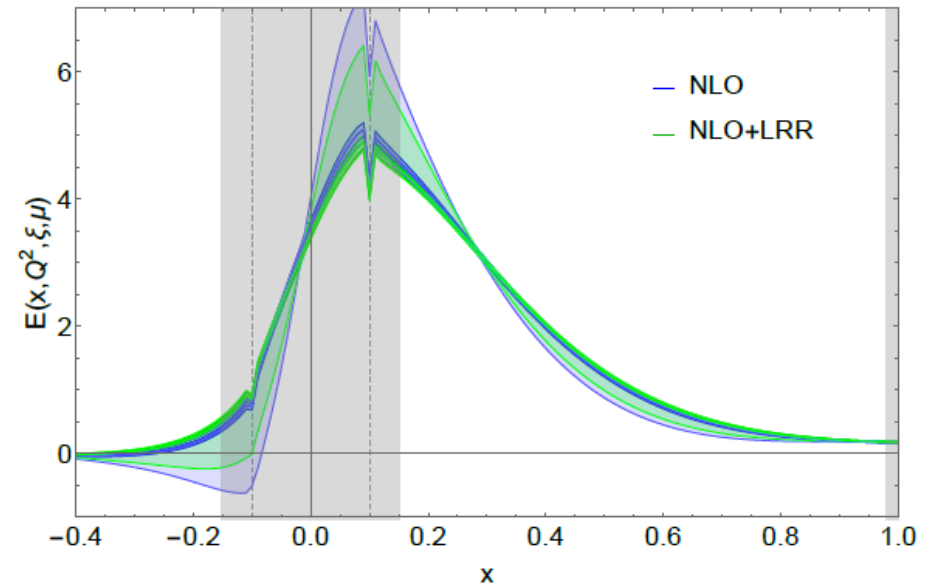
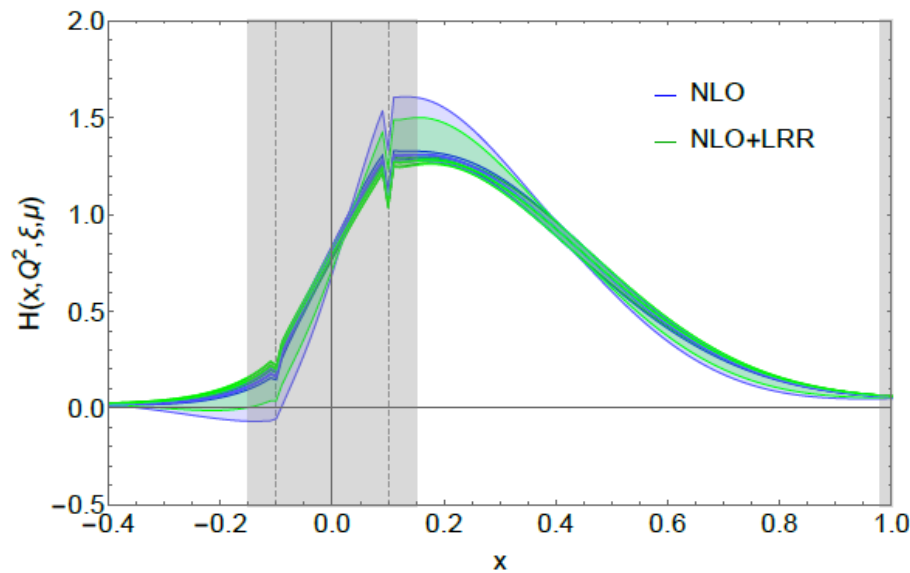


Quantities
that can be
calculated on
the lattice

J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

$\xi \neq 0$ GPDs

§ Physical pion mass; NLO $P_z \approx 2$ GeV, $\xi = 0.1$, $Q^2 = 0.23$ GeV²



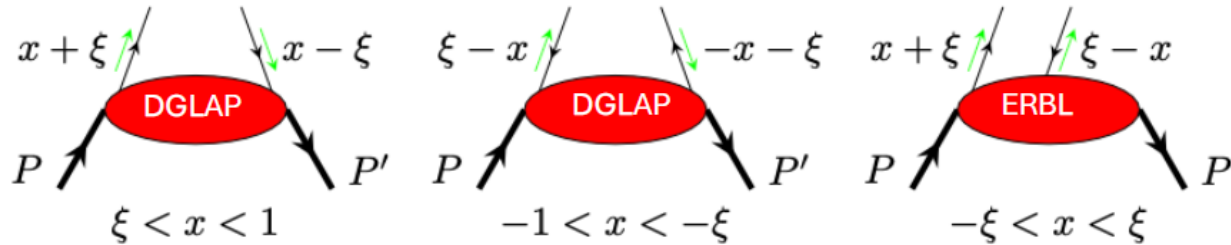
J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

Large logarithms in Matching

J. Holligan, HL, Rui Zhang, Y. Zhao, JHEP 207 (2025) 241



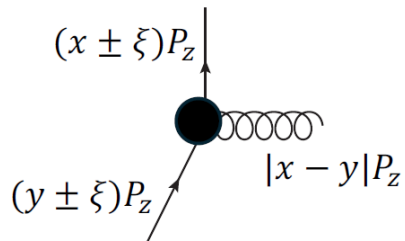
Slide by Rui Zhang (MIT)



Difficulty

Multi-scale nature

- Outgoing quark $(x + \xi)P_z$
- Incoming quark $(x - \xi)P_z$
- Gluon emission $|x - y|P_z$



Three different logarithms cannot be resummed with only one RG equation

Irrelevant logarithms

Quark log can be suppressed

- $\frac{|\xi+x|}{(\xi+y)} \ln \left(\frac{4(\xi+x)^2 P_z^2}{\mu^2} \right) \xrightarrow{\xi+x \rightarrow 0} 0$
- $\frac{|\xi-x|}{(\xi+y)} \ln \left(\frac{4(\xi-x)^2 P_z^2}{\mu^2} \right) \xrightarrow{\xi-x \rightarrow 0} 0$

ERBL region:

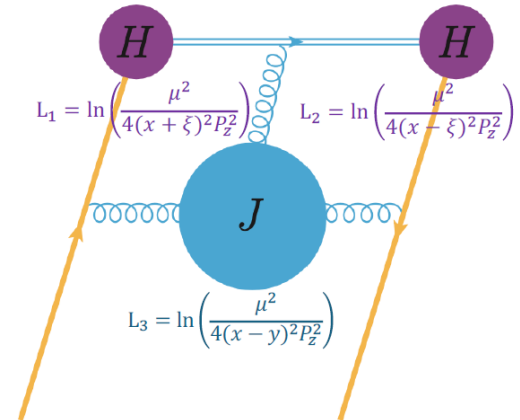
Only in the threshold limit (soft gluon emission $x \rightarrow y$), the log terms diverge

DGLAP region:

Logs diverge when $\xi \rightarrow 0$, but same as above when $\mu = 2xP_z$

Threshold Factorization

$$C(x, y, \xi, P_z) \xrightarrow{x \rightarrow y} J(|x - y|P_z) \otimes H((x \pm \xi)P_z)$$



Threshold log: soft gluon $x \rightarrow y$

Large logarithms in Matching

§ Application on toy GPD model

Slide by Rui Zhang (MIT)

- Initial matching

- $C_{\text{DGLAP}}(\mu) = \hat{V}_C(2xP, \mu)C(2xP)$
- $C_{\text{ERBL}}(\mu) = \hat{V}_C(2\xi P, \mu)C(2\xi P)$

- Separate all scales

- $C \xrightarrow{x \rightarrow y} H \otimes J$ (or $J \otimes H$)

- Solving RG equations

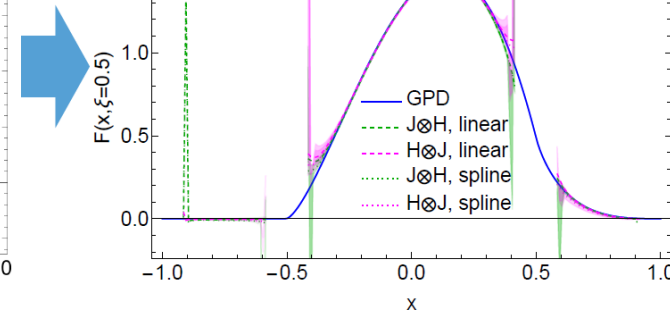
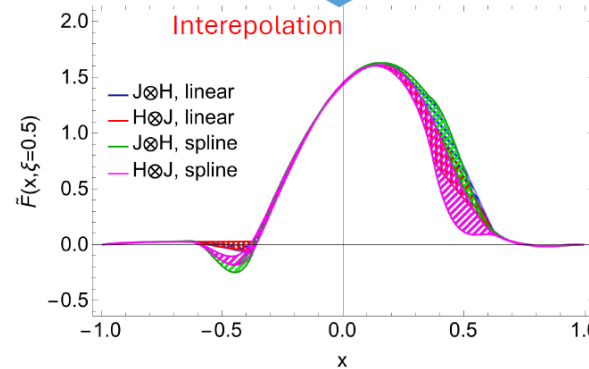
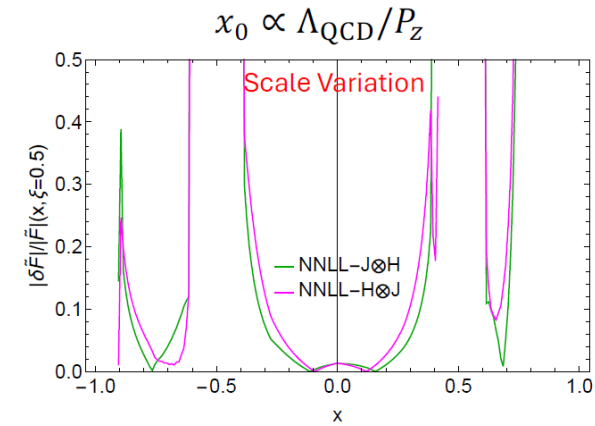
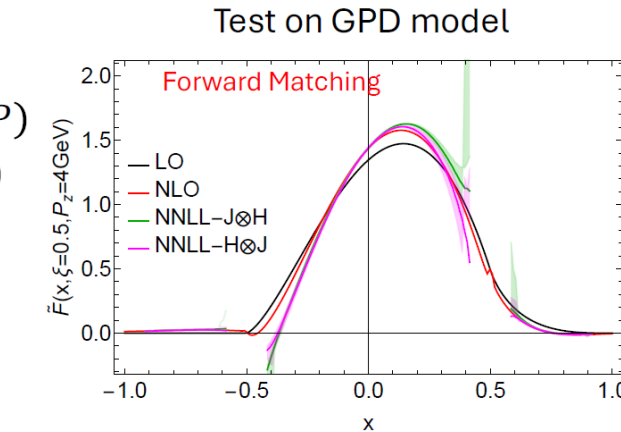
- $H_{\text{TR}}(\mu) = \hat{V}_H(\mu_h, \mu)H(\mu_h)$
- $J_{\text{TR}}(\mu) = \hat{V}_J(\mu_i, \mu)J(\mu_i)$

- Identifying physical scales

- $\mu_{h_1} = 2|\xi + x|P$
- $\mu_{h_2} = 2|\xi - x|P$
- $\mu_i = 2 \min\{|x|, |\xi \pm x|\}P$

- Correct Full Matching

$$C_{\text{TR}} = (H \otimes J)_{\text{TR}} \otimes (H \otimes J)_{\text{NLO}}^{-1} \otimes C_{\text{NLO}}$$

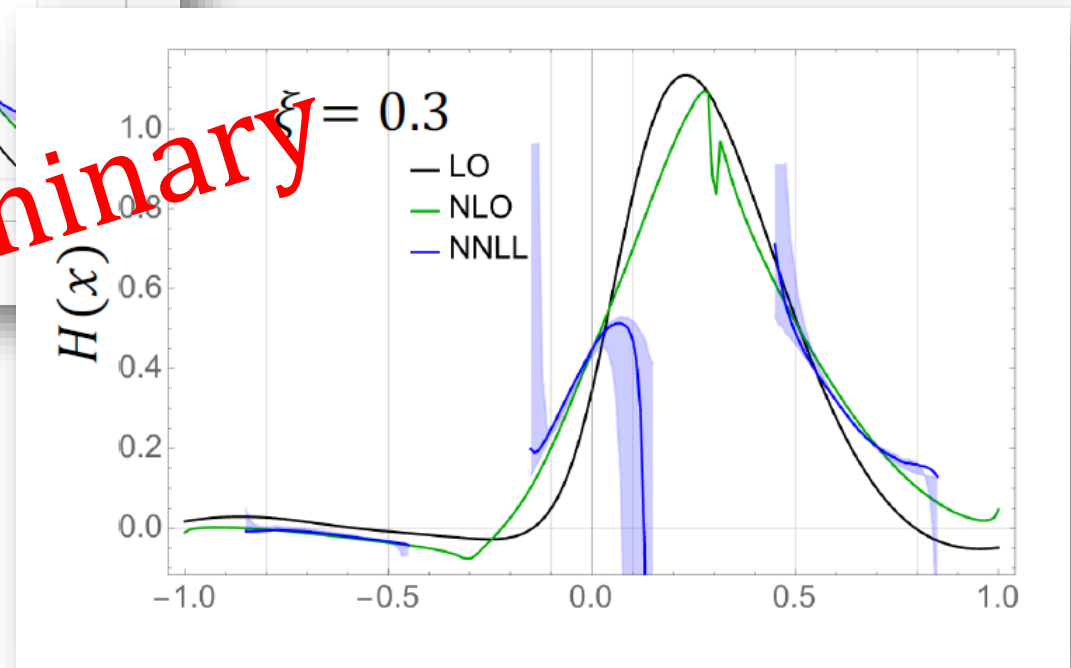
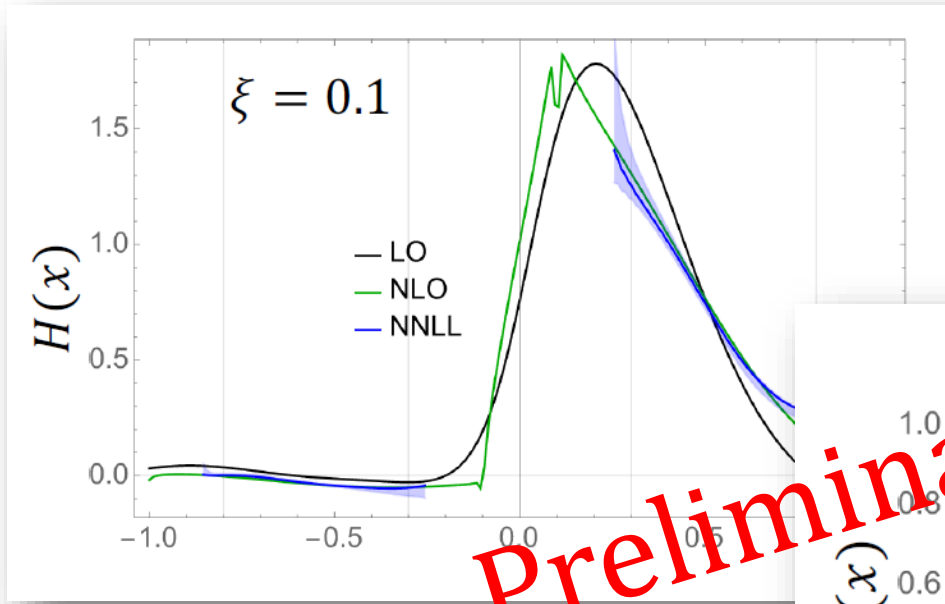


J. Holligan, HL, Rui Zhang, Y. Zhao, JHEP 207 (2025) 241

Large logarithms in Matching

§ First application to lattice data at physical pion mass

plots by Rui Zhang (MIT)



Lattice Progress & Challenges

§ Not enough time to cover the following

- ⌘ Nucleon isovector form factors and generalized form factors (GFFs) using pseudo-PDF and traditional OPE
- ⌘ (LO) Twist-3 GPDs by ETMC

§ Challenges ahead for precision PDFs/GPDs

- ⌘ Large momentum P_z is essential
 - ⌘ With sufficient statistics nucleons may reach 5 GeV
- ⌘ Large non-zero skewness without dropping P_z
- ⌘ Methods for signal-to-noise improvement
 - ⌘ Gluonic observables, new ideas for large momentum
- ⌘ Access small- x physics; some methods have inverse problem in PDF extraction, more computational resources, etc.

For more details and references, refer to 2202.07193

Summary and Outlook

§ Exciting era using LQCD to study nucleon and pion GPDs

⌘ Overcoming longstanding obstacles: Bjorken- x dependent GPDs

§ Much progress in GPDs on the lattice

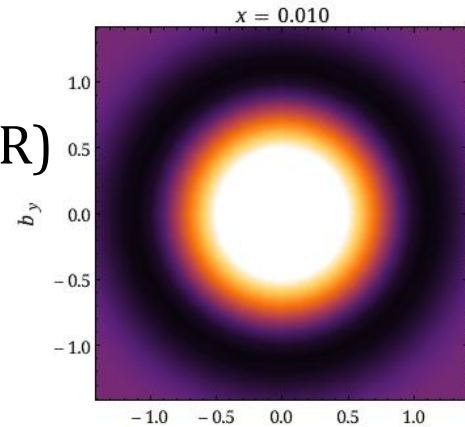
⌘ Widely studied with LaMET and its variants

⌘ Improvements: NNLO GPDs and systematics (LRR+RGR)

⌘ Recent progress made in nonzero-skewness GPDs

§ Precision and progress are limited by resources

⌘ Challenges = new opportunities



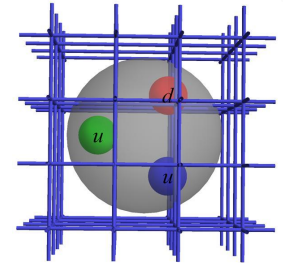
**Theory
Input**



**Global Analysis
of GPDs**

**Exp't
Input**

+



EXCLAIM

Thanks to MILC collaboration for sharing their 2+1+1 HISQ lattices & USQCD/NSF/DOE for computational resources
The work of HL is sponsored by NSF grant PHY 2209424 & 1653405, DOE grant DE-SC0024053 & RCSA Cottrell Scholar Award

Students Wanted

LGT4HEP website: <https://lgt4hep.github.io/>



High Energy Physics Computing Traineeship for Lattice Gauge Theory

Apply now:

Visit lgt4hep.github.io to learn more and where to apply for the traineeship graduate school program.



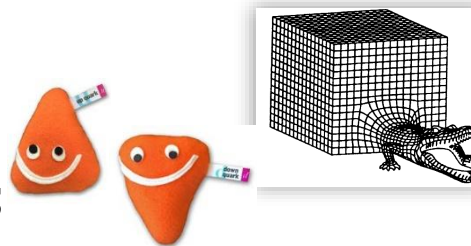
Backup Slides



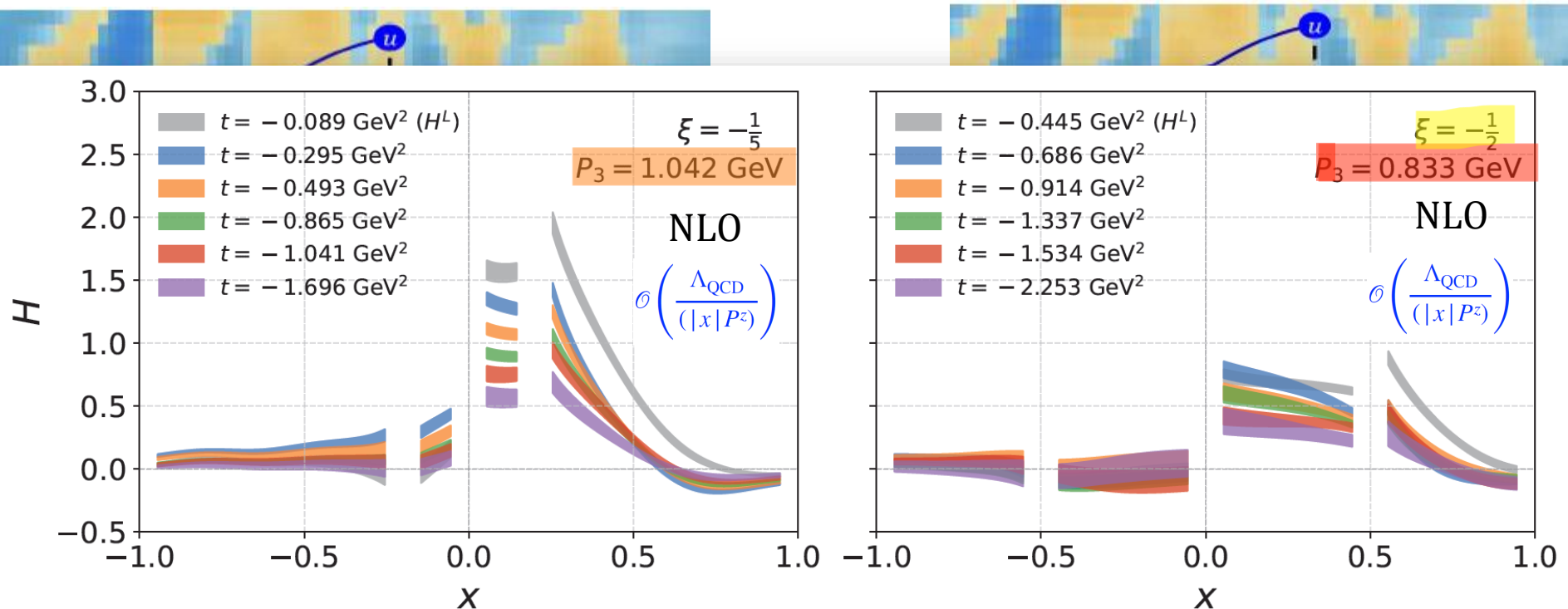
Asymmetric-Frame GPD

§ New calculations by ETMC using asymmetric frame

- 2+1+1 twisted-Wilson, 0.09 fm,
- 260-MeV** pion, one source-sink used
- $\xi \neq 0$ isovector nucleon GPD results



finite-volume,
discretization,
heavy quark,
excited-state, ...



ETMC, [2508.17998](https://arxiv.org/abs/2508.17998)

Nucleon Polarized GPDs

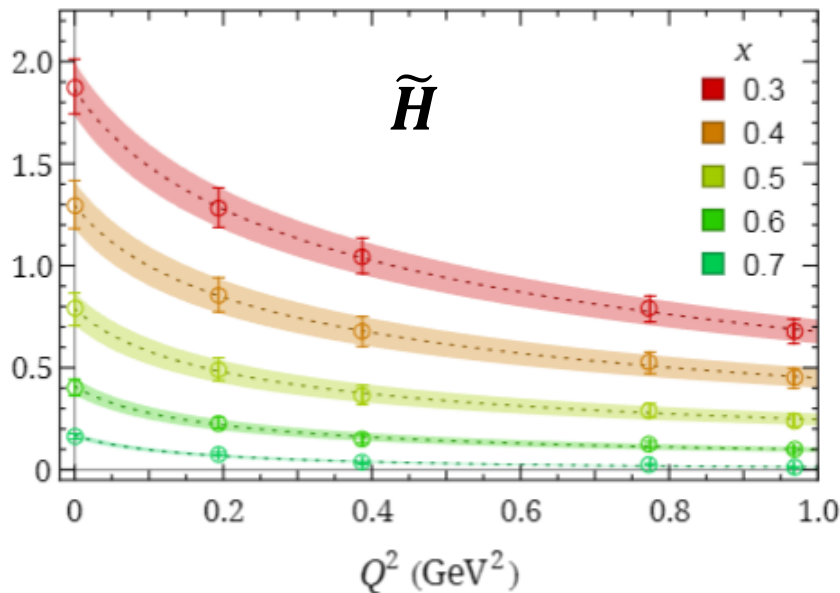
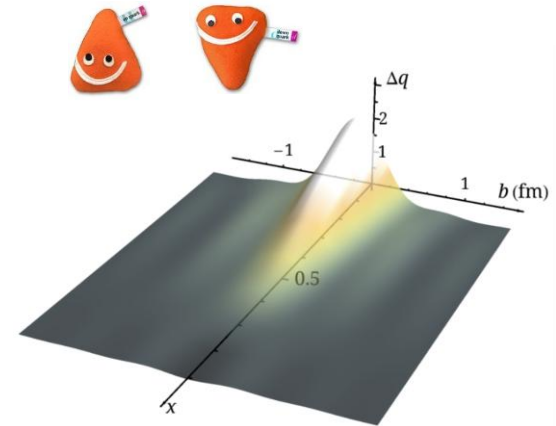
§ Helicity GPD ($\tilde{H}$) using quasi-PDFs at physical pion mass

⌘ MSULat: clover/2+1+1 HISQ

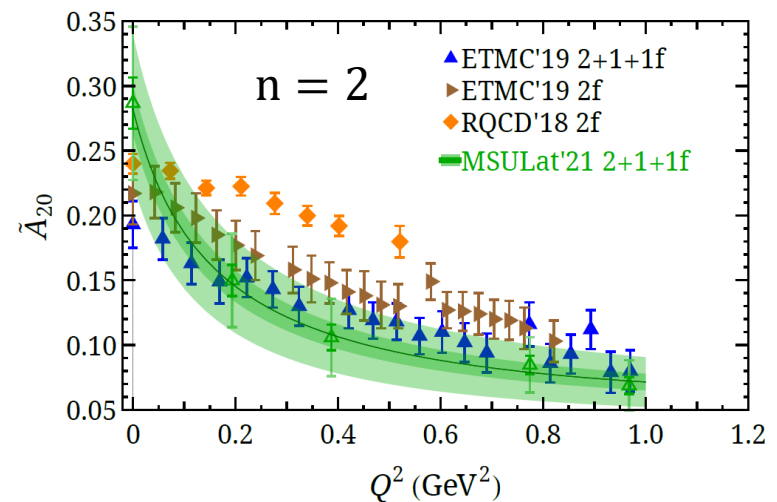
0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV

⌘ $\xi = 0$ isovector nucleon (quasi-)GPD results

HL (MSULat), Phys.Lett.B 824 (2022) 136821



⌘ Take the integral to form moments



Hadron Tomography

§ Lots of progress on tomography by many collaborations

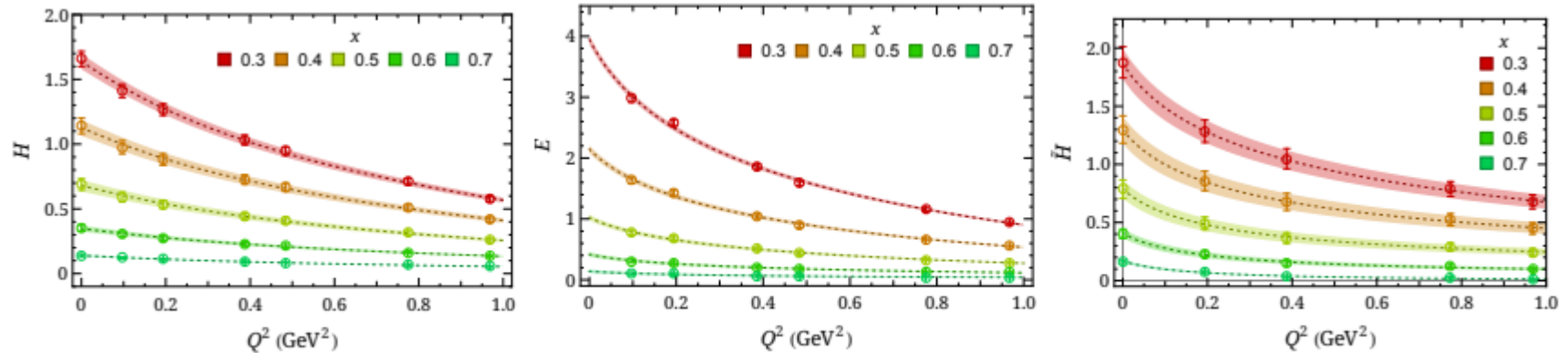
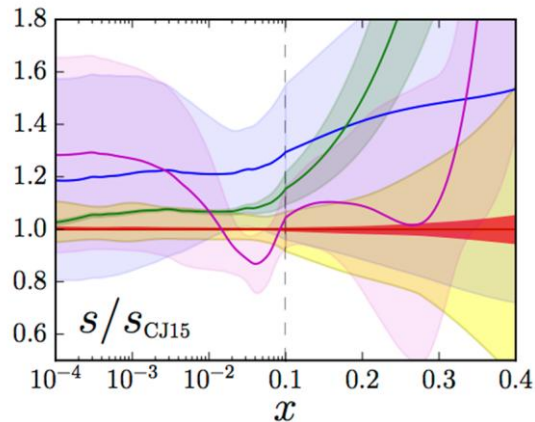


Figure 4.1: Nucleon isovector H (left), E (middle) and $\tilde{H}$ (right) GPDs at $\xi = 0$ with z -expansion to Q^2 at selected x values.

HL, Prog.Part.Nucl.Phys. 144 (2025) 104177

First Lattice Strange PDF

§ Large uncertainties in global PDFs



— CJ15 ($T = 10$)
— CJ15 ($T = 1$)
— MMHT14
— HERAPDF1.5
— NNPDF3.0

∞ Assumptions imposed
due to lack of precision data

$$s = \bar{s} = \kappa(\bar{u} + \bar{d})$$

CTEQ-JLAB <https://www.jlab.org/theory/cj/>



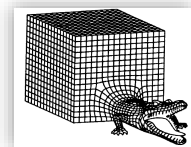
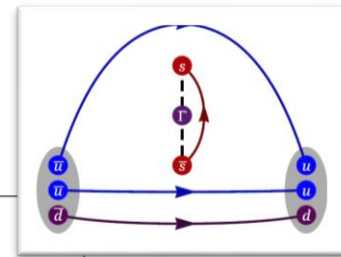
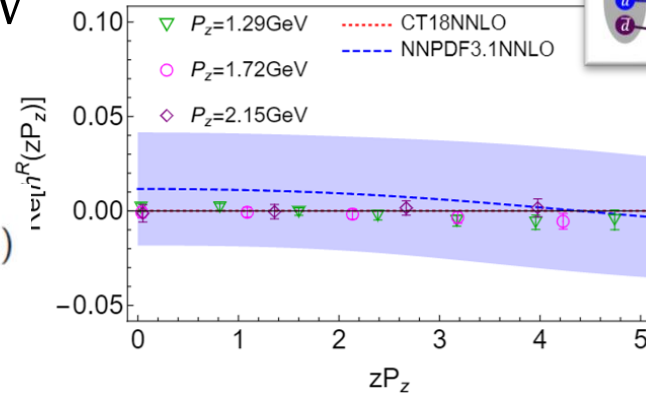
First Lattice Strange PDF

§ Results by MSULat/quasi-PDF method

- ∞ Clover on 2+1+1 HISQ, 0.12-fm 310-MeV QCD vacuum
- ∞ Extrapolated to $M_\pi \approx 140$ MeV

2005.01124, R. Zhang et al
(MSULat)

$$\text{Re}[h(z)] \propto \int dx (s(x) - \bar{s}(x)) \cos(xzP_z)$$



Quantities
that can be
calculated on
the lattice



Lattice Strangeness Asymmetry Impact

§ Results by MSULat/quasi-PDF method

- ∞ Clover on 2+1+1 HISQ, 0.12-fm 310-MeV QCD vacuum
- ∞ Extrapolated to $M_\pi \approx 140$ MeV, $P_z \approx 1.7$ GeV

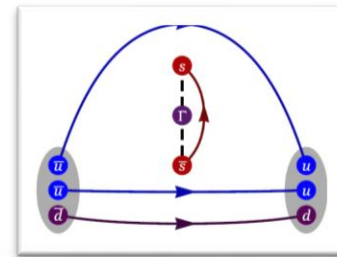
R. Zhang et al (MSULat),
2005.01124

$$\text{Re}[h(z)] \propto \int dx (s(x) - \bar{s}(x)) \cos(xz)$$

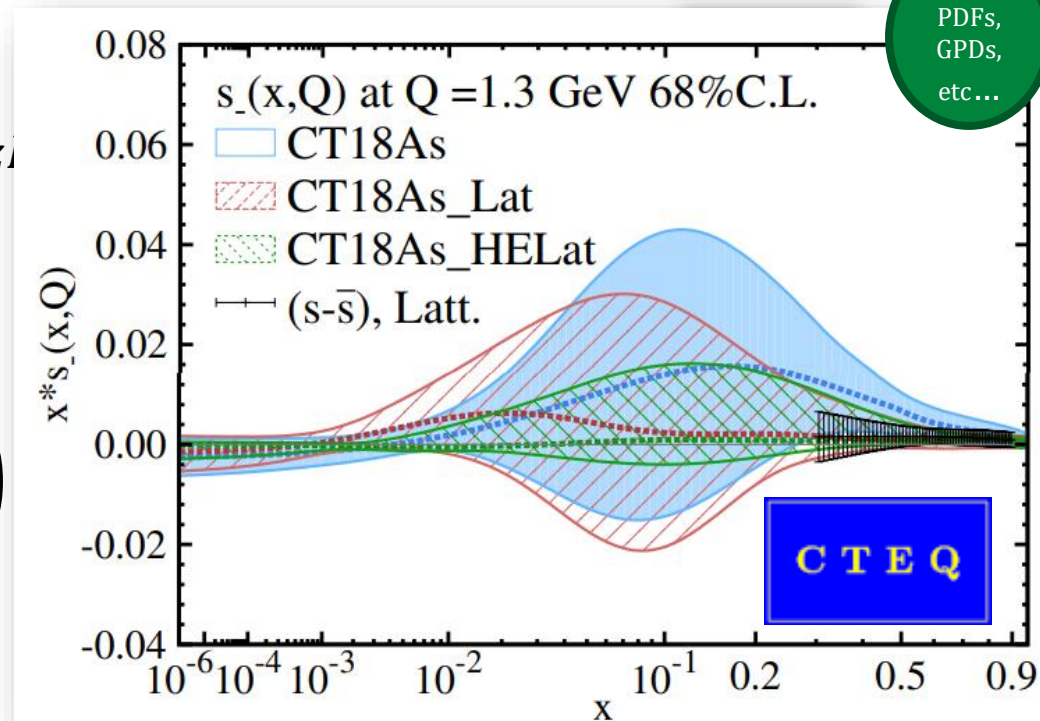
§ From quasi-PDF to PDF

$$\tilde{f}_q(x, P_z) = \int_{-1}^1 \frac{dy}{|y|} f_q(y) C_{q/q}(x, y, P_z, \mu) + O\left(\frac{\Lambda_{\text{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(1-x)^2 P_z^2}\right)$$

T. Hou, HL, M. Yan, C. Yuan,
2211.11064



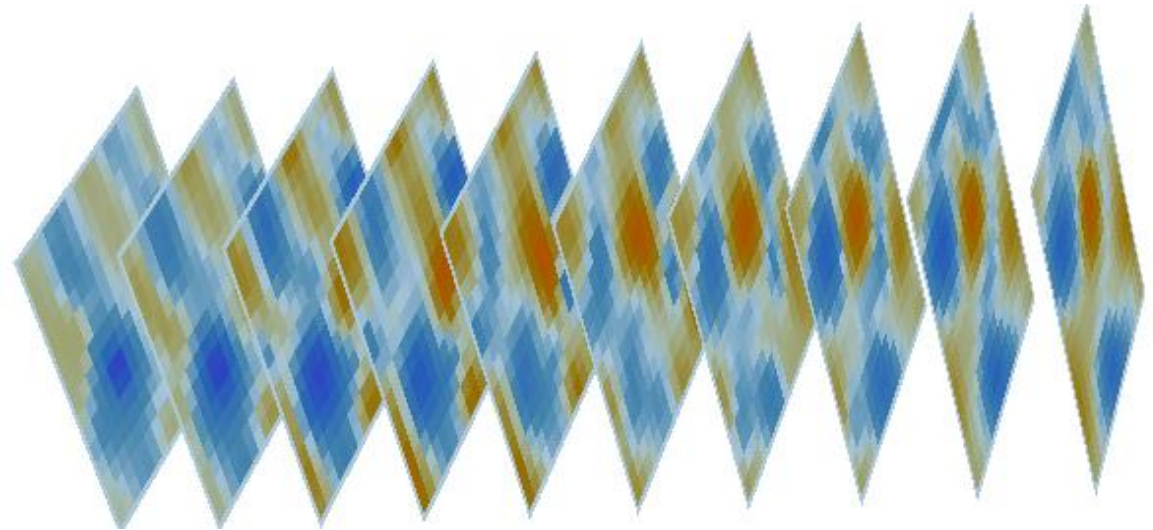
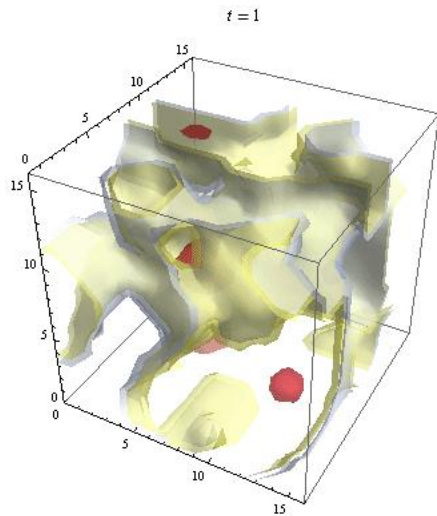
Wanted
PDFs,
GPDs,
etc...



§ The strangeness asymmetry $s(x, Q) - \bar{s}(x, Q)$ at $x > 0.2$ is difficult to measure, but can be predicted in lattice QCD

Anatomy of a Lattice Calculation

1. Start with QCD Vacuum (gauge configurations)

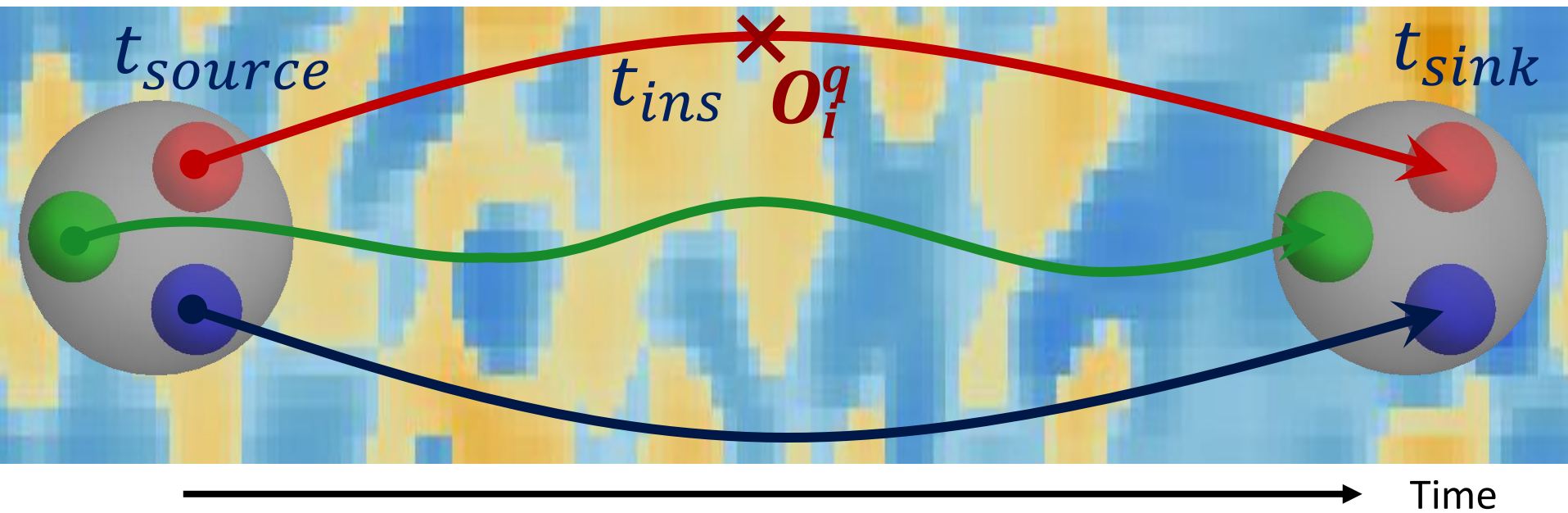


§ Pick a QCD vacuum

↻ Gauge/fermion actions, flavor (2, 2+1, 2+1+1), m_π , a , L , ...

Anatomy of a Lattice Calculation

Lattice-QCD calculation of $\langle N | \bar{q} \Gamma q | N \rangle$



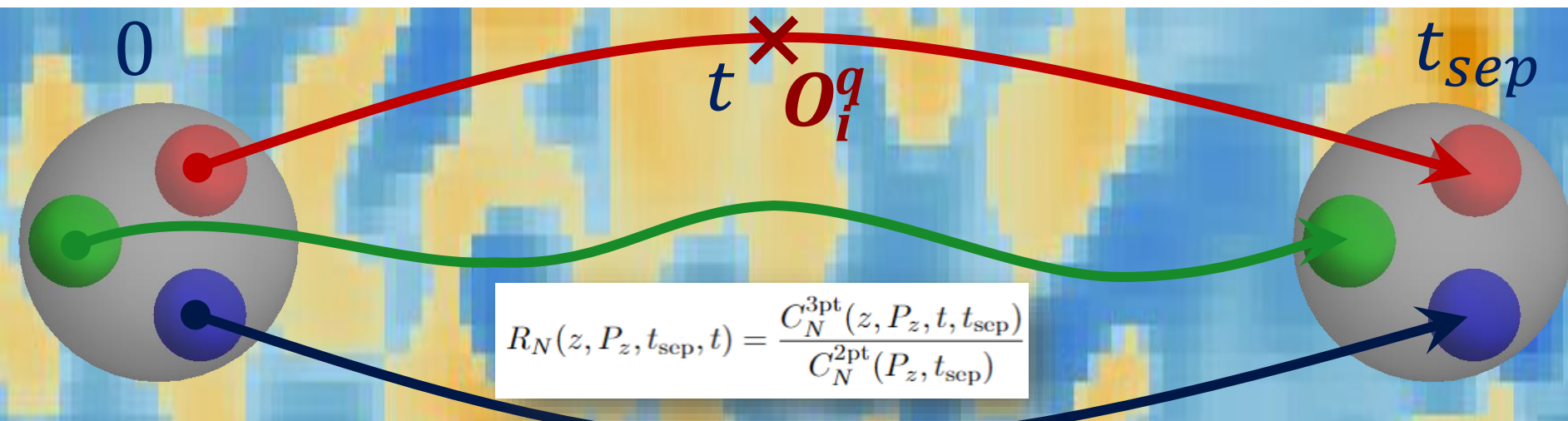
§ Construct correlators (hadronic observables)

⌘ Requires “quark propagator”

Invert Dirac-operator matrix (rank $O(10^{12})$)

Anatomy of a Lattice Calculation

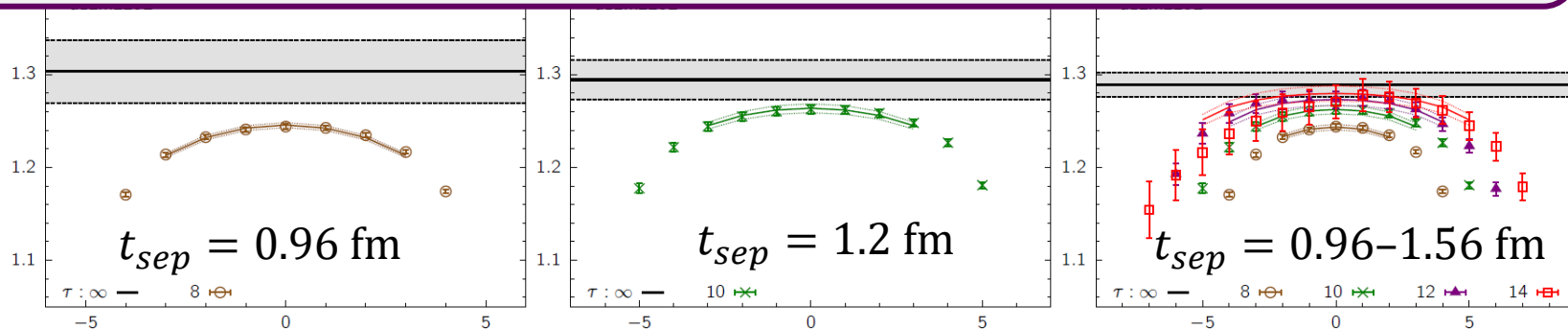
Lattice-QCD calculation of $\langle N | \bar{q} \Gamma q | N \rangle$



§ Careful analysis needed to remove systematics

⌘ Wrong results if **excited-state systematic** is not under control

Ratio
Plot



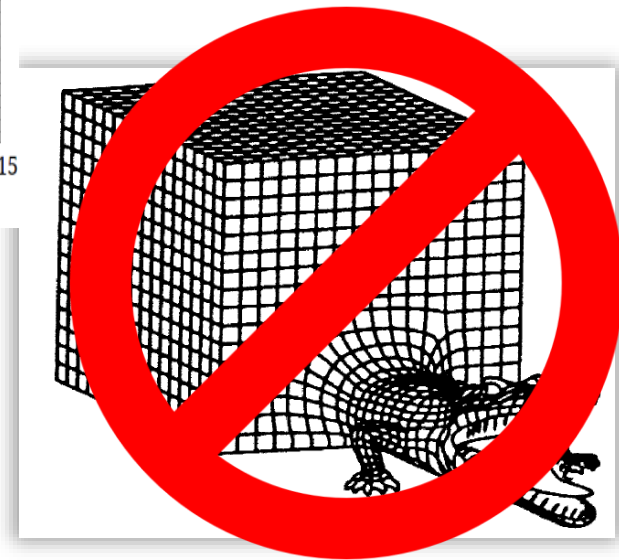
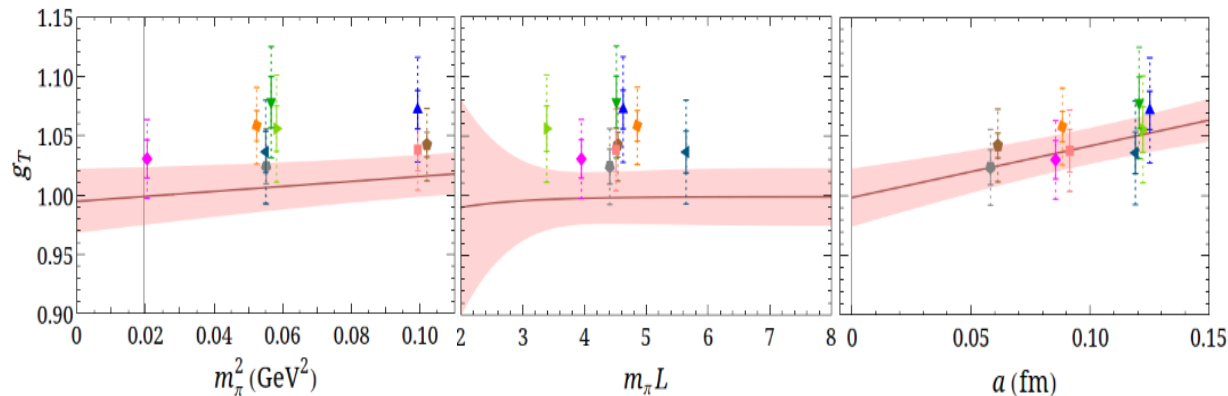
Anatomy of a Lattice Calculation

4. Systematic uncertainty (nonzero a , finite L , etc.)

⌘ Nonperturbative renormalization, etc

⌘ Extrapolation to the continuum limit

$$(m_\pi \rightarrow m_\pi^{\text{phys}}, L \rightarrow \infty, a \rightarrow 0)$$



Moments of PDFs

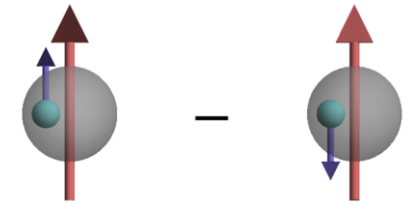
§ PDG-like rating system or average

§ LatticePDF Workshop

↻ Lattice representatives came together and devised a rating system

§ Lattice QCD/global fit status

$$\langle x^{n-1} \rangle_{\delta q} = \int_{-1}^1 dx x^{n-1} \delta q(x)$$



LatticePDF Report, 1711.07916, 2006.08636

Moment	Collaboration	Reference	N_f	DE	CE	FV	RE	ES	Value	Global Fit
g_T	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	** 0.926(32)	0.10 — 1.1
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	* 0.989(32)(10)	
	χ QCD 20	(Horkel <i>et al.</i> , 2020)	2+1	■	★	○	★	★	† 1.096(30)	
	LHPC 19	(Hasan <i>et al.</i> , 2019)	2+1	○	★	○	★	★	* 0.972(41)	
	Mainz 19	(Harris <i>et al.</i> , 2019)	2+1	★	○	★	★	★	0.965(38)(⁺¹³ ₋₄₁)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	1.08(3)(3)(9)	
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2	■	★	○	★	★	** 0.974(33)	
	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	1.004(21)(02)(19)	
$\langle 1 \rangle_{\delta u^-}$	RQCD 14	(Bali <i>et al.</i> , 2015)	2	○	★	★	★	■	1.005(17)(29)	-0.14 — 0.91
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	** 0.716(28)	
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	* 0.784(28)(10)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	0.85(3)(2)(7)	
$\langle 1 \rangle_{\delta d^-}$	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	0.782(16)(2)(13)	-0.97 — 0.47
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	** -0.210(11)	
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	* -0.204(11)(10)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	-0.24(2)(0)(2)	
$\langle 1 \rangle_{\delta s^-}$	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	-0.219(10)(2)(13)	N/A
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	** -0.0027(58)	
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	* -0.0027(16)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	-0.012(16)(8)	
	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	-0.00319(69)(2)(22)	

Moments of PDFs

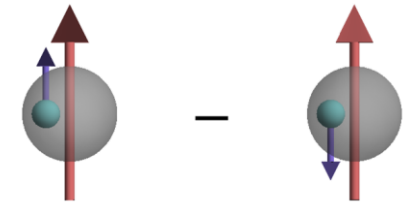
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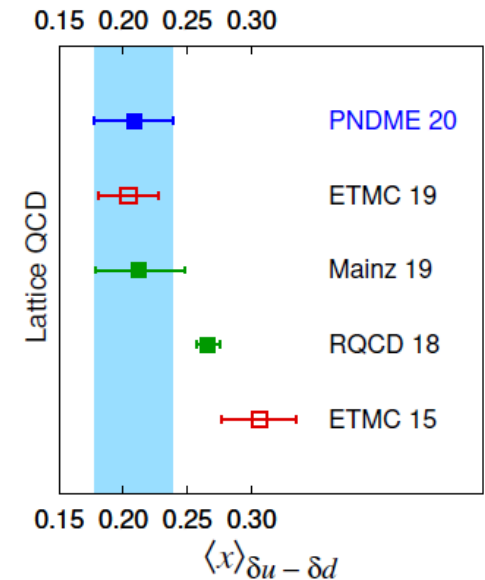
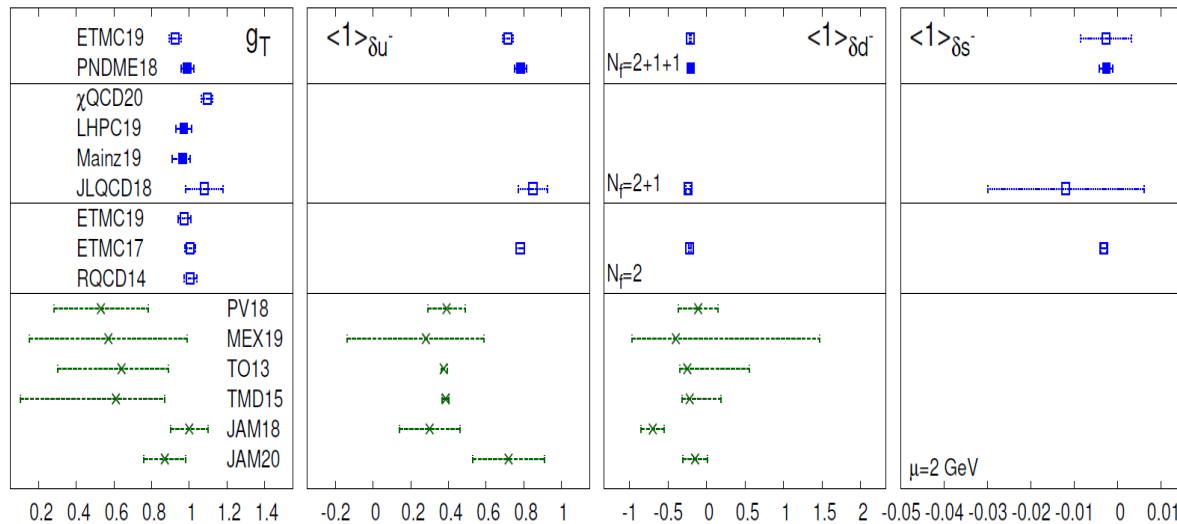
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LatticePDF Report, 1711.07916, 2006.08636

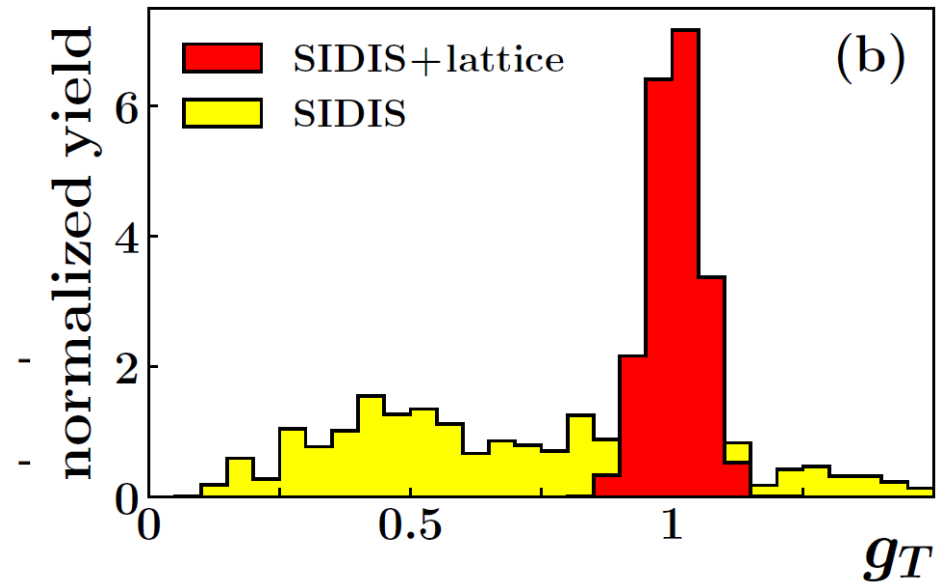
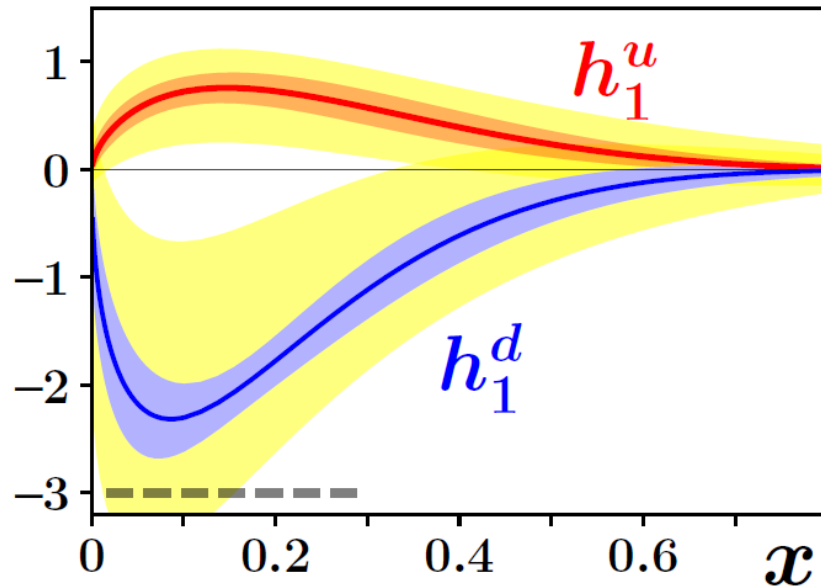


S. Mondal et al (PNDME), 2005.13779

From Charges to PDFs

§ Improved transversity distribution with LQCD g_T

- ⌘ Global analysis with 12 extrapolation forms: $g_T = 1.006(58)$
- ⌘ Use to constrain the global analysis fits to SIDIS $\pi^\pm$ production data from proton and deuteron targets

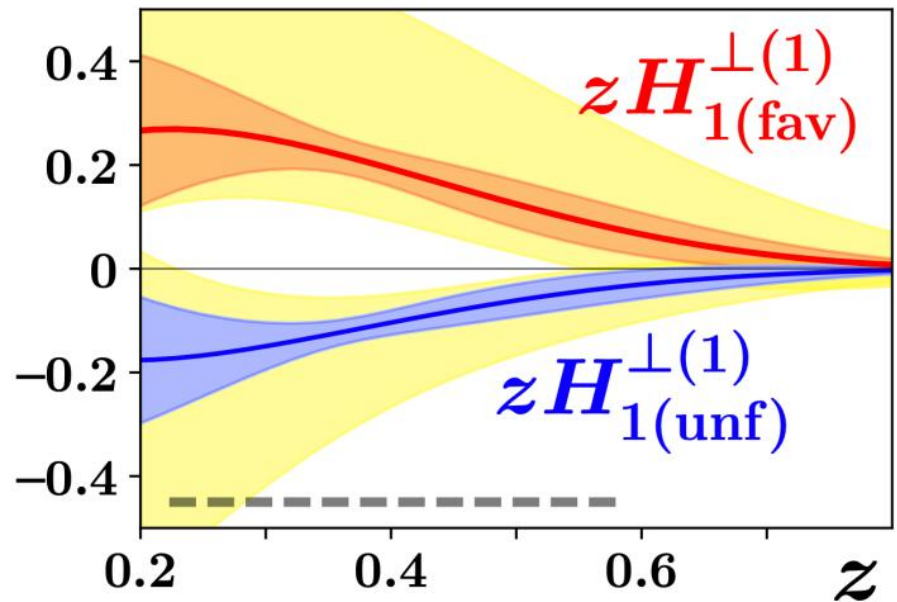
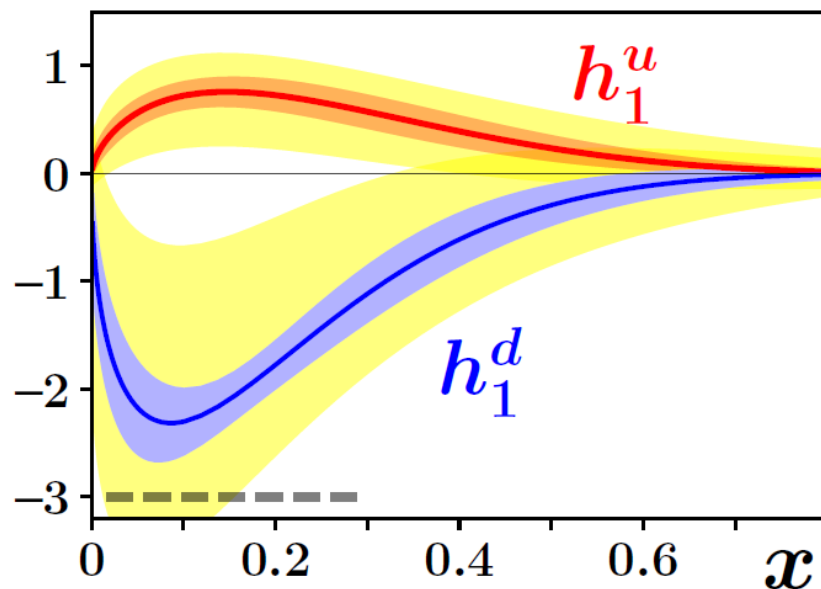


Lin, Melnitchouk, Prokudin, Sato, 1710.09858, Phys. Rev. Lett. 120, 152502 (2018)

From Charges to PDFs

§ Improved transversity distribution with LQCD g_T

- ↻ Global analysis with 12 extrapolation forms: $g_T = 1.006(58)$
- ↻ Use to constrain the global analysis fits to SIDIS $\pi^\pm$ production data from proton and deuteron targets

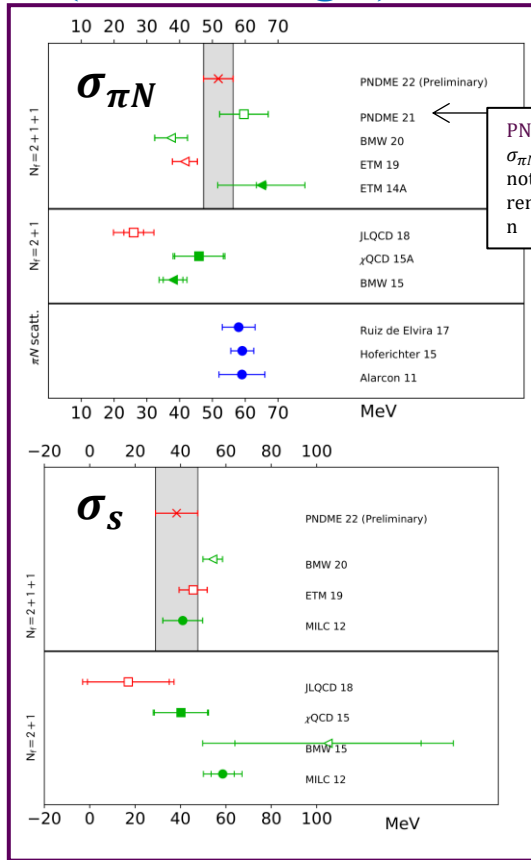


Lin, Melnitchouk, Prokudin, Sato, 1710.09858, Phys. Rev. Lett. 120, 152502 (2018)

Nucleon Flavor Diagonal Charges

Comparison with FLAG 2021 results

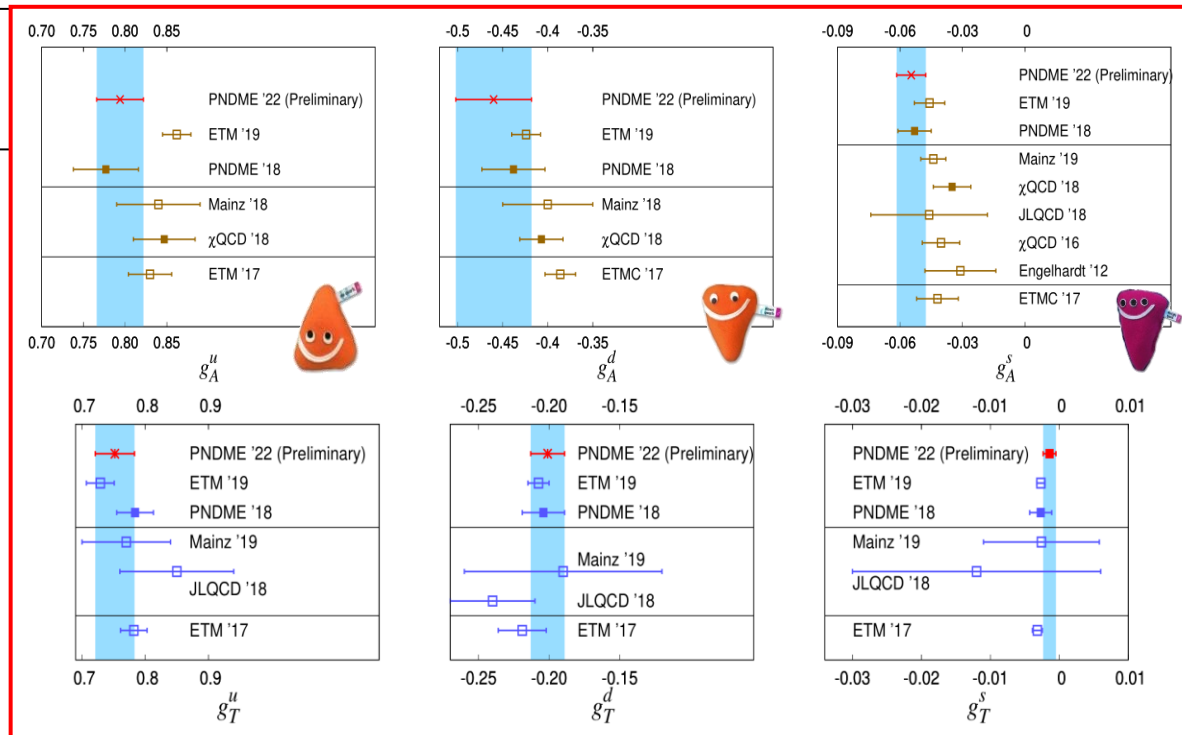
Nucleon sigma terms (Scalar charges)



[PNDME, Lattice 2022 update, preliminary]

- Clover fermion on $N_f = 2 + 1 + 1$ HISQ ensembles
- Flavor mixing calculated nonperturbatively
- **Chiral-Continuum extrapolation** including a data at M_π^{Phys}

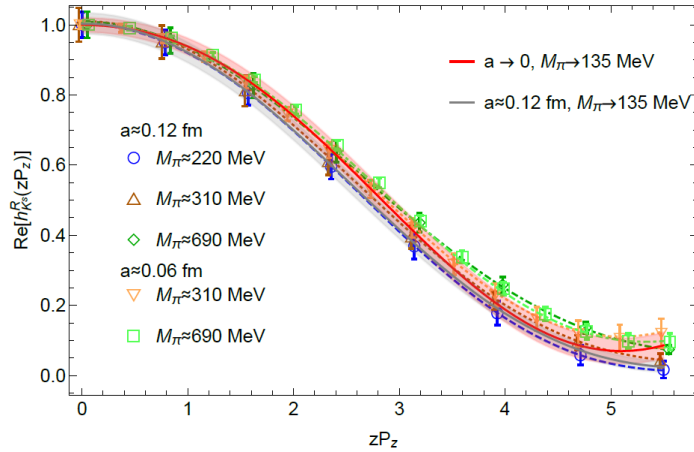
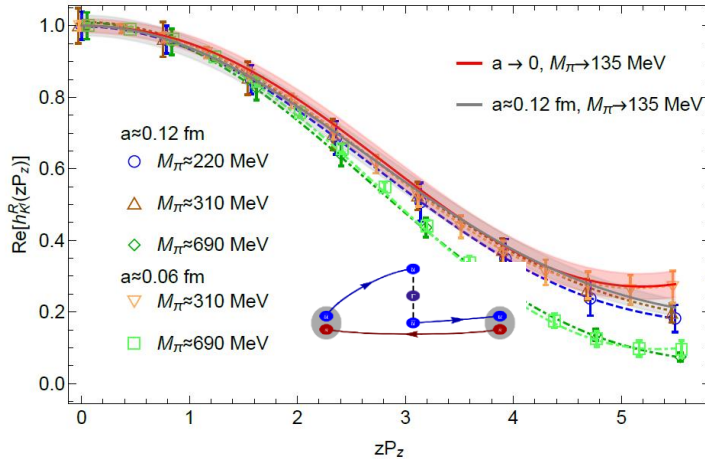
Axial and Tensor charges



Plots by Sungwoo Park

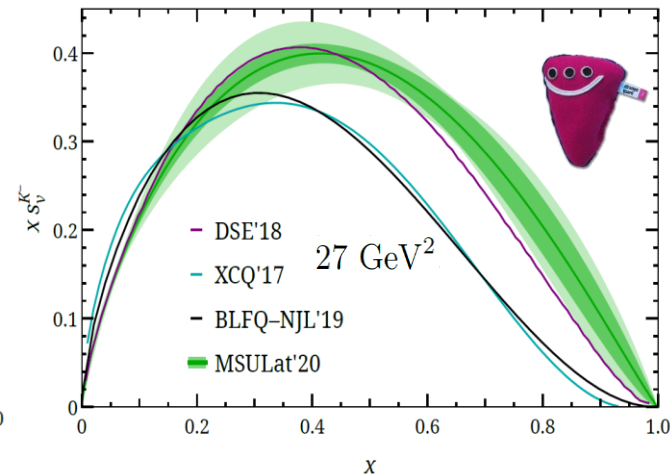
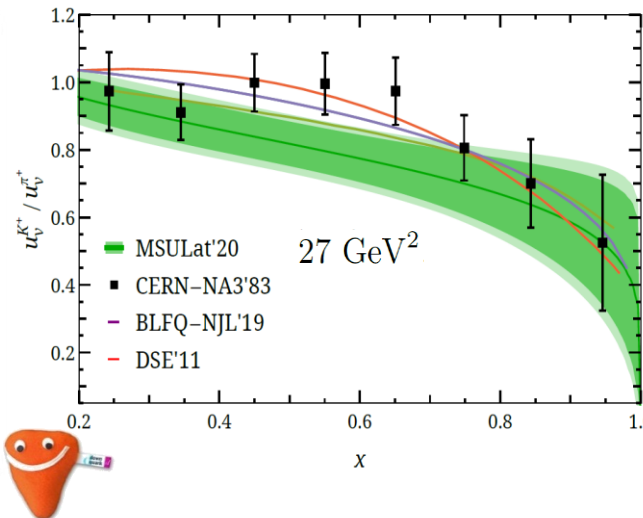
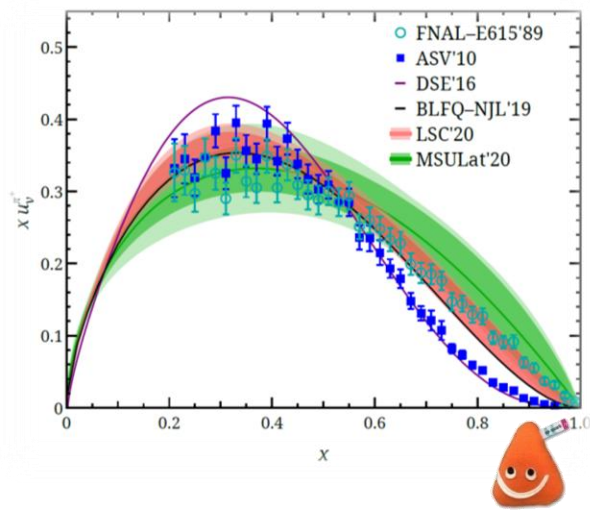
Meson Valence-quark PDFs

§ Pion/Kaon PDFs using quasi-PDF in the continuum limit



Quantities that can be calculated on the lattice

Wanted PDFs, GPDs, etc...

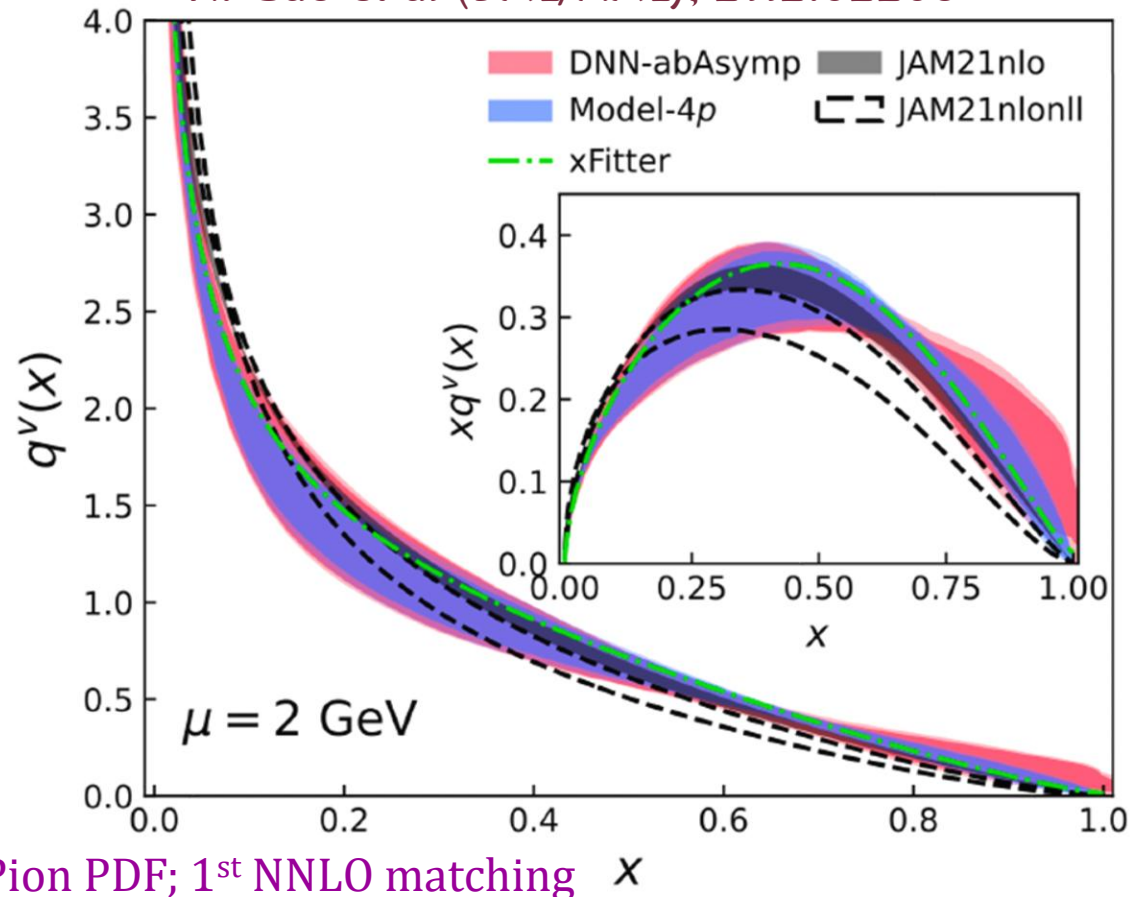
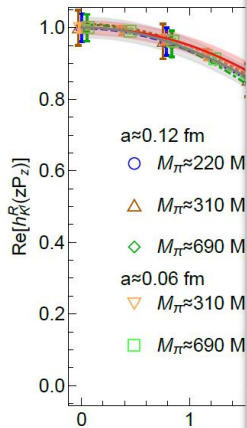


Meson Valence-quark PDFs

§ Pion/Kaon

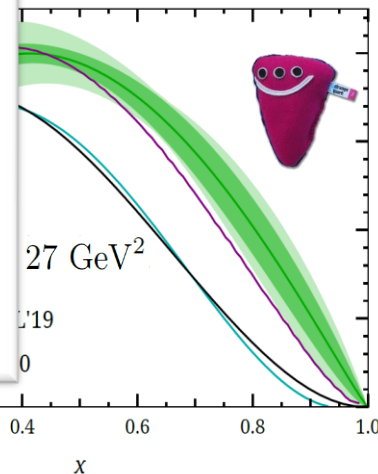
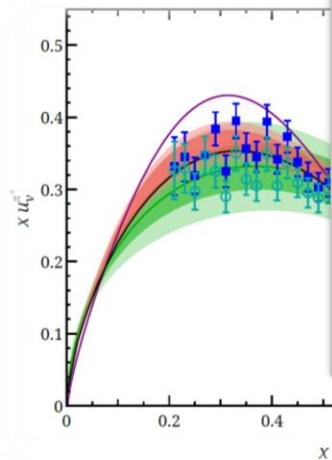
X. Gao et al (BNL/ANL), 2112.02208

um limit



Quantities
that can be
calculated on
the lattice

Wanted
PDFs,
GPDs,
etc...



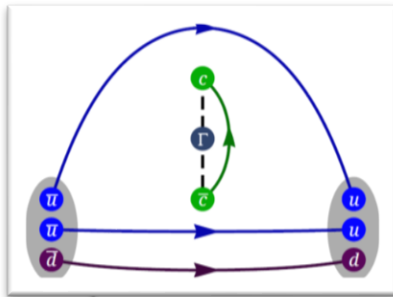
First Lattice Charm PDF

§ Large uncertainties in global PDFs

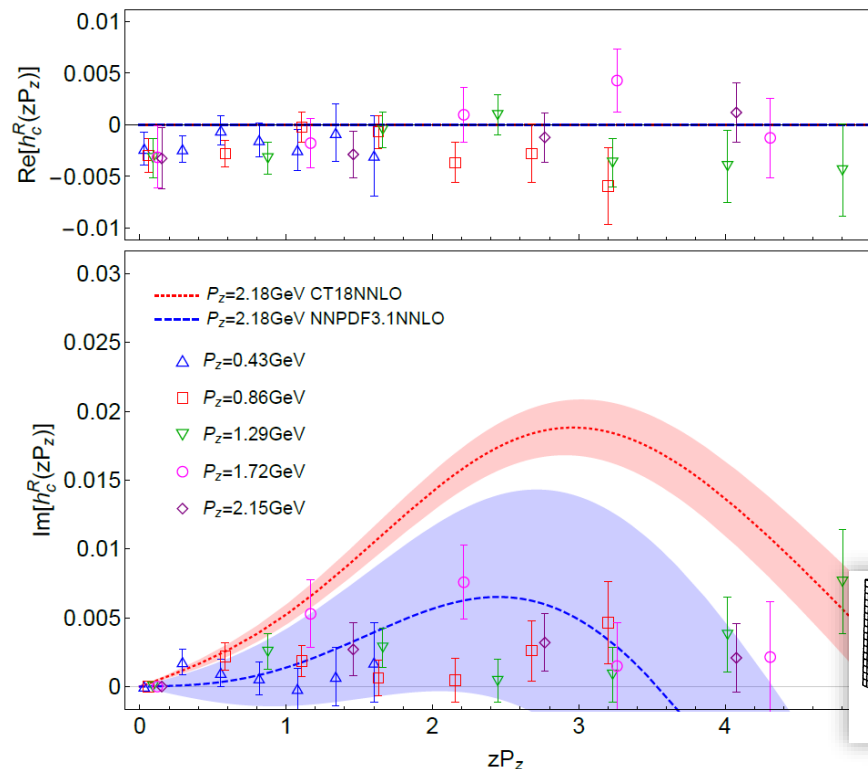
§ Results by MSULat/quasi-PDF method

☞ Clover on 2+1+1 HISQ 0.12-fm 310-MeV QCD vacuum

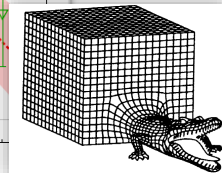
2005.01124, R. Zhang et al (MSULat)



- suggest a symmetric $c - \bar{c}$ distribution
- much smaller than strange PDF



Quantities that can be calculated on the lattice



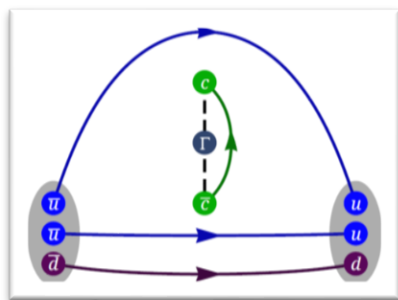
First Lattice Charm PDF

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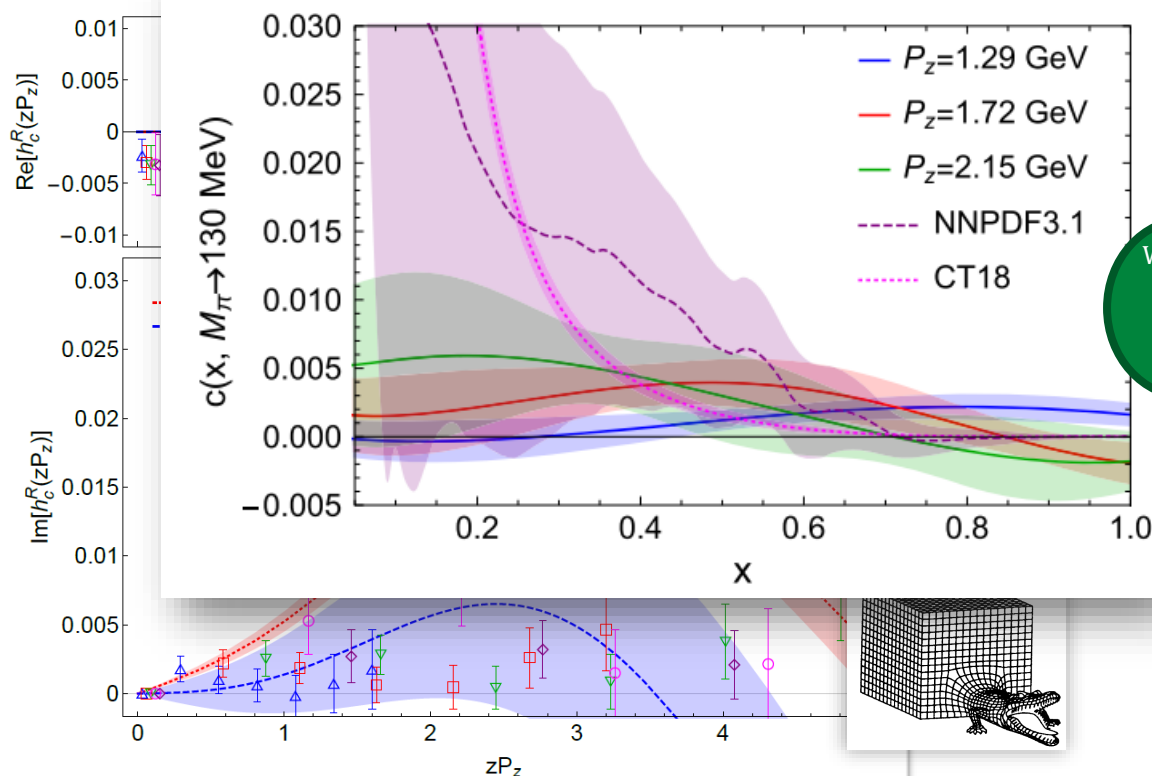
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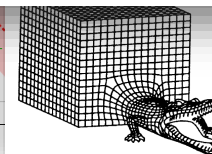
2005.01124, R. Zhang et al (MSULat)



- suggest a symmetric $c - \bar{c}$ distribution
- much smaller than strange PDF



Wanted
PDFs,
GPDs,
etc...



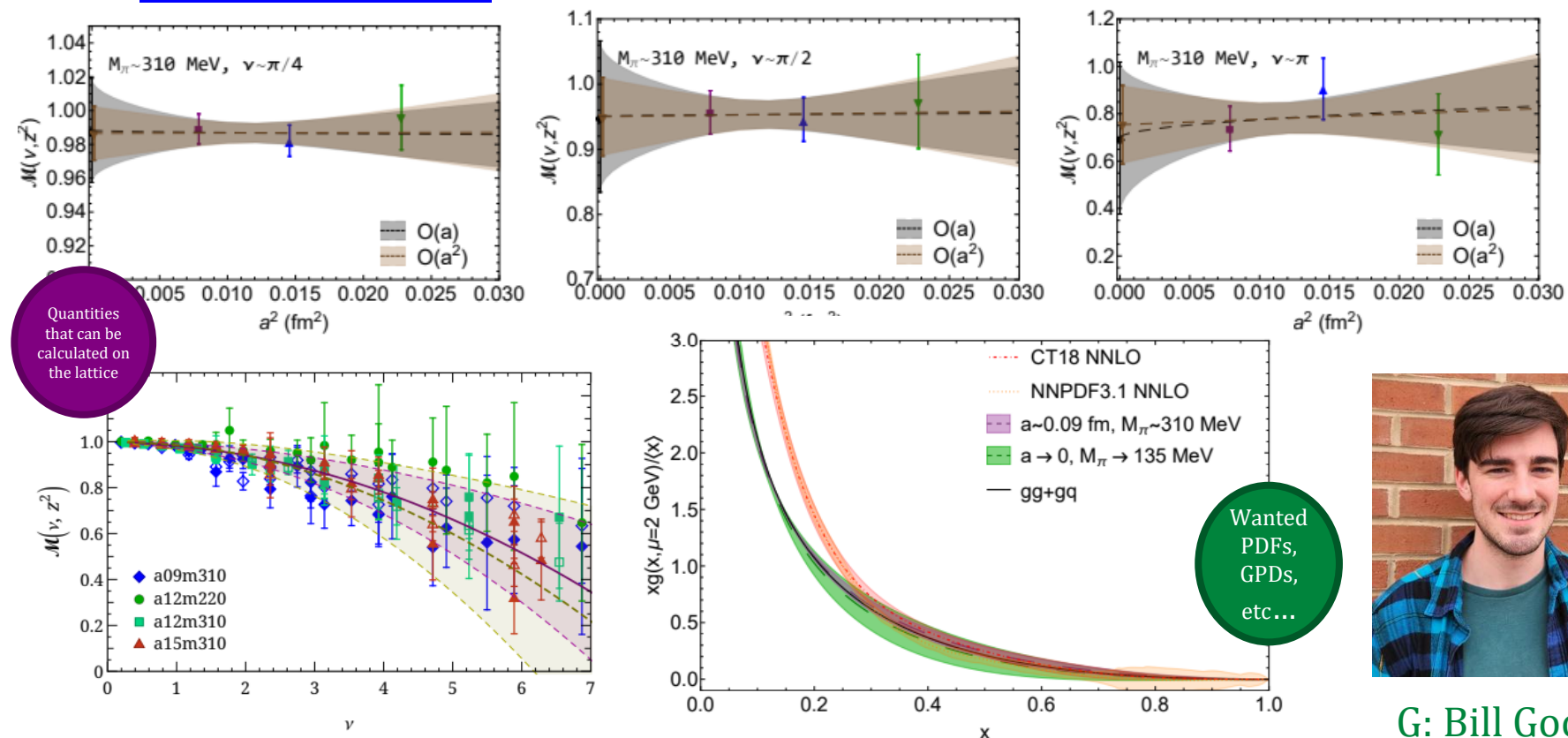
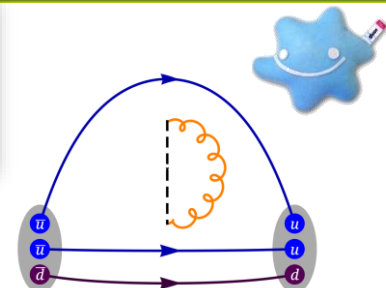
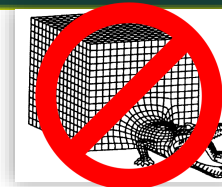
Gluon PDF in Nucleon

§ Continuum Gluon PDF w/ pseudo-PDF

$\propto$ 2+1+1 HISQ {0.09, 0.12, 0.15} fm,

[220,310,700]-MeV pion, 10^5 - 10^6 statistics

[arXiv:2210.09985](https://arxiv.org/abs/2210.09985)



G: Bill Good

Moments of PDFs

§ Transversity first moments are most commonly done

§ State-of-the art example

↻ 2 physical pion mass ensembles

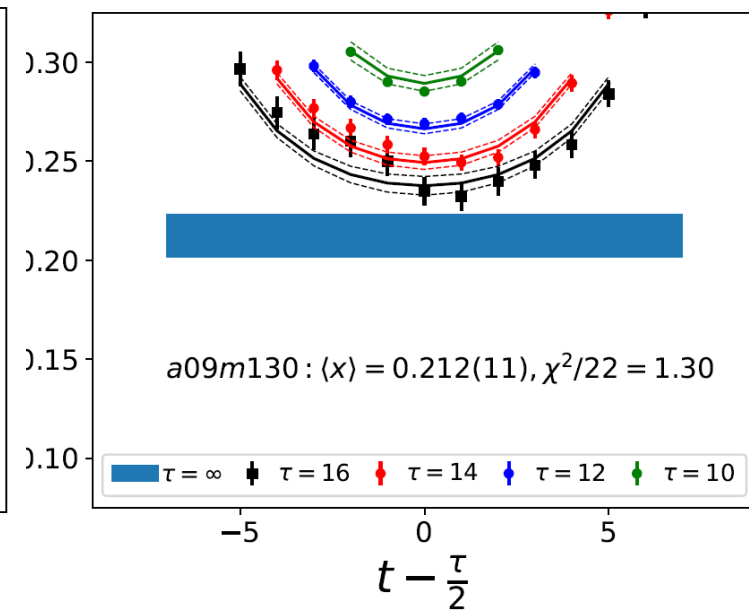
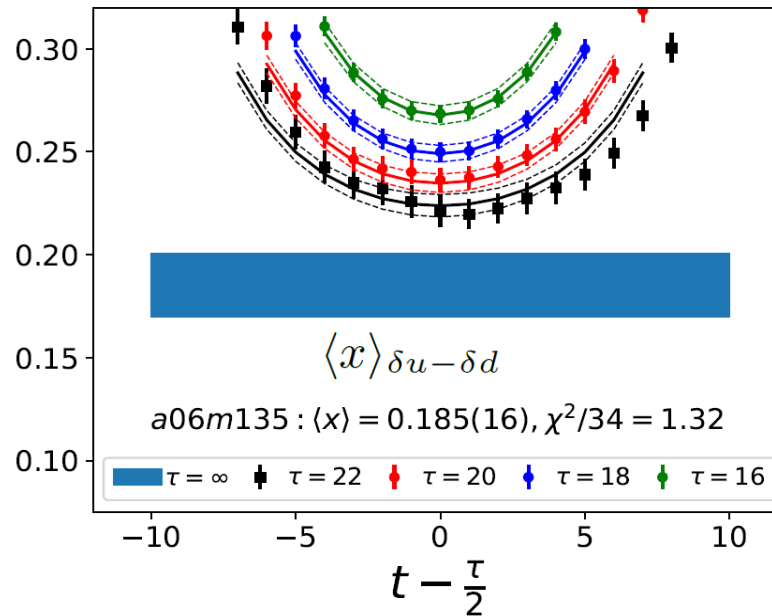
↻ Extrapolate to the physical limit

Santanu Mondal et al (PNDME collaboration), 2005.13779

$$\langle x^n \rangle_{\delta q} = \int_{-1}^1 dx x^n \delta q(x)$$



Plot by
Santanu Mondal



Moments of PDFs

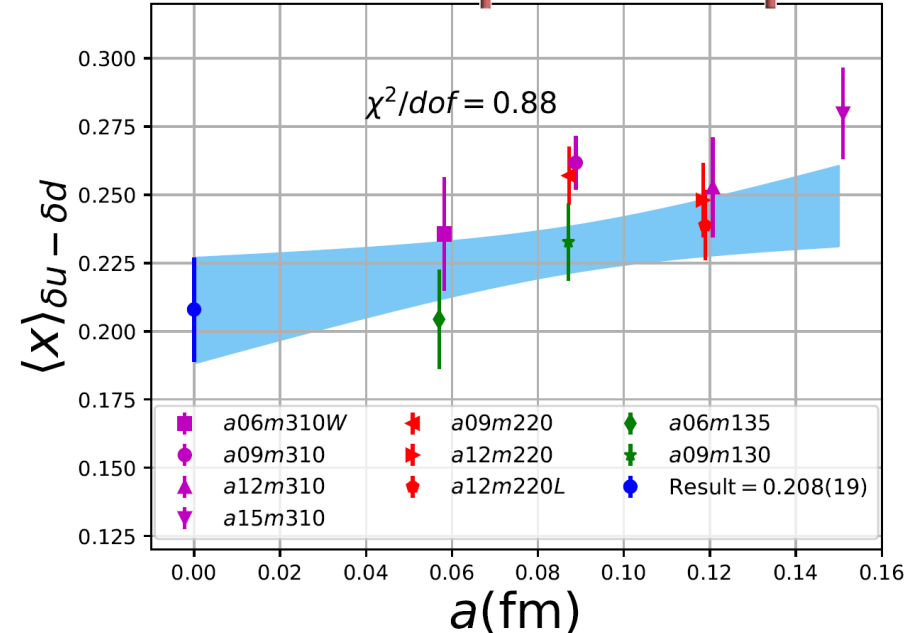
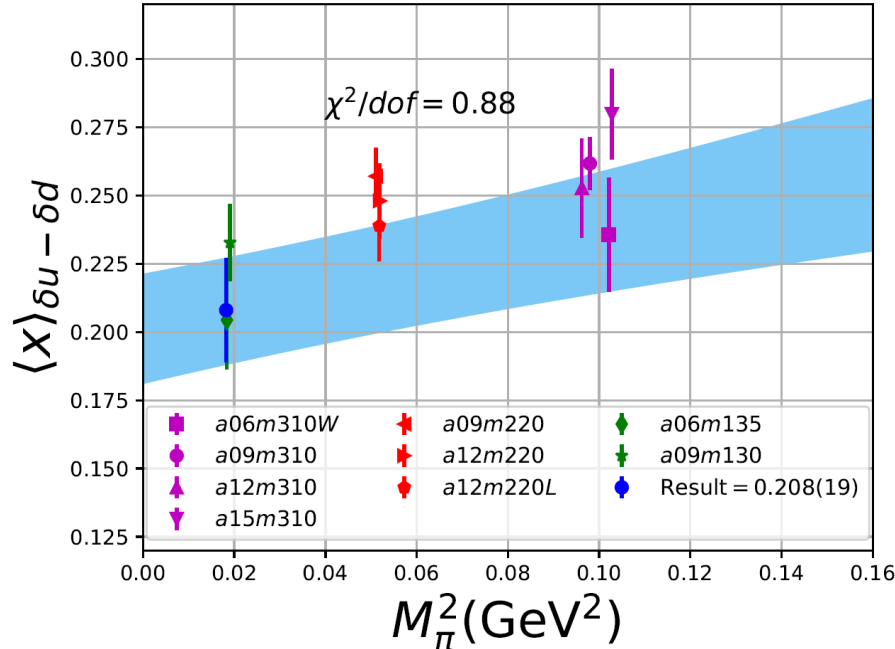
§ Only lowest few moments

§ State-of-the art example

↻ Extrapolate to the physical limit

Santanu Mondal et al (PNDME collaboration), 2005.13779

$$\langle x^{n-1} \rangle_{\delta q} = \int_{-1}^1 dx x^{n-1} \delta q(x)$$



§ Usually more than one LQCD calculation

↻ Sometimes LQCD numbers do not even agree with each other...

Moments of PDFs

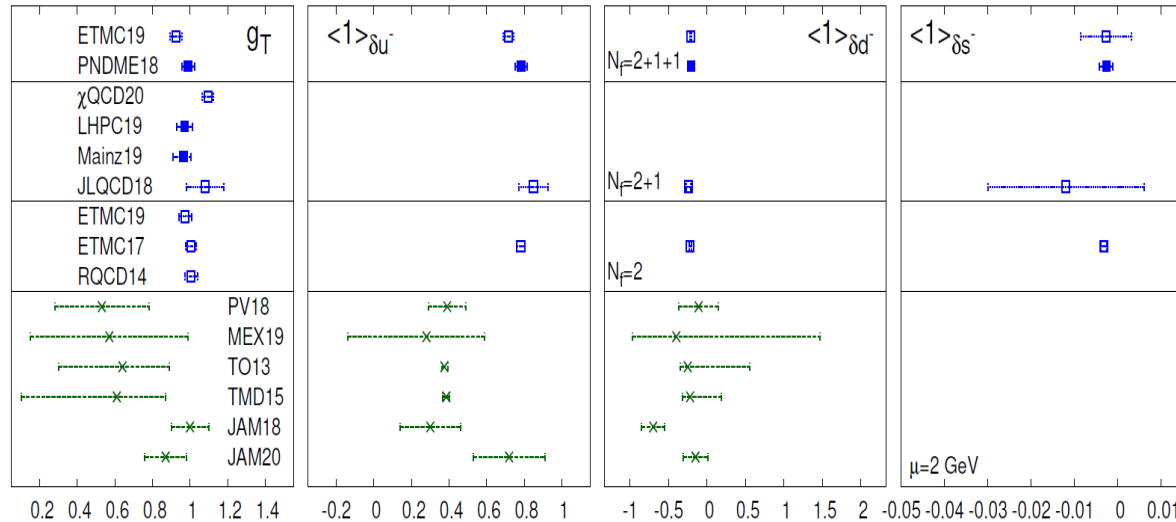
§ PDG-like rating system or average

§ LatticePDF Workshop

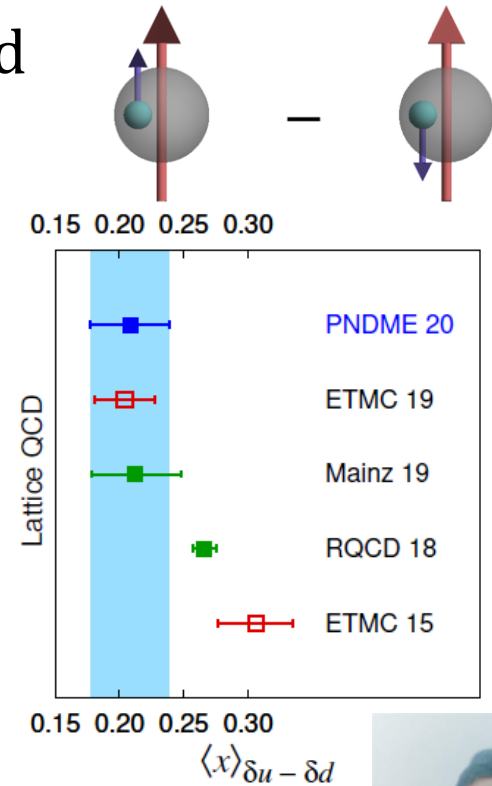
∞ Lattice representatives came together and devised a rating system

§ Recent lattice QCD/global fit status

LatticePDF Report, 1711.07916, 2006.08636



$$\langle x^{n-1} \rangle_{\delta q} = \int_{-1}^1 dx x^{n-1} \delta q(x)$$

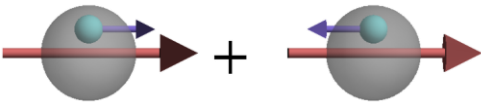


S. Mondal et al
2005.13779

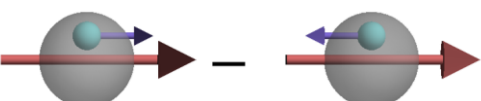


PDFs on the Lattice

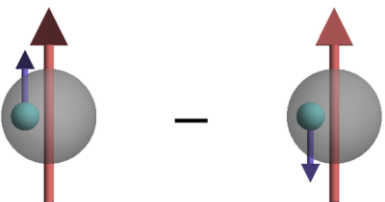
§ Traditional lattice calculations rely on operator product expansion, only provide moments



spin-averaged/unpolarized



spin-dependent
longitudinally polarized



spin-dependent
transversely polarized

$$\langle x^{n-1} \rangle_q = \int_{-1}^1 dx x^{n-1} q(x)$$

$$\langle x^{n-1} \rangle_{\Delta q} = \int_{-1}^1 dx x^{n-1} \Delta q(x)$$

$$\langle x^{n-1} \rangle_{\delta q} = \int_{-1}^1 dx x^{n-1} \delta q(x)$$

most well known



very poorly known

§ True distribution can only be recovered with **all** moments

Moments of PDFs

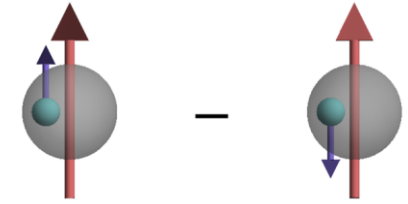
§ PDG-like rating system or average

§ LatticePDF Workshop

↻ Lattice representatives came together and devised a rating system

§ Lattice QCD/global fit status

$$\langle x^{n-1} \rangle_{\delta q} = \int_{-1}^1 dx x^{n-1} \delta q(x)$$



LatticePDF Report, 1711.07916, 2006.08636

Moment	Collaboration	Reference	N_f	DE	CE	FV	RE	ES	Value	Global Fit
g_T	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	0.926(32)	0.10 — 1.1
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	0.989(32)(10)	
	χ QCD 20	(Horkel <i>et al.</i> , 2020)	2+1	■	★	○	★	★	1.096(30)	
	LHPC 19	(Hasan <i>et al.</i> , 2019)	2+1	○	★	○	★	★	0.972(41)	
	Mainz 19	(Harris <i>et al.</i> , 2019)	2+1	★	○	★	★	★	0.965(38)(⁺¹³ ₋₄₁)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	1.08(3)(3)(9)	
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2	■	★	○	★	★	0.974(33)	
	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	1.004(21)(02)(19)	
$\langle 1 \rangle_{\delta u^-}$	RQCD 14	(Bali <i>et al.</i> , 2015)	2	○	★	★	★	■	1.005(17)(29)	-0.14 — 0.91
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	0.716(28)	
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	0.784(28)(10)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	0.85(3)(2)(7)	
$\langle 1 \rangle_{\delta d^-}$	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	0.782(16)(2)(13)	-0.97 — 0.47
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	-0.210(11)	
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	-0.204(11)(10)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	-0.24(2)(0)(2)	
$\langle 1 \rangle_{\delta s^-}$	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	-0.219(10)(2)(13)	N/A
	ETMC 19	(Alexandrou <i>et al.</i> , 2019b)	2+1+1	■	★	○	★	★	-0.0027(58)	
	PNDME 18	(Gupta <i>et al.</i> , 2018)	2+1+1	★	★	★	★	★	-0.0027(16)	
	JLQCD 18	(Yamanaka <i>et al.</i> , 2018)	2+1	■	○	○	★	★	-0.012(16)(8)	
	ETMC 17	(Alexandrou <i>et al.</i> , 2017d)	2	■	★	■	★	★	-0.00319(69)(2)(22)	

Forward-Limit Case: PDF

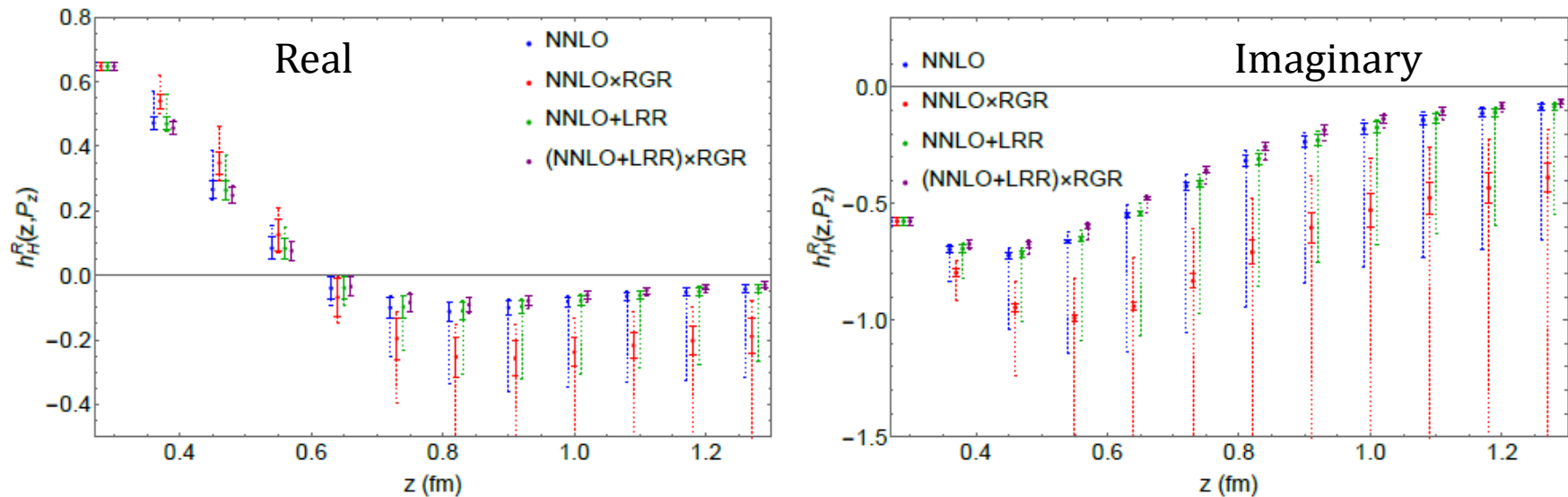
§ NNLO hybrid-ratio renormalized matrix elements

$$h^R(z, P_z) = \begin{cases} N \frac{h^B(z, P_z)}{h^B(z, P_z=0)} & \text{for } z < z_s \\ Ne^{(\delta m + m_0)(z-z_s)} \frac{h^B(z, P_z)}{h^B(z, P_z=0)} & \text{for } z \geq z_s \end{cases}$$

Remove the **linear divergence** & **renormalon ambiguity** at large distances

∞ Vary the scale within [0.75, 1.5]: $\approx 15\%$ variation $\alpha_s(\mu = 2.0 \text{ GeV})$

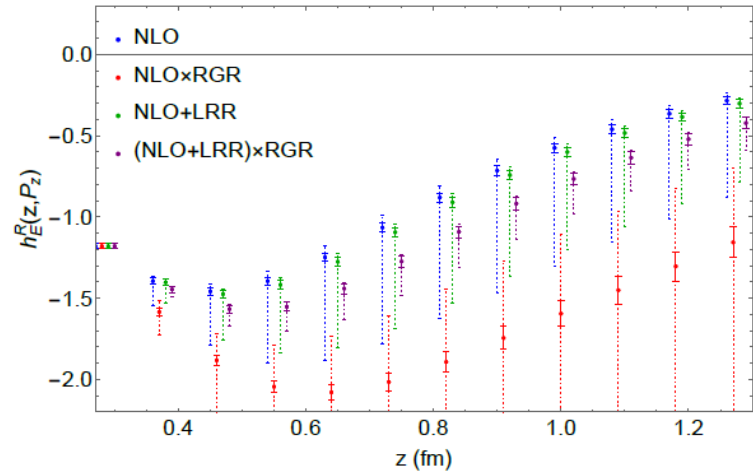
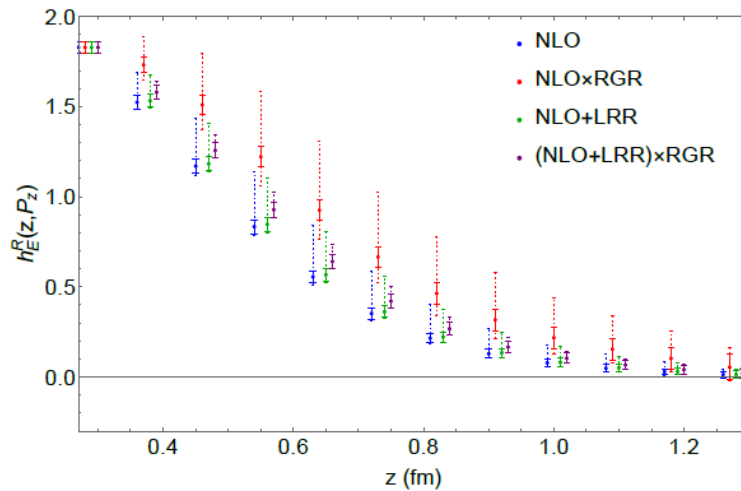
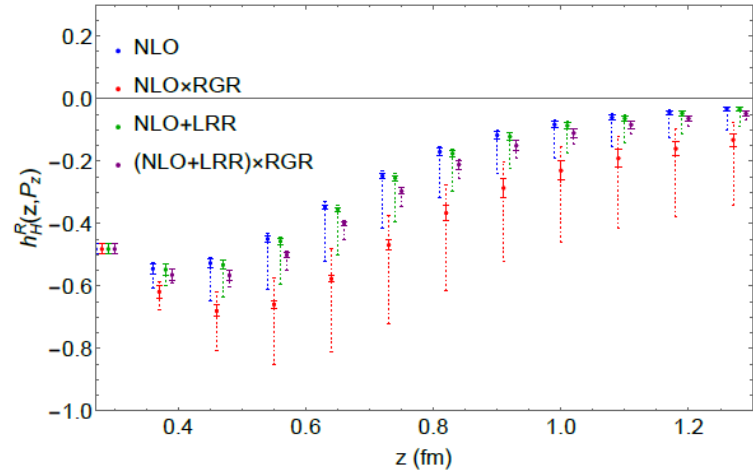
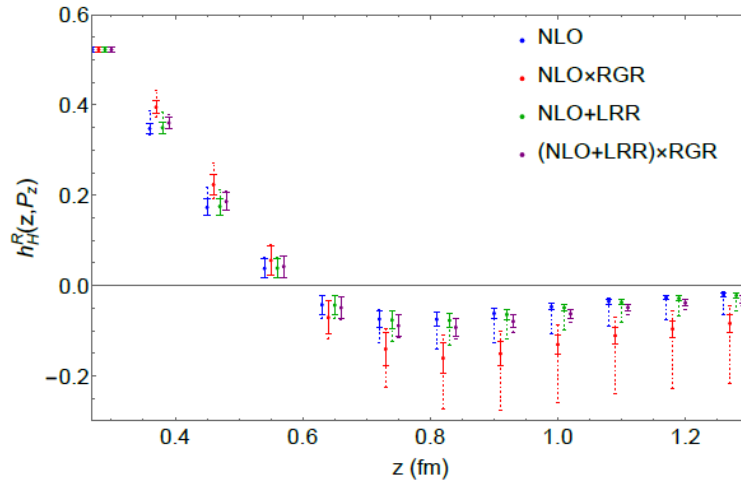
∞ Systematic errors shown below:



J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

$\xi \neq 0$ GPDs

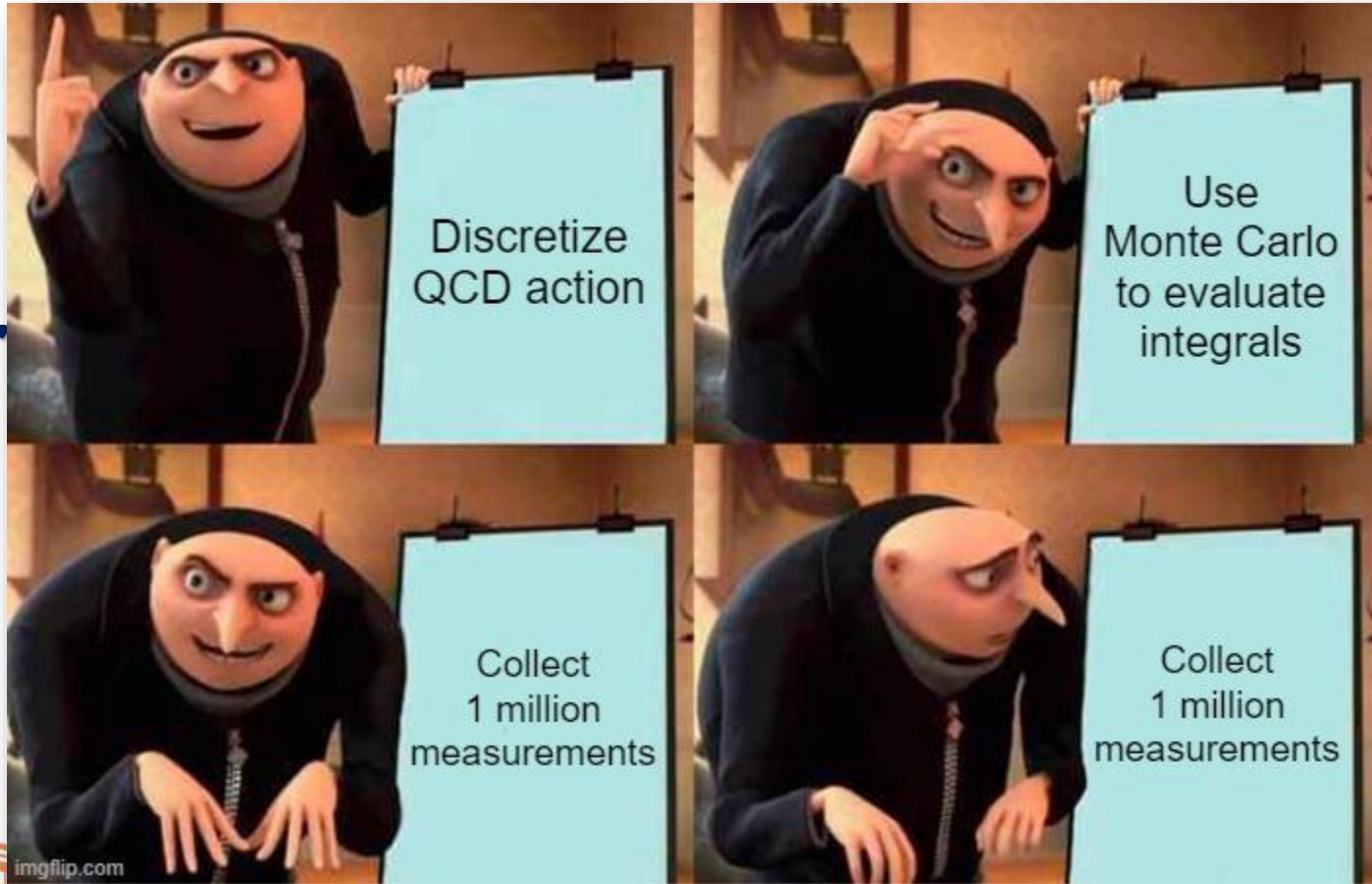
ξ NLO $\xi = 0.1$, $Q^2 = 0.23 \text{ GeV}^2$



Quantities
that can be
calculated on
the lattice

J. Holligan, HL (MSULat), 2312.10829 [hep-lat]

Str



DFs

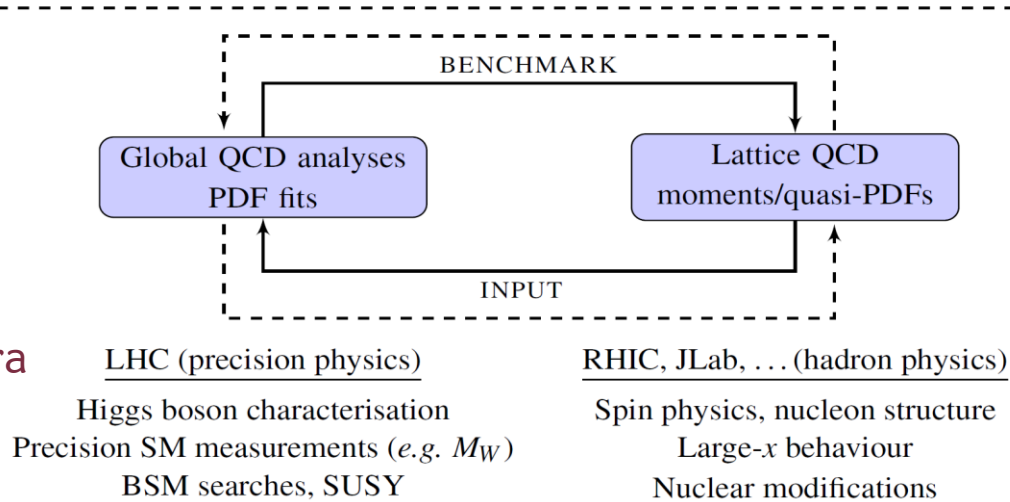


results



How Can Lattice Help?

THE PDFLATTICE2017 WORKSHOP



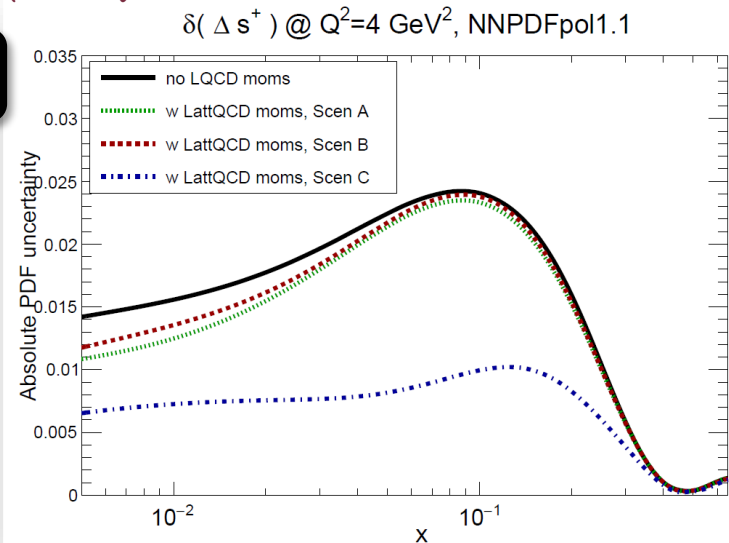
Plot by
E. Nocera

Example study

Whitepaper , Progress in Par. and Nuc. Phys. 100, 106 (2018)

A: 70% B: 50% C: 20%

§ Is there one quantity for which LQCD can achieve a precision at which it can make a significant difference?



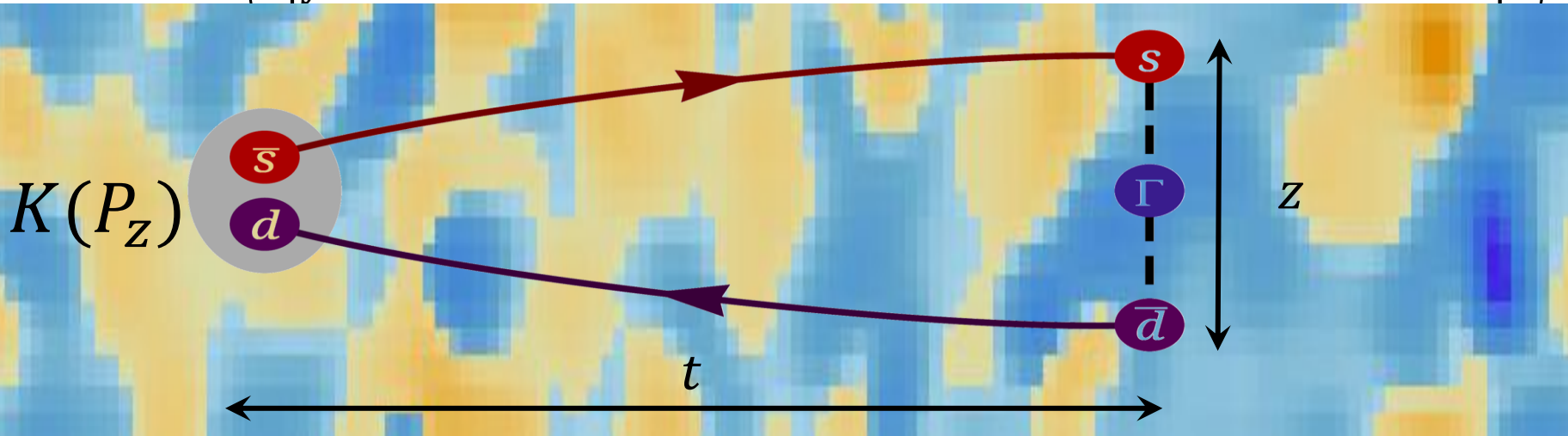
Application on Inverse Problem



R. Zhang, C. Honkala, et al. (MSULat), 2005.13955

Example: Pion/Kaon Distribution Amplitude

$$C_M^{\text{DA}}(z, P, t) = \left\langle 0 \left| \int d^3y e^{i\vec{P}\cdot\vec{y}} \bar{\psi}_1(\vec{y}, t) \gamma_z \gamma_5 U(\vec{y}, \vec{y} + z \hat{z}) \psi_2(\vec{y} + z \hat{z}, t) \bar{\psi}_2(0, 0) \gamma_5 \psi_1(0, 0) \right| 0 \right\rangle$$



Application on Inverse Problem

Quantities
that can be
calculated on
the lattice
today

=

Σ

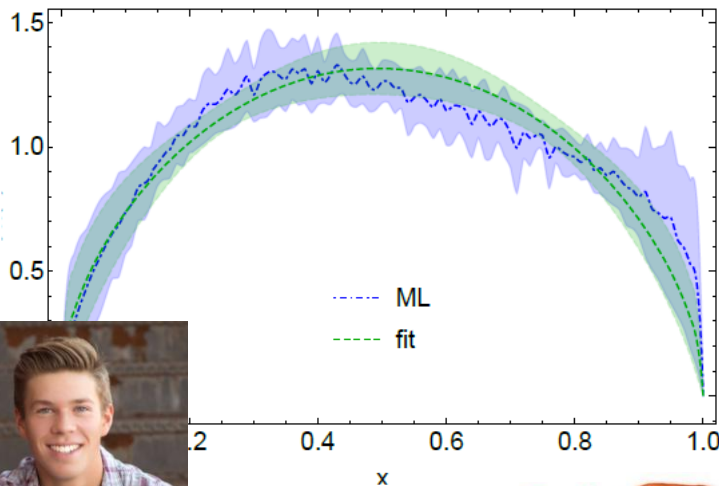
Wanted
PDFs,
GPDs,
etc.

$\times$

pQCD-
calculat
ed
kernel

R. Zhang, C. Honkala, et al. (MSULat), 2005.13955

Pion Distribution Amplitude

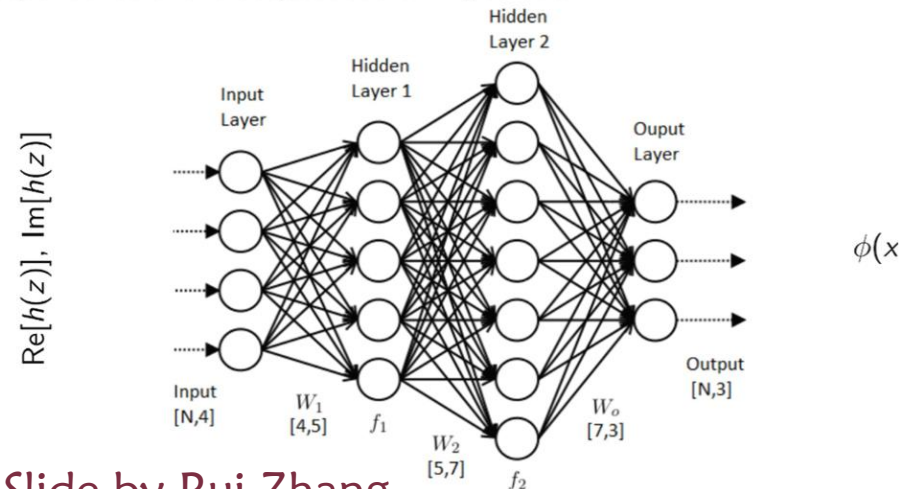


UG: Carson Honkala



Machine Learning - A Promising Solution?

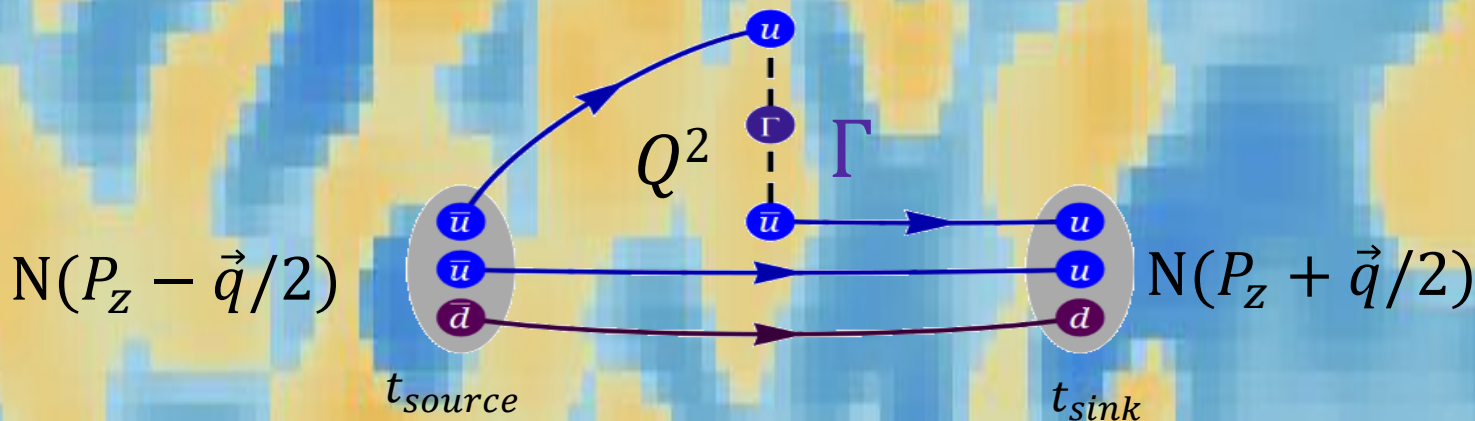
Machine learning models are effective in extracting complicated dependence of the output data on input data.



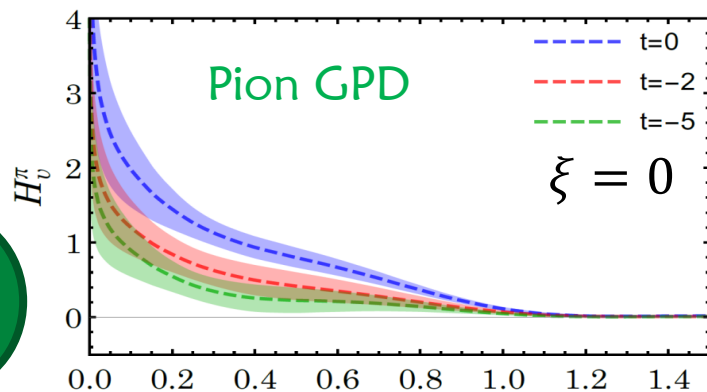
Slide by Rui Zhang

Generalized Parton Distributions

§ On the lattice, one needs to calculate the following

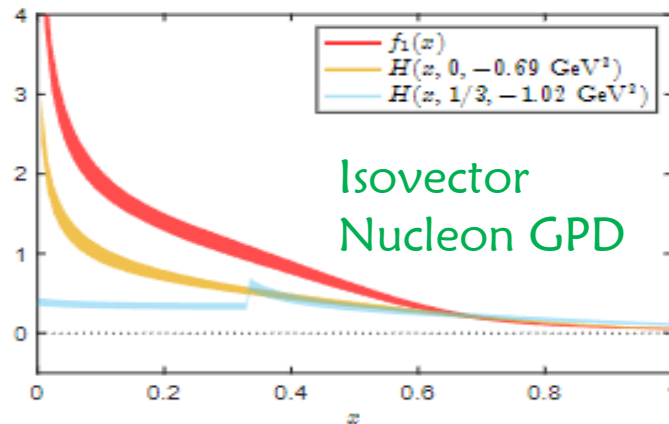


§ Heavy pion-mass results



Wanted
PDFs,
GPDs,
etc...

J. Chen, HL, J. Zhang, 1904.12376



C. Alexandrou et al, 2008.10573



Isvector Nucleon GPDs

§ Nucleon GPD using quasi-PDFs at physical pion mass

∞ MSULat: clover/2+1+1 HISQ

0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV

$$\begin{aligned} F^q(x, \xi, t) &= \int \frac{dz^-}{4\pi} e^{-ixP^+z^-} \langle p' | \bar{q}(z^-/2) \gamma^+ q(-z^-/2) | p \rangle \\ &= \frac{1}{2P^+} \left[H^q(x, \xi, t) \bar{u}(p') \gamma^+ u(p) - E^q(x, \xi, t) \bar{u}(p') \frac{i\sigma^{+\alpha} \Delta_\alpha}{2m} u(p) \right] \end{aligned}$$



Nucleon Polarized GPDs

§ Helicity GPD ($\tilde{H}$) using quasi-PDFs at **physical pion mass**

↻ MSULat: clover/2+1+1 HISQ

0.09 fm, 135-MeV pion mass, $P_z \approx 2$ GeV

$$\begin{aligned}\tilde{F}^q(x, \xi, t) &= \int \frac{dz^-}{4\pi} e^{-ixP^+z^-} \langle p' | \bar{q}(z^-/2) \gamma^+ \gamma_5 q(-z^-/2) | p \rangle \\ &= \frac{1}{2P^+} \left[\tilde{H}^q(x, \xi, t) \bar{u}(p') \gamma^+ \gamma_5 u(p) - \tilde{E}^q(x, \xi, t) \bar{u}(p') \frac{\gamma_5 \Delta^+}{2m} u(p) \right]\end{aligned}$$

