Dispersive analysis of two photon reactions

Igor Danilkin

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in collaboration with
O. Deineka, V. Ermolina, M. Vanderhaeghen
(on $\gamma^{(*)}\gamma^{(*)} \to M_1 M_2$ and $a_\mu[\text{S-wave}, I=1]_{\pi\eta, K\bar{K}}$)
P. Stoffer, M. Hoferichter
(on $a_\mu[\text{S-wave}, I=0]_{\pi\pi, K\bar{K}}$)
M. Lellmann, A. Denig, M. Vanderhaeghen, et al.
(on new MC generator)

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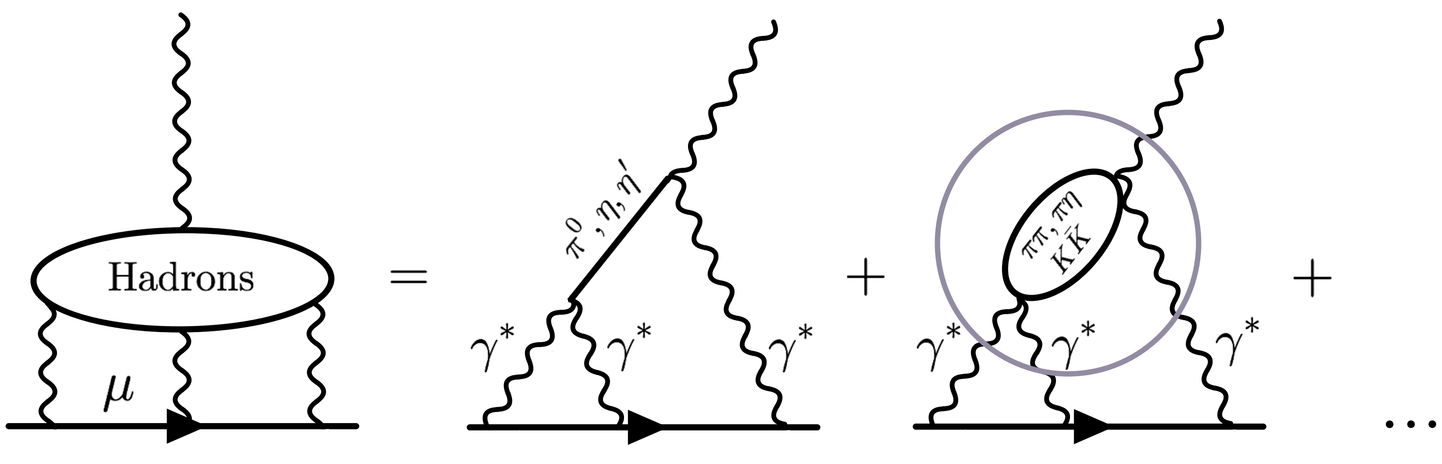
Motivation



Ingredients for HLbL: $\gamma^*\gamma^* \to \pi\pi, \pi\eta, K\bar{K}\ldots$
for spacelike γ^*: $q = -Q^2 < 0$

Contribution	Our estimate
π^0, η, η'-poles	93.8(4.0)
π, K-loops/boxes	−16.4(2)
S-wave $\pi\pi$ rescattering	−8(1)
subtotal	69.4(4.1)
scalars }	− 1(3)
tensors }	
axial vectors	6(6)
u, d, s-loops / short-distance	15(10)
c-loop	3(1)
total	92(19)

[White paper (2020)]

$\boxed{\begin{array}{l}\pi\pi_{I=0}, \pi\pi_{I=2} \\ f_0(500)\end{array}}$ $\rightarrow$ $a_\mu^{\mathrm{HLbL}}[\text{scalars}] = -9.1(1.0) \times 10^{-11}$

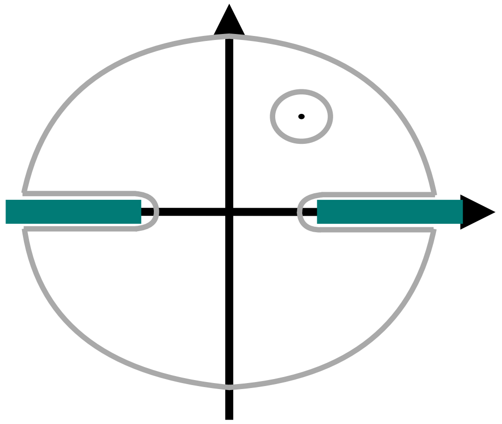
$\boxed{\begin{array}{l}\pi\pi/K\bar{K}_{I=0}, \pi\eta/K\bar{K}_{I=1}, \pi\pi_{I=2} \\ f_0(500)/f_0(980)/a_0(980)\end{array}}$

[White paper (2025)]

[Colangelo et al. (2014-2017),
I.D. et al. (2021),
Deineka et al. (2025)]

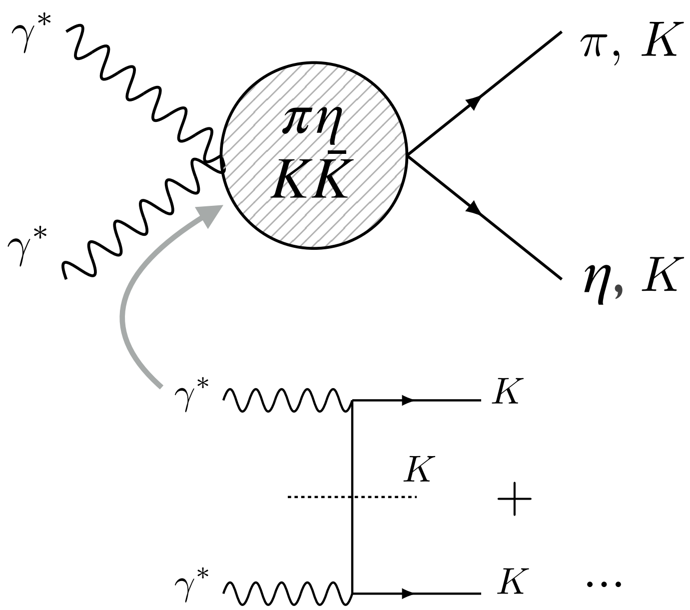
Framework

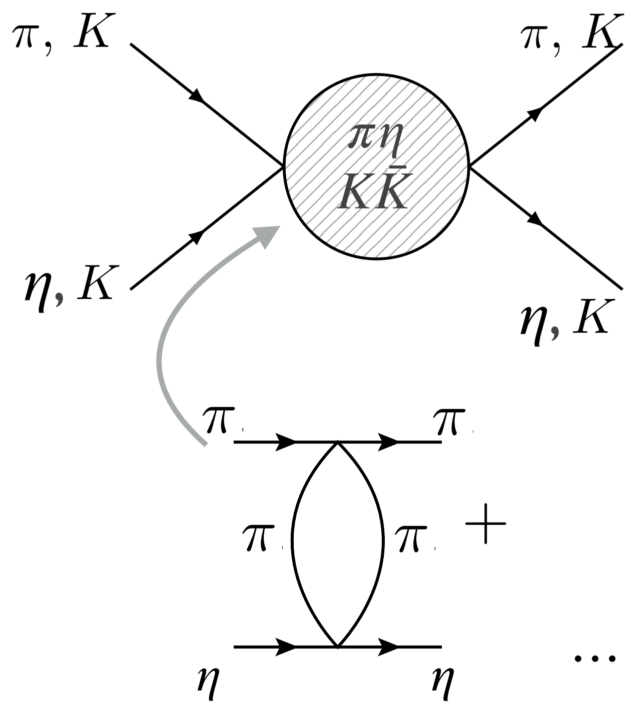
- **Strategy**: apply dispersion relations to the **kinematically unconstrained** p.w. amplitudes



$$t_{ab}(s) = \int_L \frac{ds'}{\pi} \frac{\mathrm{Im}\, t_{ab}(s')}{s'-s} + \sum_c \int_R \frac{ds'}{\pi} \frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$

- Key features:
  1) **Left-hand cuts differ** for each channel
  2) The coupled-channel system $\gamma^*\gamma^*/\pi\eta/K\bar{K}$ can be reduced to to the off-diagonal $\gamma^*\gamma^* \to \pi\eta/K\bar{K}$, which requires the hadronic rescattering $\pi\eta/K\bar{K}$ input





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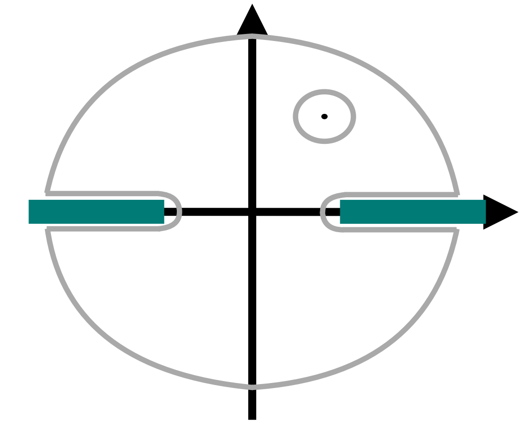

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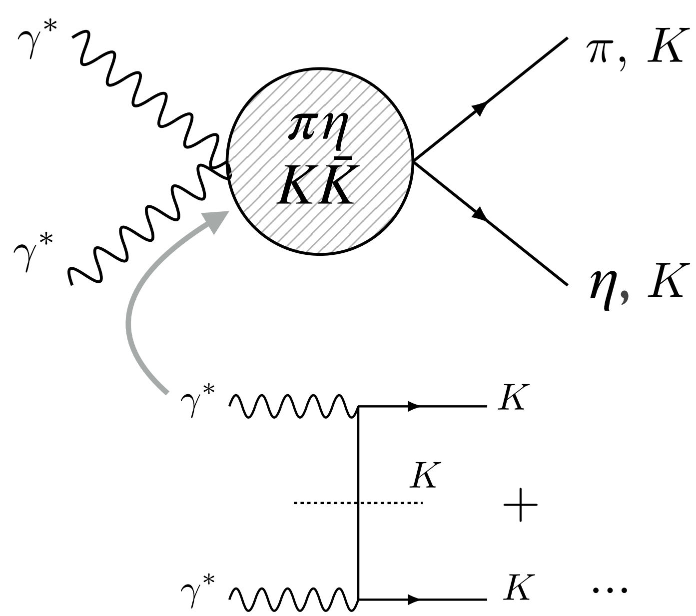
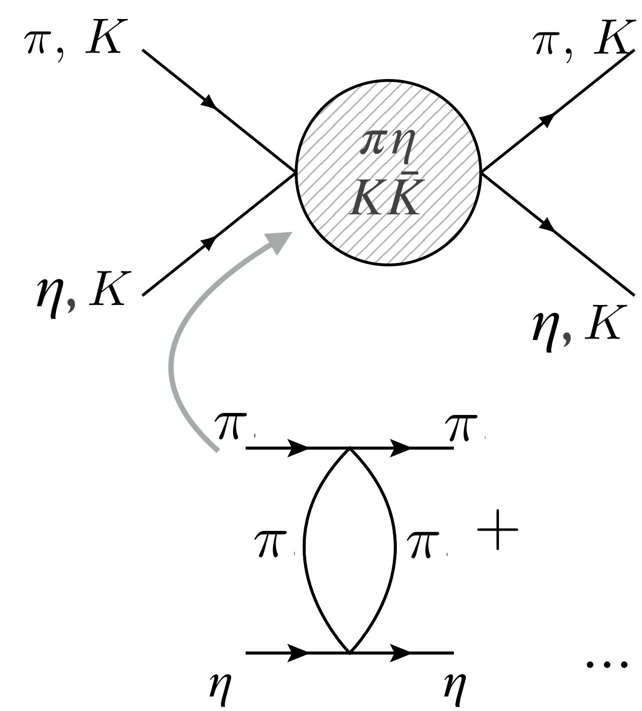
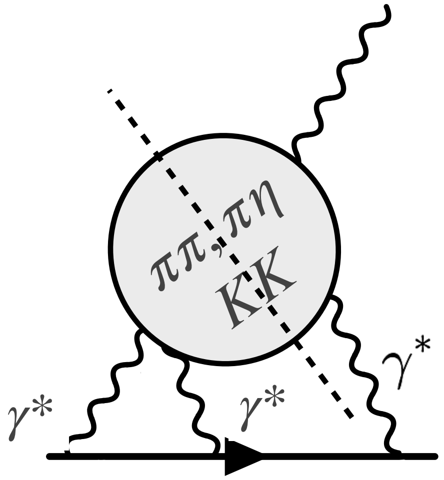


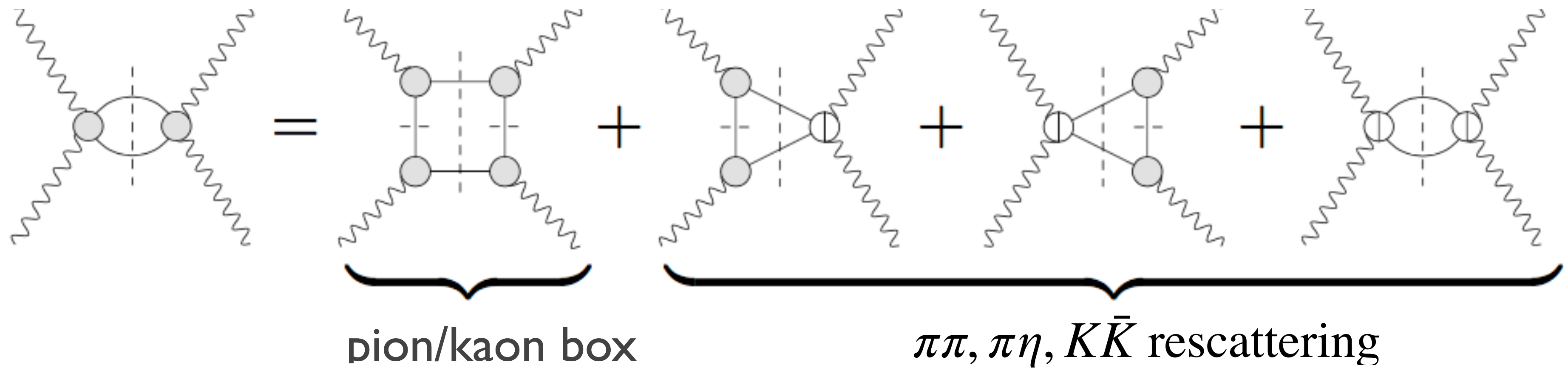


- Details of the analysis:
  1) $\pi\eta/K\bar{K}$ solved via **N/D method** with **conformal mapping** for left-hand cuts (subtracted)
  2) $\gamma^*\gamma^* \to \pi\eta/K\bar{K}$ solved using **modified Muskhelishvili-Omnès formalism** (subtracted and unsubtr.)

[Chew et al. (1960), Gasparyan et al. (2010), Muskhelishvili (1953), Omnès (1958), Garcia-Matrin et al. (2010)]

Challenges





[Colangelo et al. (2014-2017), I.D. et al. (2019, 2021)]

$$a_\mu[\text{S-wave}, I=0]_{\pi\pi} = -9.3(0.9) \times 10^{-11} \qquad f_0(500)$$

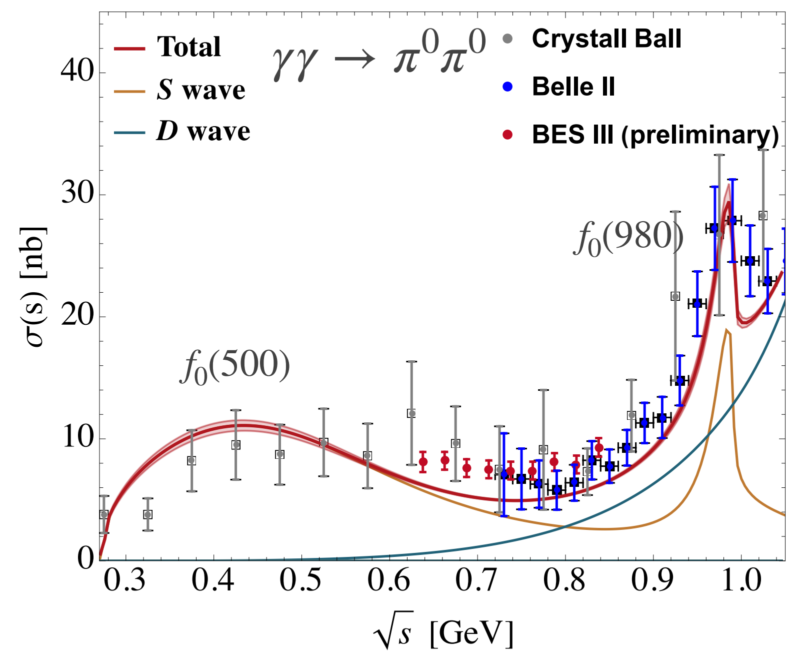
$$a_\mu[\text{S-wave}, I=0]_{\pi\pi, K\bar{K}} = -9.8(1.0) \times 10^{-11} \quad f_0(500)/f_0(980)$$

Unsubtracted dispersion relation for $\gamma^*\gamma^* \to \pi\pi/K\bar{K}$

Left-hand cuts: π/K pole with vector form factors $F_{\pi,K}(Q^2)$

Data used: $\pi\pi/K\bar{K}$ scattering data (Roy-like analyses)

$\gamma\gamma \to \pi^0\pi^0$ used to justify left-hand cut approximation



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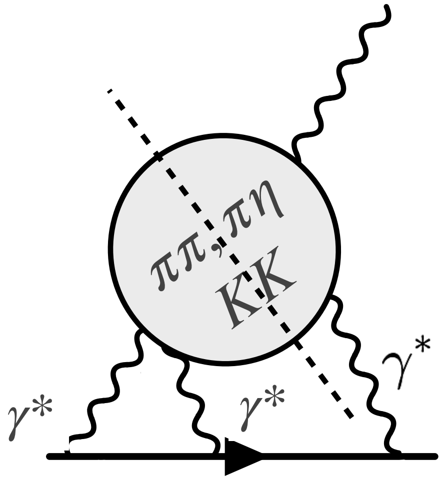

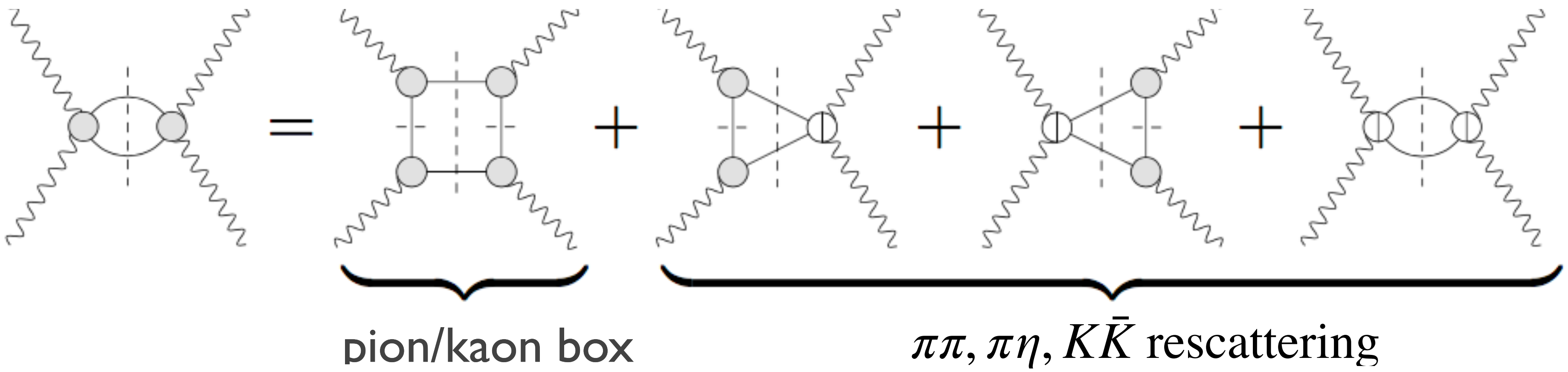

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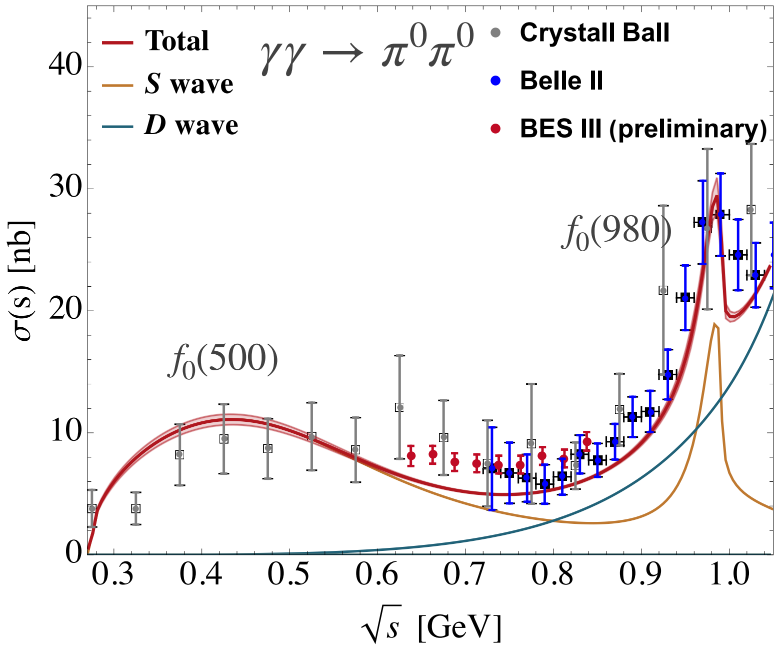


$a_\mu[\text{S-wave}, I=1]_{\pi\eta, K\bar{K}} = ?$ $\quad a_0(980)$

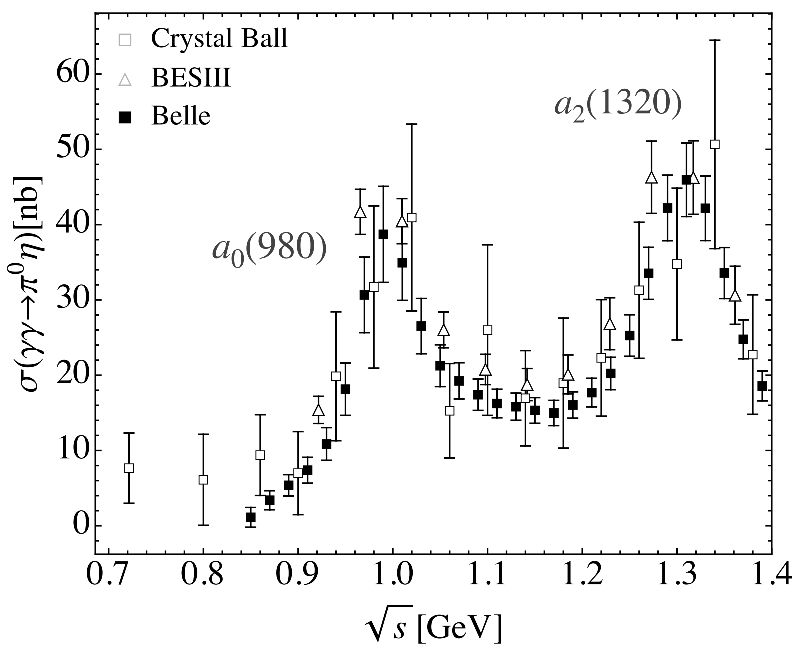
Unsubtracted dispersion relation for $\gamma^*\gamma^* \to \pi\eta/K\bar{K}$

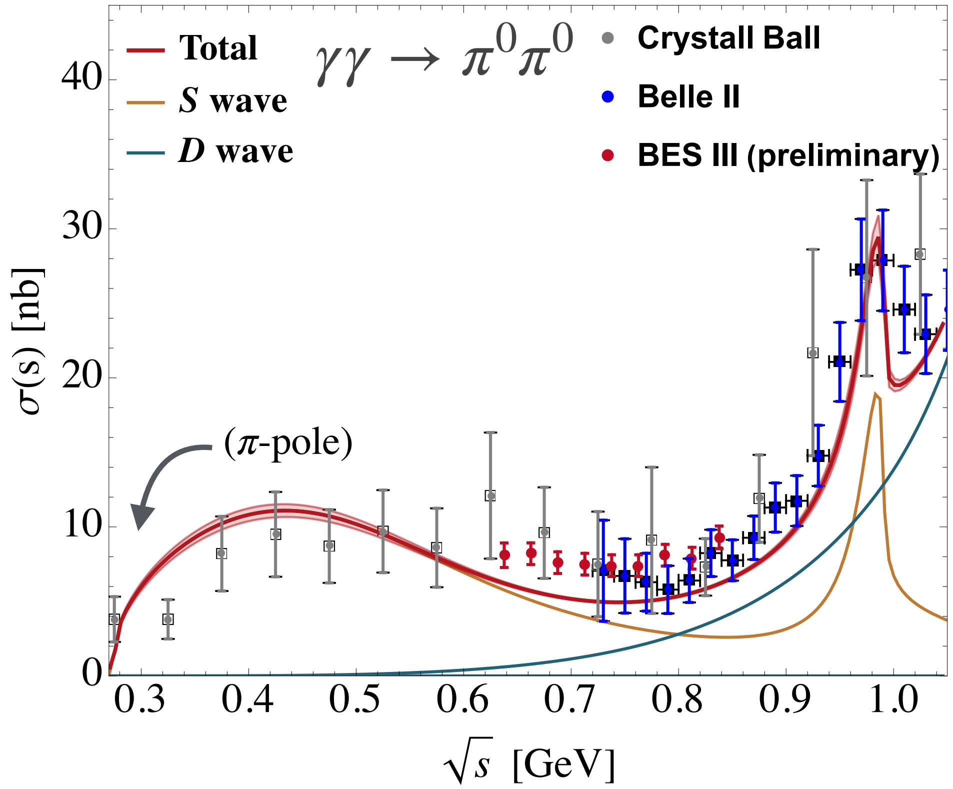
Left-hand cuts: K pole with vector form factor $F_K(Q^2)$

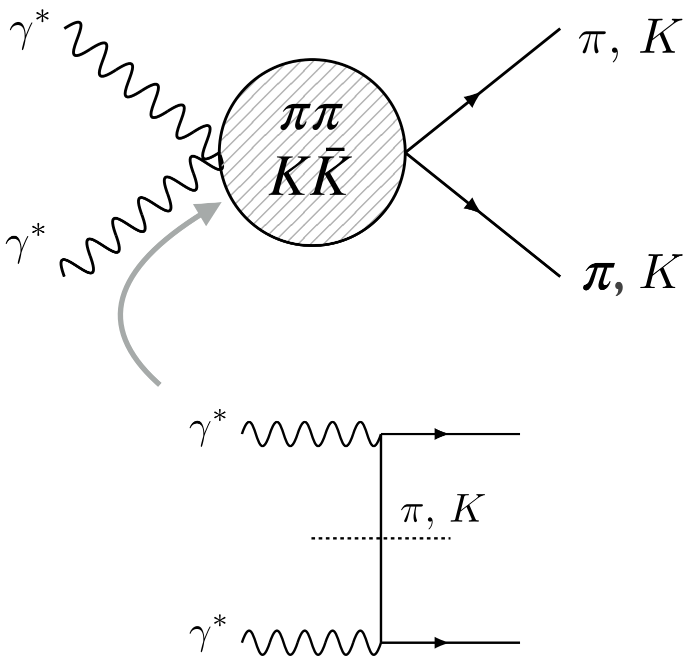
Challenge: ! no direct $\pi\eta/K\bar{K}$ scattering data

$\gamma\gamma \to \pi\eta/K\bar{K}$ data used to constraint $\pi\eta/K\bar{K}$ amplitude

Required check: assess importance of heavier left-hand cuts





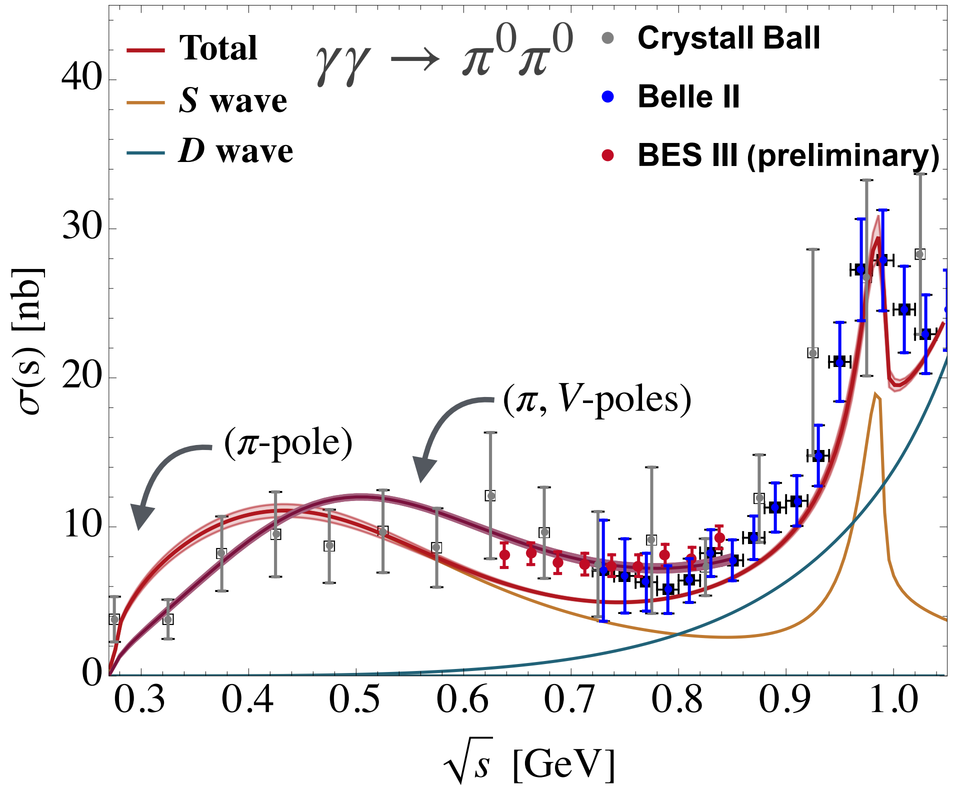


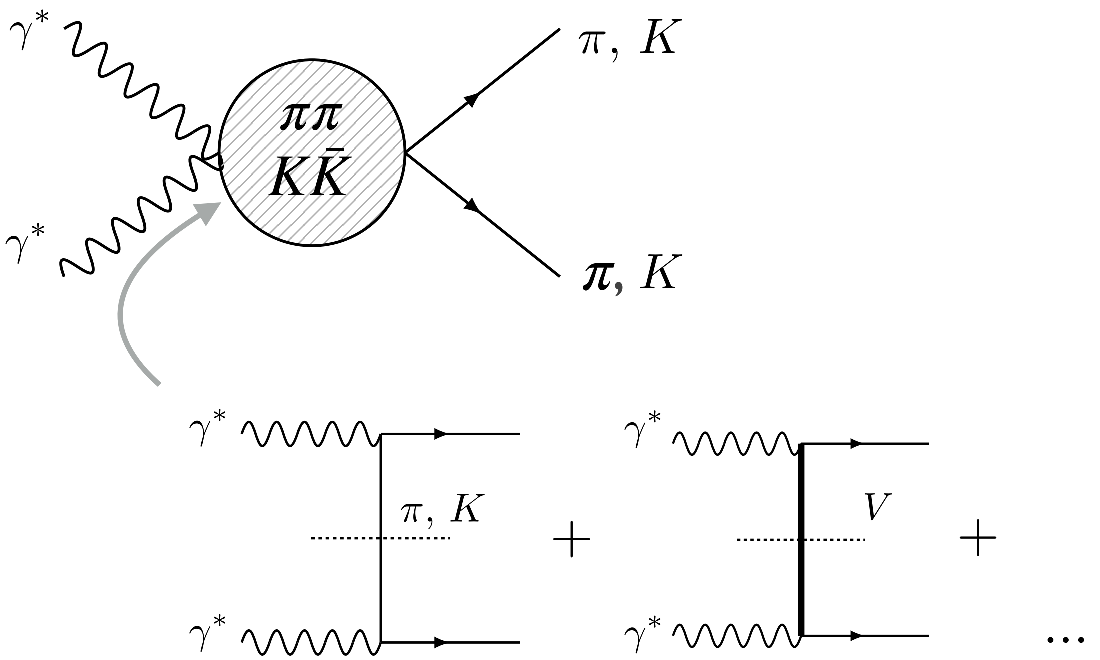
Unsubtracted dispersion relation for $\gamma\gamma \to \pi\pi/K\bar{K}$ (S wave)

- **Left-hand cuts**: π/K pole
- $\Gamma_{\gamma\gamma}(f_0(500), f_0(980))$ consistent with other analyses
- $(\alpha_1 - \beta_1)_{\pi^\pm}$ consistent with ChPT
- $(\alpha_1 - \beta_1)_{\pi^0} \sim 9 \times 10^{-4}\,\text{fm}^3$ (no Adler zero $\gamma\gamma \to \pi^0\pi^0$) vs $(\alpha_1 - \beta_1)^{\chi PT}_{\pi^0} = -1.9(2) \times 10^{-4}\,\text{fm}^3$

[Colangelo et al. (2017)]
[I.D. et al. (2019)]

Pion polarizabilities





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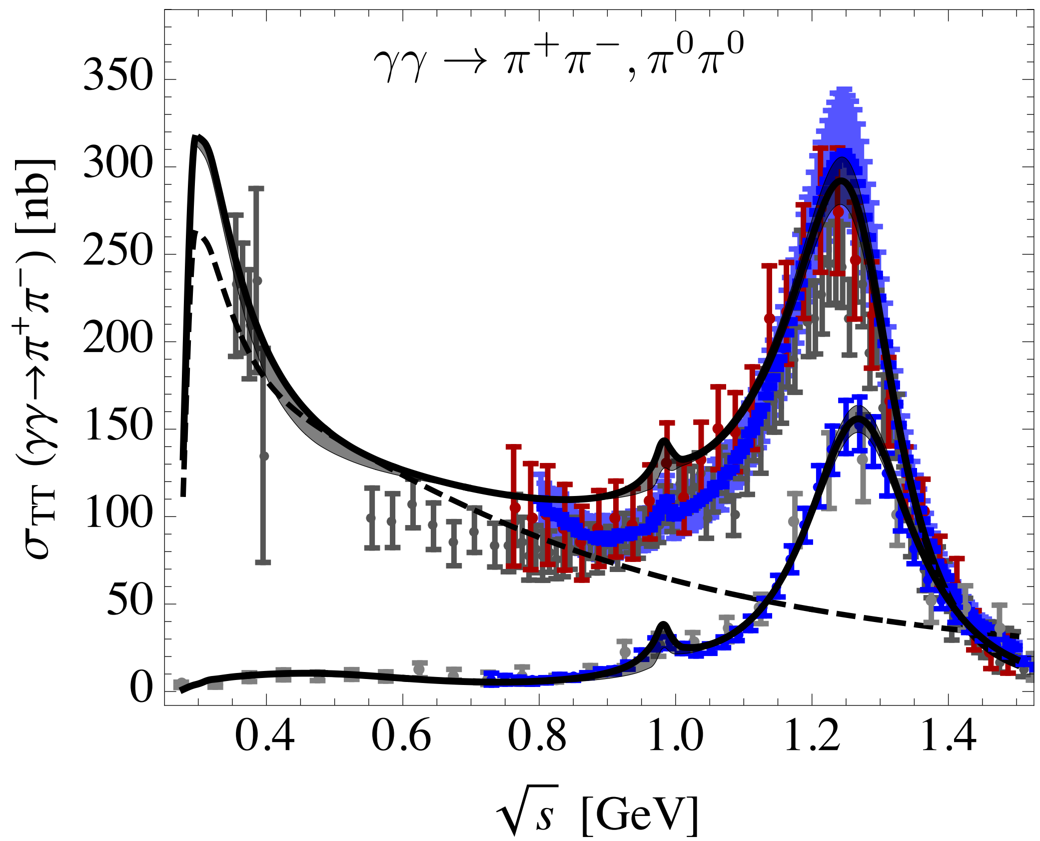
[Colangelo et al. (2017)]
[I.D. et al. (2019)]

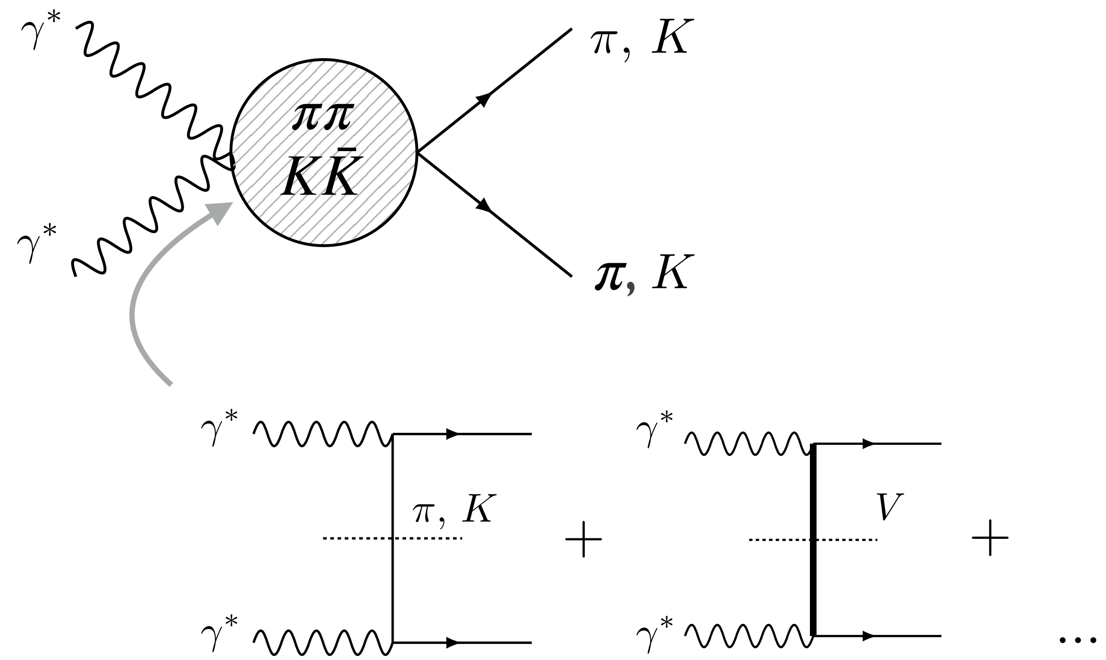
Subtracted dispersion relation for $\gamma\gamma \to \pi\pi/K\bar{K}$ (S wave)

- **Left-hand cuts**: π/K pole + V pole (ω exchange dominates)
- Accurate fit to $\gamma\gamma \to \pi^0\pi^0$ via subtraction constants
- Cure $(\alpha_1 - \beta_1)_{\pi^0}$ by inclusion of the Adler zero
- **!** $d\sigma/d\cos\theta$ from BESIII is crucial for pion polarizabilities
- **!** $\gamma\gamma^* \to \pi\pi$ data required to determine Q^2 dependence of $(\alpha_1 - \beta_1)_\pi$

[Garcia-Matrin et al. (2010)]
[Dai, Pennington (2016)]
[Ermolina et al. EPJ Web Conf (2024)]

Results for $\gamma^*\gamma^* \to \pi\pi$



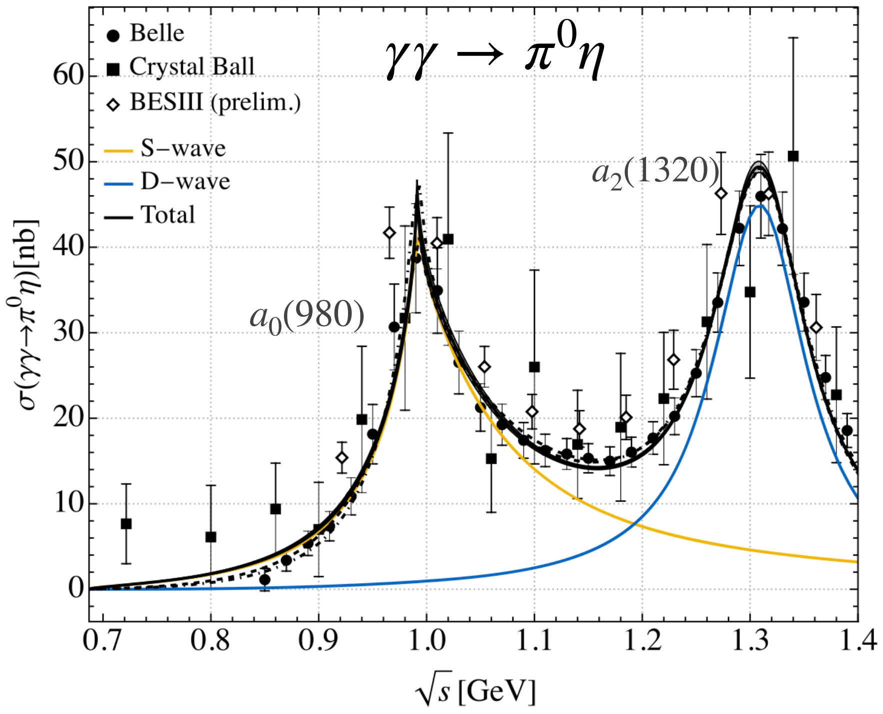


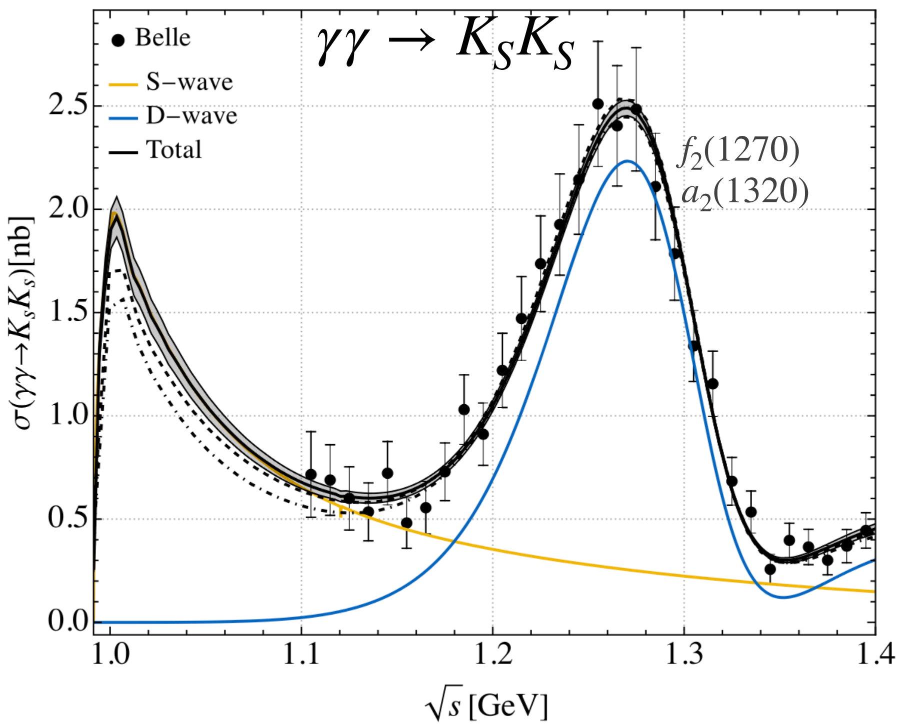
Unsubtracted dispersion relation for $\gamma\gamma \to \pi\pi/K\bar{K}$ (D wave)

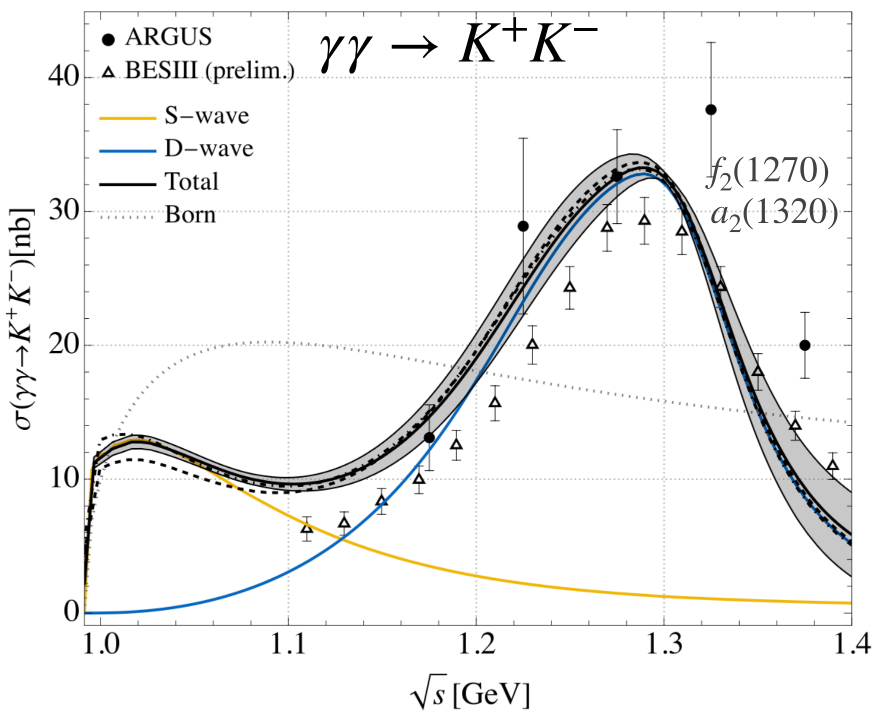
- **Left-hand cuts**: π/K pole + V pole (ω exchange dominates)
- Heavier cuts absolutely necessary for the dispersive description of the D-wave region ($f_2(1270)$)
- Approximate heavier cuts by V-exch, $C_{\rho^{\pm,0}\pi^{\pm,0}\gamma} \simeq C_{\omega\pi^0\gamma}/3 \approx g^{eff}_{VP\gamma} = 0.33\,\mathrm{GeV}^{-1},\ g^{PDG}_{VP\gamma} = 0.37(2)\,\mathrm{GeV}^{-1}$

[Garcia-Matrin et al. (2010)]
[I.D. et al. (2019)]

Results $\gamma\gamma \to \pi\eta/K\bar{K}$







Chiral constraints on $\pi\eta/K\bar{K}$:

- Adler zero $\pi\eta \to K\bar{K}$
- $t_{\pi\eta\to\pi\eta}(s_{th}), t_{\pi\eta\to K\bar{K}}(s_{th})$

D-wave

- Breit-Wigner parametrization for $a_2(1320), f_2(1270)$

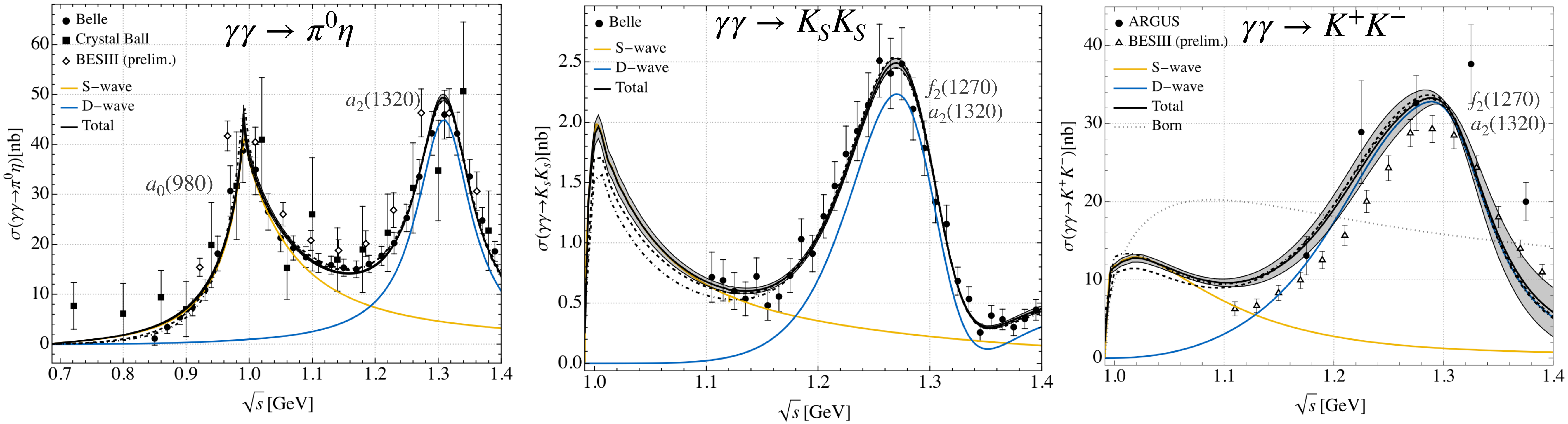
Unsubtracted dispersion relation (S-wave):

- **Left-hand cuts**: K pole
- no Adler zero $\gamma\gamma \to \pi^0\eta$
- Prediction for $\gamma^*\gamma^* \to \pi\eta/K\bar{K}_{I=1}$

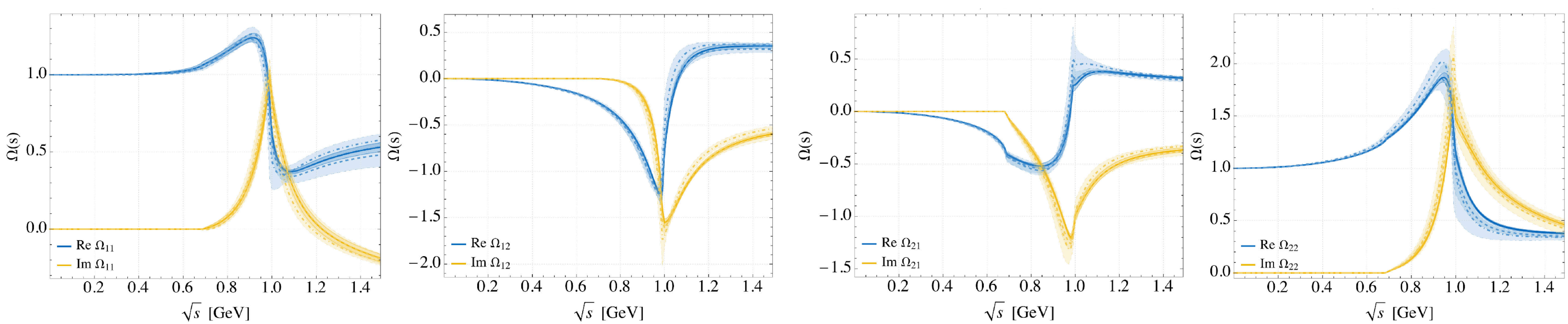
Subtracted dispersion relation (S-wave):

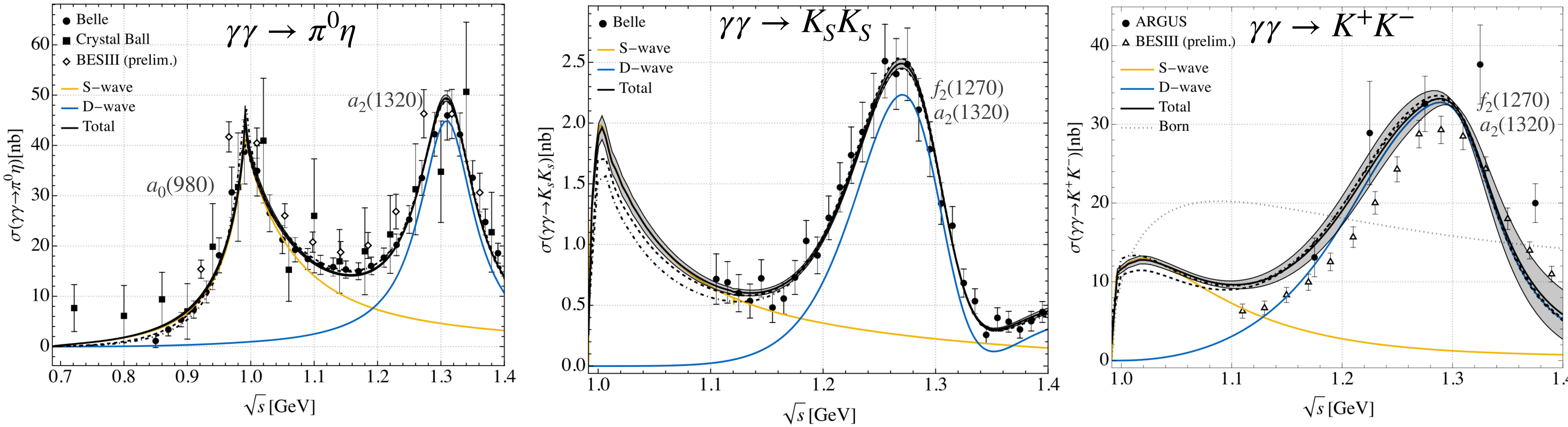
- **Left-hand cuts**: K pole + V pole
- Adler zero $\gamma\gamma \to \pi^0\eta$
- Require $\gamma^*\gamma^* \to \pi\eta/K\bar{K}$ data to determine Q^2 dependence of the subtraction constants

Results $\gamma\gamma \to \pi\eta/K\bar{K}$



- **Unsubtracted** and **Subtracted** dispersion relations for $\gamma\gamma \to \pi\eta/K\bar{K}$ yield very similar $\pi\eta/K\bar{K}$ amplitudes





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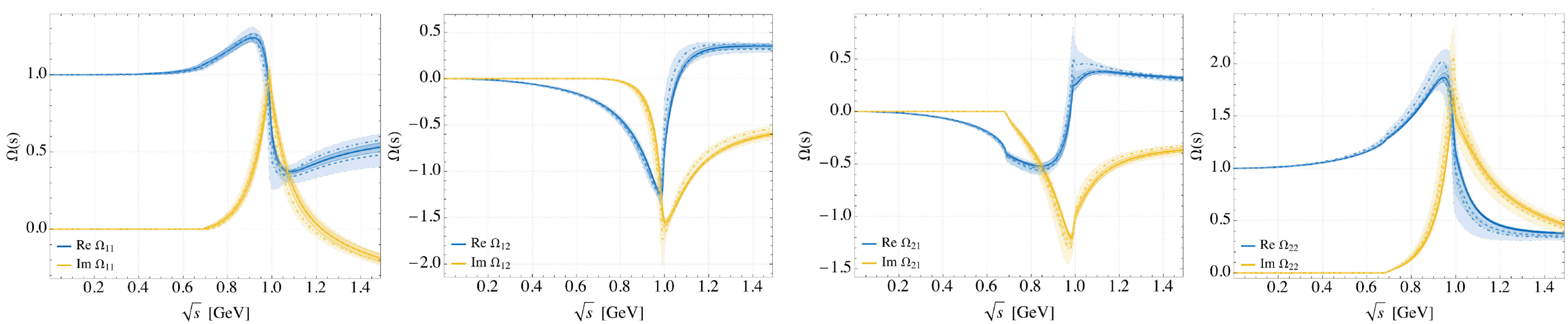
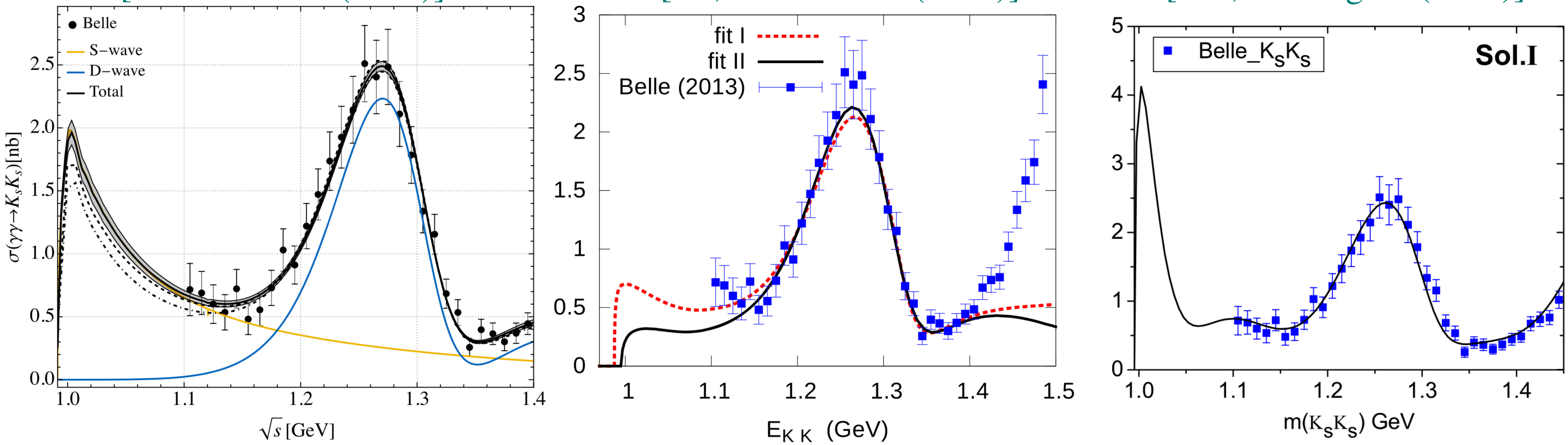


		Pole Position (MeV)	$\pi\eta$ (GeV)	$K\bar{K}$ (GeV)	$\gamma\gamma$ (MeV)
Deineka et al. 2024	RSII	$1047(18) - i\,72(17)$	3.8(3)	5.2(4)	7.3(5)
	RSIII	$930(25) - i\,80(10)$	2.9(1)	2.0(1)	8.9(3)
Lu et al. 2020	RSII	$1000^{+13}_{-1} - i\,37^{+13}_{-3}$	$2.2^{+0.6}_{-0.2}$	$4.0^{+0.3}_{-0.2}$	$5.0^{+0.9}_{-0.5}$

PDG (MeV):
(960...1030) – i(20...70)

Differences from other theoretical approaches

[Deineka et al. (2025)] [Lu, Moussallam (2020)] [Dai, Pennington (2014)]



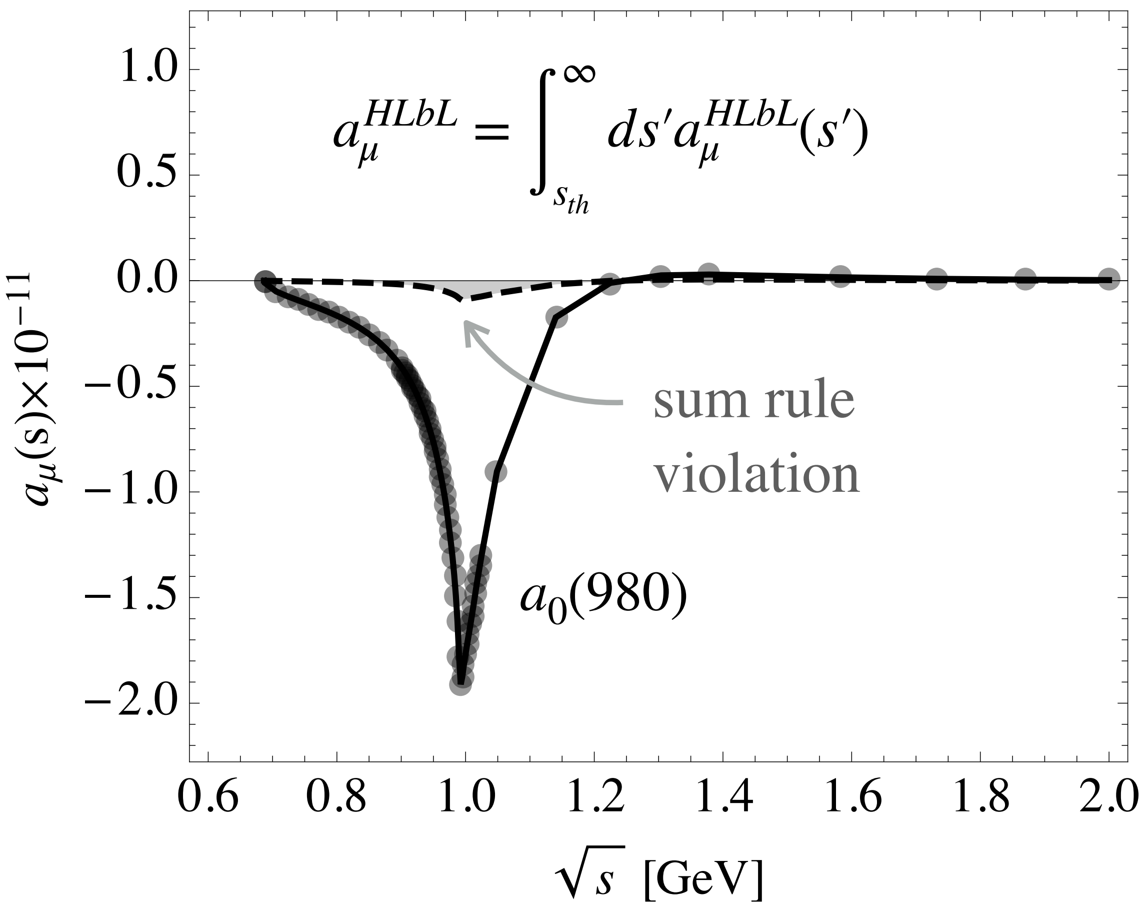
- [Oller et al. (1998)] Summed loops using unitarized tree-level ChPT
- [Danilkin et al. (2012, 2017)] Dispersive (N/D) method with Omnès matrix fixed by ChPT
  No soft-photon constraint or Adler zero for $\gamma\gamma \to \pi^0\eta$
- [Dai, Pennington (2014)] Amplitude analysis of $\gamma\gamma \to \pi\pi/K\bar{K}$
  Isovector channel parametrized phenomenologically by polynomials in s
- [Lu, Moussallam (2020)] Direct Omnès solution, **heavily** based on ChPT (only partly relies on $\gamma\gamma$ data)
  $\Omega_{ab} \sim O(1/s) \Rightarrow \delta_1 + \delta_2 \to 2\pi$, included also $a_0(1450)$ resonance
  Only subtracted DR for $\gamma\gamma \to \pi^0\eta/K\bar{K}$ is possible
  No prediction for of $\gamma^*\gamma^* \to \pi^0\eta/K\bar{K}$

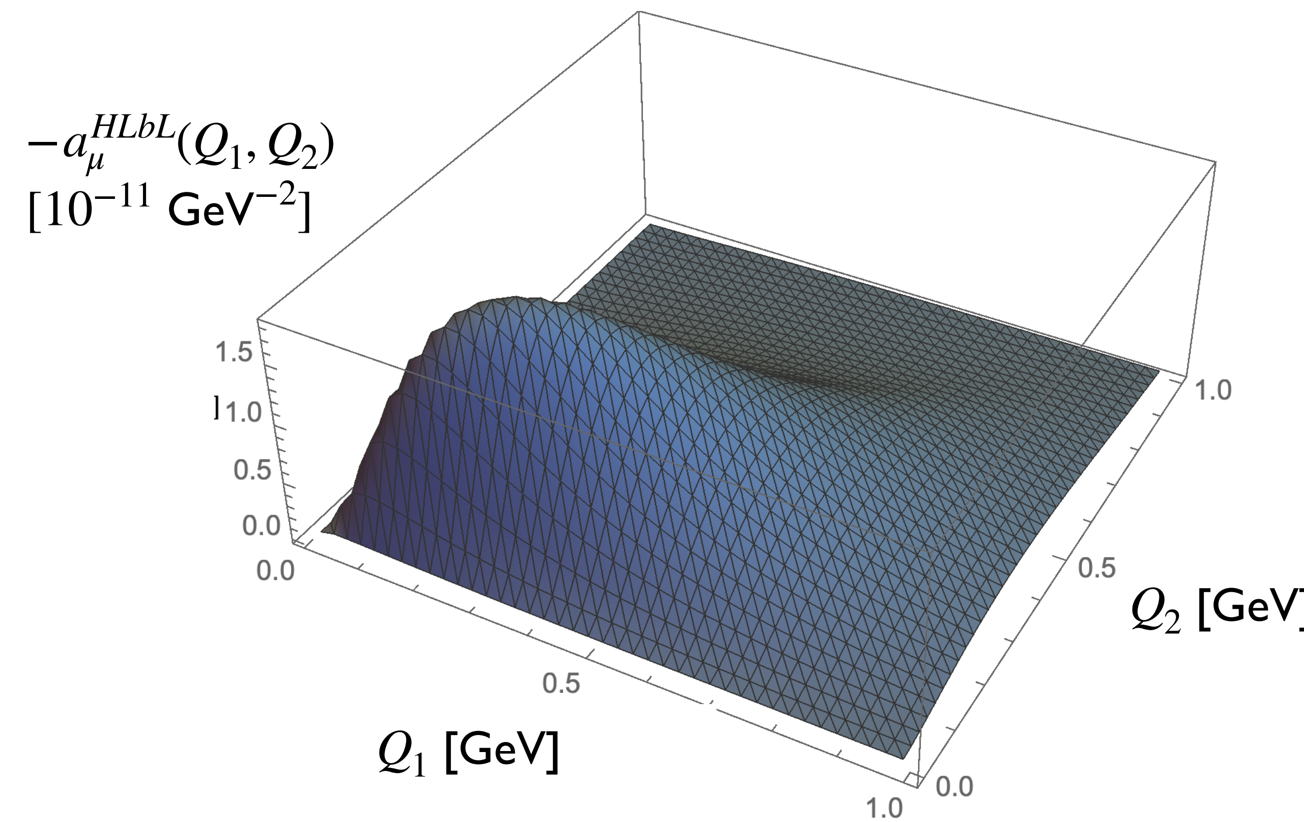
$$\Omega_{ab}(s) = \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{t^*_{ac}(s')\,\rho_c(s')\,\Omega_{cb}(s')}{s'-s}$$

$a_0(980)$ contribution to HLbL piece of a_μ

$$a_\mu^{HLbL} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1,Q_2,Q_3)\bar{\Pi}_i(Q_1,Q_2,Q_3)$$

[Colangelo et al. (2014-2017)]





- Error budget: $\pi\eta/K\bar{K}$, TFF input, sum rule violation

$$a_\mu[\text{S-wave}, I=1]_{\pi\eta,K\bar{K}} = -\,0.44(3)(3)(2)\times 10^{-11}$$ [Deineka et al. (2024)]

$$a_\mu[\text{NWA}]_{a_0(980)} = -\left([0.3,0.6]^{+0.2}_{-0.1}\right)\times 10^{-11}$$ [I.D. et al. (2021), Schuler et al. (1998)]

- Upcoming BESIII data $\gamma\gamma^* \to \pi\pi, \pi^0\eta\ (Q^2 = 0.2-2.0\,\mathrm{GeV}^2)$

new two-photon MC generator

HadroTOPS = event generator for Hadron production in Two-Photon Scattering

Theory contribution to HadroTOPS: $\pi\pi/\pi\eta$ channels

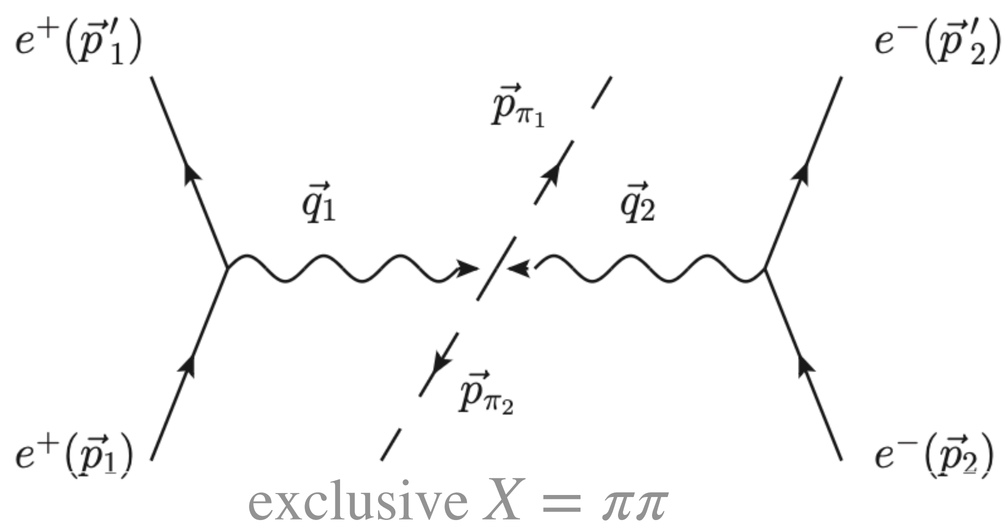
- The cross section for the **exclusive** process $e^+e^- \to e^+e^-\pi\pi$ is given by

$$d\sigma_{h_1h_2} = \frac{1}{F}\, d\mathrm{Lips} \sum_{h_1',h_2'} |\mathscr{M}|^2 = \frac{1}{F}\, d\mathrm{Lips}\, \frac{e^4}{Q_1^4 Q_2^4} \times \begin{cases} L_{1,\mu\mu'}\, L_{2,\nu\nu'}\, H^{\mu\nu}\, (H^{\mu'\nu'})^* \\ \displaystyle\sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'} \rho_{h_1}^{\lambda_1\lambda_1'}\, \rho_{h_2}^{\lambda_2\lambda_2'}\, H_{\lambda_1\lambda_2}\, H^*_{\lambda_1'\lambda_2'} \end{cases}$$

Lorentz-covariant form (used in **Ekhara 3.2**) [Czyz et al.]

equivalent form (used in **HadroTOPS**) [Lellmann et al. (2025)]

where $H_{\lambda_1\lambda_2} \equiv \epsilon^\mu(q_1,\lambda_1)\, \epsilon^\nu(q_2,\lambda_2)\, H_{\mu\nu}$ are the helicity amplitudes for $\gamma^*\gamma^* \to \pi\pi$



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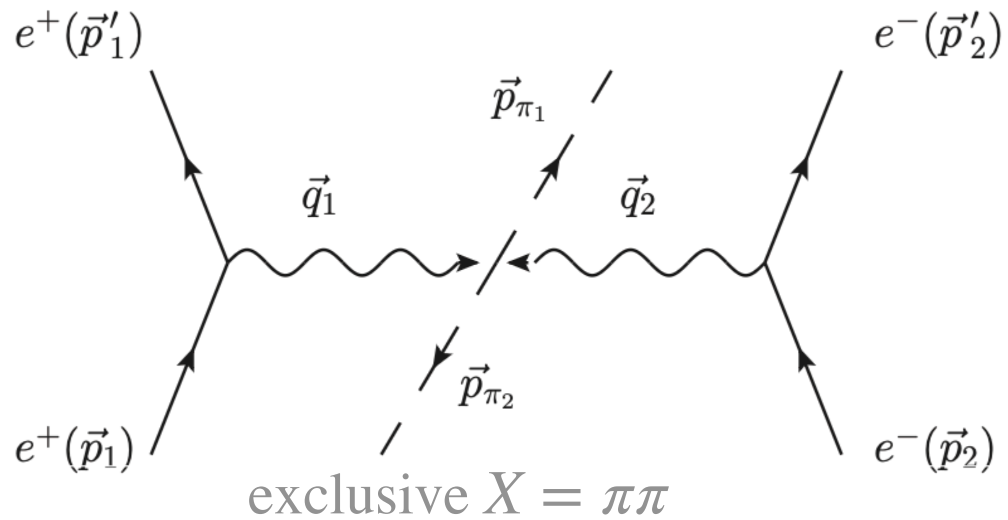


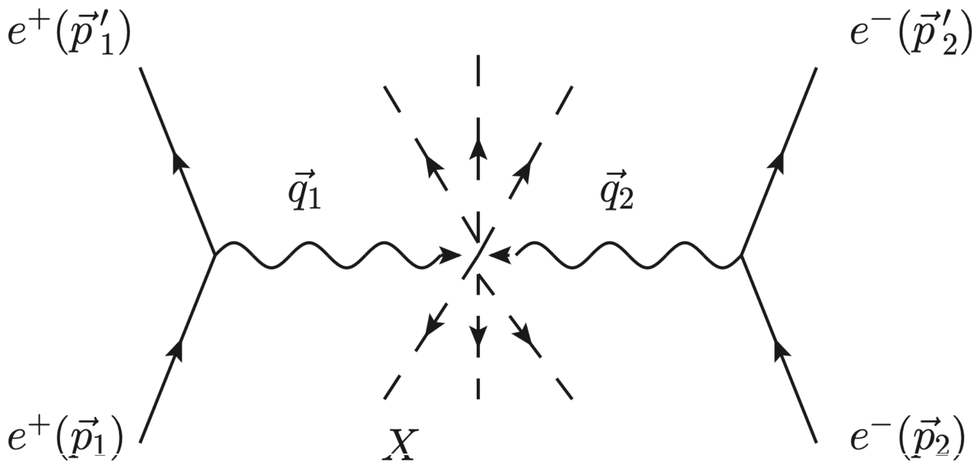
- The central object in the **HadroTOPS** framework is the imaginary part of the **forward LbL amplitude** $\text{Im}\, M_{\lambda_1'\lambda_2',\lambda_1\lambda_2}$, which, according to unitarity, can be written as a sum over all intermediate hadronic states

$$\text{Im}\, M_{\lambda_1'\lambda_2',\lambda_1\lambda_2} = \frac{1}{2} \sum_X \int d\Gamma_X\, (2\pi)^4\, \delta^4(q_1 + q_2 - p_X)\, H_{\lambda_1\lambda_2}\, H^*_{\lambda_1'\lambda_2'} = \sum_{X=\pi,\ldots} + \sum_{X=\pi\pi,\ldots} + \ldots$$

Theory contribution to HadroTOPS: $\pi\pi/\pi\eta$ channels

- The **inclusive** $e^+e^- \to e^+e^-X$ cross-section can be expressed compactly in terms of 8 independent response functions $\mathrm{Im}\, M_{++,++}, \ldots, \mathrm{Im}\, M_{++,00}$ [Bonneau et al. (1975), Budnev et al. (1975)]

$$d\sigma_{h_1h_2} = \frac{\alpha^2}{8\pi^4 Q_1^2 Q_2^2} \frac{\sqrt{X}}{s(1-4m_e^2/s)^{1/2}} \frac{d^3\vec{p}_1{}'}{E_1'} \frac{d^3\vec{p}_2{}'}{E_2'} \Big\{ 4\rho_1^{++}\rho_2^{++}\frac{1}{2}(\sigma_0+\sigma_2) + \rho_1^{00}\rho_2^{00}\sigma_{LL}$$
$$+2\rho_1^{++}\rho_2^{00}\sigma_{TL} + 2\rho_1^{00}\rho_2^{++}\sigma_{LT} + 2\left(\rho_1^{++}-1\right)\left(\rho_2^{++}-1\right)\cos(2\tilde{\phi})\,\tau_{TT}$$
$$+8\left[\left(\rho_1^{00}+1\right)\left(\rho_2^{00}+1\right)\left(\rho_1^{++}-1\right)\left(\rho_2^{++}-1\right)\right]^{1/2}\cos\tilde{\phi}\,\frac{1}{2}(\tau_0+\tau_1) + h_1h_2(\ldots)\Big\}$$



inclusive (sum over X)

$$\sigma_0 = \frac{1}{2\sqrt{X}}\,\mathrm{Im}\, M_{++,++} = \sum_X \ldots$$

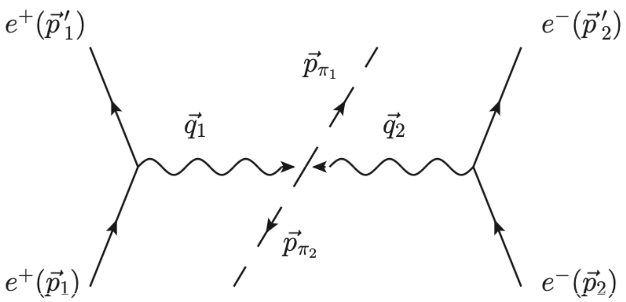
$$\ldots$$

$$\tau_0 = \frac{1}{2\sqrt{X}}\,\mathrm{Im}\, M_{++,00} = \sum_X \ldots$$

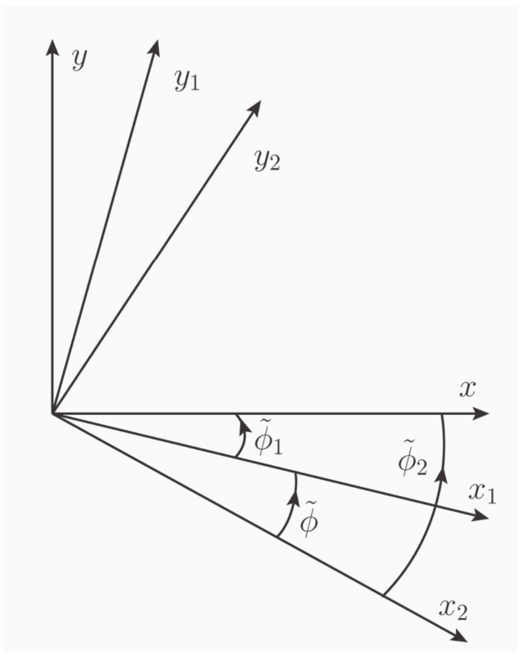
for $X = \pi\pi$
it is complex

Theory contribution to HadroTOPS: $\pi\pi/\pi\eta$ channels

- For the **exclusive** process $e^+e^- \to e^+e^-\pi\pi$ the expressed becomes much longer $(15^{(\text{unpol})} + 10 = 25$ differential hadronic response functions) with additional differential dependence on azimuthal angles [Lellmann et al. (2025)]



$$
\begin{aligned}
d\sigma^{(unpol)} = {} & \frac{\alpha^2}{8\pi^4 Q_1^2 Q_2^2} \frac{\sqrt{X}}{s(1-4m^2/s)^{1/2}} \frac{d^3\vec{p}_1^{\,\prime}}{E_1'} \frac{d^3\vec{p}_2^{\,\prime}}{E_2'} \frac{d\Omega_\pi}{2\pi} \frac{4}{(1-\varepsilon_1)(1-\varepsilon_2)} \\
& \times \left\{ \frac{1}{2}\left(\frac{d\sigma_0}{d\cos\theta_\pi} + \frac{d\sigma_2}{d\cos\theta_\pi} \right) + \left[\varepsilon_1 + \frac{2m^2}{Q_1^2}(1-\varepsilon_1)\right]\left[\varepsilon_2 + \frac{2m^2}{Q_2^2}(1-\varepsilon_2)\right] \frac{d\sigma_{LL}}{d\cos\theta_\pi} \right. \\
& + \left[\varepsilon_2 + \frac{2m^2}{Q_2^2}(1-\varepsilon_2)\right]\left(1+\varepsilon_1\cos\left(2\tilde{\phi}_1\right)\right) \frac{d\sigma_{TL}}{d\cos\theta_\pi} + \left[\varepsilon_1 + \frac{2m^2}{Q_1^2}(1-\varepsilon_1)\right]\left(1+\varepsilon_2\cos\left(2\tilde{\phi}_2\right)\right) \frac{d\sigma_{LT}}{d\cos\theta_\pi} \\
& + \frac{1}{2}\varepsilon_1\varepsilon_2 \left[\cos 2(\tilde{\phi}_2 - \tilde{\phi}_1)\frac{d\sigma_0}{d\cos\theta_\pi} + \cos 2(\tilde{\phi}_1 + \tilde{\phi}_2)\frac{d\sigma_2}{d\cos\theta_\pi}\right] - \left[\varepsilon_1\cos\left(2\tilde{\phi}_1\right) + \varepsilon_2\cos\left(2\tilde{\phi}_2\right)\right]\frac{d\tau_{T2}}{d\cos\theta_\pi} \\
& + \left[\varepsilon_1(1+\varepsilon_1) + \frac{4m^2}{Q_1^2}\varepsilon_1(1-\varepsilon_1)\right]^{1/2}\left[\varepsilon_2(1+\varepsilon_2) + \frac{4m^2}{Q_2^2}\varepsilon_2(1-\varepsilon_2)\right]^{1/2} \\
& \left. \times \left[\cos\left(\tilde{\phi}_2 - \tilde{\phi}_1\right)\left(\frac{d\tau_0}{d\cos\theta_\pi} + \frac{d\tau_1}{d\cos\theta_\pi}\right) + \cos\left(\tilde{\phi}_1 + \tilde{\phi}_2\right)\left(\frac{d\tau_1}{d\cos\theta_\pi} - \frac{d\tau_{L2}}{d\cos\theta_\pi}\right)\right] \cdots \right\}
\end{aligned}
$$



- **HadroTOPS** results are consistent with **Ekhara 3.2**

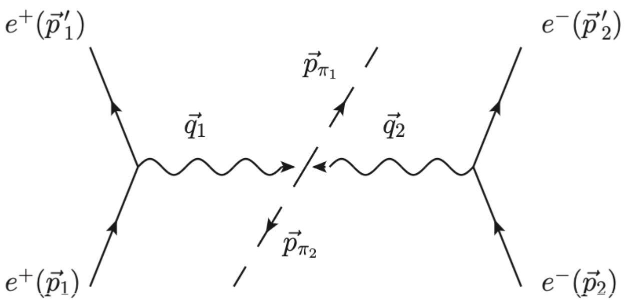
Planes: $e^+e^+, e^-e^-, \pi\pi$

$\tilde{\phi} = \tilde{\phi}_2 - \tilde{\phi}_1$

$d^3\vec{p}_1^{\,\prime} = |\vec{p}_1^{\,\prime}|^2 d|\vec{p}_1^{\,\prime}|\, d\cos\theta_1\, d\tilde{\phi}_1$

$d^3\vec{p}_2^{\,\prime} = |\vec{p}_2^{\,\prime}|^2 d|\vec{p}_2^{\,\prime}|\, d\cos\theta_2\, d\tilde{\phi}_2$

$\phi_\pi = 0$ (choice)

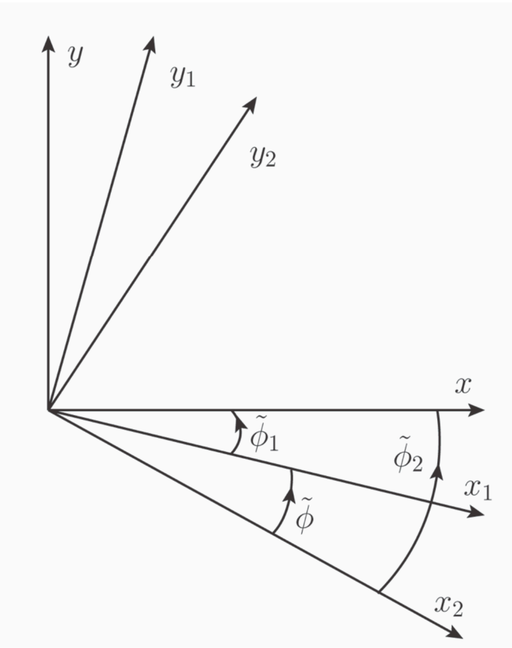


- Integrating over $\tilde{\phi}_2$ and taking the limit $Q_2 \to 0$

$$d\sigma^{(unpol)}|_{Q_2^2\to 0} = \frac{\alpha^2}{8\pi^4 Q_1^2 Q_2^2} \frac{\sqrt{X}}{s(1-4m^2/s)^{1/2}} \frac{d^3\vec{p}_1'}{E_1'} \frac{d^3\vec{p}_2'}{E_2'} \frac{d\Omega_\pi}{2\pi} \frac{4}{(1-\varepsilon_1)(1-\varepsilon_2)}$$

$$\left\{ \frac{1}{2}\left(\frac{d\sigma_0}{d\cos\theta_\pi} + \frac{d\sigma_2}{d\cos\theta_\pi} \right) + \left[\varepsilon_1 + \frac{2m^2}{Q_1^2}(1-\varepsilon_1) \right] \frac{d\sigma_{LT}}{d\cos\theta_\pi} - \varepsilon_1 \cos(2\tilde{\phi}_1) \frac{d\tau_{T2}}{d\cos\theta_\pi} \right.$$

$$\left. + \left[\varepsilon_1(1+\varepsilon_1) + \frac{4m^2}{Q_1^2}\varepsilon_1(1-\varepsilon_1) \right]^{1/2} \cos\tilde{\phi}_1 \left(\frac{d\tau_{-12}}{d\cos\theta_\pi} - \frac{d\tau_{-1T}}{d\cos\theta_\pi} \right) \right\}$$



- **Novel way** of to experimentally **extract additional information** about response functions of $\gamma^*\gamma^* \to \pi\pi$. For example

$$\frac{d\sigma_0}{d\cos\theta_\pi} \equiv \frac{\beta_\pi}{64\pi\sqrt{X}} \left|H_{++}\right|^2$$

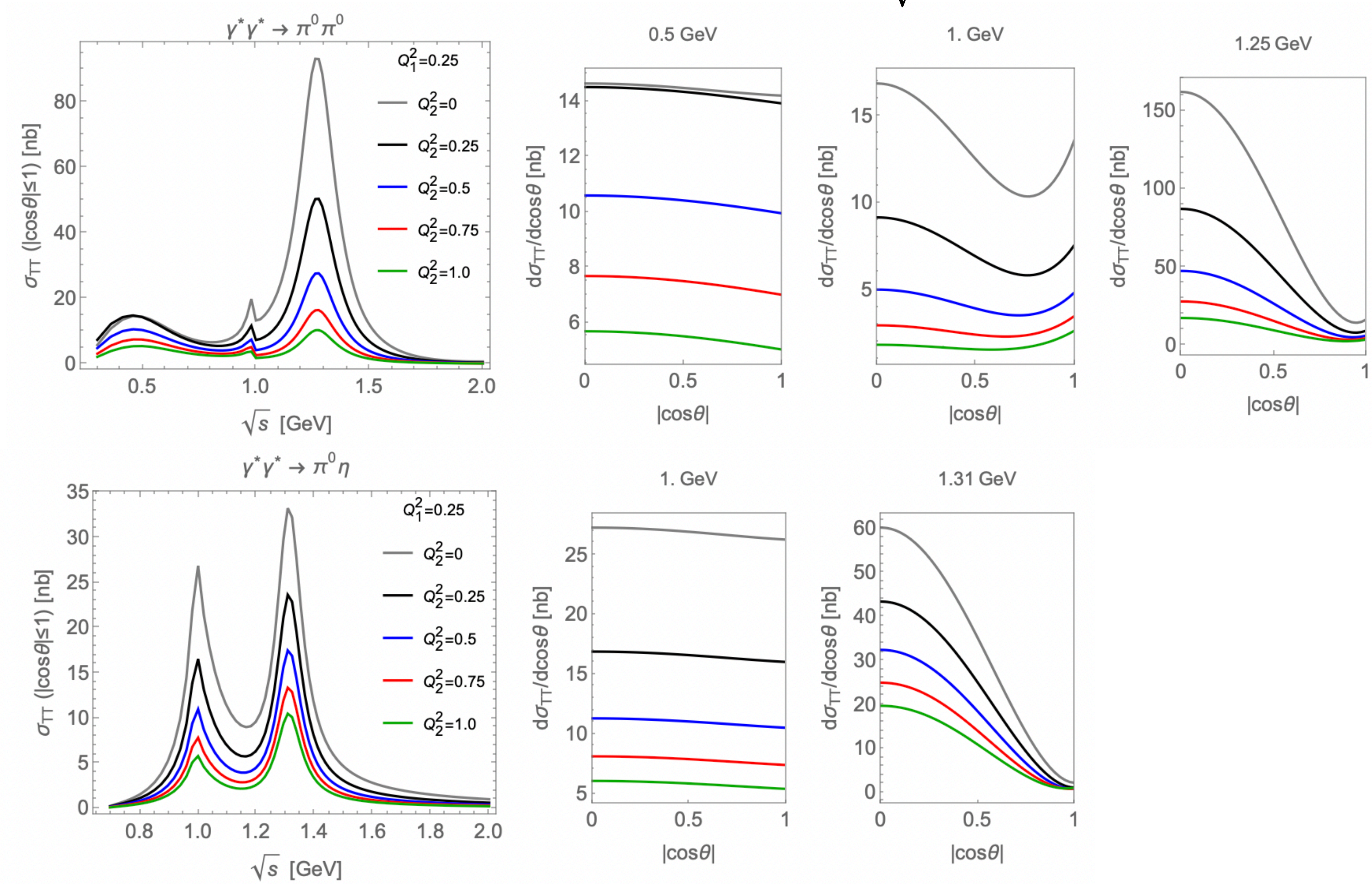
$$\frac{d\sigma_2}{d\cos\theta_\pi} \equiv \frac{\beta_\pi}{64\pi\sqrt{X}} \left|H_{+-}\right|^2$$

$$\frac{d\tau_{T2}}{d\cos\theta_\pi} \equiv \frac{\beta_\pi}{64\pi\sqrt{X}} \mathrm{Re}\left(H_{++}^* H_{+-}\right)$$

Theory contribution to HadroTOPS: $\pi\pi/\pi\eta$ channels

- Example of input provided to HadroTOPS

$$\frac{d\sigma_{TT}}{d\cos\theta_\pi} \equiv \frac{\beta_\pi}{32\pi\sqrt{X}}\left(\left|H_{++}\right|^2 + \left|H_{+-}\right|^2\right)$$



- For $\gamma^*\gamma^* \to \pi^0\pi^0$: $J = 0{,}2$ dispersive result from [I.D et al. (2020)] (see also [Hoferichter et al. (2019)])
- For $\gamma^*\gamma^* \to \pi^0\eta$: $J = 0$ dispersive result from [Deineka et al. (2025)]

  $J = 2$ BreitWigner approx. with FFs fixed from the quark model [Schuler et al. (1997)] adopted using the $T \to \gamma^*\gamma^*$ basis from [Hoferichter et al. (2020)]

Summary and Outlook

- $(g-2)$
  Dispersive analyses enable a controlled treatment of the $\pi\pi/\pi\eta/K\bar{K}$ channels.
  Using unsubtracted DRs, in addition to the dominant $f_0(500)$ contribution, the $f_0(980)$ and $a_0(980)$ contributions to the HLbL part of the muon $(g-2)$ are **now quantified**.
  Upcoming BESIII data on $\gamma\gamma^* \to \pi^+\pi^-, \pi^0\pi^0, \pi^0\eta$ with $Q^2 = 0.2 - 2.0\,\mathrm{GeV}^2$
  Dispersive $\gamma\gamma^* \to \pi\pi$ amplitudes serve as input to VCS on a nucleon (Poster of Matteo Ronchi)

- $a_0(980)$
  A major challenge in the $\pi\eta/K\bar{K}_{I=1}$ system is the lack of direct scattering data.
  To constrain $a_0(980)$ we employed **different dispersive analyses** (subtracted and unsubtracted) and left-hand cut approximations, together with Adler zero ChPT constraints.
  Resulting resonance parameters of $a_0(980)$ **improve** the current PDG values.
  The **hadronic Omnès output** is applicable to different processes with $\pi\eta/K\bar{K}$ final states.
  New measurements on $\gamma\gamma \to K\bar{K}$ near/below 1.1 GeV from Belle II and BESIII are **essential**.

- Two-photon MC (HadroTOPS)
  We provided the expression for the **exclusive process** $e^+e^- \to e^+e^-\pi\pi$ cross section, together with the $\sigma_0, \sigma_2\ldots$ response functions. These results have been tested by comparison with those from Ekhara.
  We also outlined an **azimuthal dependence**, which provides a way to experimentally extract additional information about the response functions.

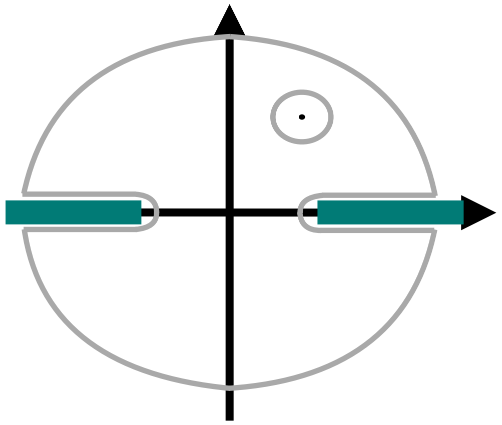
Thank you!

EXTRA

Coupled-channel Omnès matrix

- Once-subtracted p.w. dispersion relation

$$t_{ab}(s) = \underbrace{t_{ab}(0) + \frac{s}{\pi}\int_L \frac{ds'}{s'}\frac{\text{Im}\, t_{ab}(s')}{s'-s}}_{U_{ab}(s)} + \frac{s}{\pi}\sum_c \int_R \frac{ds'}{s'}\frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$



can be solved using the N/D method with **input from** $U_{ab}(s)$ above threshold [Chew, Mandelstam (1960)] [Luming (1964)] [Johnson, Warnock (1981)]

$$t_{ab}(s) = \sum_c D^{-1}_{ac}\,N_{cb}(s)$$

$$N_{ab}(s) = U_{ab}(s) + \frac{s}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'}\frac{N_{ac}(s')\,\rho_c(s')\,(U_{cb}(s') - U_{cb}(s))}{s'-s} \qquad \textbf{(left-hand cuts)}$$

$$D_{ab}(s) = \delta_{ab} - \frac{s}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'}\frac{N_{ab}(s')\,\rho_b(s')}{s'-s} \qquad \textbf{(right-hand cuts)}$$

- The Omnès function fulfils the unitarity relation on the right-hand cut and is analytic everywhere else. For the case of **no bound states** or **no CDD poles**:

$$\boxed{\Omega_{ab}(s) = D^{-1}_{ab}(s)}$$

We used this method to obtain data driven $\pi\pi/K\bar{K}$ and $\pi\eta/K\bar{K}$ Omnès matrices

Fitting parameters: left-hand cuts

- In general scattering problem, little is known about left-hand cuts, except their analytical structure in the complex plane. We approximate $U_{ab}(s)$ as an expansion in a **conformal mapping variable** $\xi(s)$

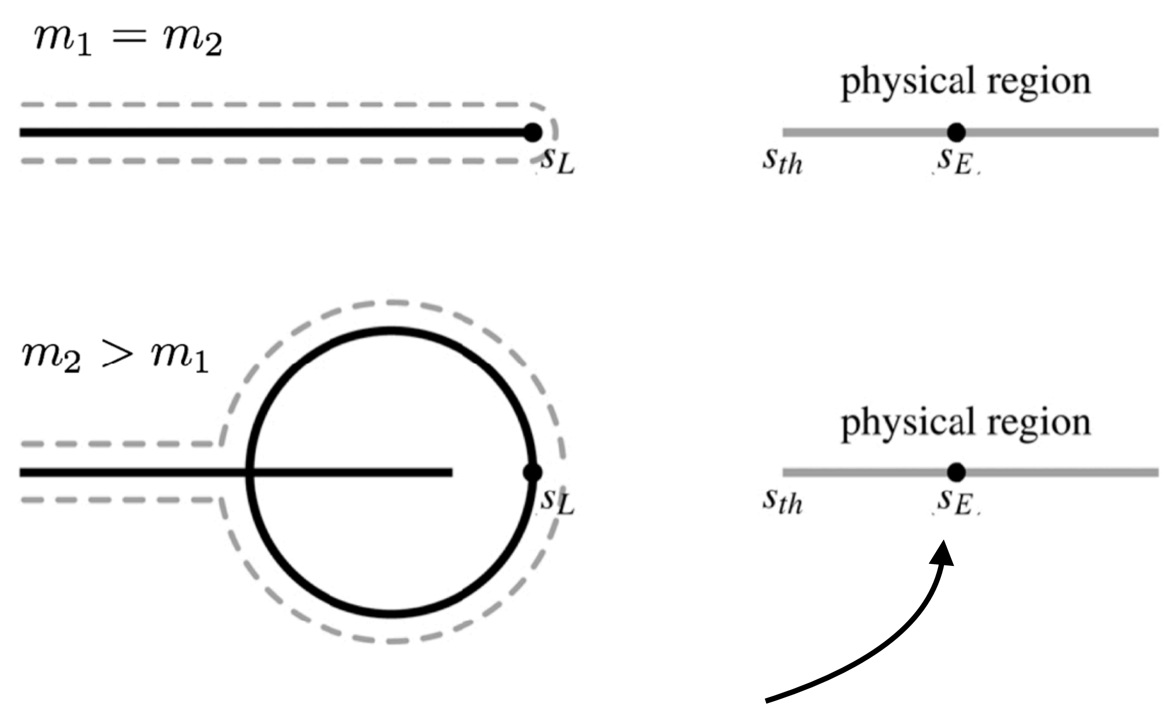
[Gasparyan, Lutz (2010)]

(asymptotically bounded
unknown function)

$$U_{ab}(s) = t_{ab}(0) + \frac{s}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'} \frac{\mathrm{Im}\, t_{ab}(s')}{s' - s}$$

$$\simeq \sum_{n=0}^{\infty} C_{ab,n}\, (\xi_{ab}(s))^n$$

unknown coefficients **fitted to data** or/and **ChPT**



$$\xi(s_E) = 0$$
$$\xi(s_L) = -1$$

$$\sqrt{s_E} = \frac{1}{2}\left(\sqrt{s_{th}} + \sqrt{s_{max}}\right)$$

source of the systematic uncertainties

Kinematic constraints: $\gamma^*\gamma^* \to P_1 P_2$

- p.w. helicity amplitudes suffer from kinematic constraints

$$h^{(J)}_{\lambda_1\lambda_2} = \int \frac{d\cos\theta}{2}\, d^J_{\lambda_1-\lambda_2,0}(\theta)\, H_{\lambda_1\lambda_2}$$

- Helicity amplitudes

$$H_{\lambda_1,\lambda_2} = \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2) \overbrace{\sum_{n=1}^{5} F_n(s,t)\, L_n^{\mu\nu}}^{H^{\mu\nu}}$$

[Bardeen et al. (1968)], [Tarrach (1975)]
[Metz et al. (1998)], [Colangelo et al. (2015)]

- Unconstrained basis for Born subtracted p.w. amplitudes $\bar{h}_i^{(J)} \equiv h_i^{(J)} - h_i^{(J),\mathrm{Born}}$

For S-wave

$$\bar{h}^{(0)}_{++} \pm \bar{h}^{(0)}_{00} \sim (s - s^{(\mp)}_{\mathrm{kin}}), \quad s^{(\pm)}_{\mathrm{kin}} \equiv -(Q_1 \pm Q_2)^2$$

$$\bar{h}^{(0)}_{i=1,2}(s) = \frac{\bar{h}^{(0)}_{++}(s) \pm \bar{h}^{(0)}_{00}(s)}{s - s^{(\mp)}_{\mathrm{kin}}}$$

[Colangelo et al. (2017)]
[Hoferichter, Stoffer (2019)]
[I.D., Deineka, Vanderhaeghen (2019)]

For D-wave

$$\bar{h}^{(J)}_i = K_{ij}\, \bar{h}^{(J)}_j \qquad j \equiv \lambda_1\lambda_2 = \{++,\, +-,\, +0,\, 0+,\, 00\}$$

K_{ij} is 5×5 matrix

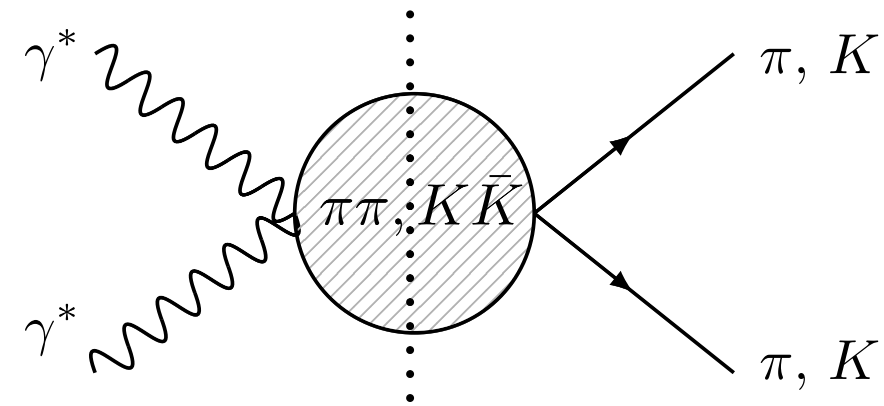
Dispersion relations: $\gamma^*\gamma^* \to P_1 P_2$

- Unsubtracted dispersion relation for kinematically unconstrained p.w. amplitudes $\bar{h}_i^{(J)} = K_{ij}\,\bar{h}_j^{(J)}$

$$j \equiv \lambda_1\lambda_2 = \{++,\, +-,\, +0,\, 0+,\, 00\}$$

$$\bar{h}_i^{(J)} = \int_{-\infty}^{0} \frac{ds'}{\pi} \frac{\mathrm{Disc}\,\bar{h}_i^{(J)}(s')}{s'-s} + \int_{4m_\pi^2}^{\infty} \frac{ds'}{\pi} \frac{\mathrm{Disc}\,\bar{h}_i^{(J)}(s')}{s'-s}$$

$$\bar{h}_i^{(J)} \equiv h_i^{(J)} - h_i^{(J),\mathrm{Born}}$$



which can be solved using modified MO method, i.e. by writing a dispersion relation for $\Omega^{(J)}(s)^{-1}\,\bar{h}_i^{(J)}(s)$

- For s-wave, I=0

$$\begin{pmatrix} \bar{h}_i^{(0)}(s) \\ \bar{k}_i^{(0)}(s) \end{pmatrix} = \Omega^{(0)}(s) \left[\int_{-\infty}^{0} \frac{ds'}{\pi} \frac{\Omega^{(0)}(s')^{-1}}{s'-s} \begin{pmatrix} \mathrm{Disc}\,\bar{h}_i^{(0)}(s') \\ \mathrm{Disc}\,\bar{k}_i^{(0)}(s') \end{pmatrix} - \int_{4m_\pi^2}^{\infty} \frac{ds'}{\pi} \frac{\mathrm{Disc}\,\Omega^{(0)}(s')^{-1}}{s'-s} \begin{pmatrix} h_i^{(0),\mathrm{Born}}(s') \\ k_i^{(0),\mathrm{Born}}(s') \end{pmatrix} \right]$$

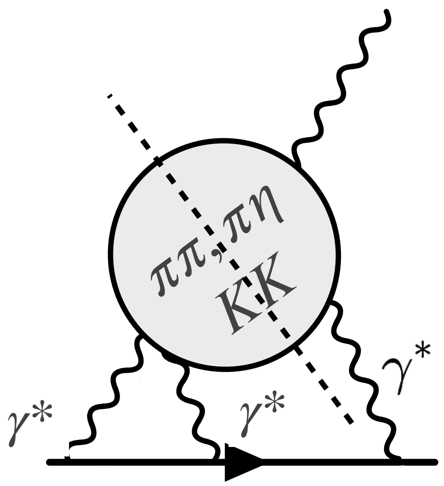
$\gamma^*\gamma^* \to \pi\pi$

$\gamma^*\gamma^* \to KK$

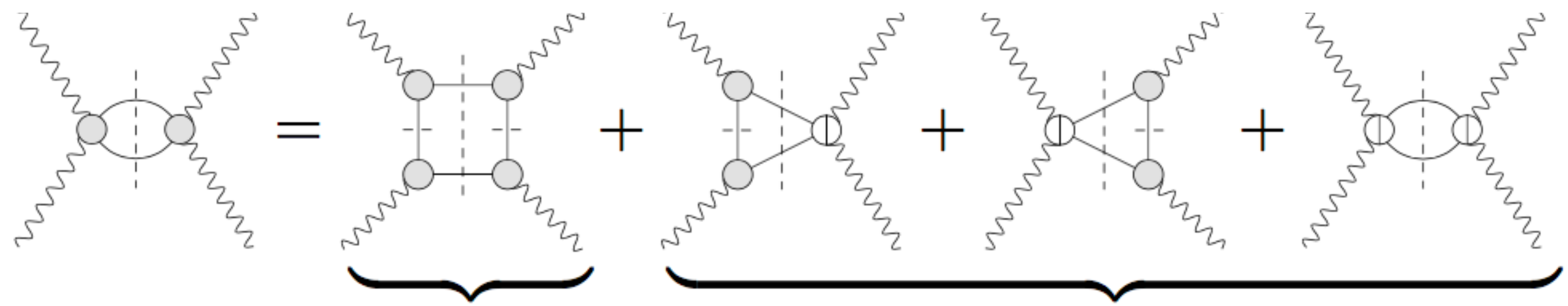
V-exch

[Omnès (1958)]
[Muskhelishvili (1953)]
[Garcia-Martin et. al (2010)]
[Hoferichter et. al. (2011,19)]
[Dai et al. (2014)]
[Moussallam (2013)]

Motivation



$$a_\mu^{HLbL} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, Q_3)\bar{\Pi}_i(Q_1, Q_2, Q_3)$$



$a_\mu[\text{box}]_{\pi\pi, K\bar{K}} = -16.4(0.2) \times 10^{-11}$ [Colangelo et al. (2014-2017)]

Input: pion (kaon) vector form factors $F_{\pi,K}(Q^2)$

$a_\mu[\text{S-wave}, I=2]_{\pi\pi} = +1.1(0.1) \times 10^{-11}$

$a_\mu[\text{S-wave}, I=0]_{\pi\pi} = -9.3(0.9) \times 10^{-11}$ (σ)

[Colangelo, Hoferichter, Procura, Stoffer (2017)]

$a_\mu[\text{S-wave}, I=0]_{\pi\pi, K\bar{K}} = -9.8(1.0) \times 10^{-11} (\sigma, f_0)$ [I.D, Hoferichter, Stoffer (2021)]

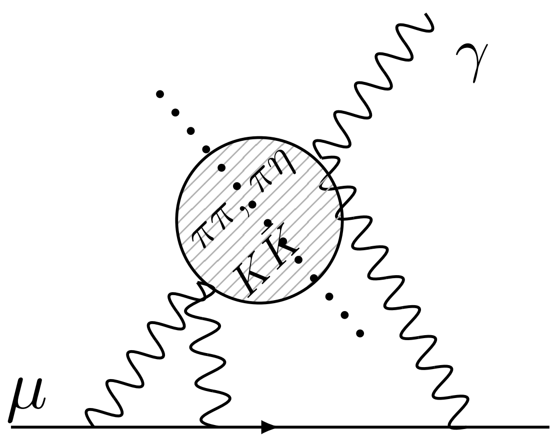
Unsubtracted dispersion relation for $\gamma^*\gamma^* \to \pi\pi/K\bar{K}$

Left-hand cuts: π/K pole with vector form factors $F_{\pi,K}(Q^2)$

Data used: $\pi\pi/K\bar{K}$ scattering data (Roy analyses)

$\gamma\gamma \to \pi^0\pi^0$ used to **justify** left-hand cut approximation

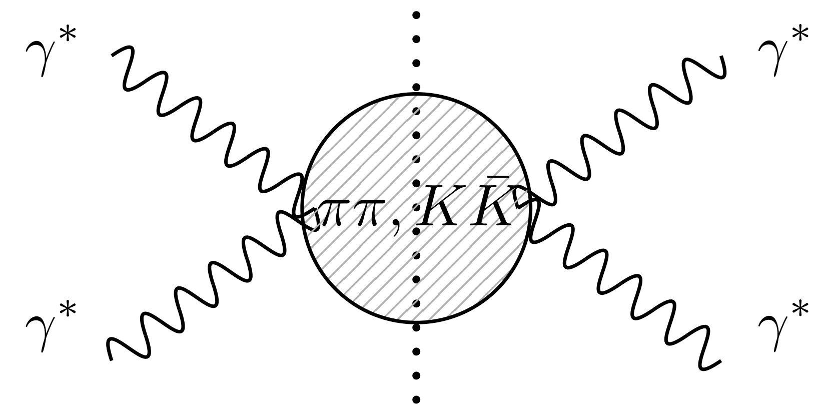
Contribution to (g-2)



Important ingredients:

$\gamma^*\gamma^* \to \pi\pi, K\bar{K}, \ldots$

$q^2 = -Q^2 < 0 \quad$ space-like γ^*



$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\, Q_1^3\, Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, Q_3)\, \bar{\Pi}_i(Q_1, Q_2, Q_3),$$

Rescattering contribution ($\bar{h} \equiv h - h^{\mathrm{Born}}$) in the S-wave

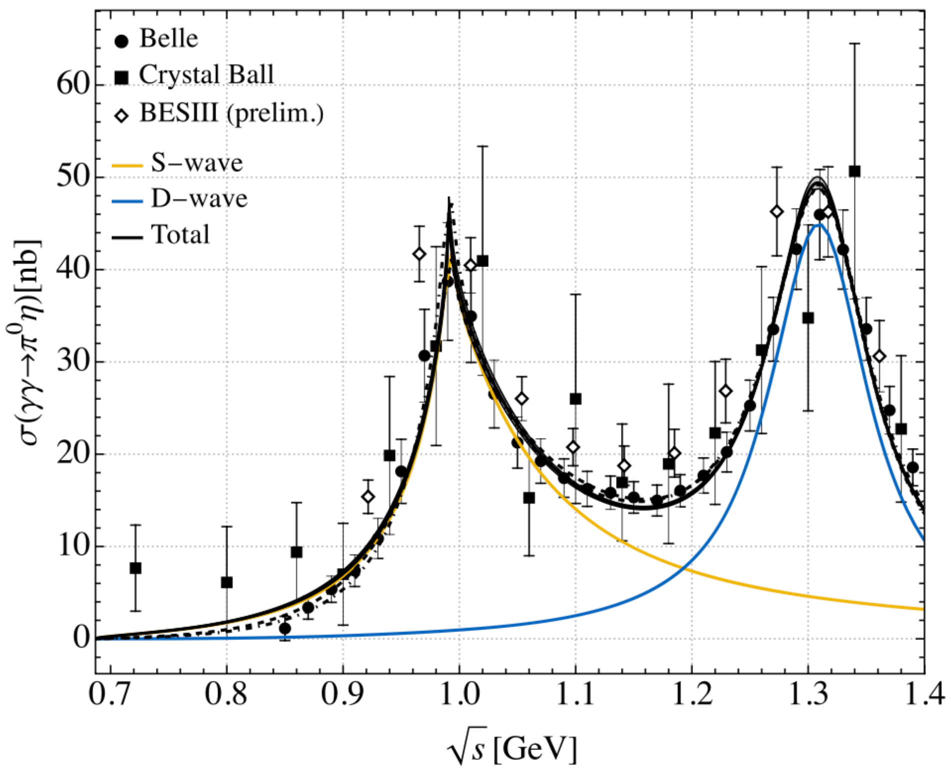
$$\bar{\Pi}_3^{J=0} = \frac{1}{\pi} \int_{4m_\pi^2}^\infty ds' \frac{-2}{\lambda_{12}(s')(s'+Q_3^2)^2} \left(4s'\, \mathrm{Im}\bar{h}^{(0)}_{++,++}(s') - (s'-Q_1^2+Q_2^2)(s'+Q_1^2-Q_2^2)\, \mathrm{Im}\bar{h}^{(0)}_{00,++}(s') \right)$$

$$\bar{\Pi}_9^{J=0} = \frac{1}{\pi} \int_{4m_\pi^2}^\infty ds' \frac{4}{\lambda_{12}(s')(s'+Q_3^2)^2} \left(2\, \mathrm{Im}\bar{h}^{(0)}_{++,++}(s') - (s'+Q_1^2+Q_2^2)\, \mathrm{Im}\bar{h}^{(0)}_{00,++}(s') \right)$$

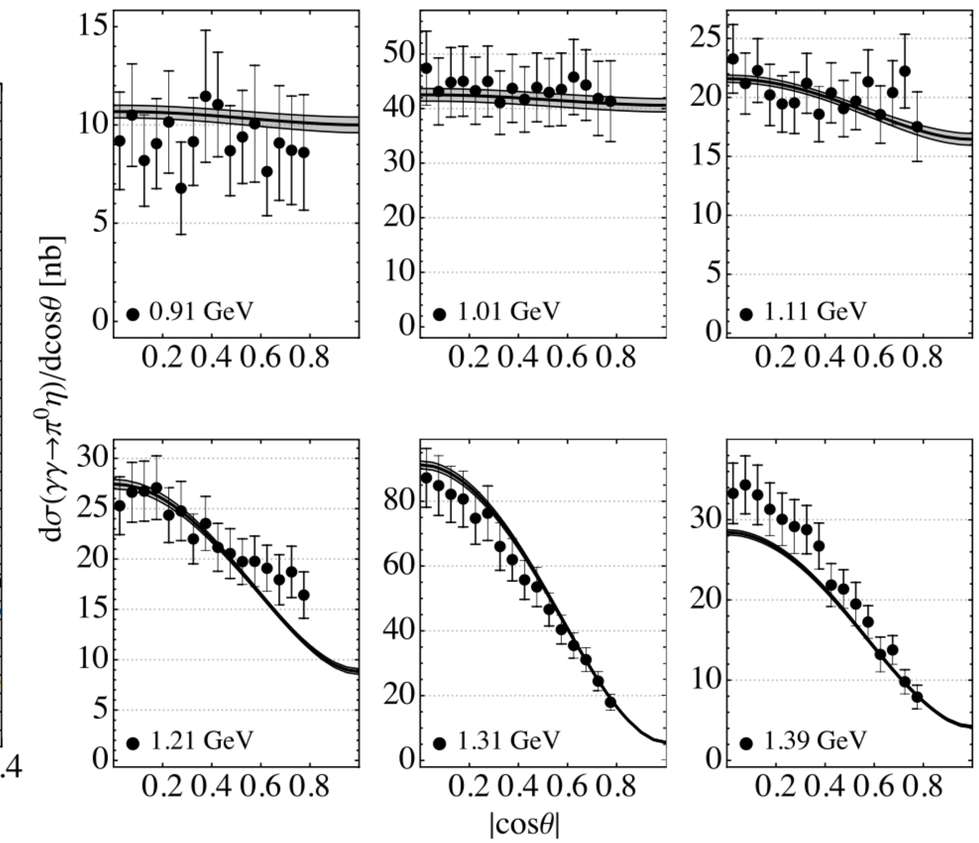
+crossed

Unitarity $\quad \gamma^*\gamma^* \to \gamma^*\gamma^* \qquad \gamma^*\gamma^* \to \pi\pi \qquad \gamma^*\gamma^* \to KK$

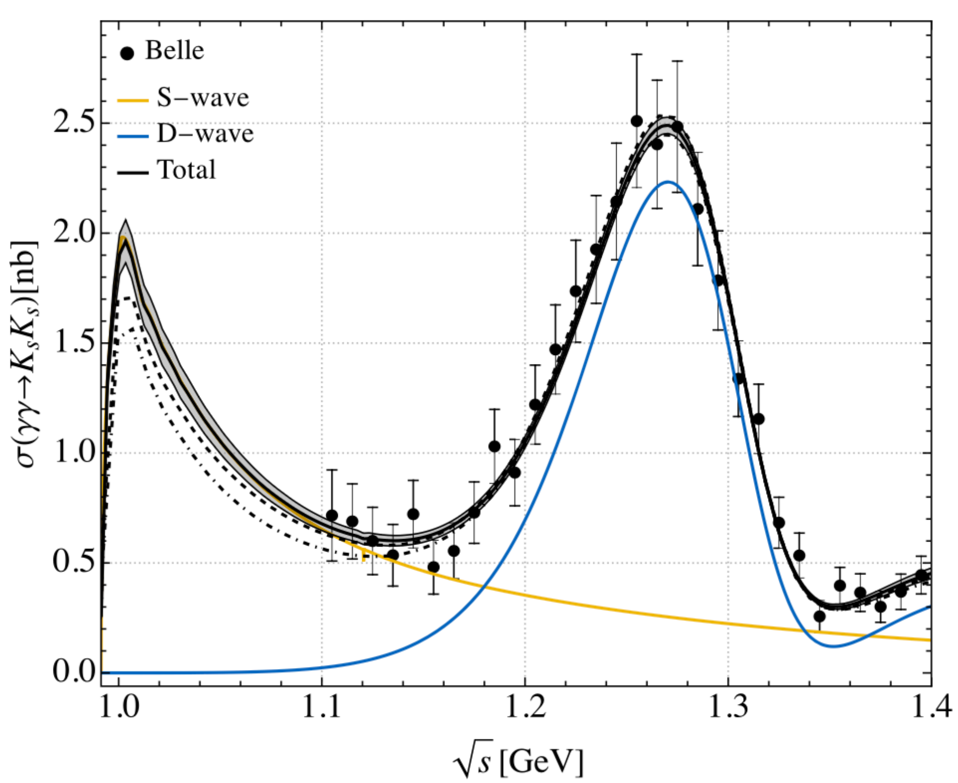
$$\mathrm{Im}\bar{h}^{(0)}_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s) = \frac{1}{2}\, \bar{h}^{(0)}_{\lambda_1\lambda_2}(s)\, \rho_\pi(s)\, \bar{h}^{(0)*}_{\lambda_3\lambda_4}(s) + \frac{1}{2}\, \bar{k}^{(0)}_{\lambda_1\lambda_2}(s)\, \rho_K(s)\, \bar{k}^{(0)*}_{\lambda_3\lambda_4}(s)$$



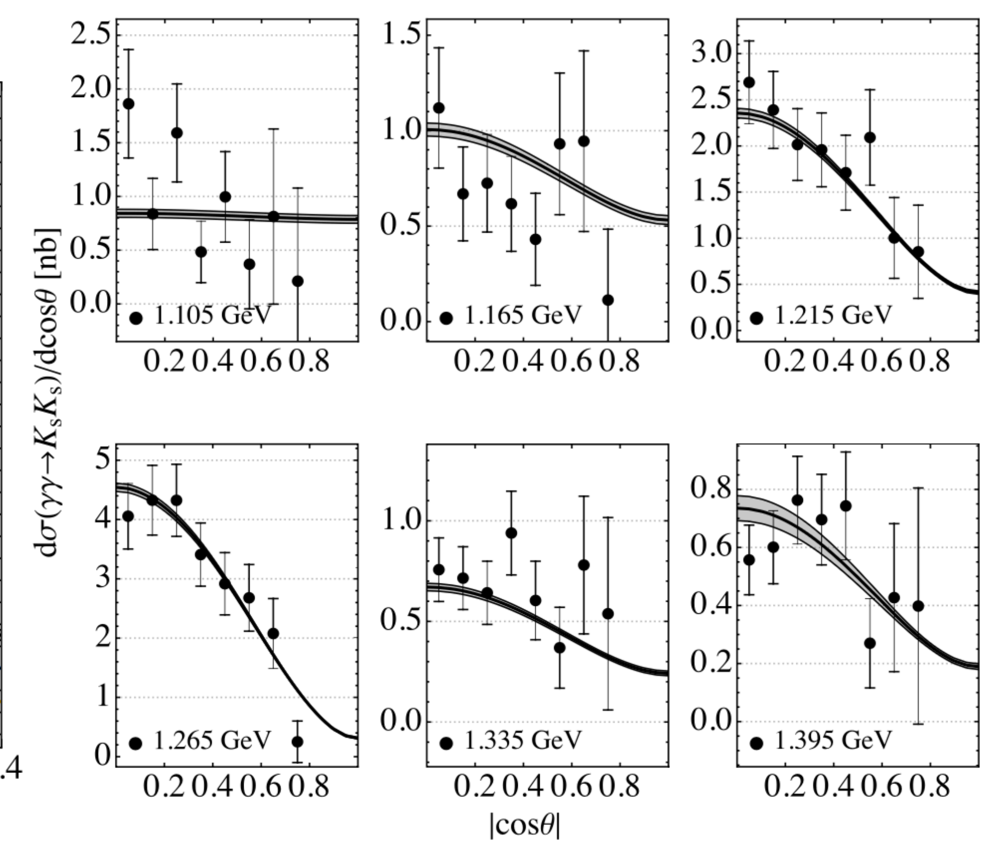
Belle
Crystal Ball
BESIII (prelim.)
S−wave
D−wave
Total
$\sigma(\gamma\gamma\to\pi^0\eta)$[nb]
$\sqrt{s}$ [GeV]



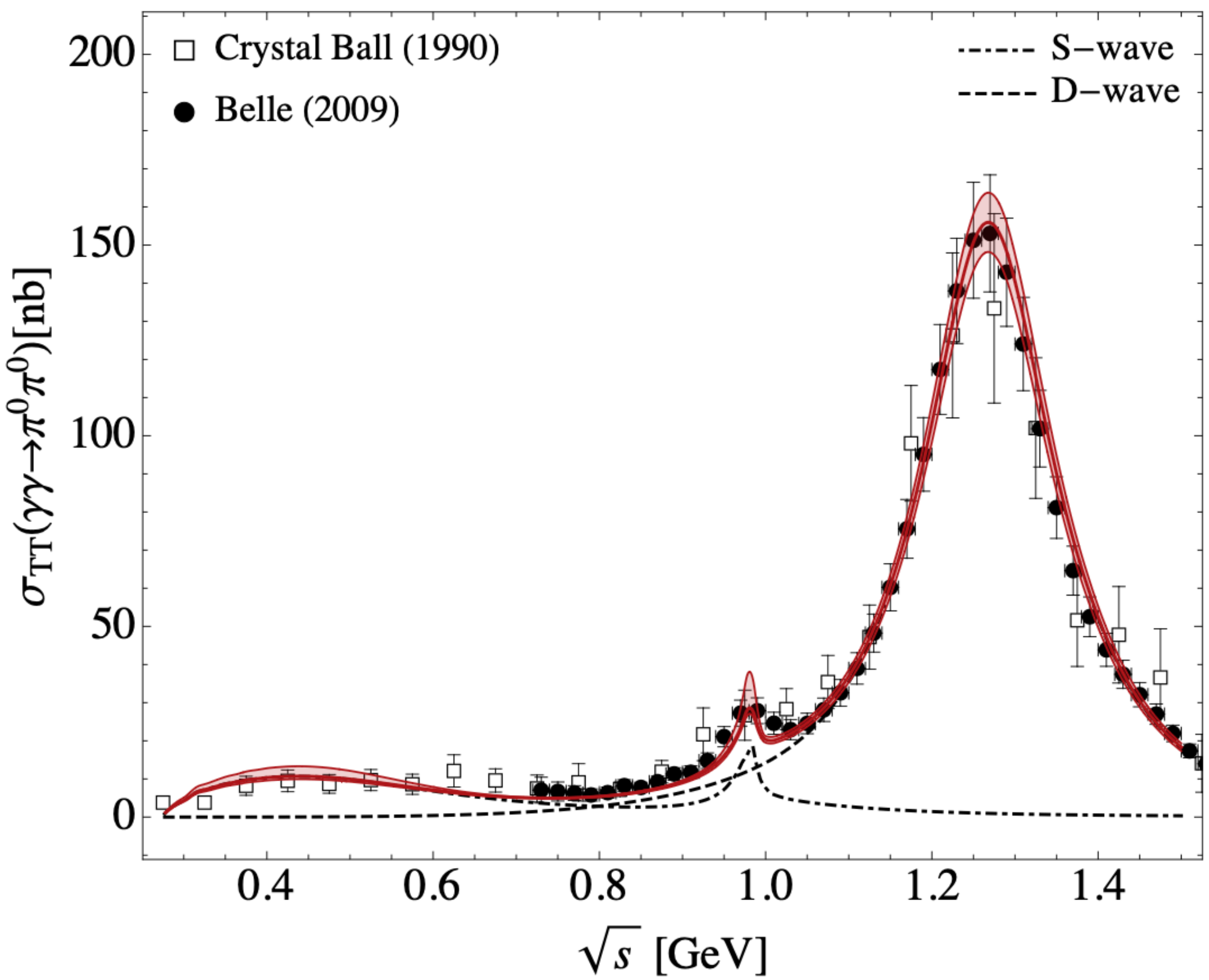
$d\sigma(\gamma\gamma\to\pi^0\eta)/d\cos\theta$ [nb]
0.91 GeV
1.01 GeV
1.11 GeV
1.21 GeV
1.31 GeV
1.39 GeV
|cosθ|



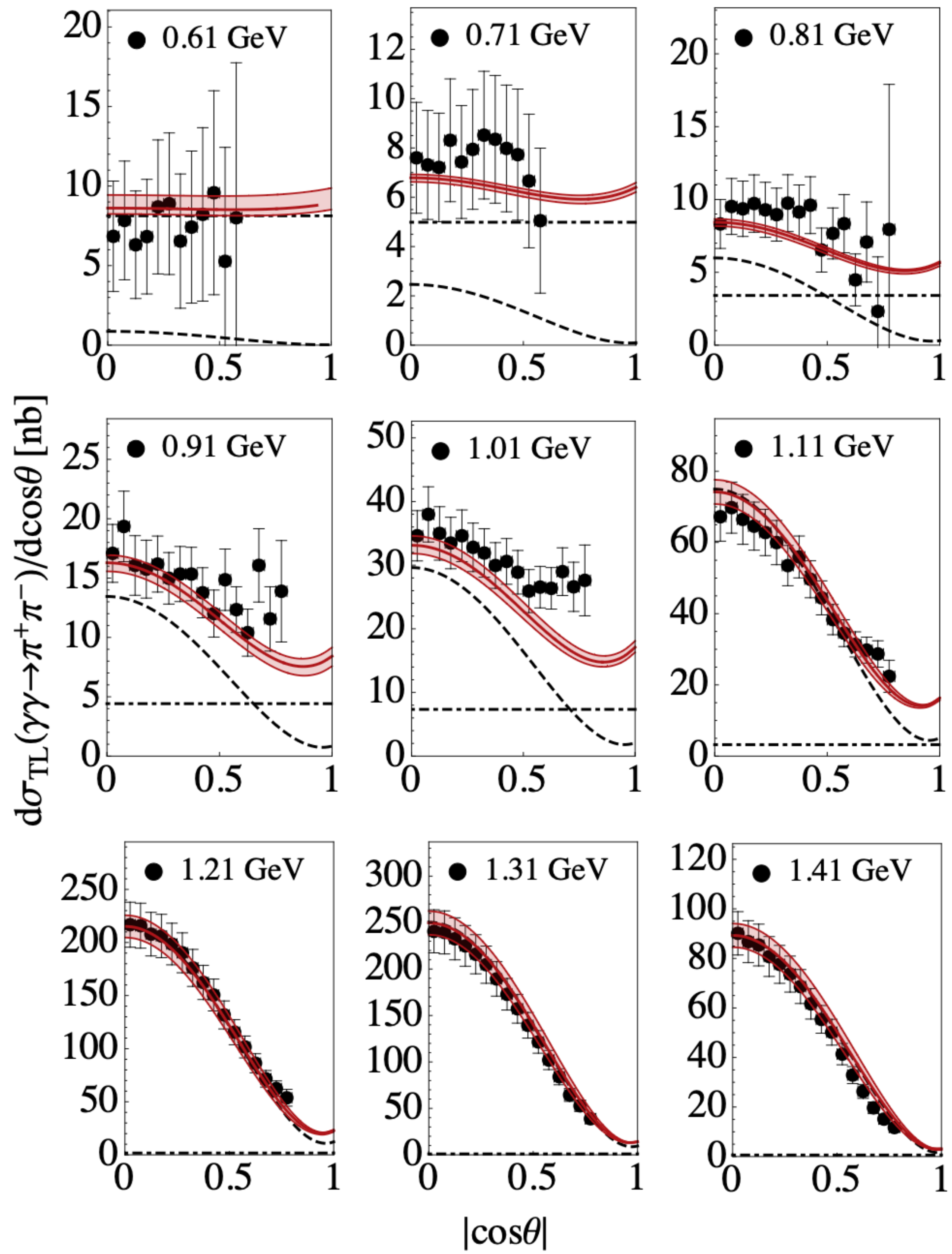
Belle
S−wave
D−wave
Total
$\sigma(\gamma\gamma\to K_sK_s)$[nb]
$\sqrt{s}$ [GeV]



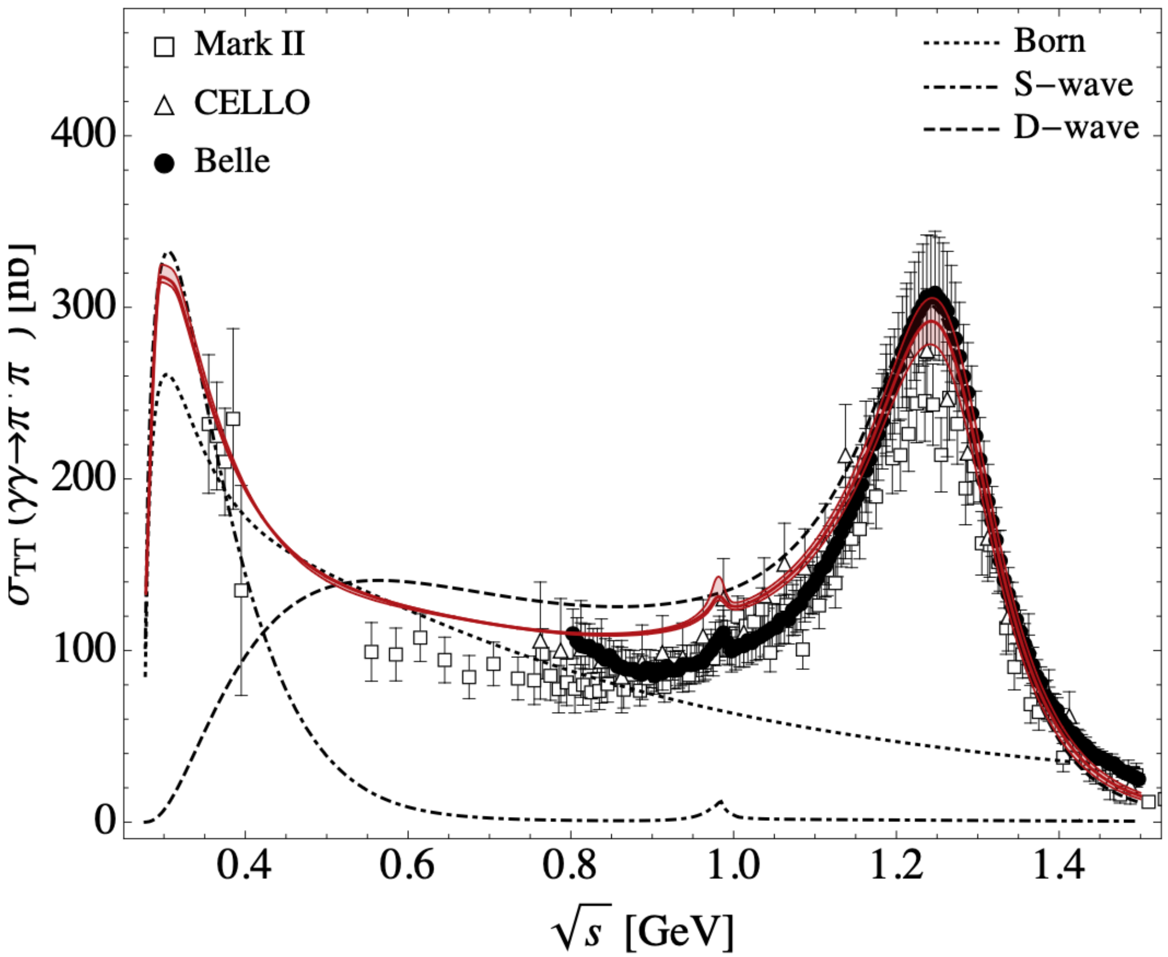
$d\sigma(\gamma\gamma\to K_sK_s)/d\cos\theta$ [nb]
1.105 GeV
1.165 GeV
1.215 GeV
1.265 GeV
1.335 GeV
1.395 GeV
|cosθ|

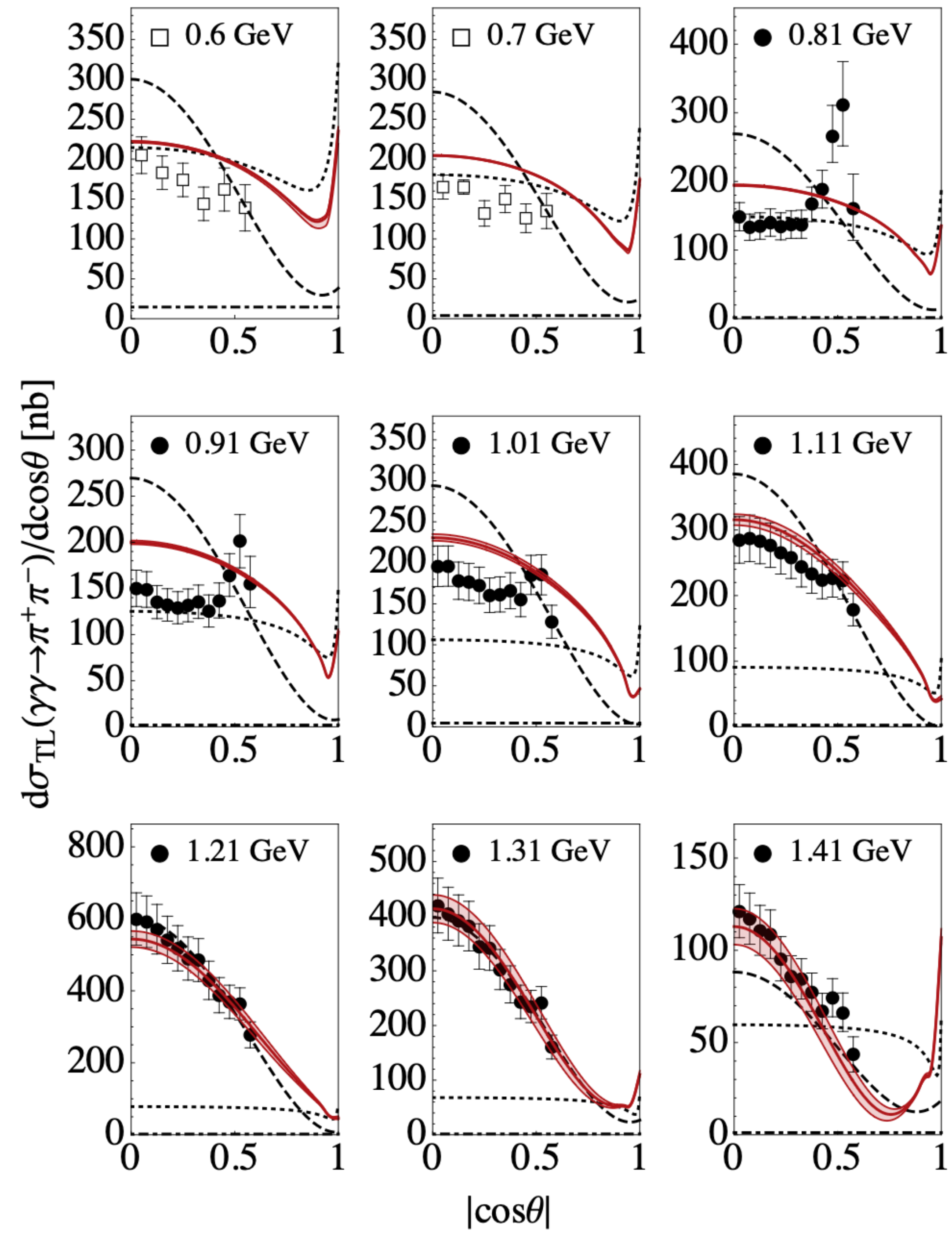


Crystal Ball (1990)
Belle (2009)
S−wave
D−wave
$\sigma_{TT}(\gamma\gamma\to\pi^0\pi^0)$[nb]
$\sqrt{s}$ [GeV]



0.61 GeV
0.71 GeV
0.81 GeV
0.91 GeV
1.01 GeV
1.11 GeV
1.21 GeV
1.31 GeV
1.41 GeV
$d\sigma_{TL}(\gamma\gamma\to\pi^+\pi^-)/d\cos\theta$ [nb]
$|\cos\theta|$

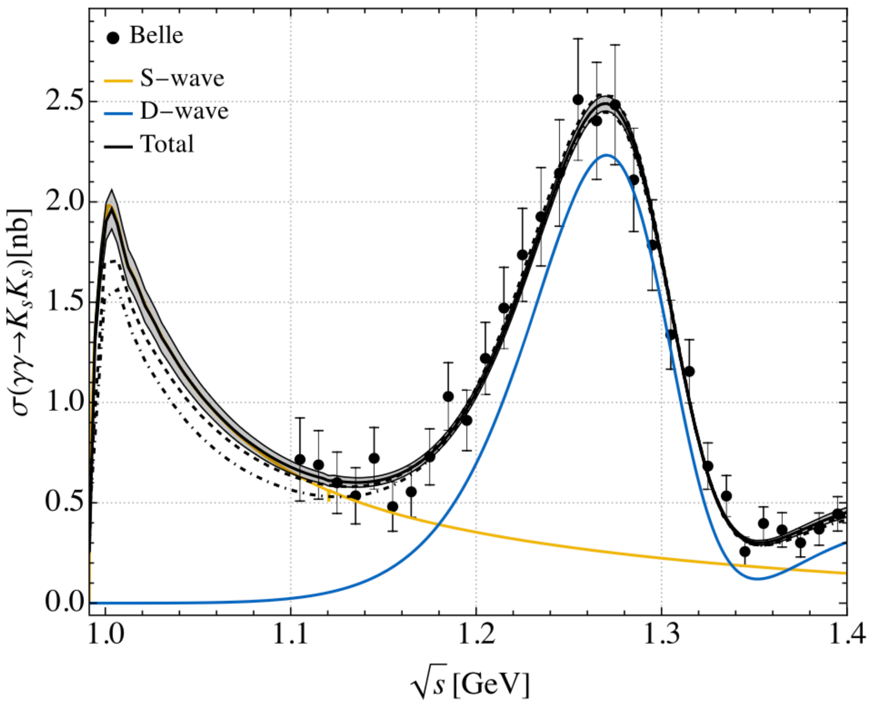

Mark II
CELLO
Belle
Born
S−wave
D−wave
$\sigma_{TT}(\gamma\gamma\to\pi^+\pi^-)$ [nb]
$\sqrt{s}$ [GeV]


0.6 GeV
0.7 GeV
0.81 GeV
0.91 GeV
1.01 GeV
1.11 GeV
1.21 GeV
1.31 GeV
1.41 GeV
$d\sigma_{TL}(\gamma\gamma\to\pi^+\pi^-)/d\cos\theta$ [nb]
$|\cos\theta|$

Differences from other theoretical approaches

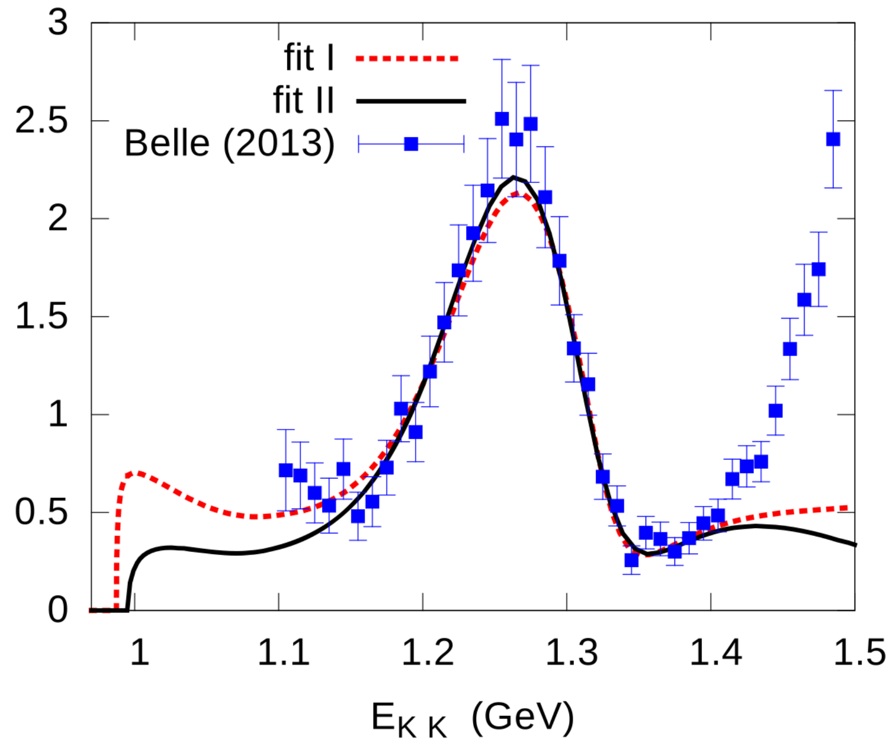
[Deineka et al. (2024)]

$|\cos\theta| \leq 0.8$



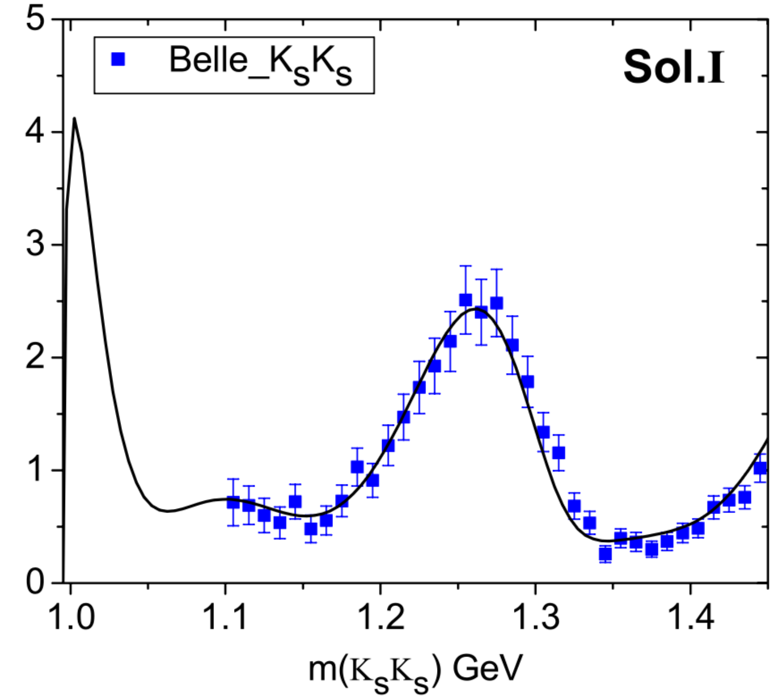
[Lu, Moussallam (2020)]

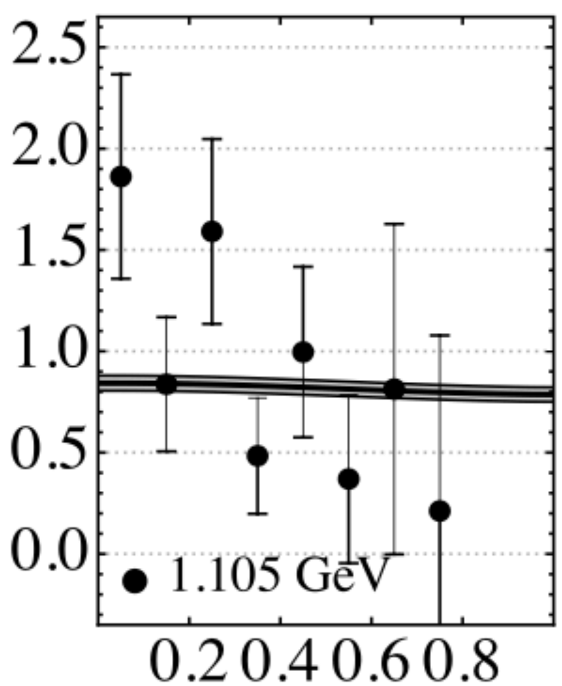
$|\cos\theta| \leq 0.8$

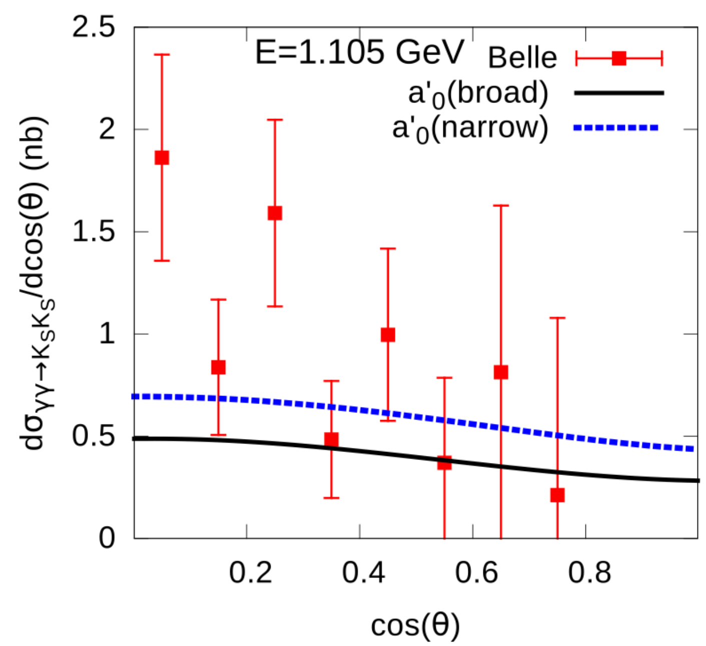


[Dai, Pennington (2014)]

$|\cos\theta| \leq 0.8$

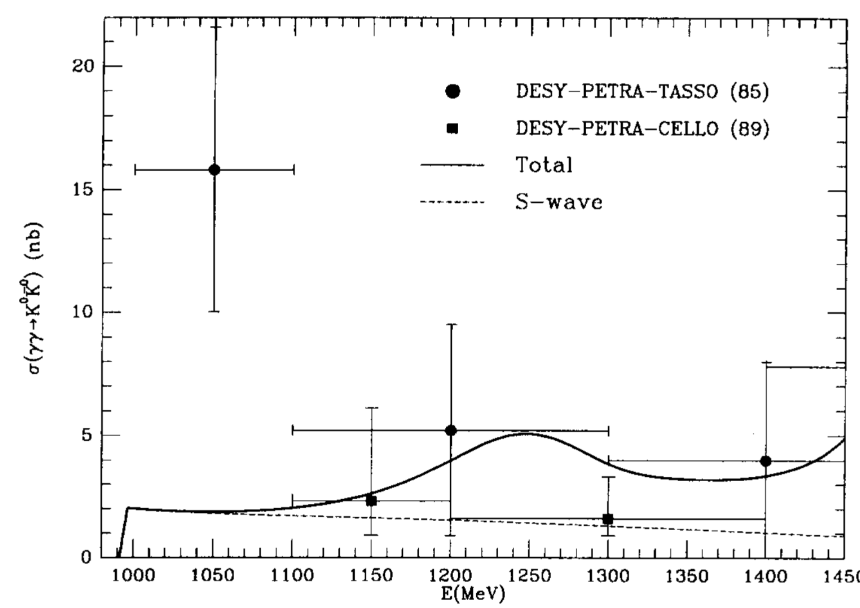




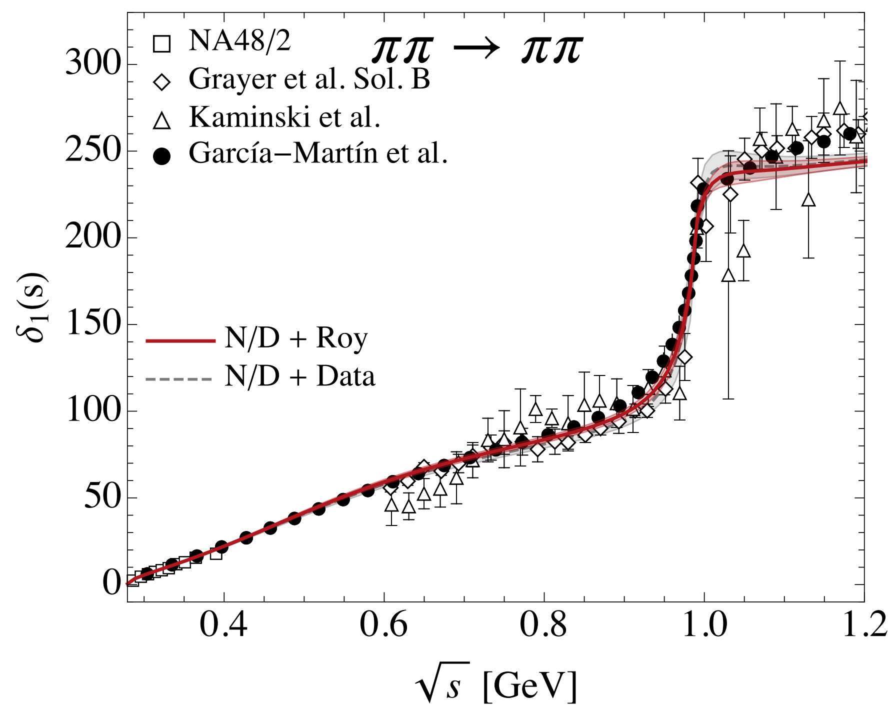


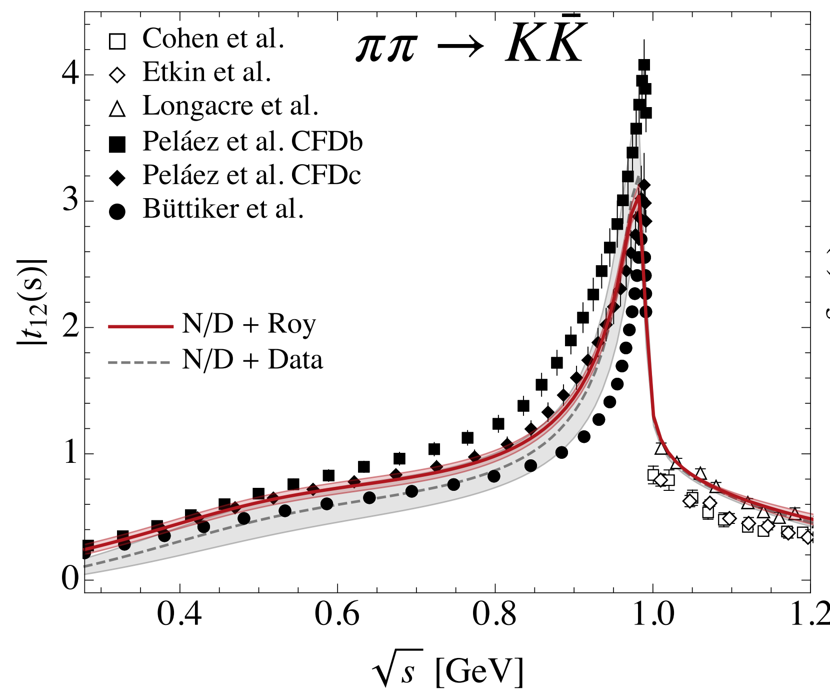
[Oller et al. (1998)]

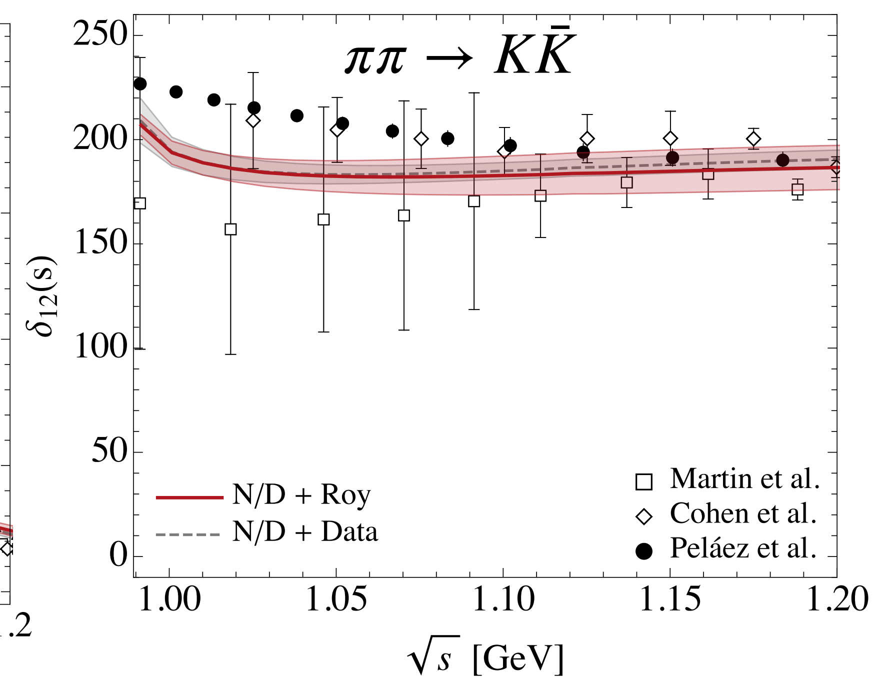
$|\cos\theta| \leq 1$



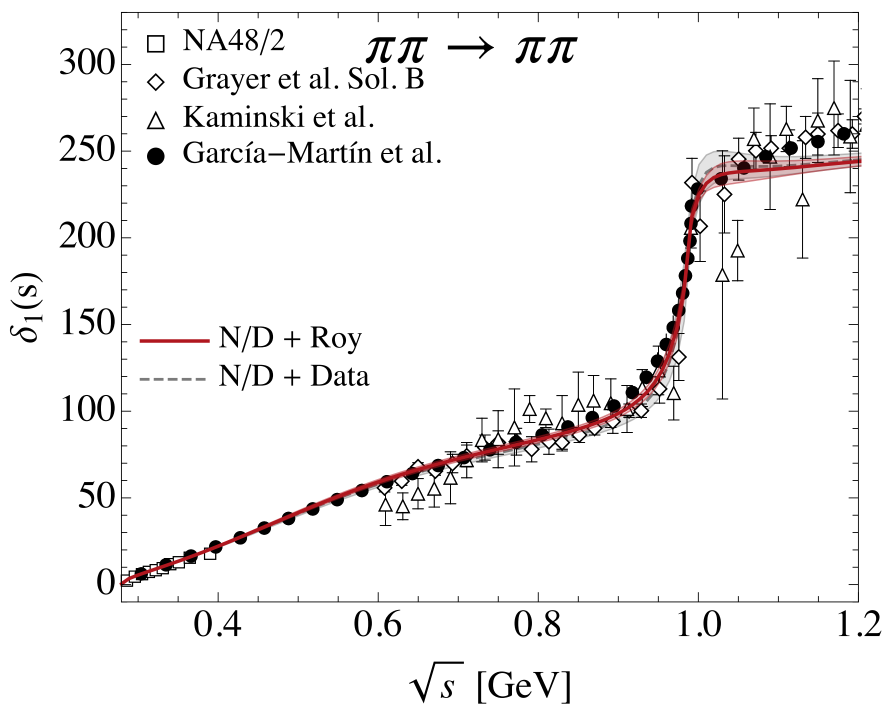
$\pi\pi/K\bar{K}$ (results of the fit)

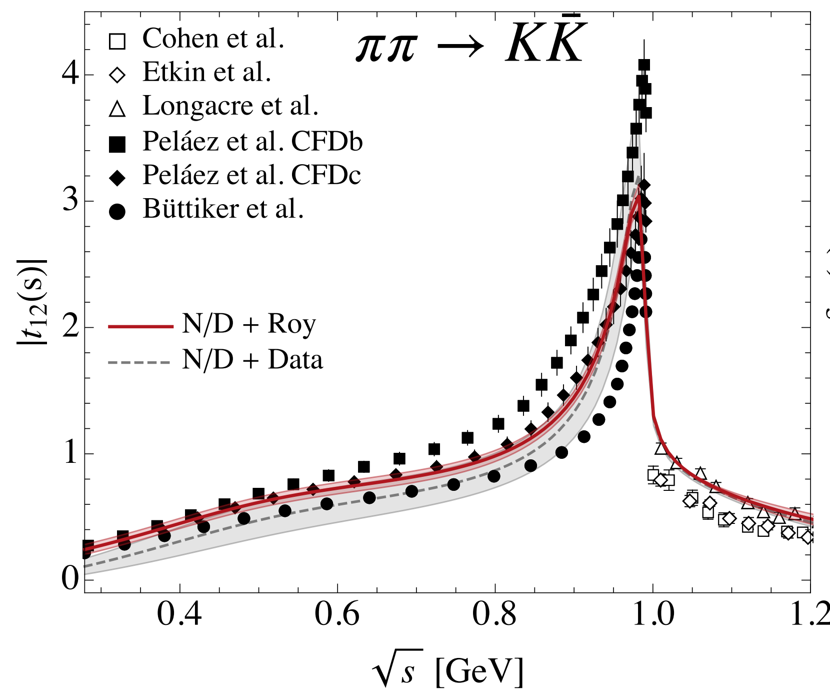


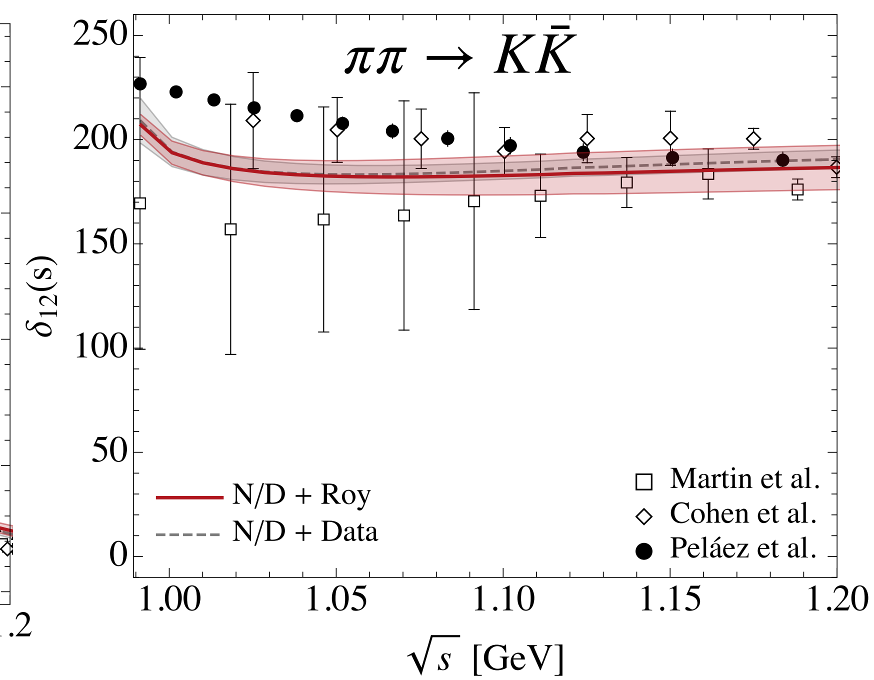




$\pi\pi/K\bar{K}$ (results of the fit)







	Our results		Roy-like analyses	
	pole position, MeV	couplings, GeV	pole position, MeV	couplings, GeV
$\sigma/f_0(500)$	$458(10)^{+7}_{-15} - i\,256(9)^{+5}_{-8}$	$\pi\pi : 3.33(8)^{+0.12}_{-0.20}$ $K\bar{K} : 2.11(17)^{+0.27}_{-0.11}$	$449^{+22}_{-16} - i\,275(15)$	$\pi\pi : 3.45^{+0.25}_{-0.29}$ $K\bar{K}$: –
$f_0(980)$	$993(2)^{+2}_{-1} - i\,21(3)^{+2}_{-4}$	$\pi\pi : 1.93(15)^{+0.07}_{-0.12}$ $K\bar{K} : 5.31(24)^{+0.04}_{-0.24}$	$996^{+7}_{-14} - i\,25^{+11}_{-6}$	$\pi\pi : 2.3(2)$ $K\bar{K}$: –