

Progress for x -dependent GPDs from lattice QCD

Krzysztof Cichy

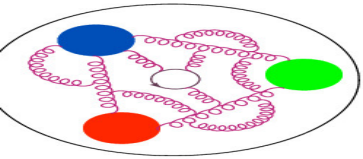
Adam Mickiewicz University, Poznań, Poland



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SONATA BIS grant No. 2016/22/E/ST2/00013 (2017-2022)
OPUS grant No. 2021/43/B/ST2/00497 (2022-2026)

Many thanks to my Collaborators for work presented here:

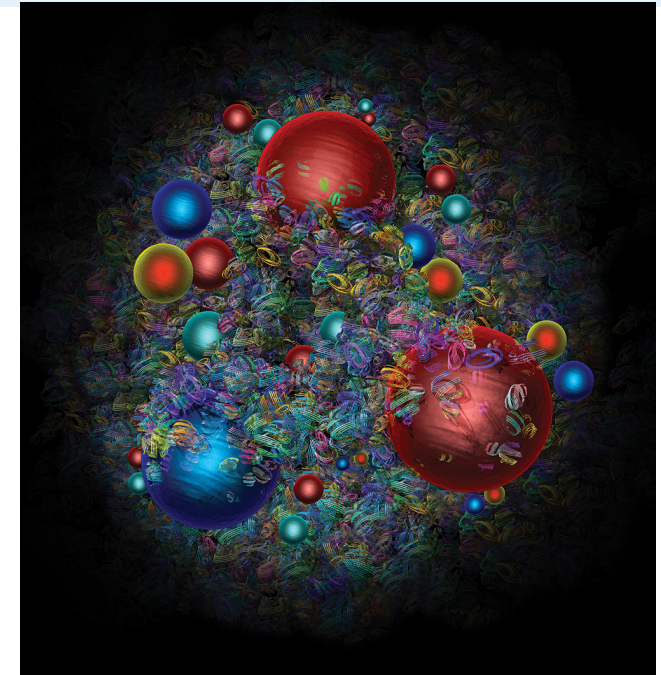
C. Alexandrou, M.C. Bañuls, S. Bhattacharya, M. Chu, M. Colaço
M. Constantinou, J. Dodson, X. Gao, K. Hadjiyiannakou, K. Jansen
D. Lin, A. Metz, J. Miller, S. Mukherjee, N. Nurminen, P. Petreczky
A. Scapellato, M. Schneider, F. Steffens, P. Sznajder, J. Wagner, Y. Zhao

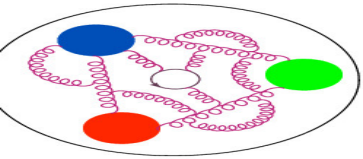


Nucleon structure and partonic distributions

One of the central aims of hadron physics:
to understand better nucleon's structure.

- How does the mass of the nucleon arise?
- How does the spin of the nucleon arise? NAS report 2018
- What are the emergent properties of dense systems of gluons?
- Answering these questions is one of the crucial expectations for the upcoming years!
- For this, we need to probe the 3D structure.

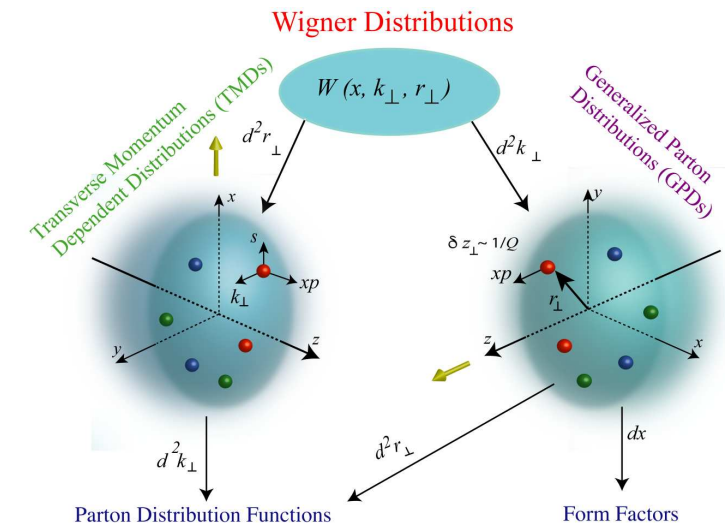
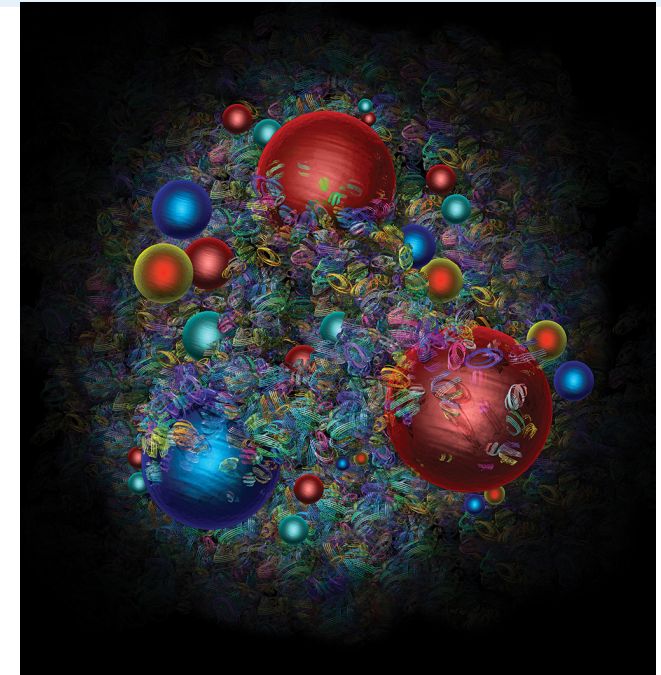


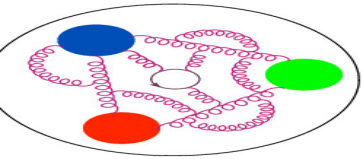


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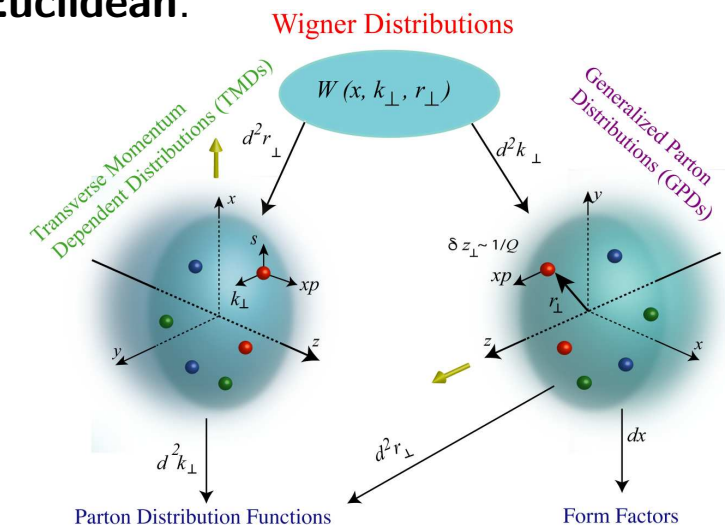
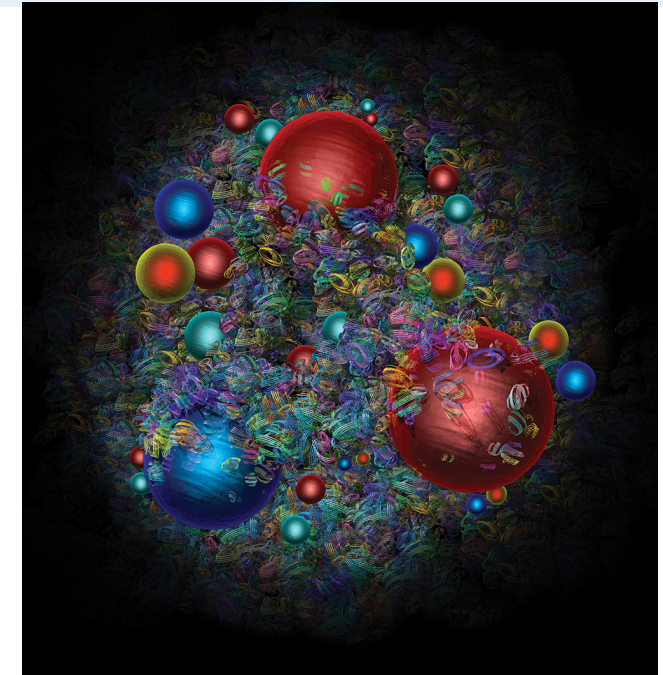
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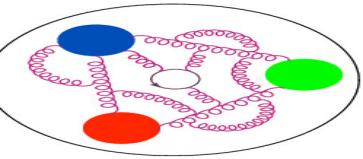
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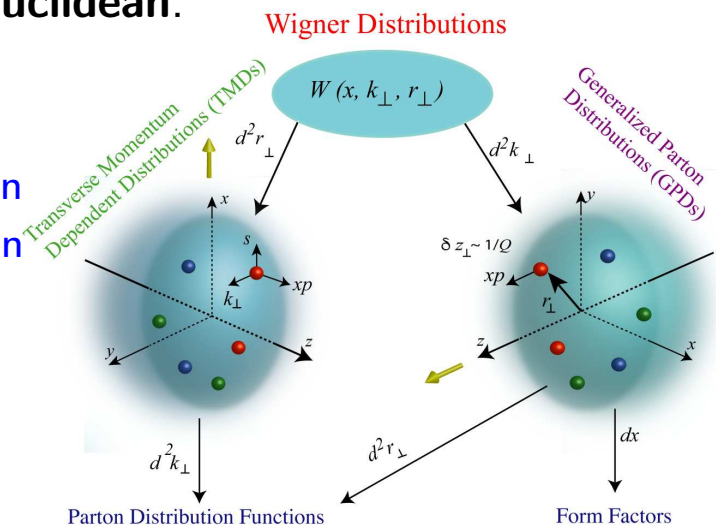
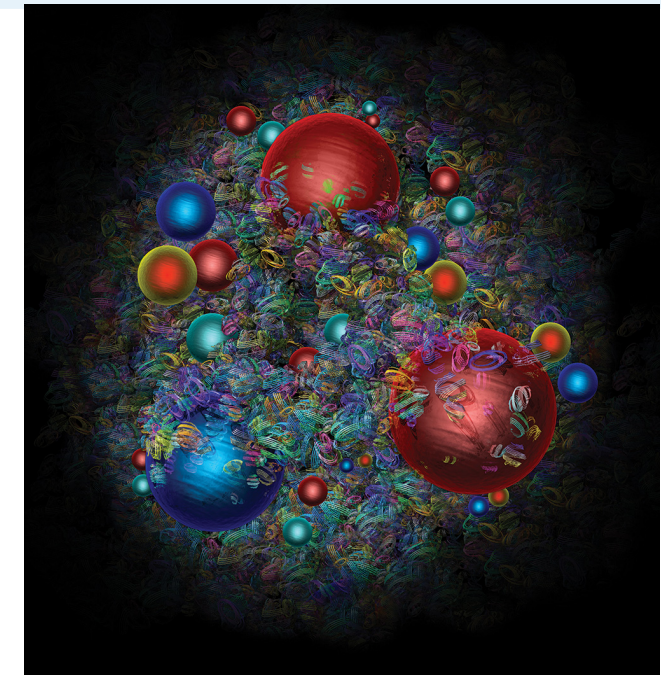
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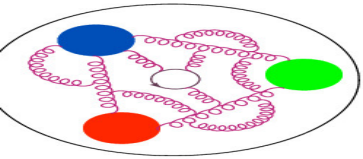
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→ **factorization**

(experiment) cross-section = perturbative-part * partonic-distribution

(lattice) lattice-observable = perturbative-part * partonic-distribution





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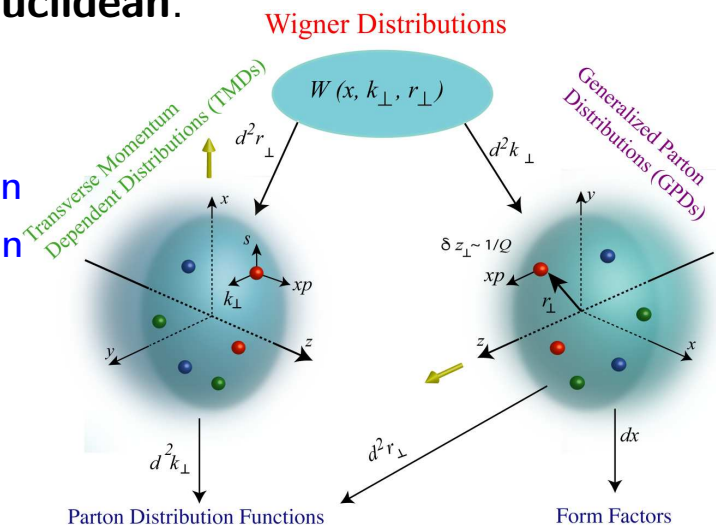
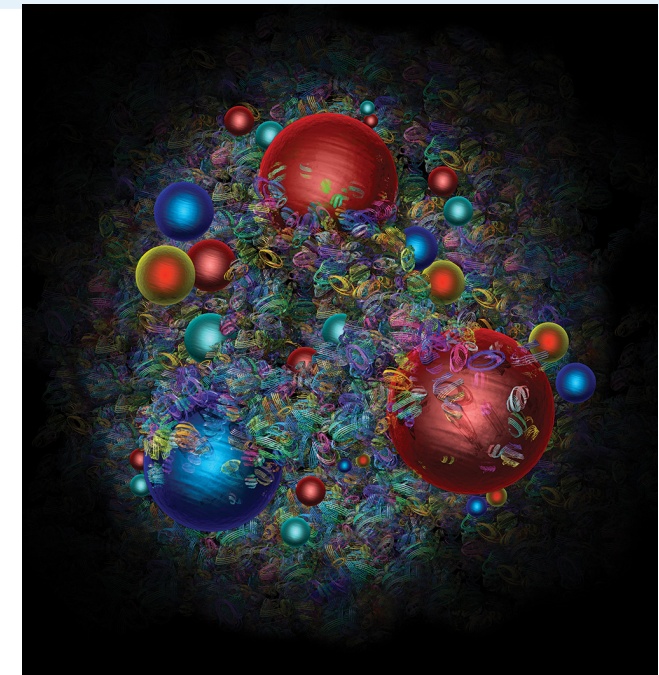
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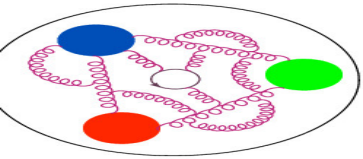
(experiment) **cross-section** = perturbative-part * partonic-distribution

(lattice) **lattice-observable** = perturbative-part * partonic-distribution

- Two most popular approaches:

- ★ **quasi-distributions** – Xiangdong Ji, 2013
- ★ **pseudo-distributions** – Anatoly Radyushkin, 2017



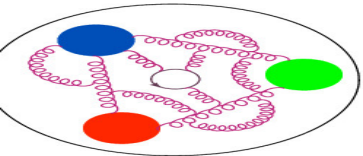


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Direct access to partonic distributions **generally possible** in lattice field theory (maybe one day in lattice QCD!) using:

- tensor network methods with
- explicit light-front evolution.



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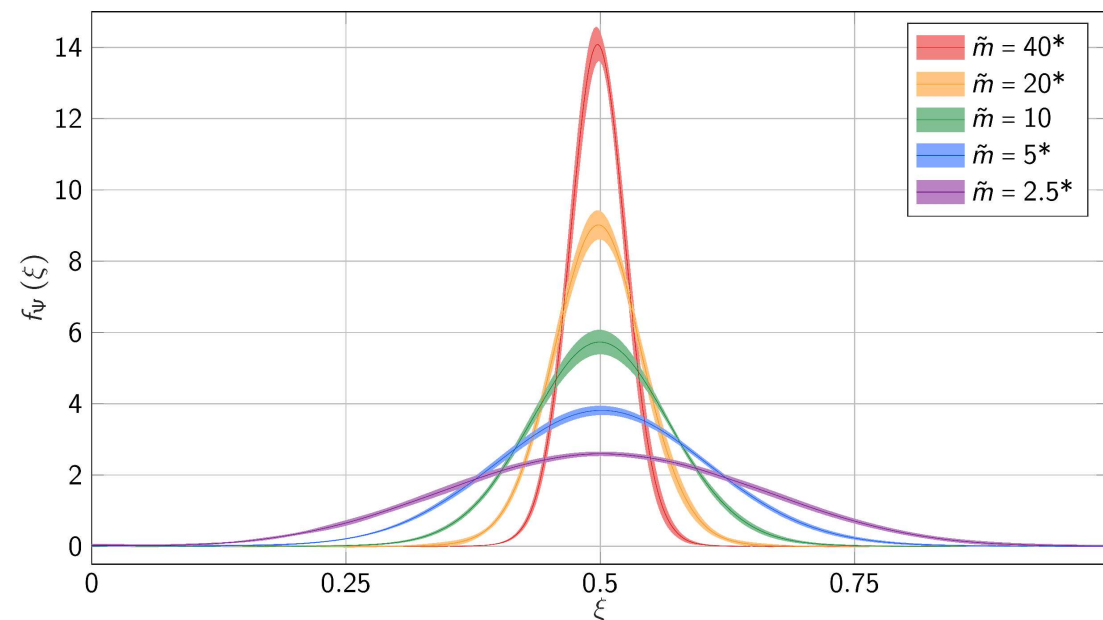
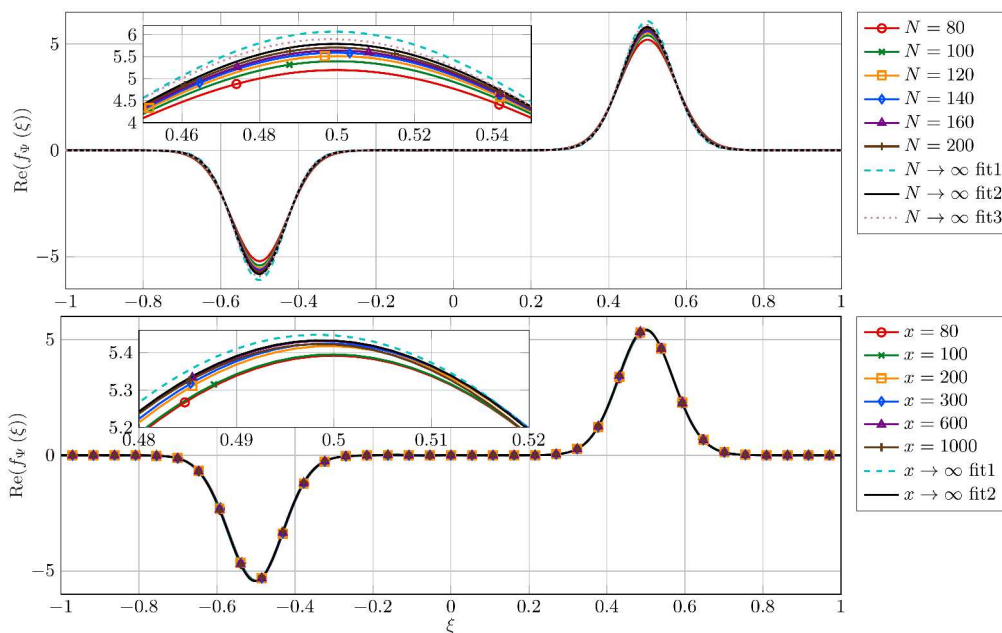
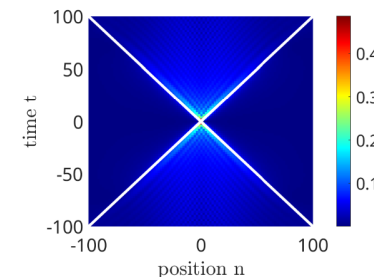
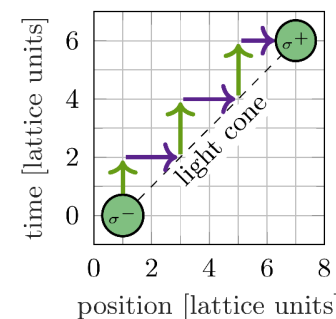
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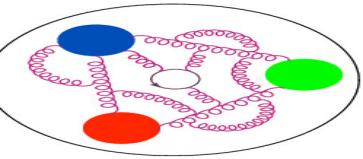
- tensor network methods with
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PDF for the **massive Schwinger model**:

M.C. Bañuls, KC, D. Lin, M. Schneider, arXiv:2504.07508

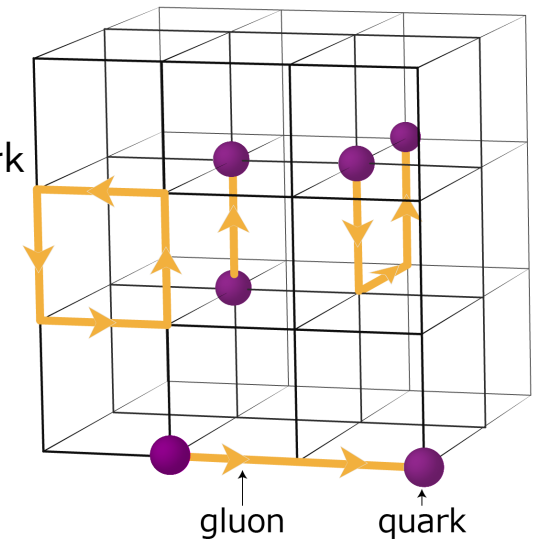
$$\mathcal{M}_{i,j}(\Delta z) = \langle h | e^{iH \frac{\Delta t}{2} \Delta z} \prod_{k < \Delta z + i} (i\sigma_k^z) \sigma_{\Delta z + i}^+ e^{-iH \Delta z - 1 \Delta t} \dots e^{-iH_3 \Delta t} e^{-iH_1 \Delta t} \prod_{k' < j} (-i\sigma_{k'}^z) \sigma_j^- | h \rangle_c$$

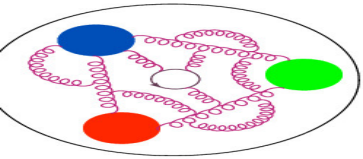




Lattice QCD – brief reminder

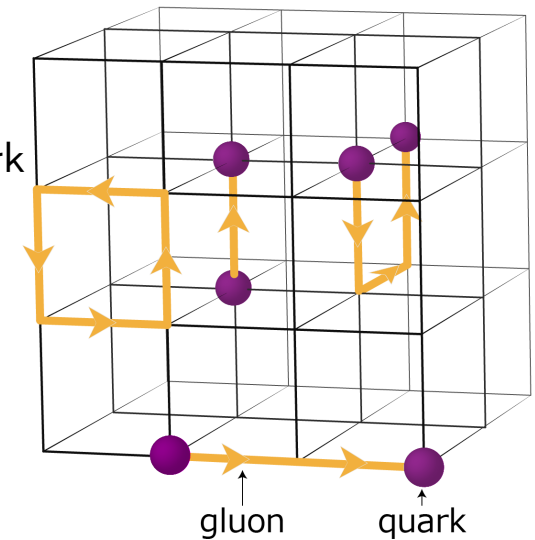
- QCD put on a **Euclidean** lattice: quarks $\rightarrow$ sites, gluons $\rightarrow$ links
- various discretizations can be used for quarks and gluons
 $\rightarrow$ **TM fermions**, **Iwasaki gluons** in our work

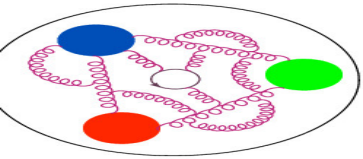




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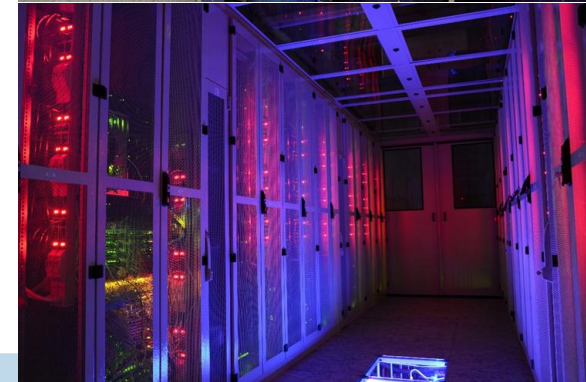
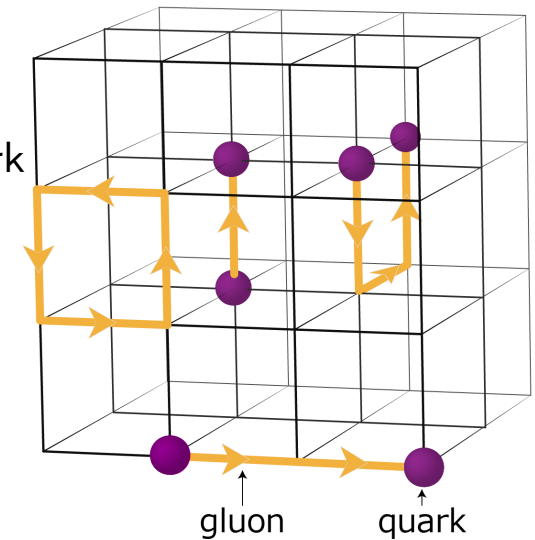
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- typical lattice parameters: $\rightarrow$ **TM fermions, Iwasaki gluons** in our work
 - * $L/a = 32, 48, 64, 80, 96, 128$ $\rightarrow L/a = 32$
 - * $a \in [0.04, 0.15]$ fm $\rightarrow a \approx 0.093$ fm
 - * $L \in [2, 10]$ fm $\rightarrow L \approx 3$ fm
 - * $m_\pi \in [135, 500]$ MeV $\rightarrow m_\pi \approx 260$ MeV
 - * $m_\pi L \geq 3 - 4$ $\rightarrow m_\pi L \approx 4$
 - * $\Rightarrow \infty$ -dim path integral $\rightarrow 10^8 - 10^{12}$ -dim integral

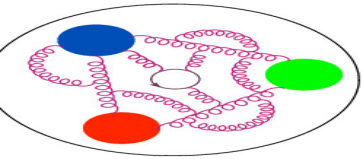




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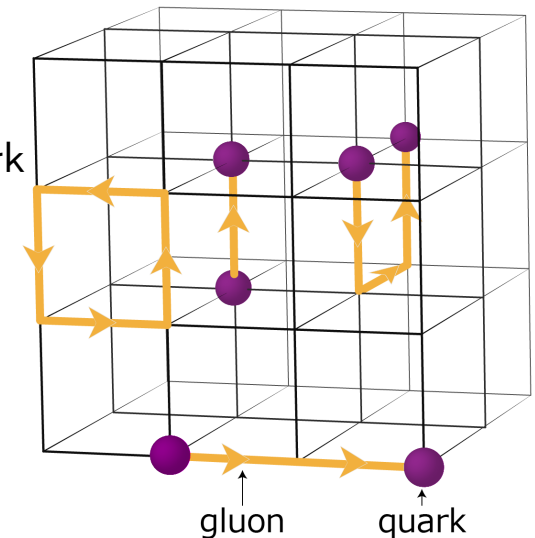
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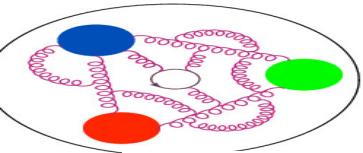




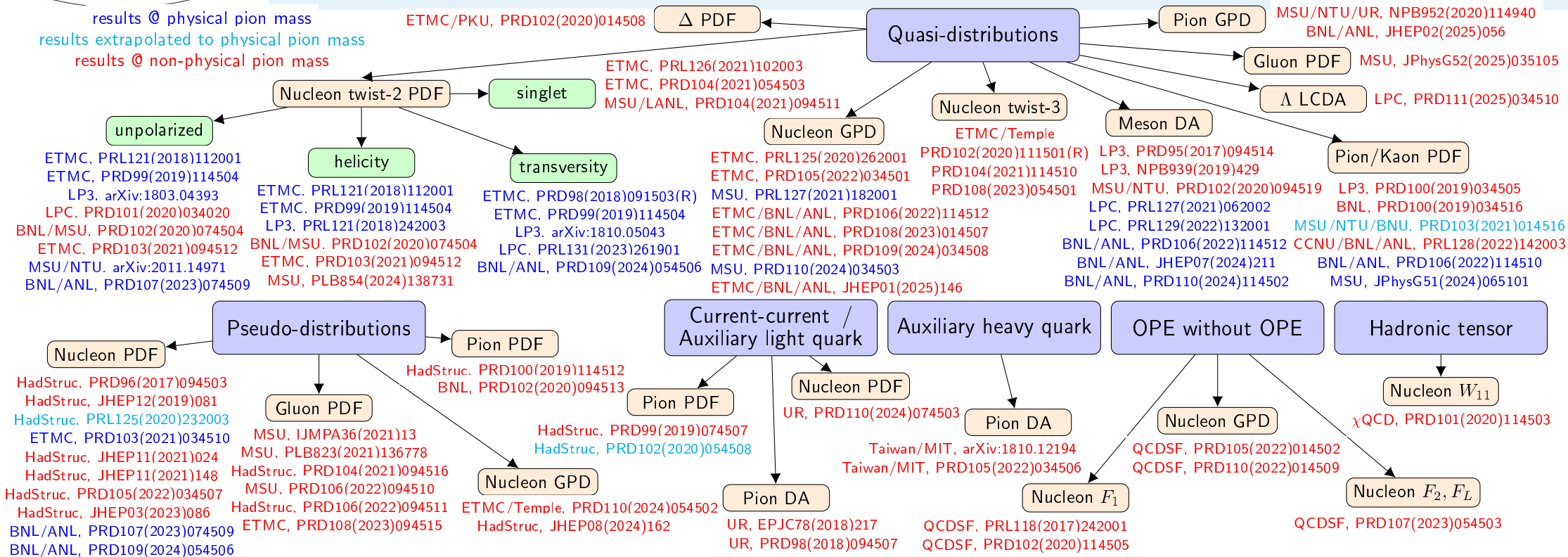
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- Lattice QCD offers a way for a careful *ab initio* study of non-perturbative aspects of QCD.
- Its huge strength: possibility to control all systematic effects: *cut-off effects, finite volume effects, renormalization quark mass effects, isospin breaking, excited states, ...*
- For many aspects, already precision results with percent/per mille total uncertainty.
- However, many aspects (the difficult ones like hadron structure!) with still only exploratory studies.



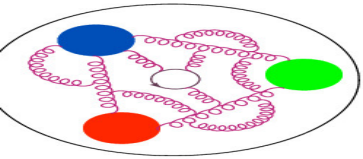


Lattice PDFs/GPDs: dynamical progress



- K. Cichy, *Progress in x -dependent partonic distributions from lattice QCD*, plenary talk LATTICE 2021, 2110.07440
- K. Cichy, *Overview of lattice calculations of the x -dependence of PDFs, GPDs and TMDs*, plenary talk of Virtual Tribute to Quark Confinement 2021, 2111.04552
- K. Cichy, M. Constantinou, *A guide to light-cone PDFs from Lattice QCD: an overview of approaches, techniques and results*, invited review for a special issue of Adv. High Energy Phys. 2019 (2019) 3036904, 1811.07248
- M. Constantinou, *The x -dependence of hadronic parton distributions: A review on the progress of lattice QCD* (would-be) plenary talk of LATTICE 2020, EPJA 57 (2021) 77, 2010.02445
- X. Ji et al., *Large-Momentum Effective Theory*, Rev. Mod. Phys. 93 (2021) 035005
- M. Constantinou et al., *Parton distributions and LQCD calculations: toward 3D structure*, PPNP 121 (2021) 103908

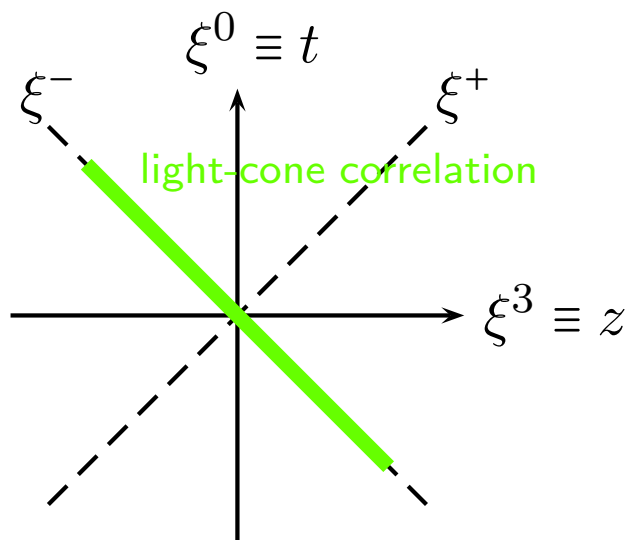
REVIEWS

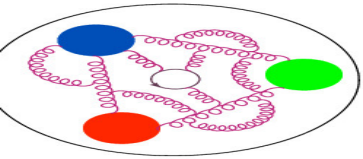


Quasi- and pseudo-distributions



X. Ji, PRL 11(2013)262002; A. Radyushkin, PRD96(2017)034025

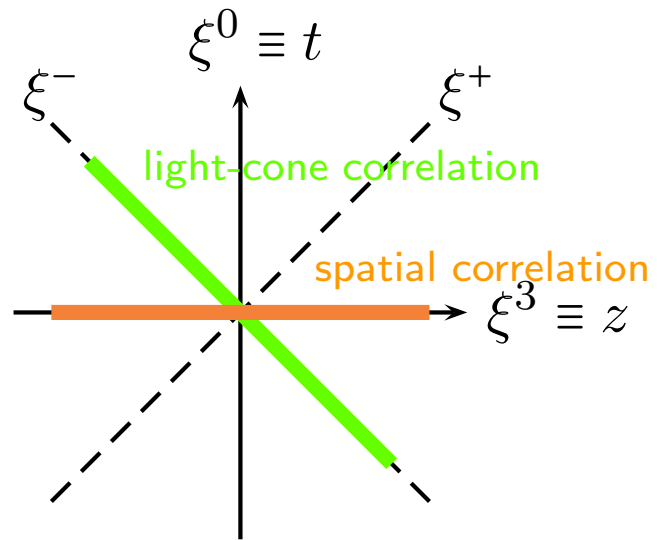


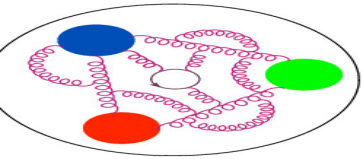


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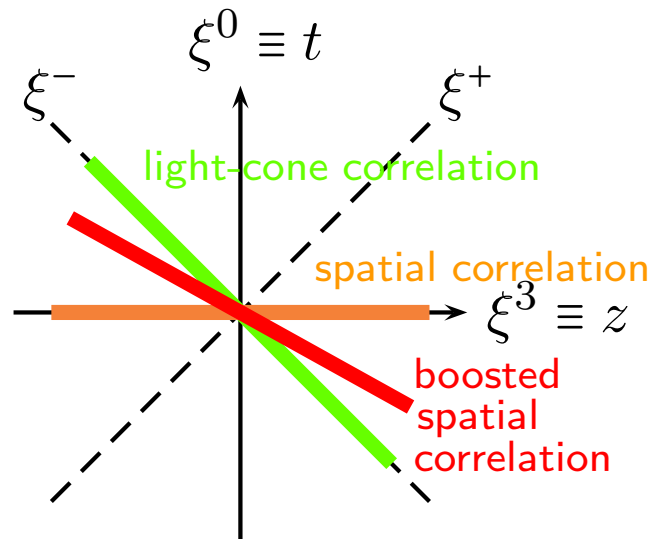


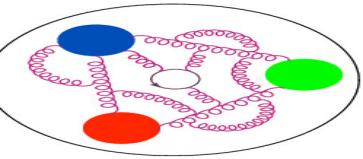


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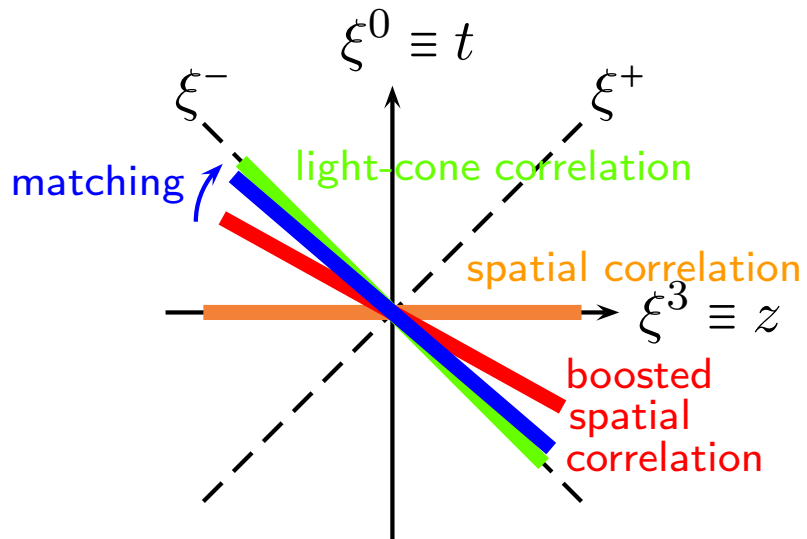


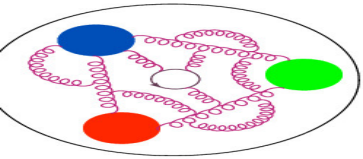


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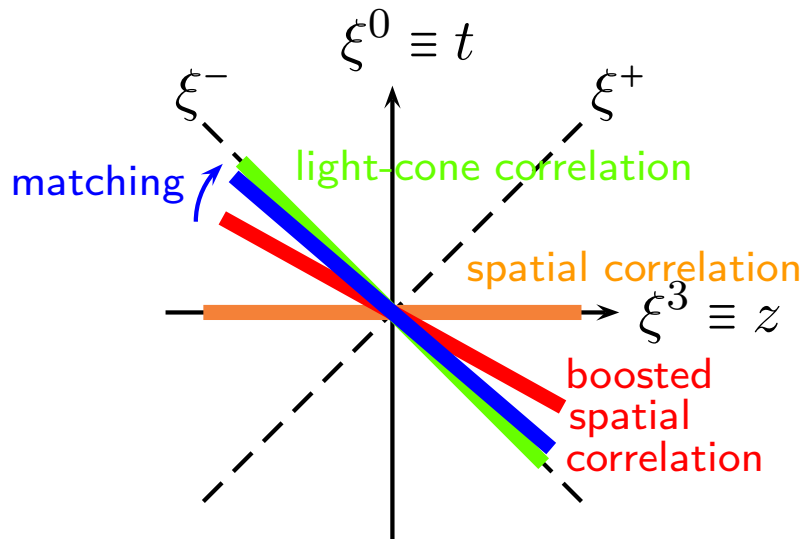
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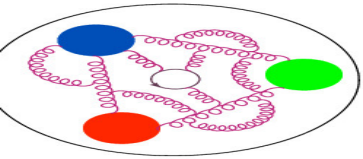
$$F(z, P_i, P_f) = \langle P_f | \bar{\psi}(z) \Gamma \mathcal{A}(z, 0) \psi(0) | P_i \rangle$$

Its Fourier transform (quasi-distribution)
can be matched onto the light-cone distribution:

(Large Momentum Effective Theory (LaMET) $\equiv$ **QUASI**)

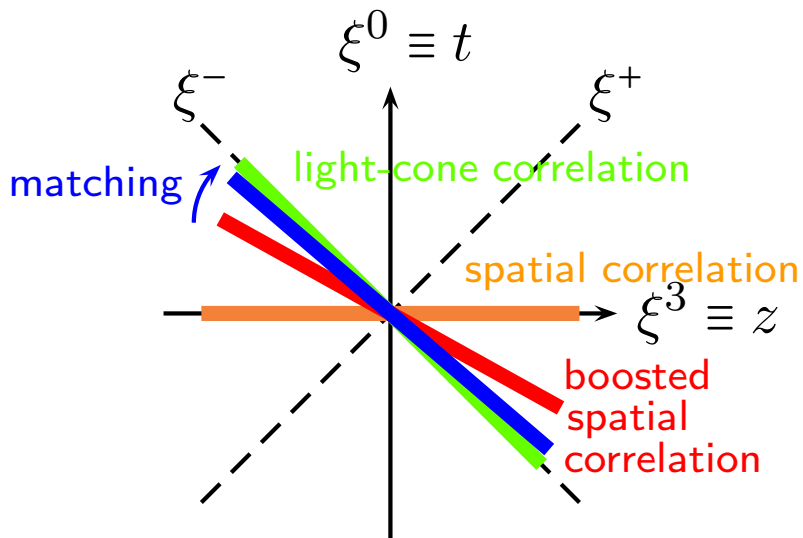
$$\tilde{q}(x, \mu, P_3) = \int_{-1}^1 \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{P_3}\right) q(y, \mu) + \mathcal{O}\left(\Lambda_{\text{QCD}}^2/P_3^2, M_N^2/P_3^2\right)$$

quasi-PDF pert.kernel matched PDF higher-twist effects



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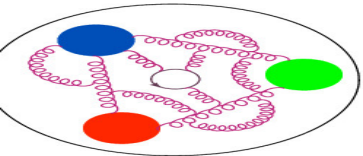
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OR the matrix element itself can be matched:

(Short-Distance Factorization (SDF) $\equiv$ **PSEUDO**)

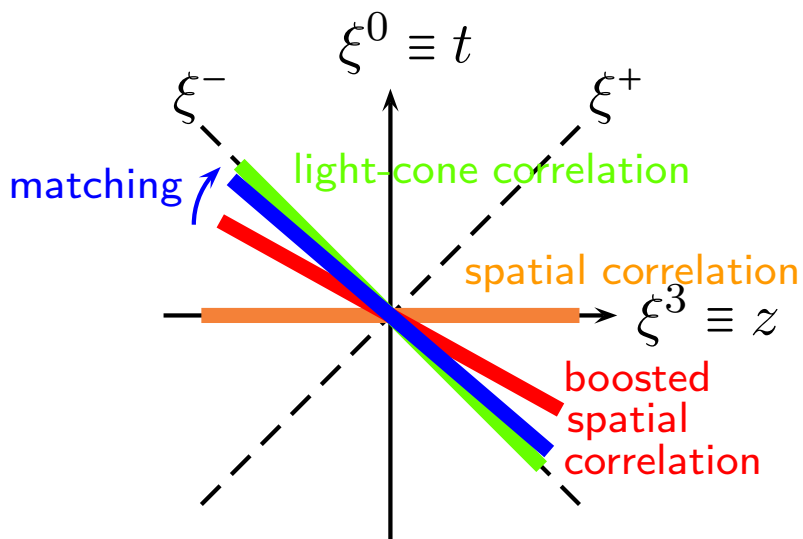
$$\bar{F}(z, P_3) = \int_{-1}^1 dy C_\nu(y) \mathcal{F}(yzP_3) + \mathcal{O}(z^2 \Lambda_{\text{QCD}}^2)$$

pseudo-ITD pert.kernel matched ITD higher-twist effects



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Dirac structures Γ for different PDFs/GPDs:

VECTOR: γ_0, γ_3 : H, E (unpolarized twist-2),

γ_1, γ_2 : G_1, G_2, G_3, G_4 (vector twist-3).

AXIAL VECTOR: $\gamma_5 \gamma_0, \gamma_5 \gamma_3$: $\tilde{H}, \tilde{E}$ (helicity twist-2),

$\gamma_5 \gamma_1, \gamma_5 \gamma_2$: $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{G}_4$ (axial vector twist-3).

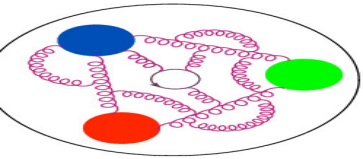
TENSOR: $\gamma_1 \gamma_3, \gamma_2 \gamma_3$: $H_T, E_T, \tilde{H}_T, \tilde{E}_T$ (transversity twist-2),

$\gamma_1 \gamma_2$: H'_2, E'_2 (tensor twist-3).

Need different projectors
to disentangle 2/4 GPDs

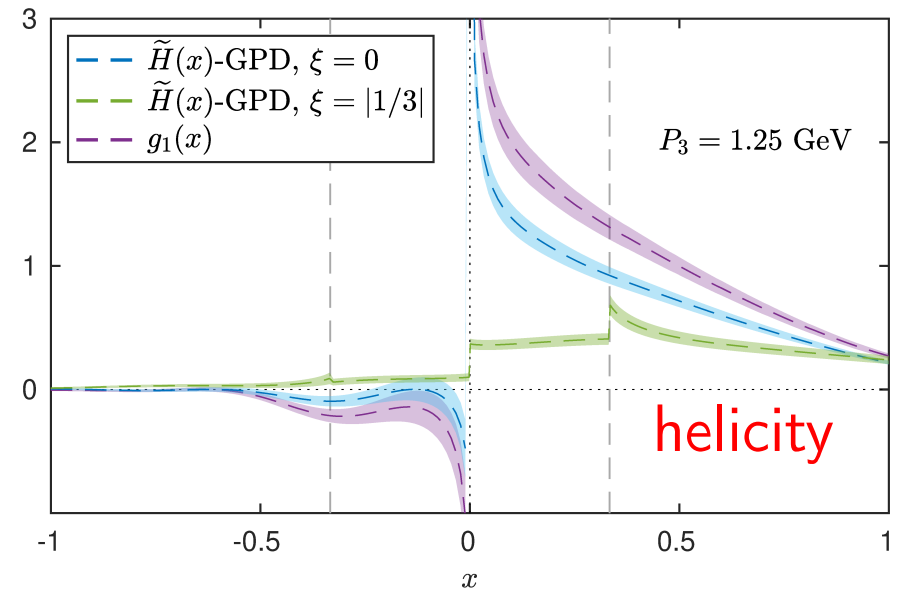
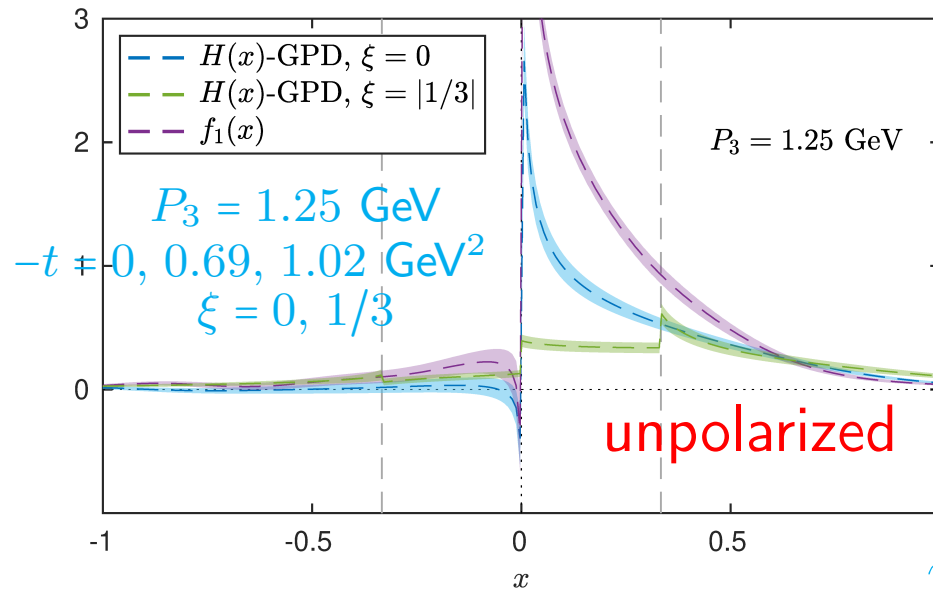
UNPOL: $\mathcal{P} = \frac{1+\gamma_0}{4}$

POL- k : $\mathcal{P} = \frac{1+\gamma_0}{4} i \gamma_5 \gamma_k$

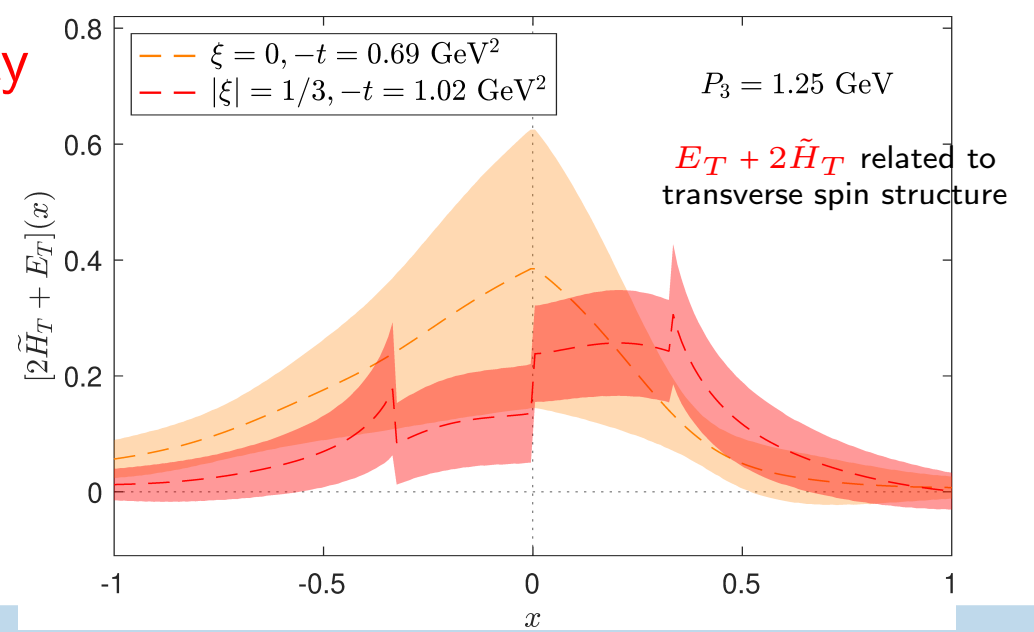
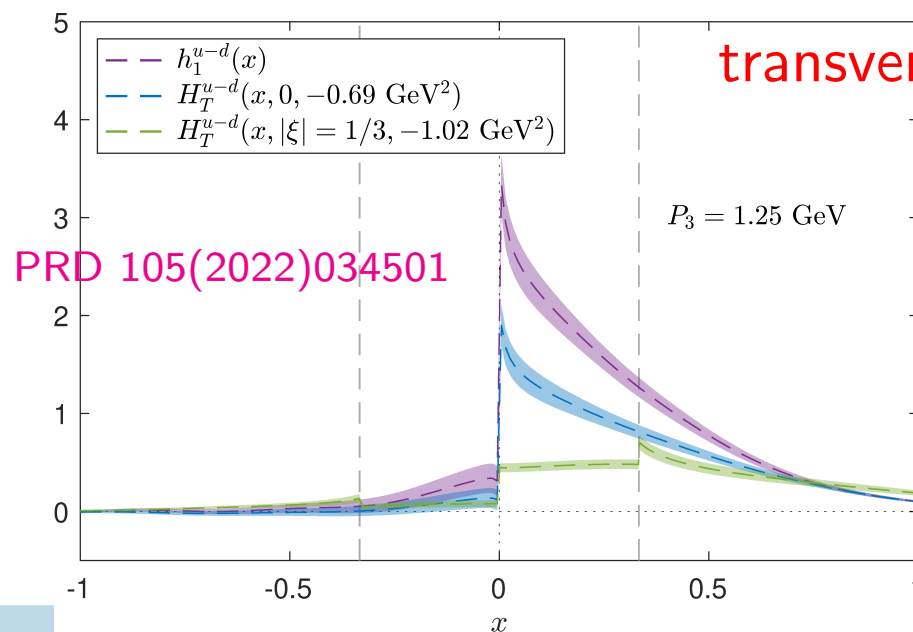


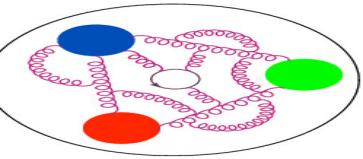
First extractions of x -dependent GPDs

C. Alexandrou, KC, M. Constantinou, K. Jansen, A. Scapellato, F. Steffens, PRL 125(2020)262001



$u - d$





Our comprehensive program for GPDs



Standard symmetric (Breit) frame:

source momentum:

$$P_i = (E, \vec{P} - \vec{\Delta}/2),$$

sink momentum:

$$P_f = (E, \vec{P} + \vec{\Delta}/2),$$

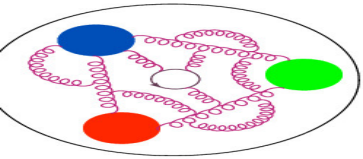
very costly, each $-t$ a separate calculation!

Asymmetric frame:

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cost efficient, many $-t$ at once!



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Main theoretical tool: Lorentz-covariant parametrization of matrix elements:

unpolarized (zero skewness): S. Bhattacharya et al., PRD106(2022)114512

$$F^\mu(z, P, \Delta) = \bar{u}(p', \lambda') \left[\frac{P^\mu}{m} A_1 + m z^\mu A_2 + \frac{\Delta^\mu}{m} A_3 + i m \sigma^{\mu z} A_4 + \frac{i \sigma^{\mu \Delta}}{m} A_5 + \frac{P^\mu i \sigma^{z \Delta}}{m} A_6 + \frac{z^\mu i \sigma^{z \Delta}}{m} A_7 + \frac{\Delta^\mu i \sigma^{z \Delta}}{m} A_8 \right] u(p, \lambda),$$

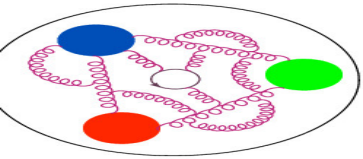
helicity (zero skewness): S. Bhattacharya et al., PRD109(2024)034508

$$F[\gamma^\mu \gamma_5] = \bar{u}(p', \lambda') \left[\frac{i \epsilon^{\mu P z \Delta}}{m} \widetilde{A}_1 + \gamma^\mu \gamma_5 \widetilde{A}_2 + \gamma_5 \left(\frac{P^\mu}{m} \widetilde{A}_3 + m z^\mu \widetilde{A}_4 + \frac{\Delta^\mu}{m} \widetilde{A}_5 \right) + m \not{z} \gamma_5 \left(\frac{P^\mu}{m} \widetilde{A}_6 + m z^\mu \widetilde{A}_7 + \frac{\Delta^\mu}{m} \widetilde{A}_8 \right) \right] u(p, \lambda)$$

transversity (zero skewness): S. Bhattacharya et al., arXiv:2505.11288 $F[i \sigma^{\mu \nu} \gamma_5] = \bar{u}(p', \lambda') \left[\sum_{i=1}^{12} \Gamma_i^{\mu \nu} A_{Ti} \right] u(p, \lambda)$

unpolarized (nonzero skewness): M. Chu et al., arXiv:2508.17988, accepted in PRD

See talk by J. Miller Wed 15:50



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also:

First nucleon unpolarized+helicity GPDs (symm. frame) C. Alexandrou et al., PRL 125(2020)262001

First nucleon transversity nucleon GPDs (symm. frame) C. Alexandrou et al., PRD 105(2022)034501

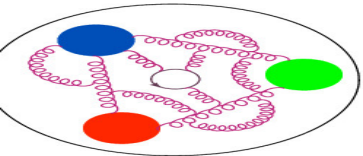
Twist-3 axial GPDs (symm. frame) S. Bhattacharya et al., PRD 108(2023)054501

Moments of unpolarized GPDs (OPE) S. Bhattacharya et al., PRD 108(2023)014507

Unpolarized pseudo-GPDs S. Bhattacharya et al., PRD110(2024)054502

Combining lattice+pheno KC, M. Constantinou, P. Sznajder, J. Wagner, PRD 110(2024)114025

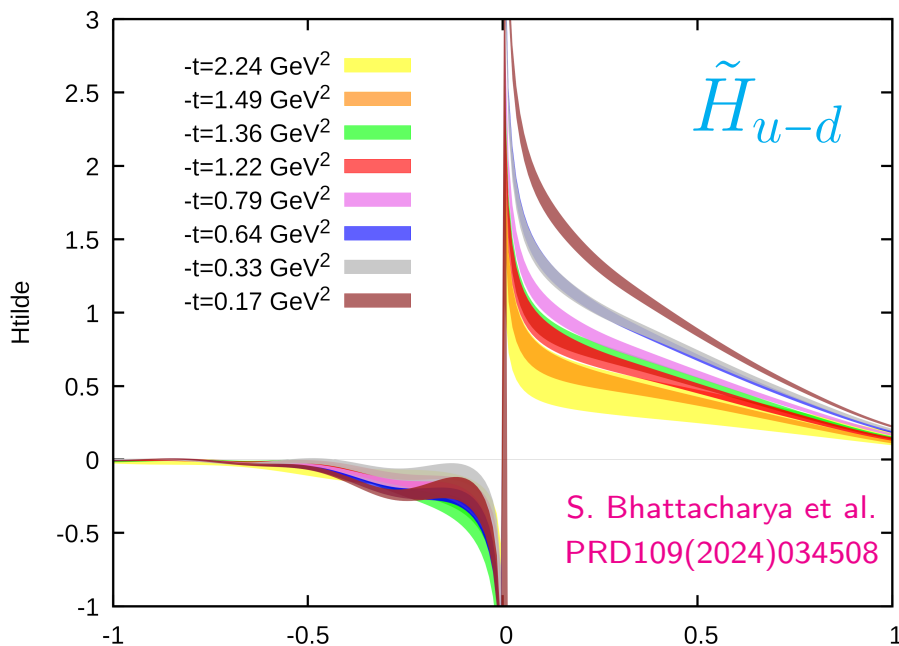
Moments of helicity GPDs (OPE) S. Bhattacharya et al., JHEP01(2025)146



t -dependence of $\tilde{H}/H/E$ GPDs (quasi, asym. frame)



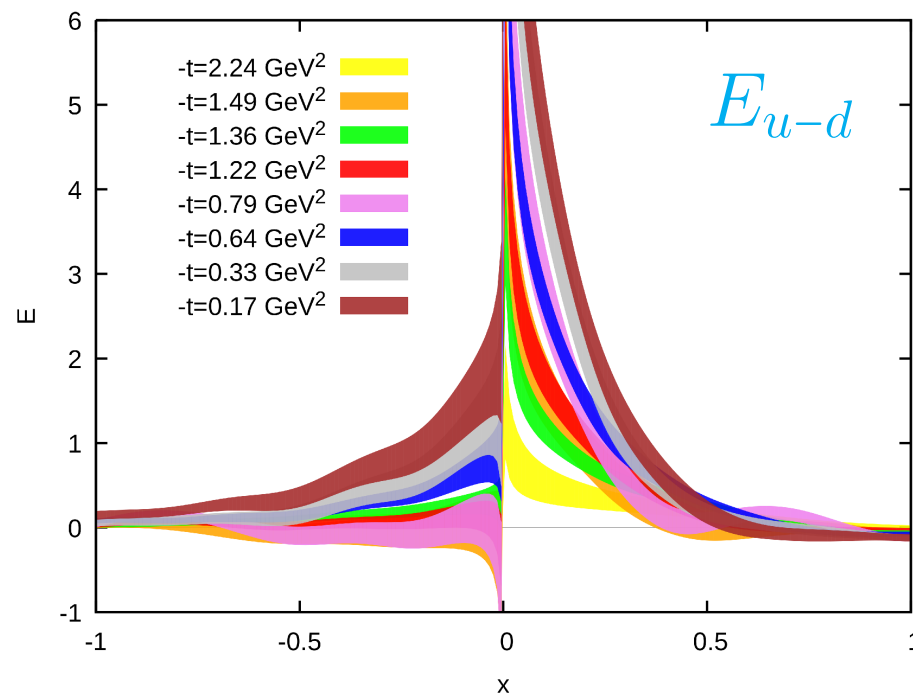
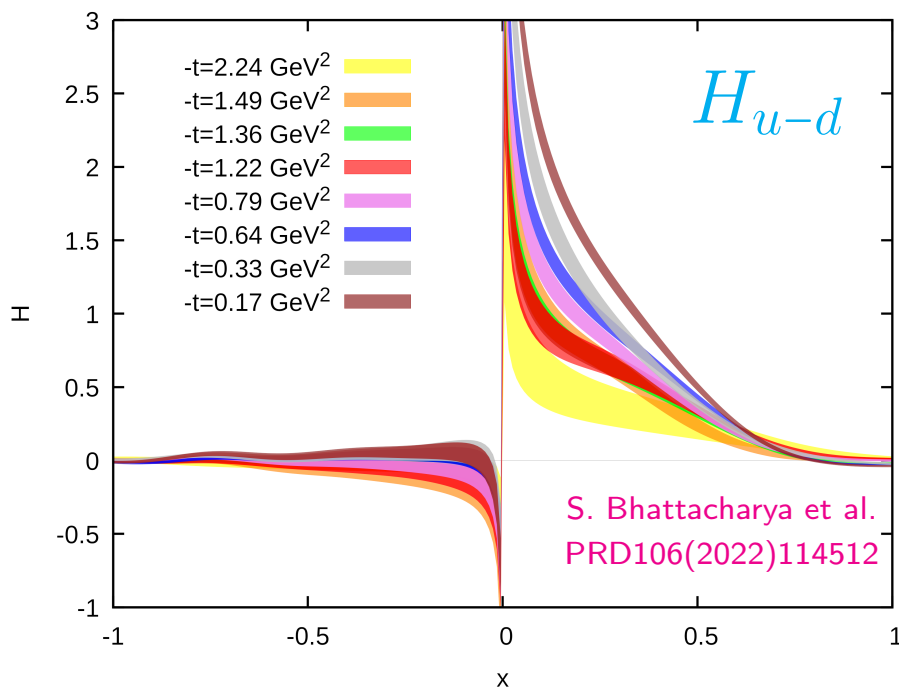
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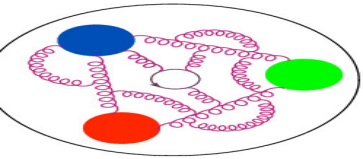


$$\begin{aligned} \Delta = (1, 0, 0) &\Rightarrow -t = 0.17 \text{ GeV}^2 \\ \Delta = (1, 1, 0) &\Rightarrow -t = 0.33 \text{ GeV}^2 \\ \Delta = (2, 0, 0) &\Rightarrow -t = 0.64 \text{ GeV}^2 \\ \Delta = (2, 1, 0) &\Rightarrow -t = 0.79 \text{ GeV}^2 \\ \Delta = (2, 2, 0) &\Rightarrow -t = 1.22 \text{ GeV}^2 \\ \Delta = (3, 0, 0) &\Rightarrow -t = 1.36 \text{ GeV}^2 \\ \Delta = (3, 1, 0) &\Rightarrow -t = 1.49 \text{ GeV}^2 \\ \Delta = (4, 0, 0) &\Rightarrow -t = 2.24 \text{ GeV}^2 \end{aligned}$$

Impact parameter distribution:

$$GPD(x, b_{\perp}) = \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} e^{-ib_{\perp} \cdot \Delta_{\perp}} GPD(x, t)$$



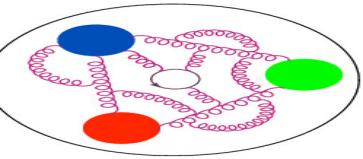


Combining lattice with pheno/exp data



Important: opportunity for **COMPLEMENTARITY**:
combine lattice data with phenomenological/experimental data

See talk by P. Sznajder Thu 10:00



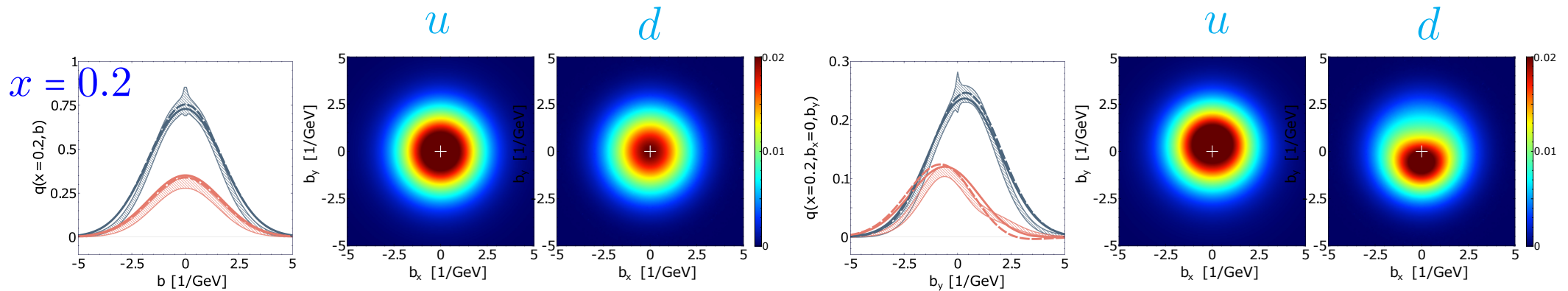
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See talk by P. Sznajder Thu 10:00

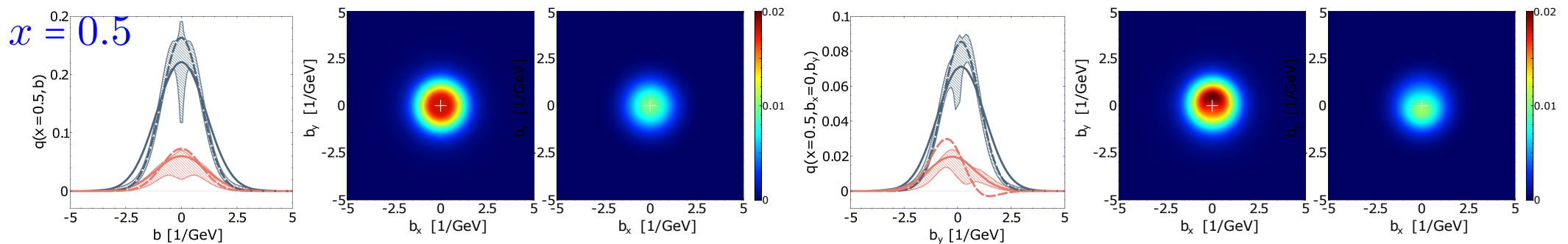
combine lattice data with phenomenological/experimental data:

- lattice: double ratios removing explicit info on PDFs and EFFs (thus, lots of systematics mildened),
- pheno/exp: proton's and neutron's magnetic and electric EFFs and their ratios.

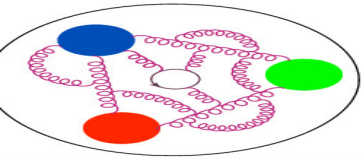


UNPOLARIZED PROTON

TRANSV. POLARIZED PROTON



KC, M. Constantinou, P. Sznajder, J. Wagner, Phys. Rev. D110 (2024) 114025

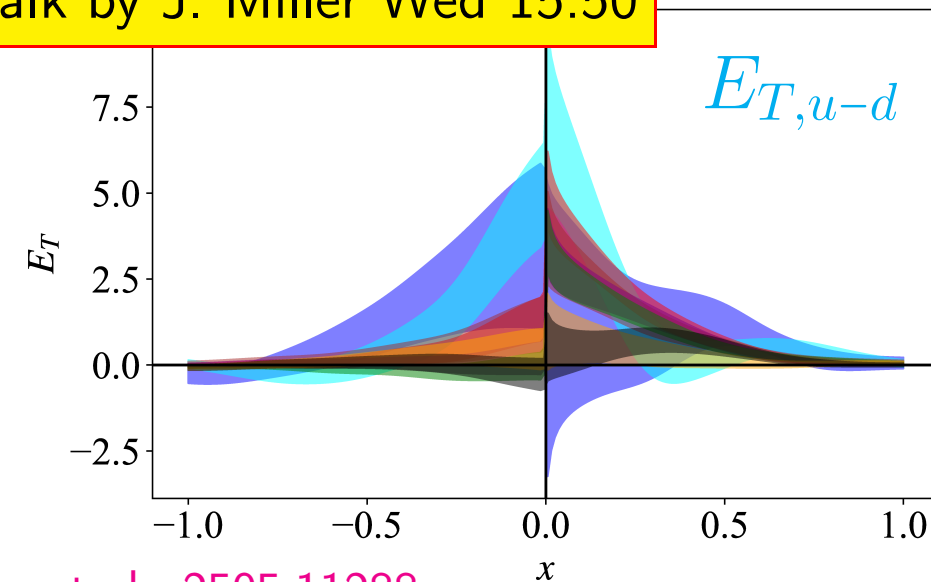
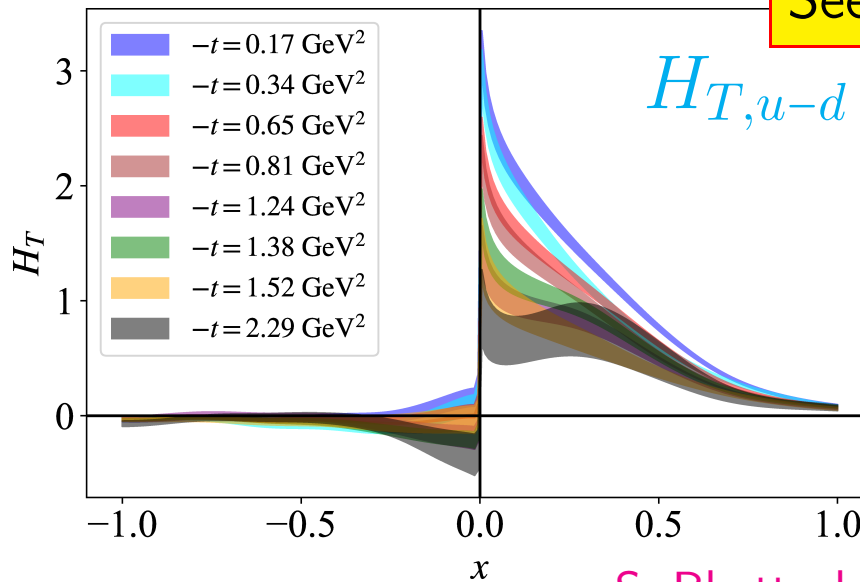


t -dependence of $H_T/E_T/\tilde{H}_T/2\tilde{H}_T + E_T$ GPDs (quasi, asym. frame)

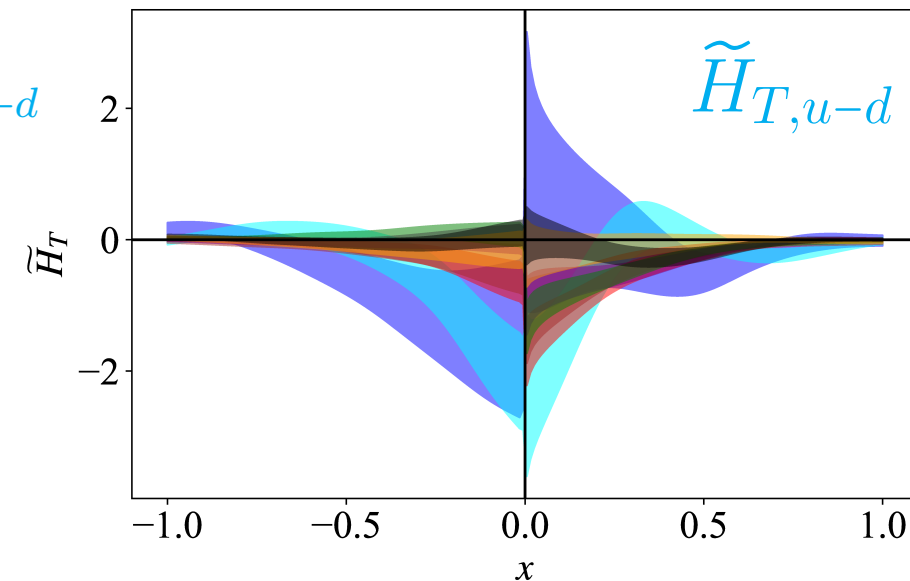
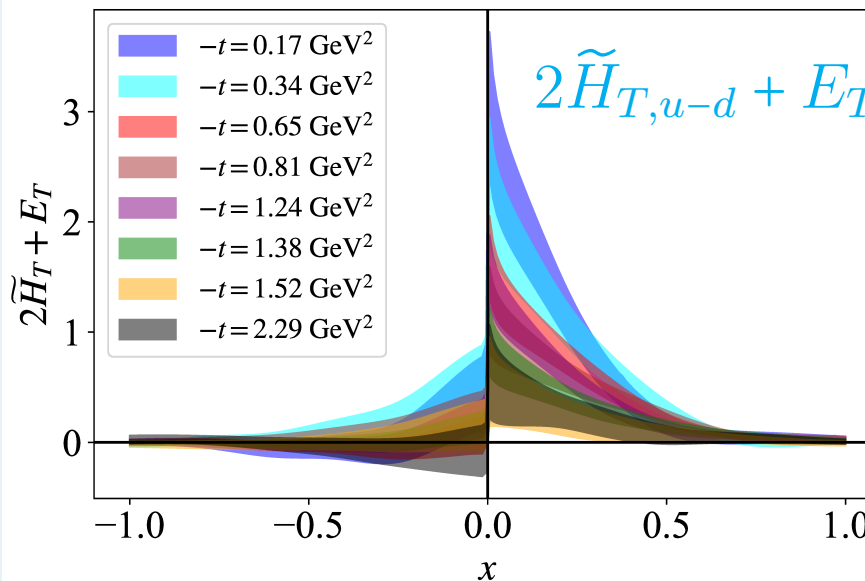


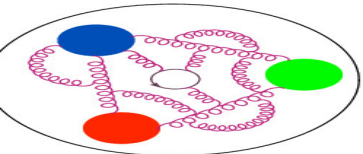
See talk by J. Miller Wed 15:50

Nucleon structure
Lattice QCD
Quasi and pseudo
First extraction
GPDs program
ANNs
 $\xi \neq 0$
Summary



S. Bhattacharya et al., 2505.11288



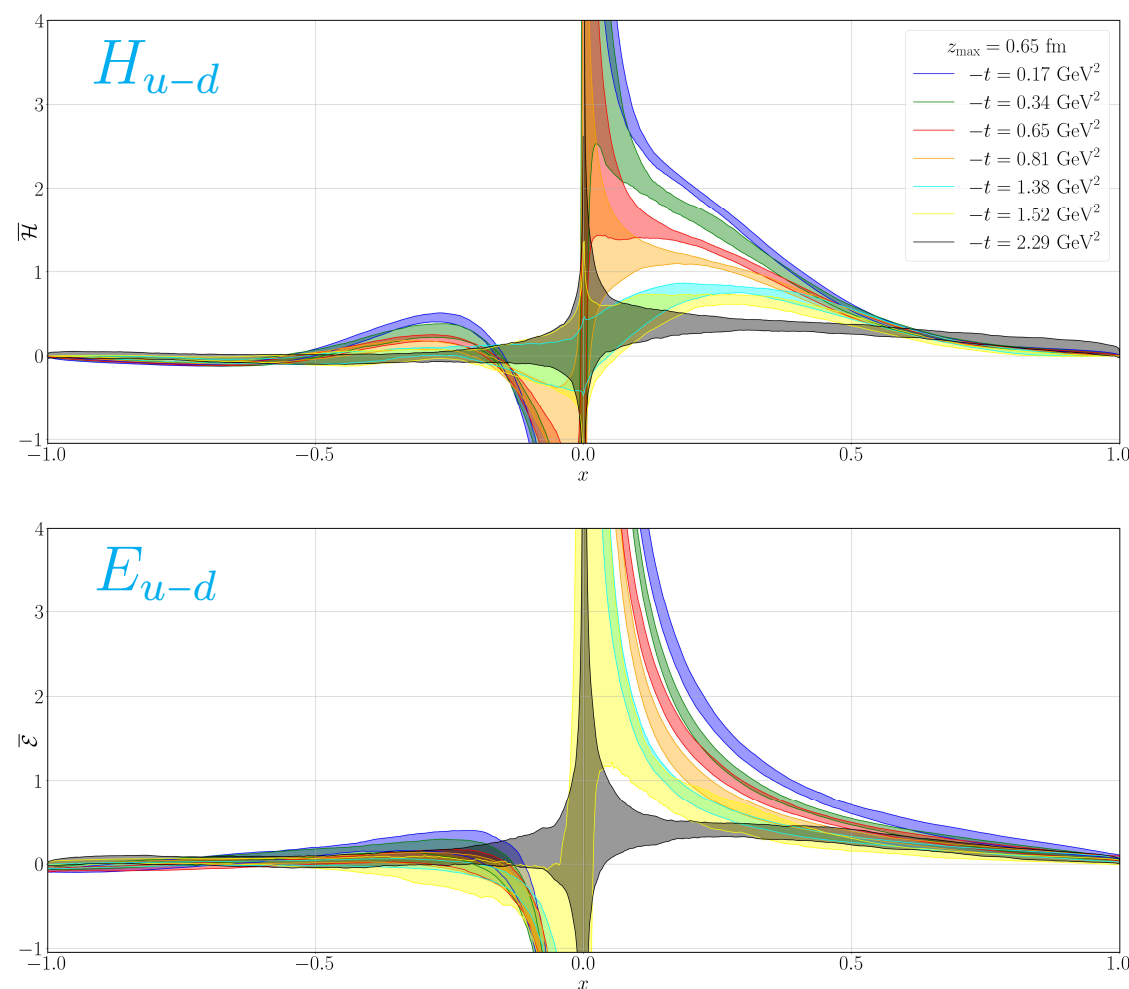


t -dependence of H/E GPDs (pseudo, asym. frame)

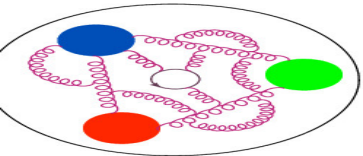


S. Bhattacharya, KC, M. Constantinou, A. Metz, N. Nurminen, F. Steffens, PRD110(2024)054502

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Qualitatively similar picture to the one from quasi-GPDs.
Quantitative differences due to different systematics.



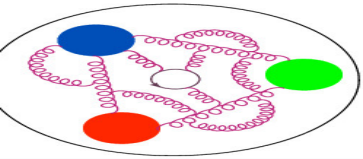
Combining quasi and pseudo approaches?



The quasi and pseudo approaches have a different status of x -dependence reconstruction:

- **quasi** – **fully-reliable** in a **limited** range of $x \in [x_{\min}, x_{\max}] \approx [0.2, 0.8]$
reason: power corrections of $\mathcal{O}(\Lambda_{\text{QCD}}^2/x^2 P_3^2)$, $\mathcal{O}(\Lambda_{\text{QCD}}^2/(1-x)^2 P_3^2)$.

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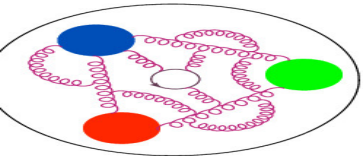
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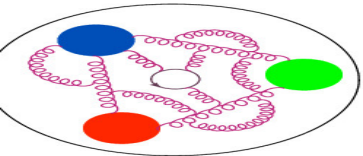


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See also: X. Ji, *Ab Initio Lattice Quantum Chromodynamics Calculations of Parton Physics in the Proton: Large-Momentum Effective Theory versus Short-Distance Expansion*, Research 2025;8, Article 0695 (arXiv:2209.09332)



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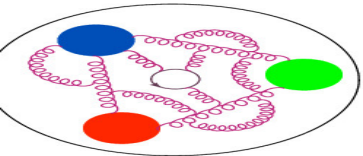
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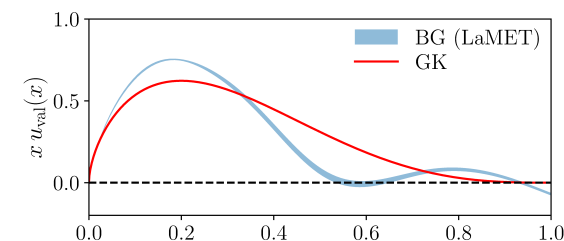
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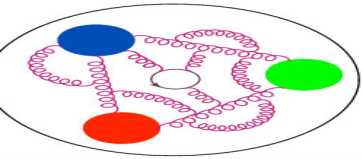
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Mock data test: generate 10 discrete points from a pheno PDF and use Backus-Gilbert to reconstruct the pheno PDF with this limited dataset



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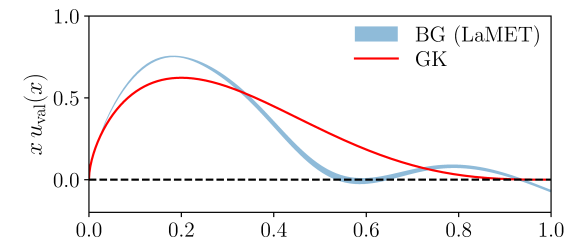
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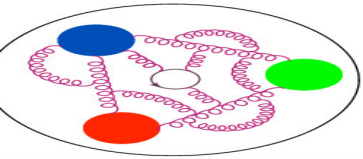
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Finally: **robust way for reconstructing the t -dependence for GPDs.**



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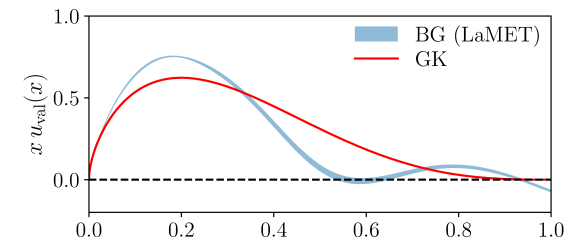
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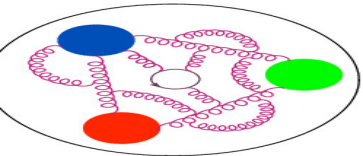
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Tool to achieve these goals:

NEURAL NETWORKS



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Our neural network framework

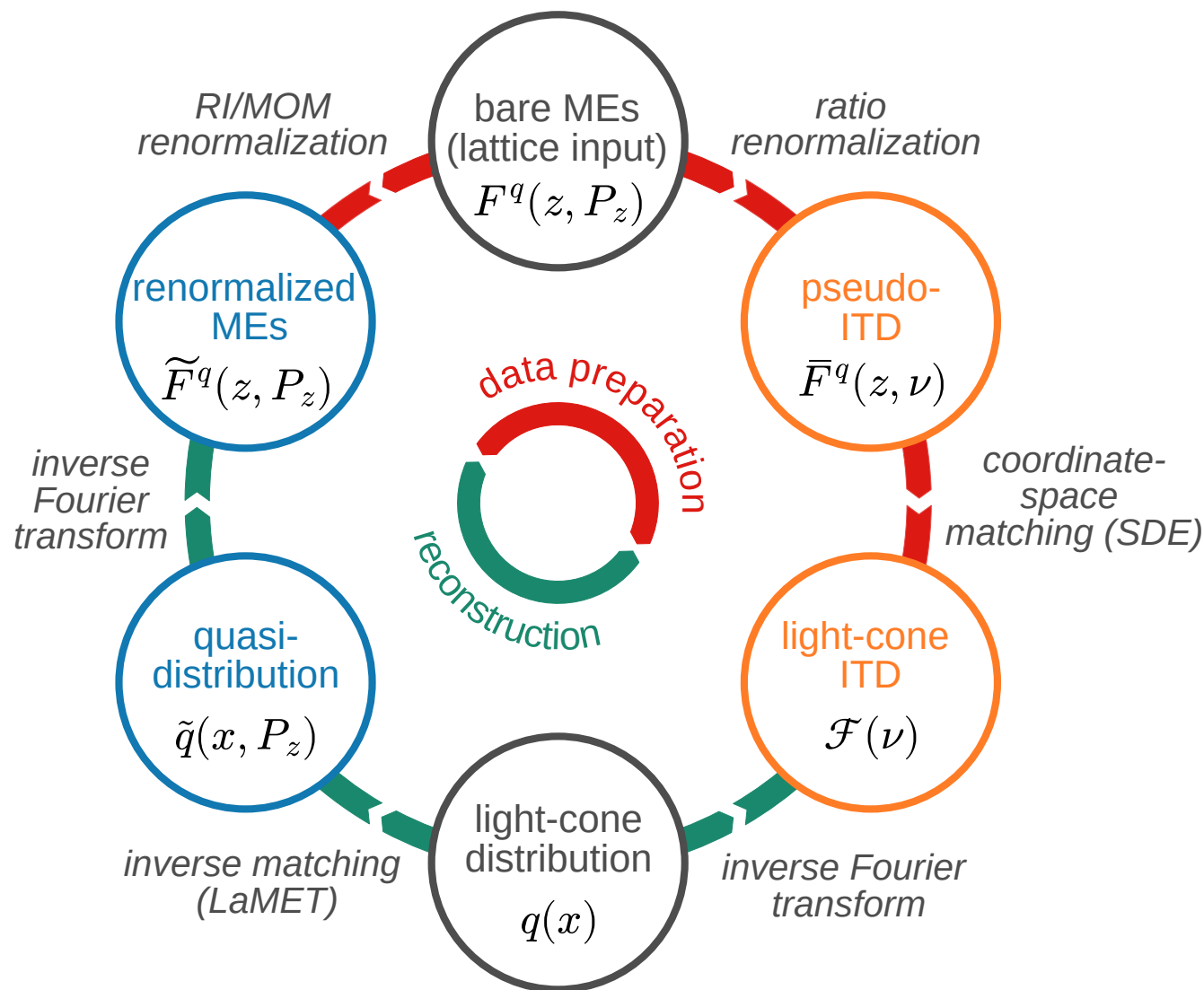


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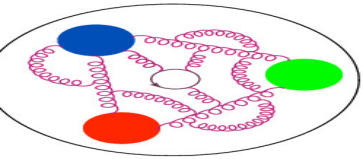
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Summary



M. Chu, KC, M. Constantinou, P. Sznajder, J. Wagner, arXiv:2509.15931

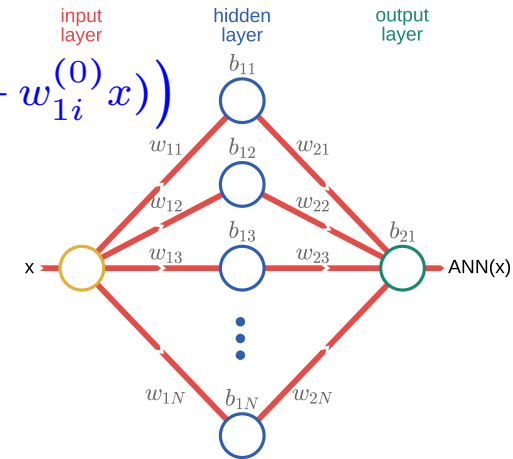


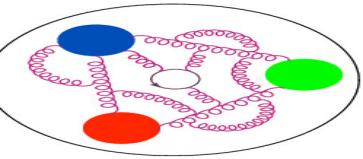
Functional form of ANN PDFs

Constructed light-cone PDF: $q^{(0)}(x) = x^{\delta^{(0)}} (1-x)^{\rho^{(0)}} \text{ANN}^{(0)}(x)$

Single artificial neural network: $\text{ANN}^{(0)}(x) = b_{21}^{(0)} + \sum_{i=1}^{N^{(0)}} \left[w_{2i}^{(0)} \ln \left(1 + \exp(b_{1i}^{(0)} + w_{1i}^{(0)} x) \right) \right]$

Weights $w_{jk}^{(0)}$ and biases $b_{jk}^{(0)}$ + powers $\delta^{(0)}, \rho^{(0)}$ constrained by lattice data.





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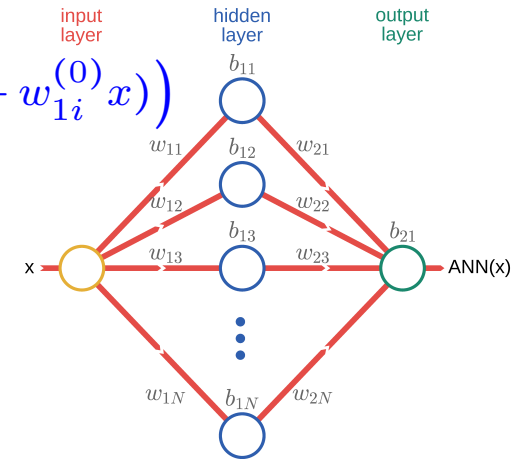
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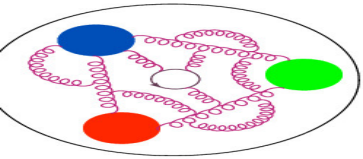
$$q^{(Q)}(x) = \begin{cases} q^{(0)}(x) + q^{(1)}(x), & 0 < x < x_{\min}, \\ q^{(0)}(x) + q^{(2)}(x), & x_{\max} < x < 1, \\ q^{(0)}(x), & \text{otherwise,} \end{cases} \quad q^{(P)}(x) = \begin{cases} q^{(0)}(x) + q^{(3)}(x), & x_{\min} < x < x_{\max}, \\ q^{(0)}(x), & \text{otherwise,} \end{cases}$$

Constraints: $q^{(1)}(x_{\min}) = 0, \quad q^{(1)}(x) = x^{\delta^{(1)}} (x_{\min} - x)^{\rho^{(1)}} \text{ANN}^{(1)}(x),$

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Functional form of ANN PDFs

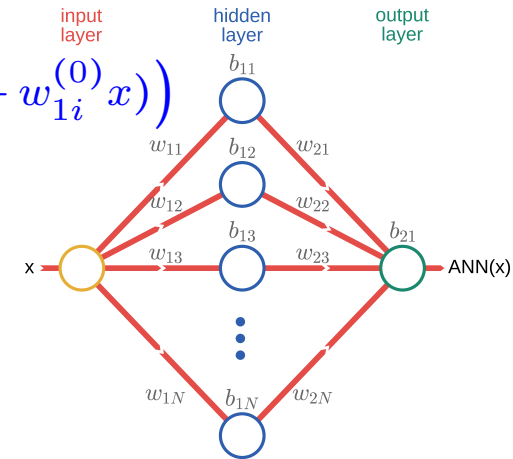
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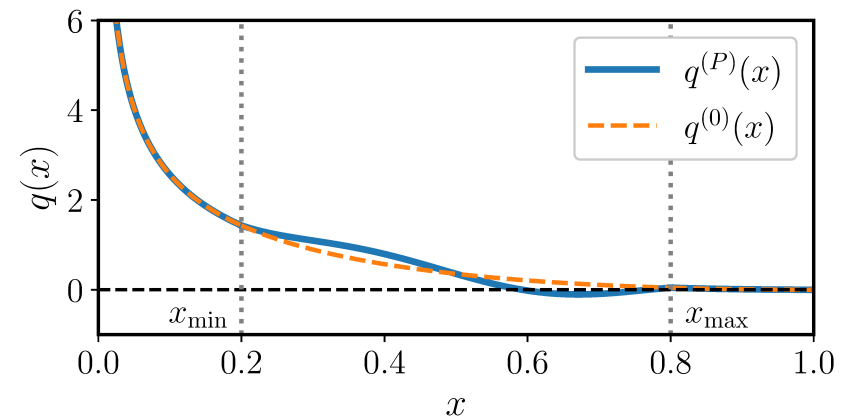
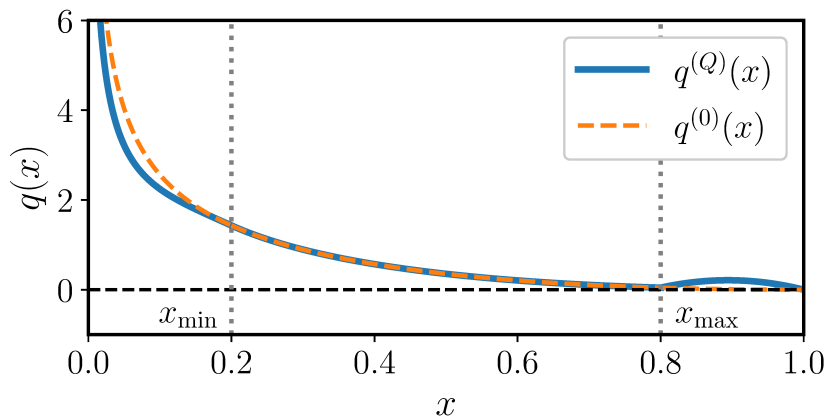
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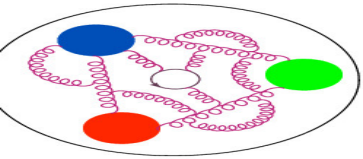
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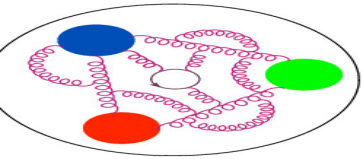




Mock data test for PDFs



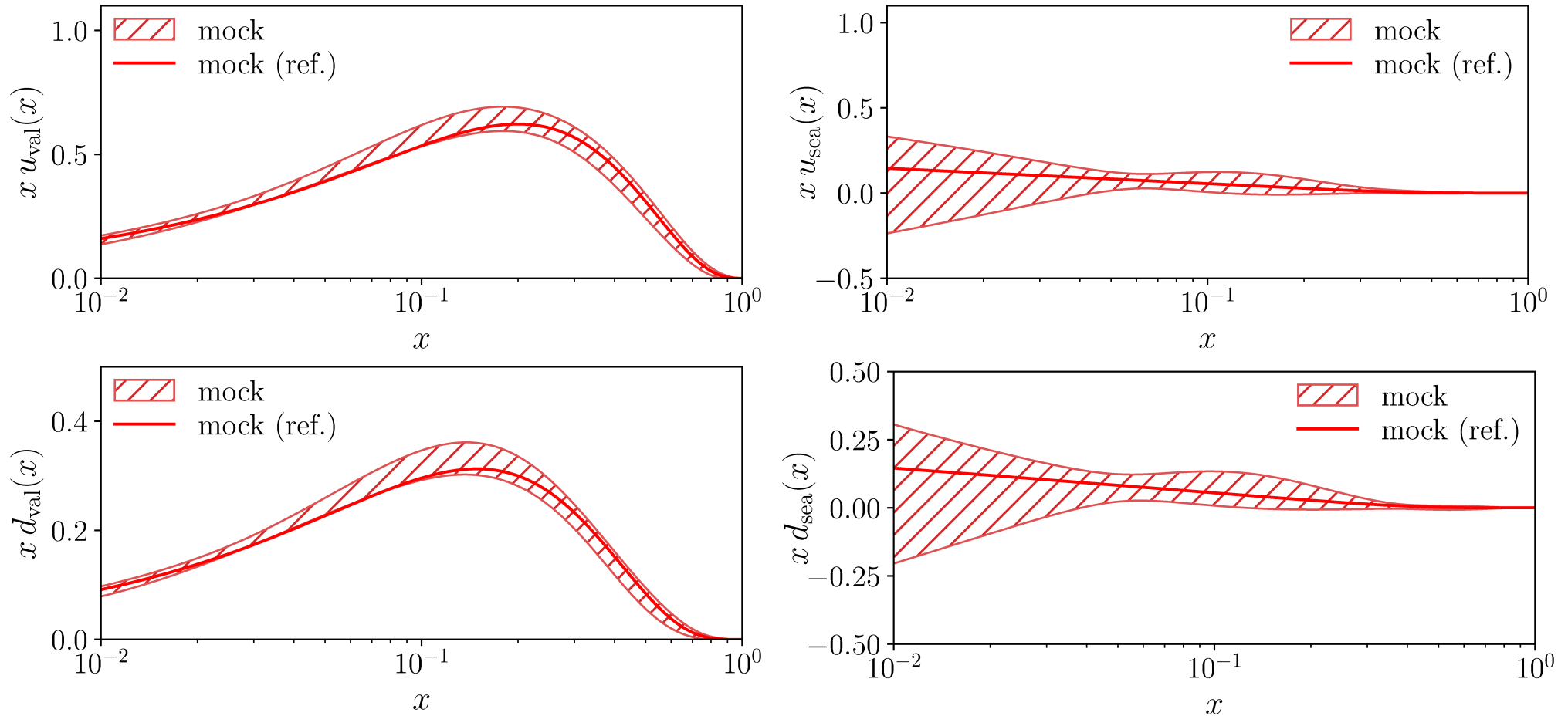
Mock data test: generate 10 discrete points from a pheno PDF and use ANN to reconstruct the pheno PDF with this limited dataset

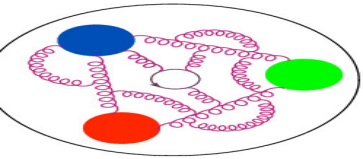


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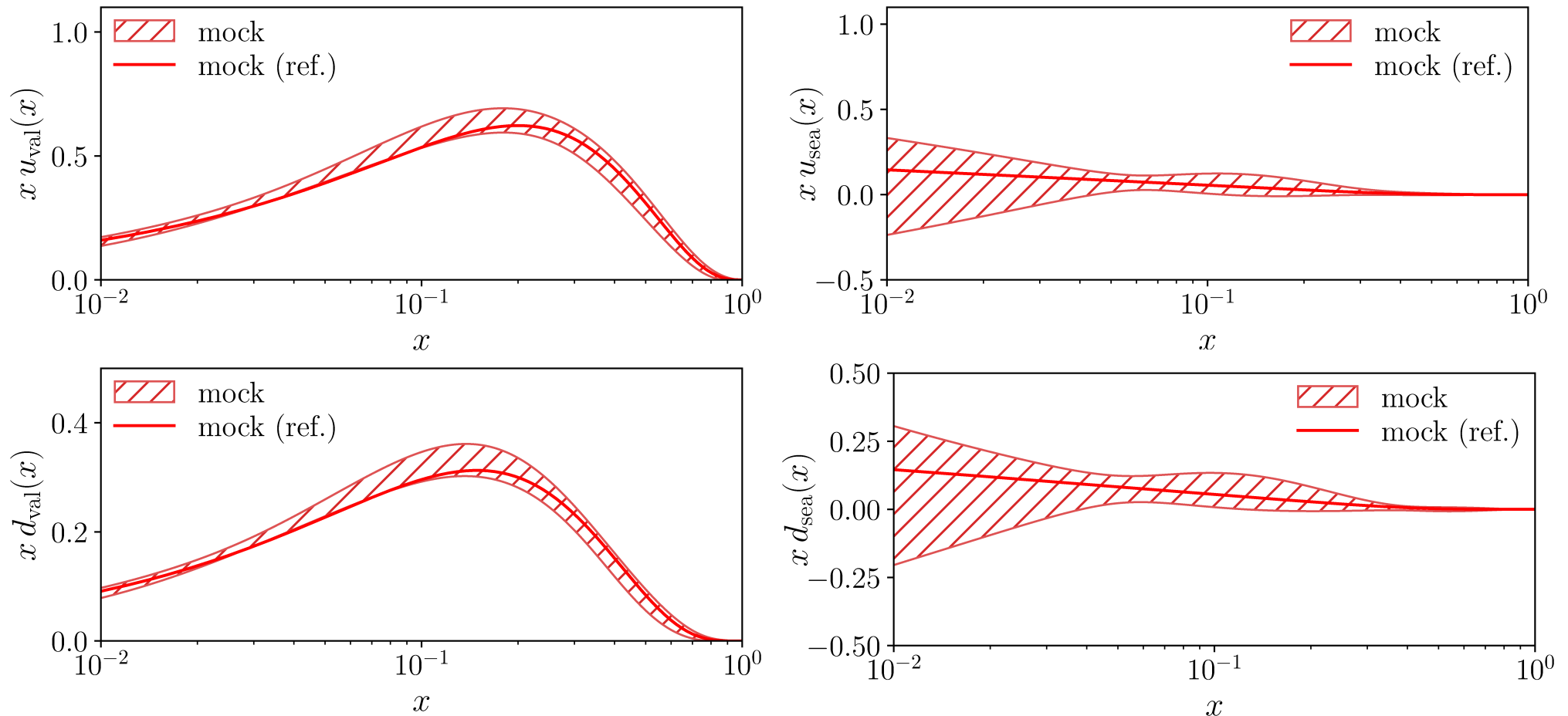




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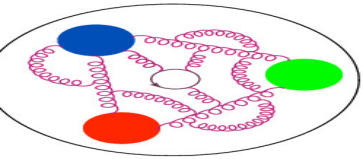
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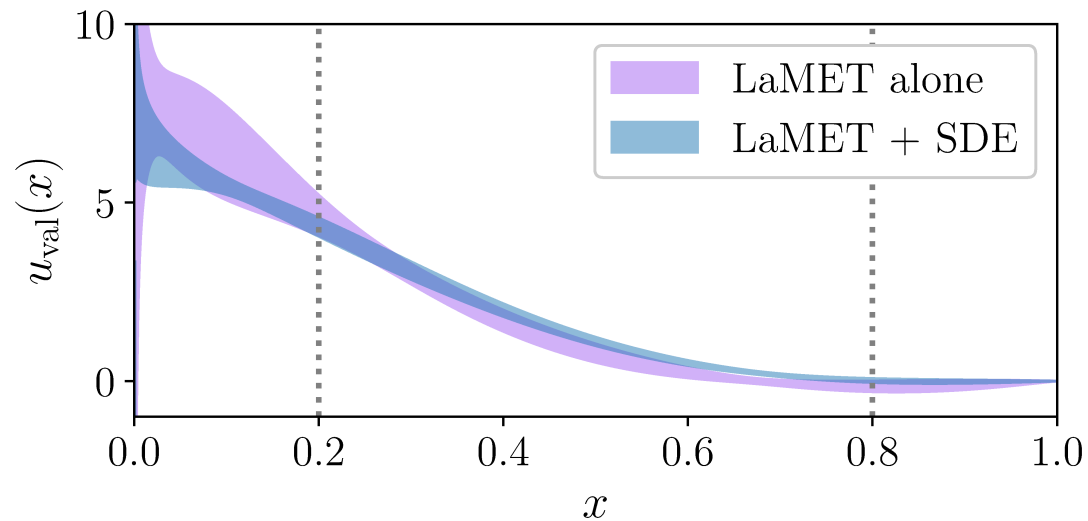
Fully successful reconstruction of the pheno PDF!

Important note: limitations of data (discreteness/truncation) translated to uncertainties

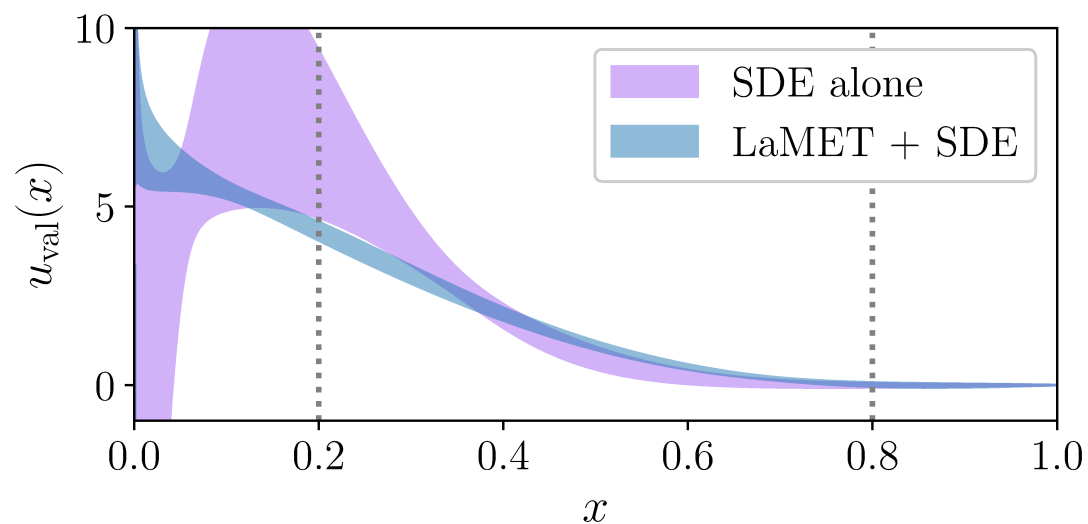


Lattice results for PDFs

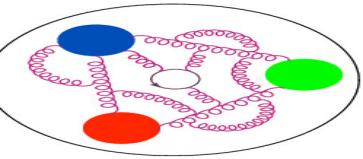
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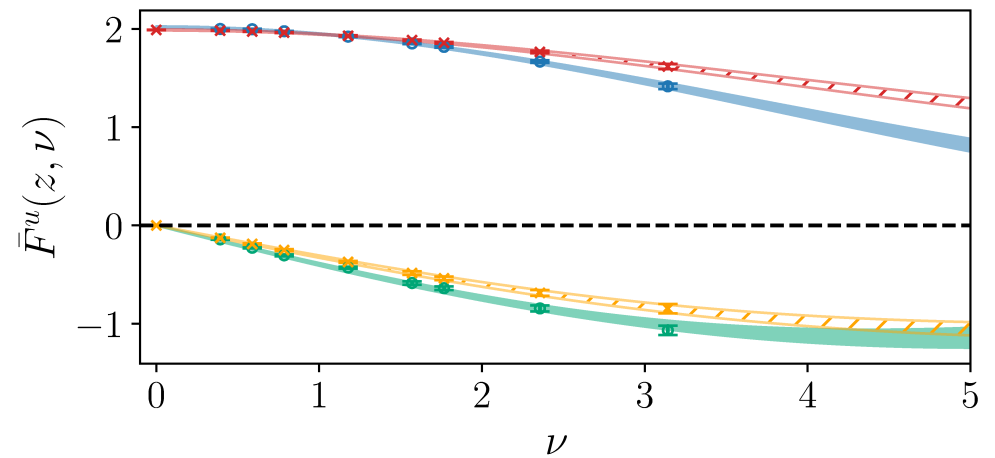
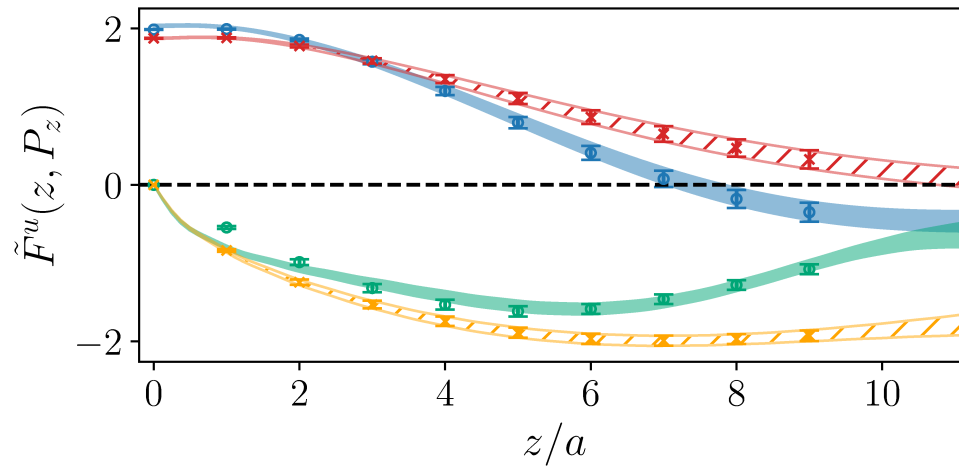
Purple bands:
fit using only quasi (LaMET)
or only pseudo (SDE)



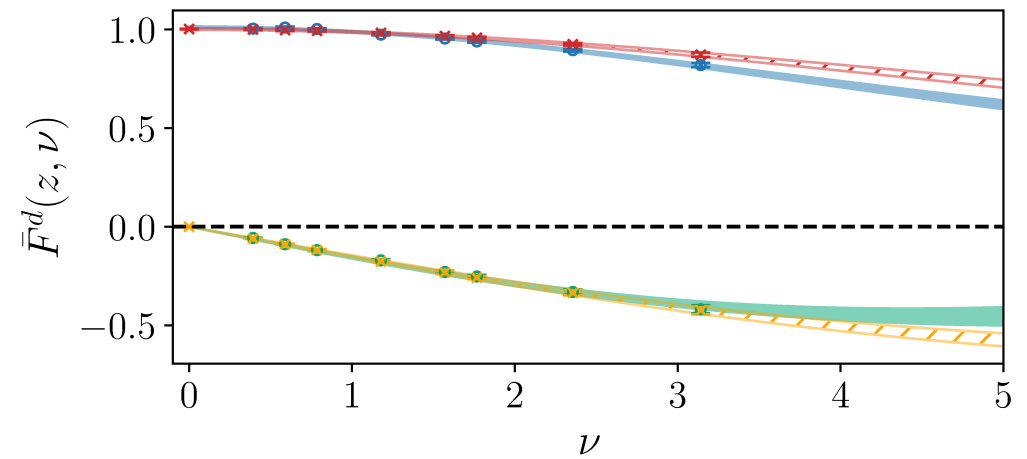
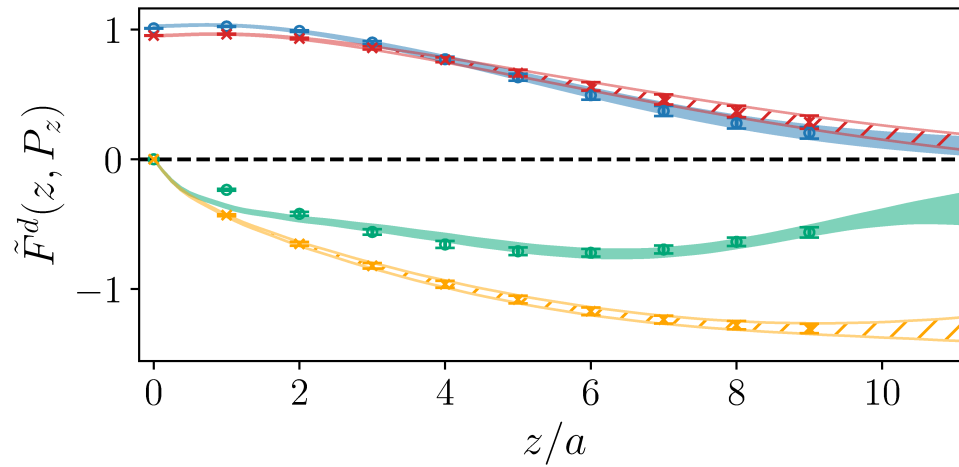
Blue bands:
combined quasi+pseudo fit
(LaMET+SDE)

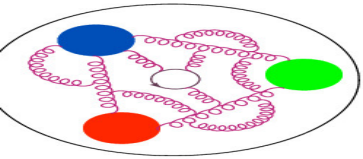


Lattice results for PDFs

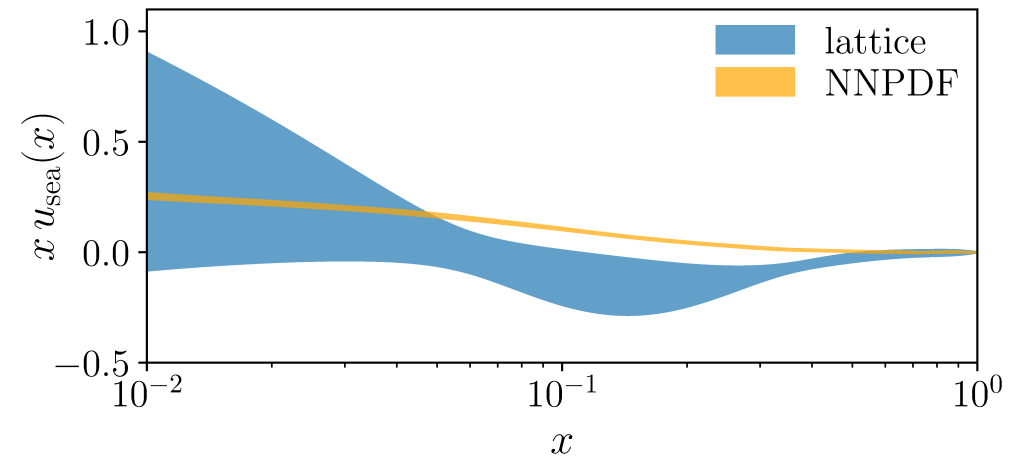
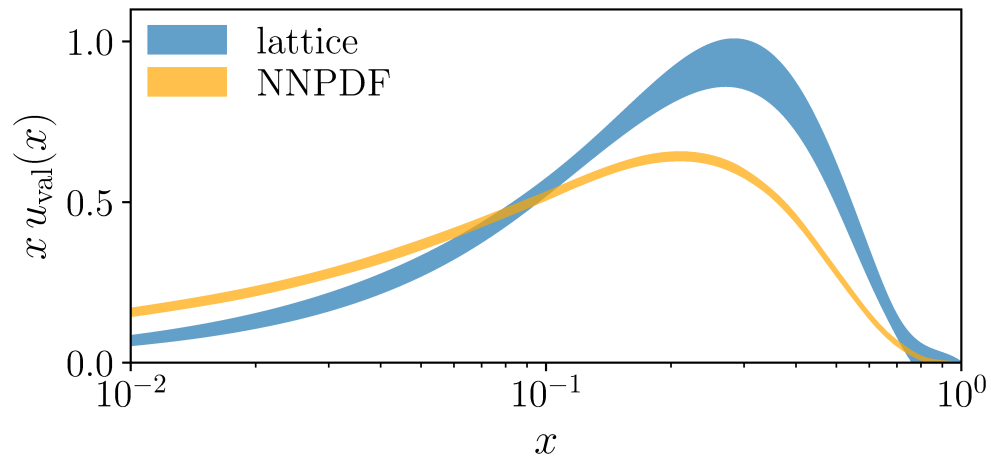


M. Chu, KC, M. Constantinou, P. Sznajder, J. Wagner, arXiv:2509.15931

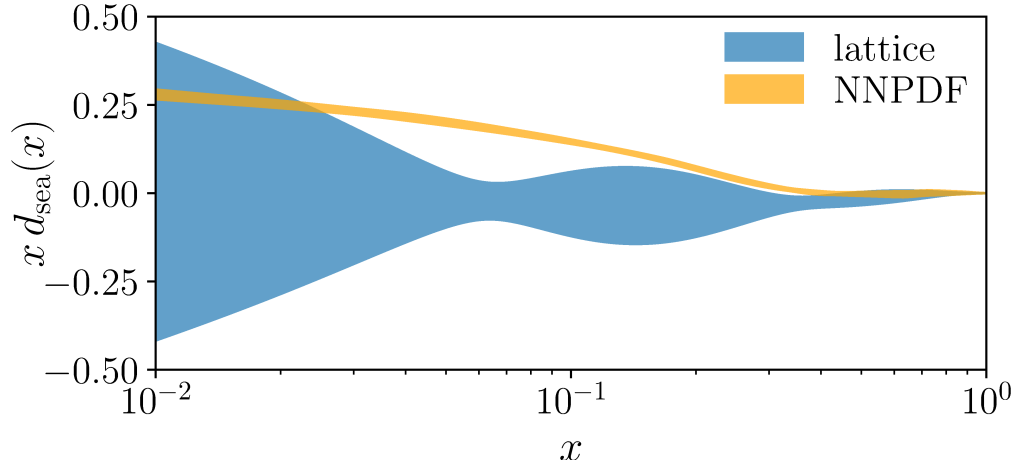
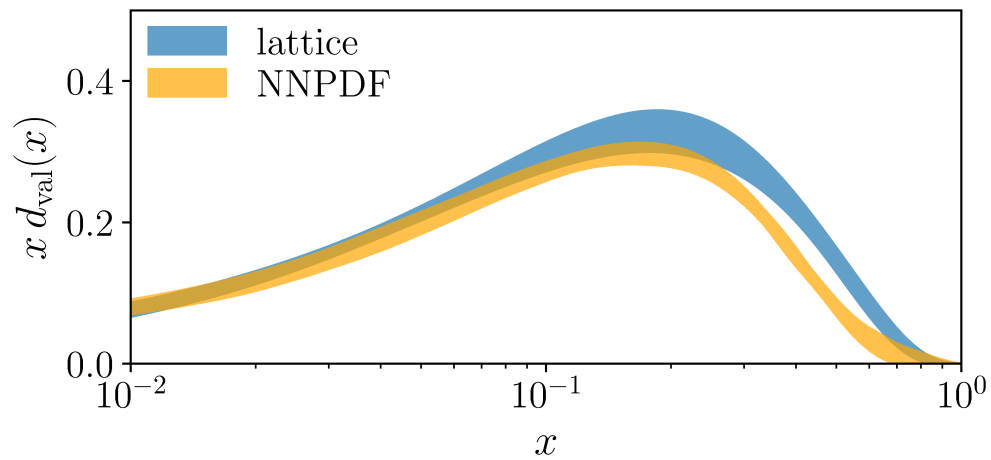


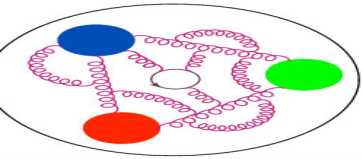


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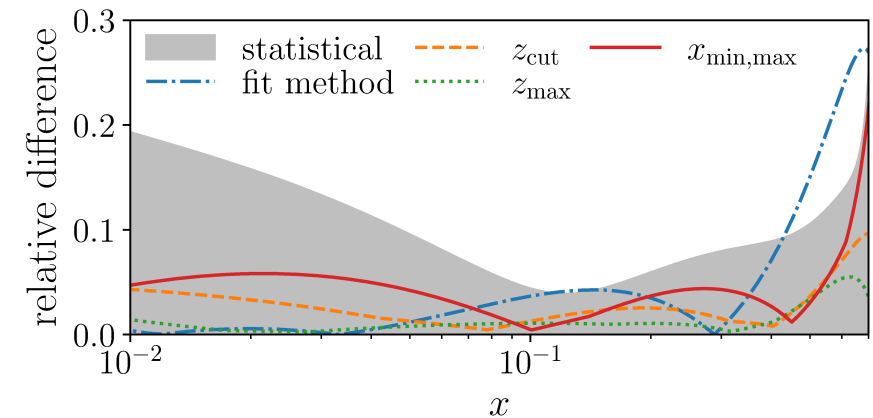
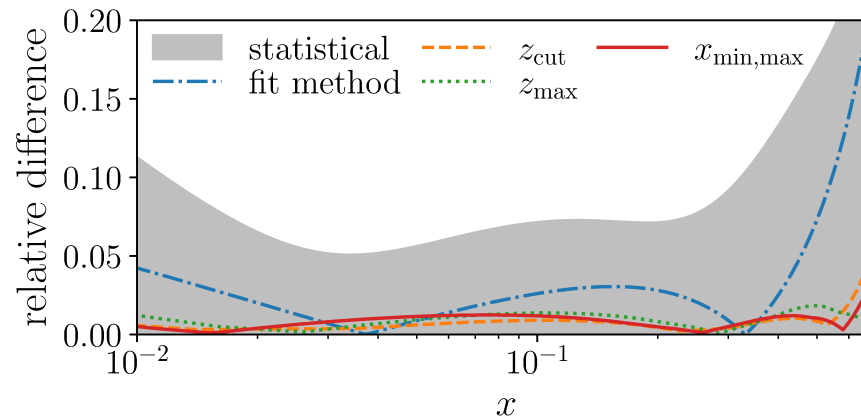
Analysis of uncertainties

Nucleon structure
Lattice QCD
Quasi and pseudo
First extraction
GPDs program
ANNs

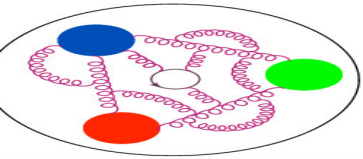
$\xi \neq 0$

Summary

- statistical errors,
- strategy of fitting real/imaginary part (**separately**/together),
- z_{cut} parameter representing the largest z values in LaMET data
 $z_{\text{cut}} = 8a, \mathbf{9a}, 10a,$
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 $z_{\text{max}} = 3a, \mathbf{4a}, 5a,$
- choice of $x_{\text{min}} = 0.15, \mathbf{0.2}, 0.25$ and $x_{\text{max}} = 0.75, \mathbf{0.8}, 0.85$.



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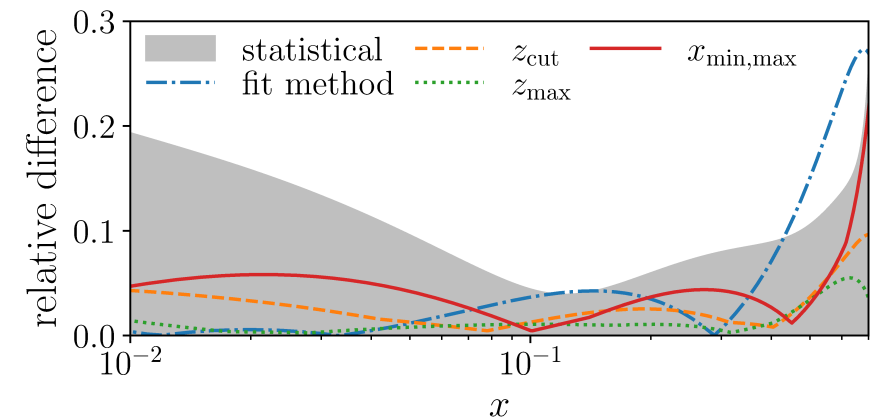
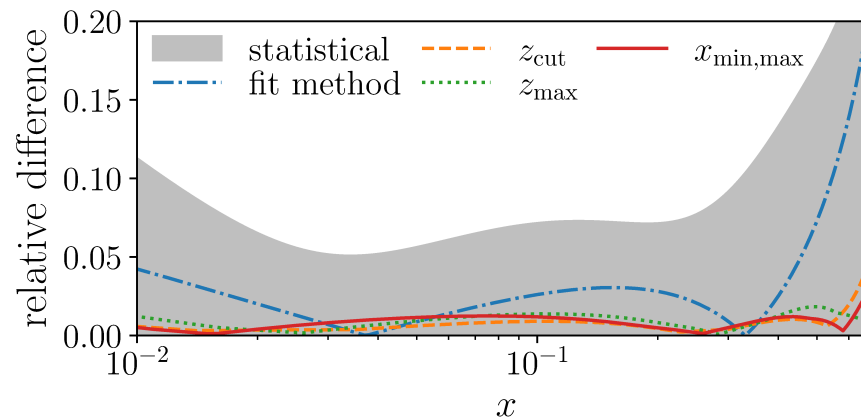
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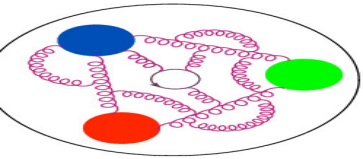
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More uncertainties to quantify: discretization effects, pion mass effects, renormalization, matching etc.

The ANN framework can naturally incorporate all these and ultimately lead to a fully quantitative result!



Generalization for GPDs



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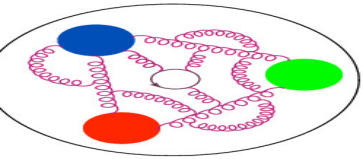
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The factor $(1 - x)^2$ introduced to satisfy the theory-driven constraints: ensures that the t dependence vanishes as $x \rightarrow 1$ (the active parton located at the origin of the reference frame).

Second power keeps the distance to the spectator system finite as $x \rightarrow 1$ (smaller/larger power would cause the nucleon to collapse/expand indefinitely).

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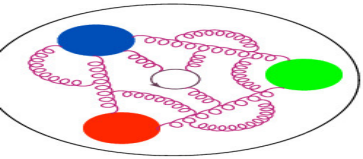
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Forward limit of $H_{\text{val}}^q(x, 0, 0)$ normalized to $F_1^{u/d} = 2/1$ for u/d quarks.

$E_{\text{val}}^q(x, 0, 0)$ – normalization NOT enforced ($F_2^{u/d} = \kappa_{u/d}$ well-known experimentally).



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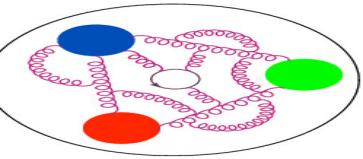
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Nucleon tomography: $\mathbf{b}_\perp$ – impact parameter (def. in center-of-momentum frame of all partons).

unpolarized proton: $q(x, \mathbf{b}_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{-i\mathbf{b}_\perp \cdot \Delta_\perp} H^q(x, 0, -\Delta_\perp^2)$.

transversely polarized proton (along the X-axis): $q_X(x, \mathbf{b}_\perp) = q(x, \mathbf{b}_\perp) - \frac{1}{2M} \frac{\partial}{\partial b_y} e_q(x, \mathbf{b}_\perp)$,
with $e_q(x, \mathbf{b}_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{-i\mathbf{b}_\perp \cdot \Delta_\perp} E_q(x, 0, -\Delta_\perp^2)$.

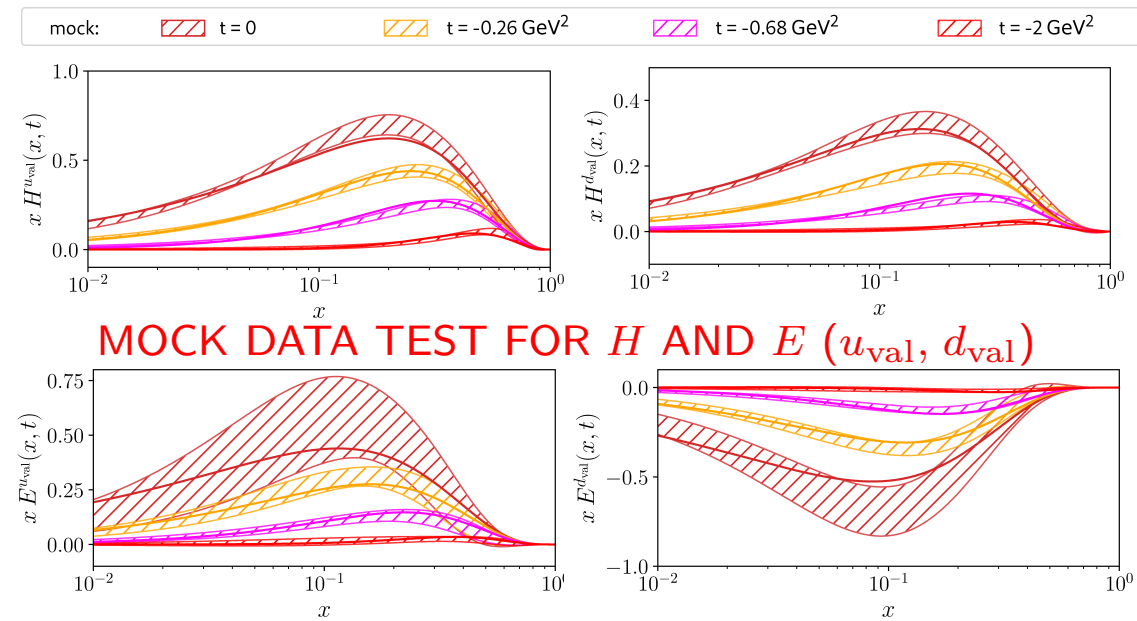
Ji's spin sum rule: $\int_{-1}^1 dx x \left(H^q(x, \xi, 0) + E^q(x, \xi, 0) \right) = 2J^q$.

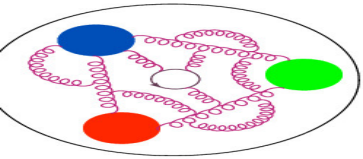


Results for GPDs



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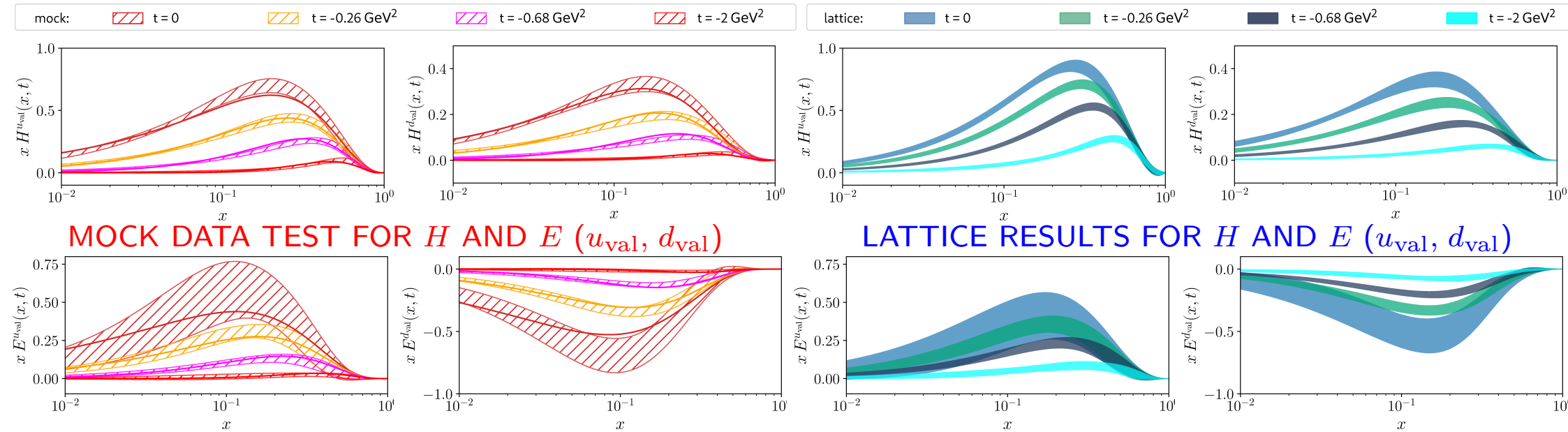


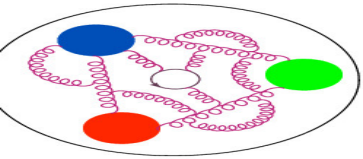


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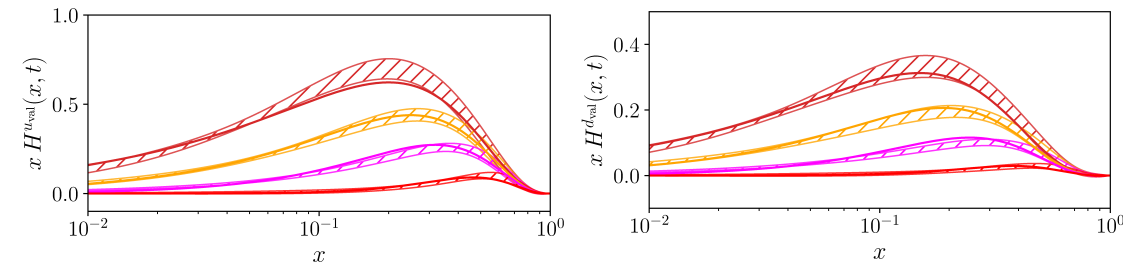
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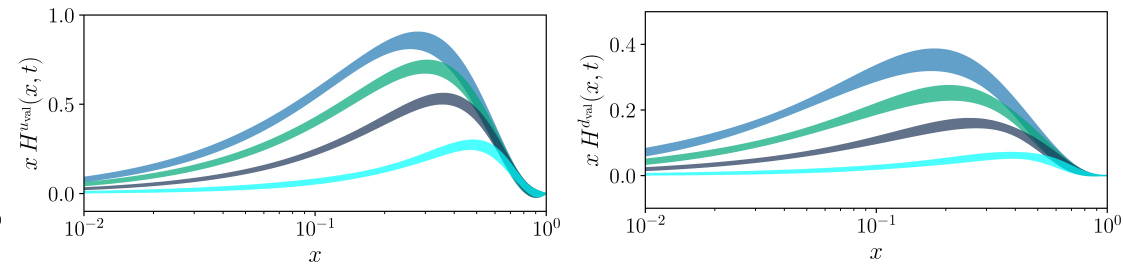
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mock: $t=0$ $t=-0.26 \text{ GeV}^2$ $t=-0.68 \text{ GeV}^2$ $t=-2 \text{ GeV}^2$

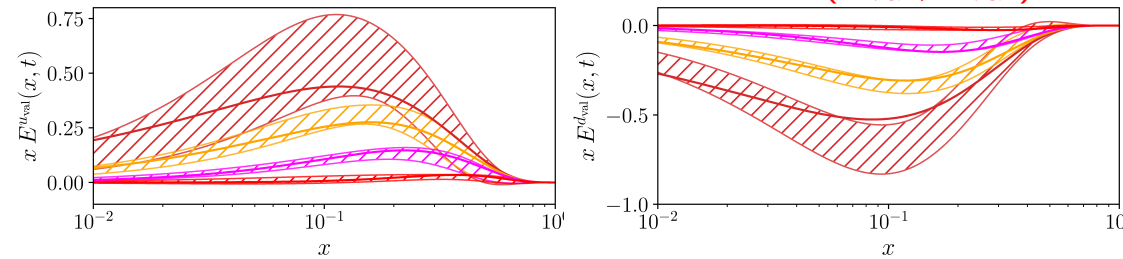
lattice: $t=0$ $t=-0.26 \text{ GeV}^2$ $t=-0.68 \text{ GeV}^2$ $t=-2 \text{ GeV}^2$



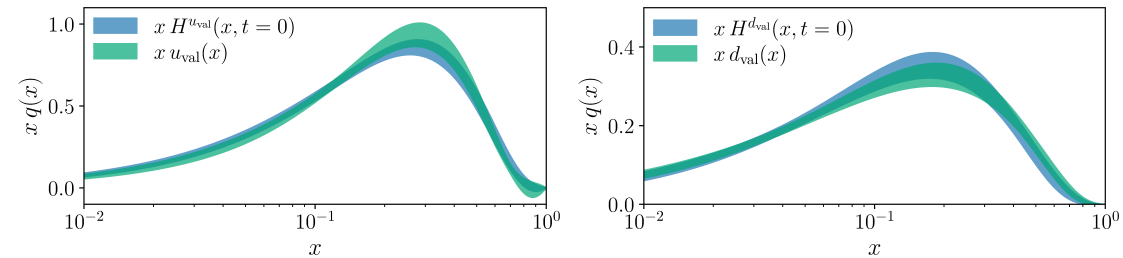
MOCK DATA TEST FOR H AND E (u_{val} , d_{val})

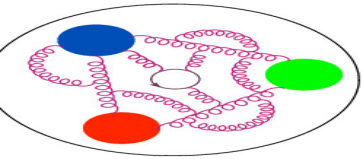


LATTICE RESULTS FOR H AND E (u_{val} , d_{val})



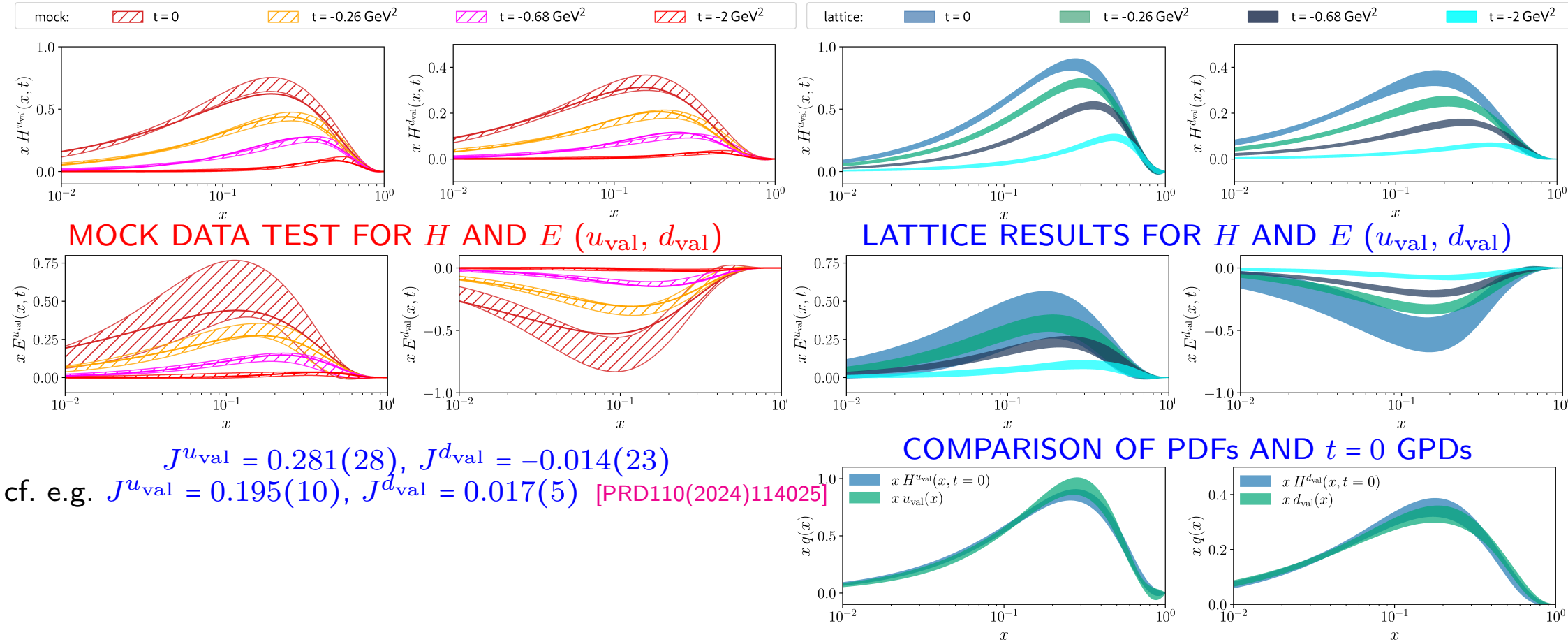
COMPARISON OF PDFs AND $t=0$ GPDs

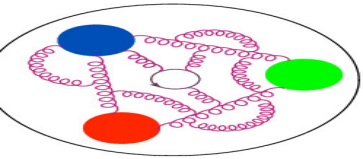




Results for GPDs

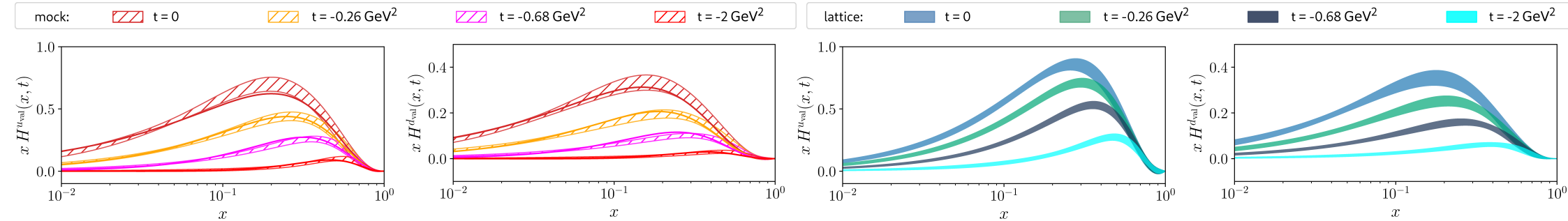
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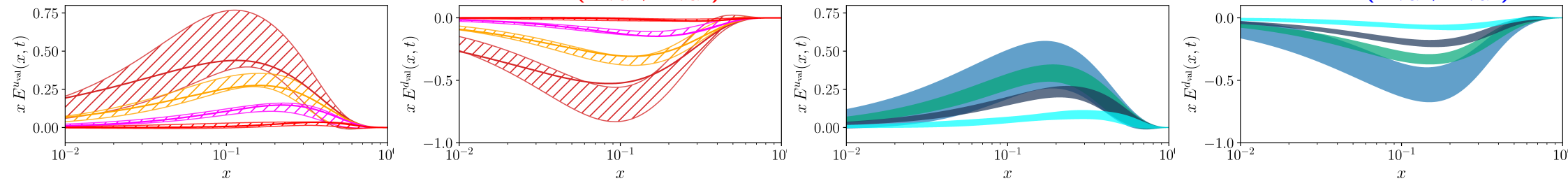
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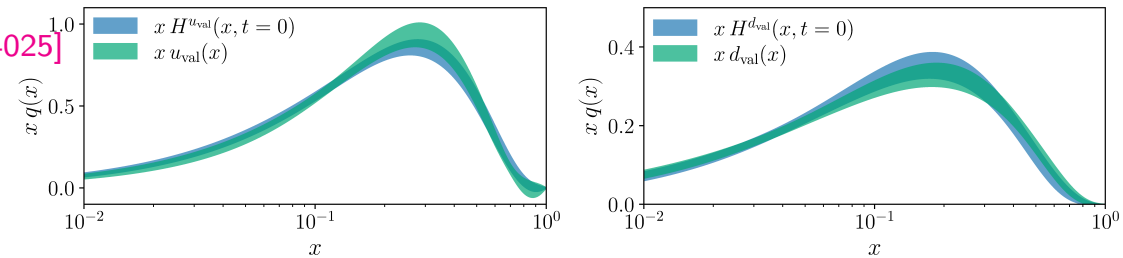
MOCK DATA TEST FOR H AND E ($u_{\text{val}}, d_{\text{val}}$)

LATTICE RESULTS FOR H AND E ($u_{\text{val}}, d_{\text{val}}$)

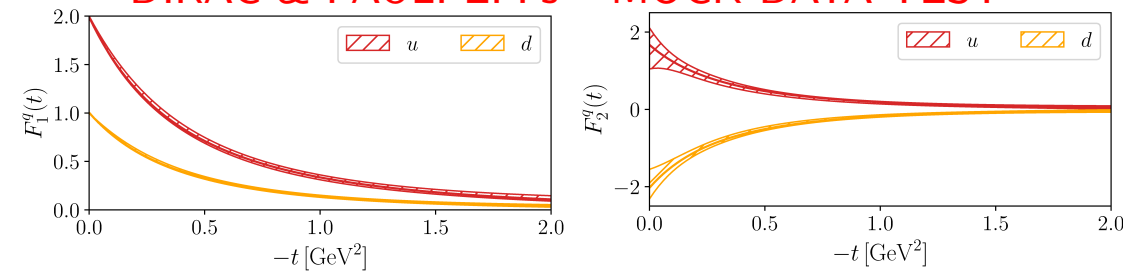


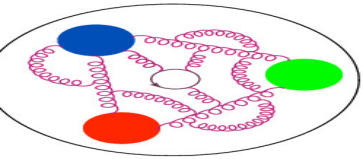
COMPARISON OF PDFs AND $t=0$ GPDs

$J^{u_{\text{val}}} = 0.281(28)$, $J^{d_{\text{val}}} = -0.014(23)$
cf. e.g. $J^{u_{\text{val}}} = 0.195(10)$, $J^{d_{\text{val}}} = 0.017(5)$ [PRD110(2024)114025]



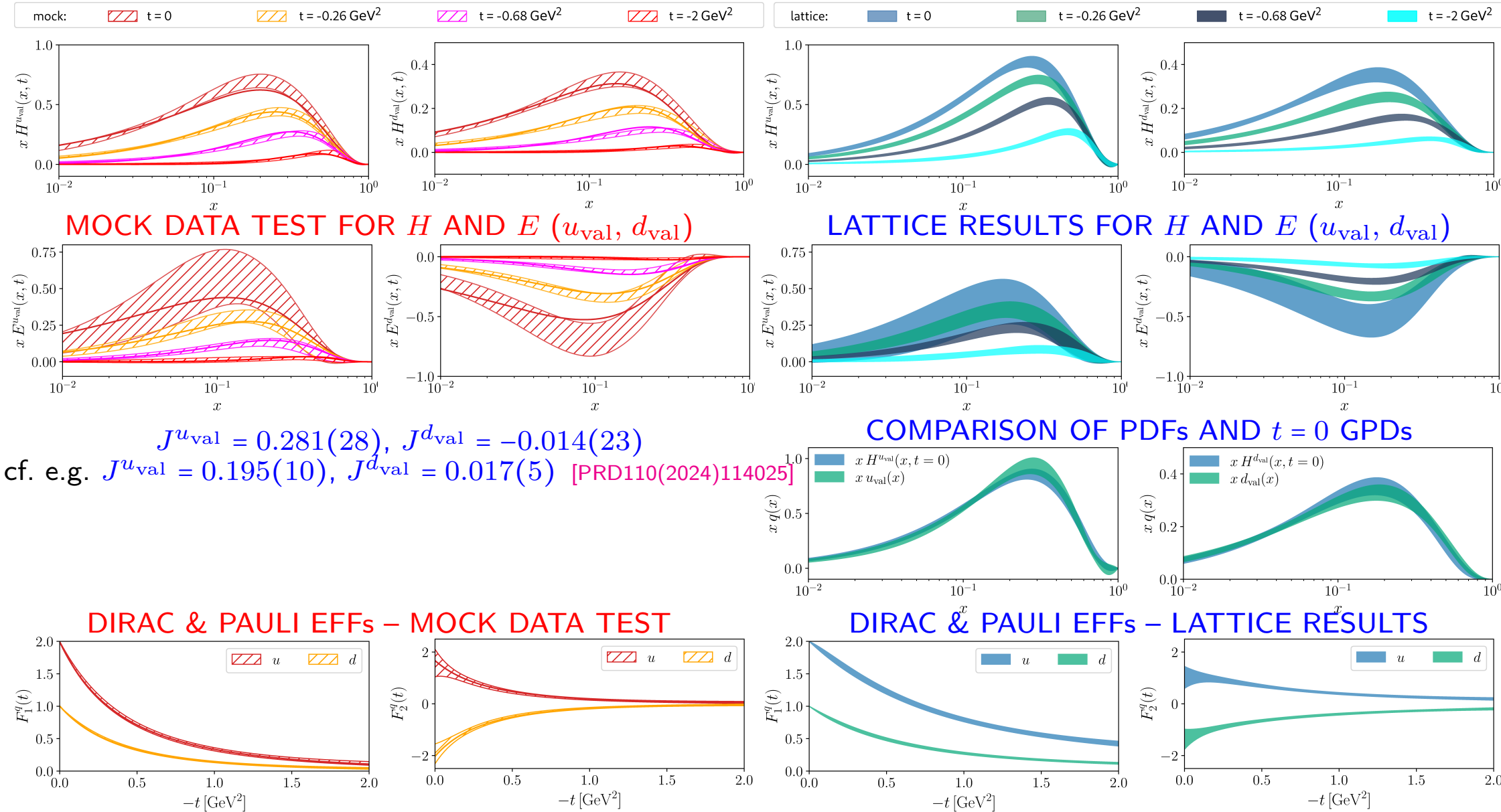
DIRAC & PAULI EFFs – MOCK DATA TEST

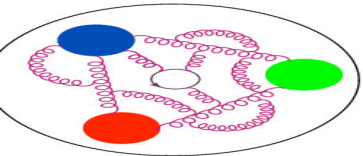




Results for GPDs

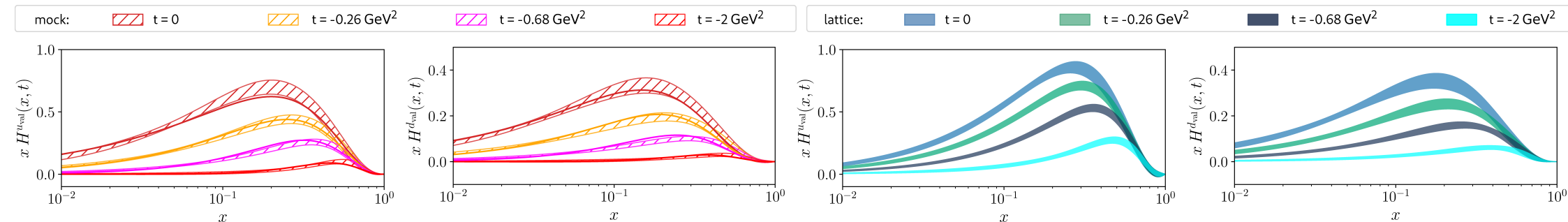
M. Chu, KC, M. Constantinou, P. Sznajder, J. Wagner, arXiv:2509.15931





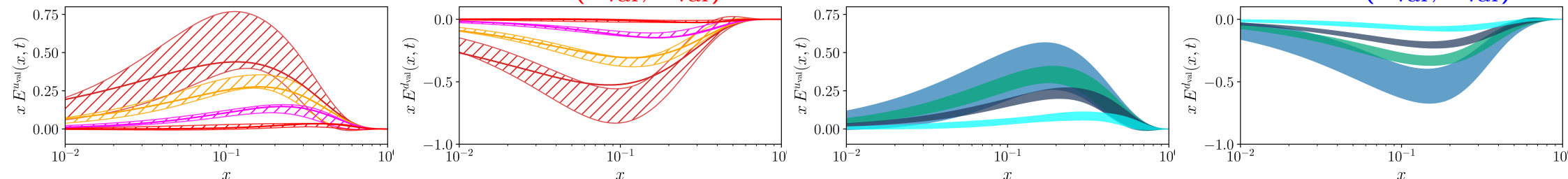
Results for GPDs

M. Chu, KC, M. Constantinou, P. Sznajder, J. Wagner, arXiv:2509.15931



MOCK DATA TEST FOR H AND E ($u_{\text{val}}, d_{\text{val}}$)

LATTICE RESULTS FOR H AND E ($u_{\text{val}}, d_{\text{val}}$)



COMPARISON OF PDFs AND $t=0$ GPDs

cf. e.g. $J^{u_{\text{val}}} = 0.281(28)$, $J^{d_{\text{val}}} = -0.014(23)$
 $J^{u_{\text{val}}} = 0.195(10)$, $J^{d_{\text{val}}} = 0.017(5)$ [PRD110(2024)114025]

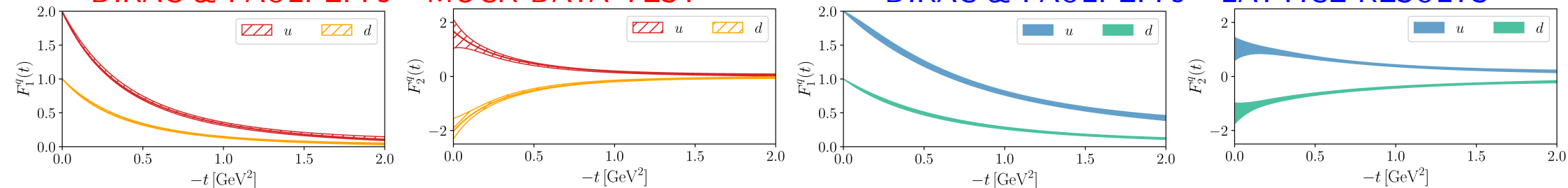
$$\kappa_p = (2/3)\kappa_u - (1/3)\kappa_d = 1.14(26)$$

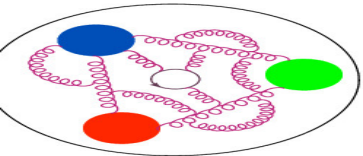
$$\kappa_u = 1.03(43), \kappa_d = -1.37(39)$$

cf. experiment $\kappa_p^{\text{exp.}} = 1.792847350(9)$ [Nature 509(2014)596]

DIRAC & PAULI EFFs – MOCK DATA TEST

DIRAC & PAULI EFFs – LATTICE RESULTS





Nucleon tomography from GPDs

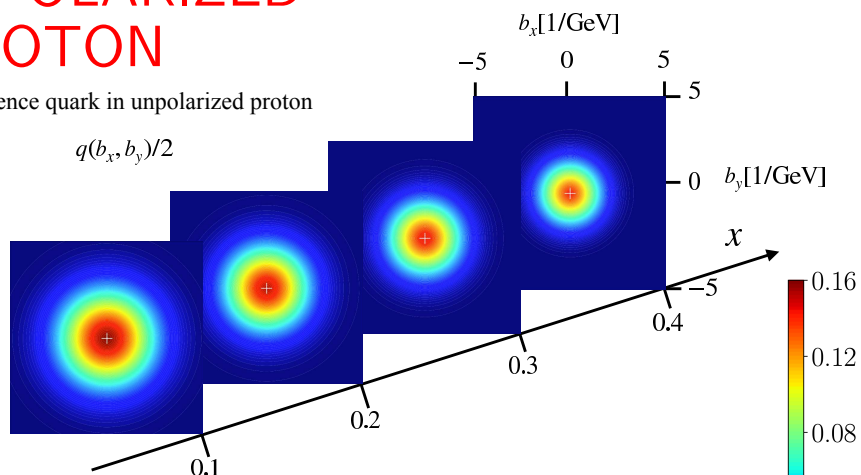


M. Chu, KC, M. Constantinou, P. Sznajder, J. Wagner, arXiv:2509.15931

UNPOLARIZED PROTON

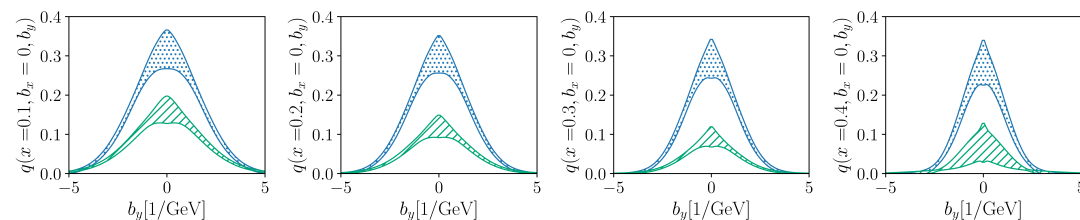
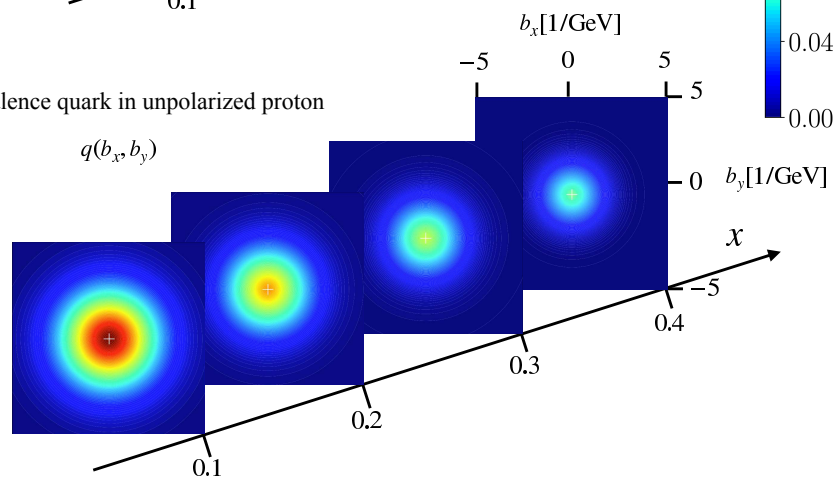
up valence quark in unpolarized proton

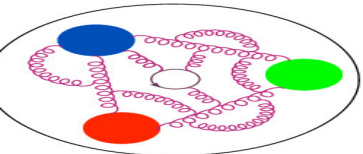
$$q(b_x, b_y)/2$$



down valence quark in unpolarized proton

$$q(b_x, b_y)$$





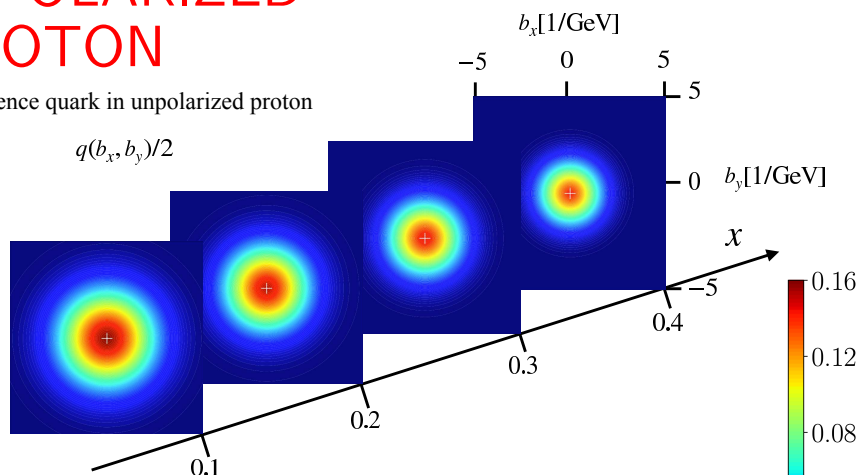
Nucleon tomography from GPDs



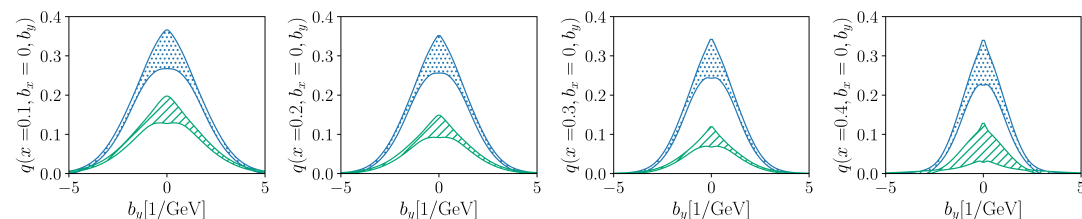
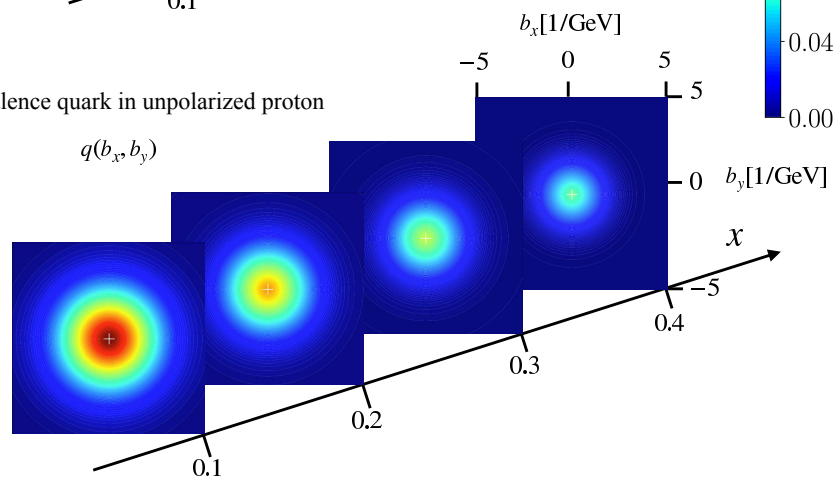
M. Chu, KC, M. Constantinou, P. Sznajder, J. Wagner, arXiv:2509.15931

UNPOLARIZED PROTON

up valence quark in unpolarized proton

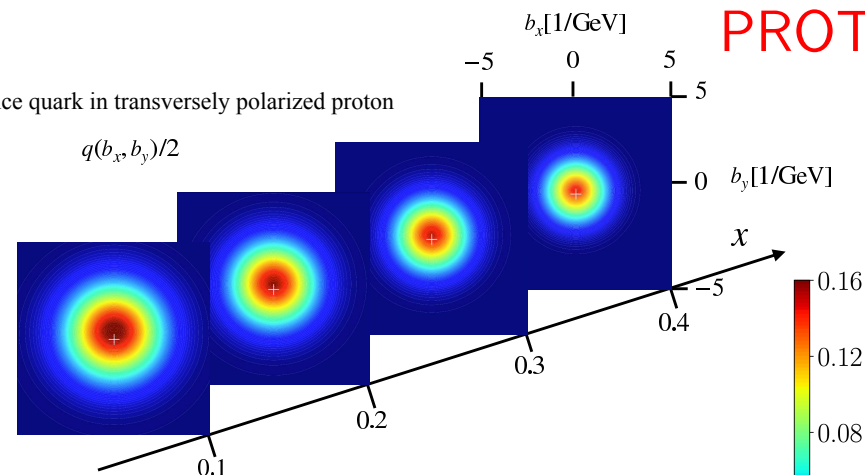


down valence quark in unpolarized proton

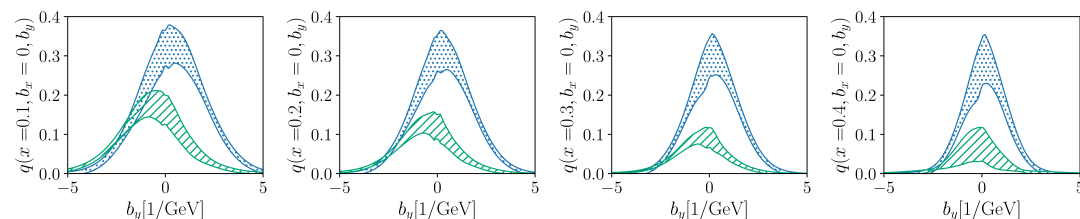
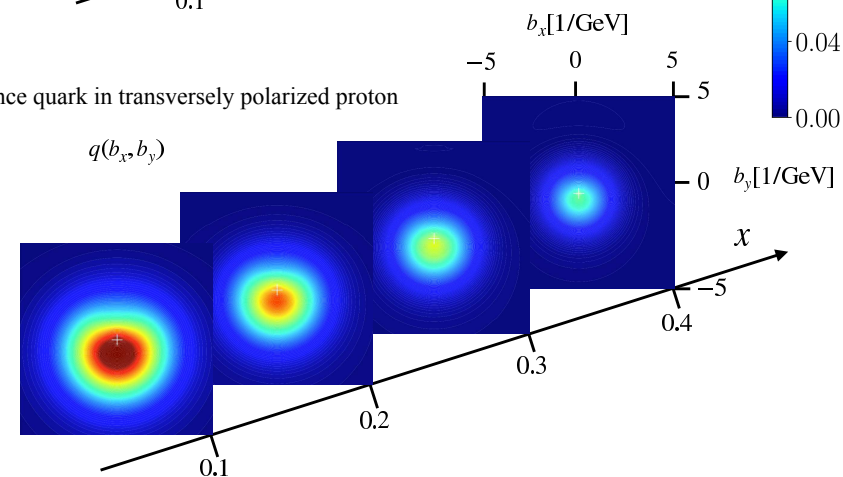


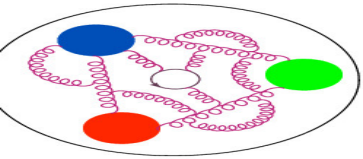
TRANSV. POLARIZED PROTON

up valence quark in transversely polarized proton



down valence quark in transversely polarized proton





Nonzero skewness GPDs – asymmetric frame



Skewness: $\xi_{\text{LC}} \equiv -\Delta^+/2P^+$, quasi-skewness: $\xi = -\Delta_3/2P_3 = \xi_{\text{LC}} + \mathcal{O}(m^2/P_3^2)$.

Two cases at $\xi \neq 0$:

- momentum transfer in transverse+longitudinal directions, $\vec{\Delta} = (\Delta_1, \Delta_2, \Delta_3)$,
- purely longitudinal momentum transfer, $\vec{\Delta} = (0, 0, \Delta_3) \equiv \Delta_L$.

General Lorentz-covariant parametrization of unpolarized MEs: S. Bhattacharya et al., PRD106(2022)114512

$$F^\mu(z, P, \Delta) = \bar{u}(p', \lambda') \left[\frac{P^\mu}{m} A_1 + m z^\mu A_2 + \frac{\Delta^\mu}{m} A_3 + i m \sigma^{\mu z} A_4 + \frac{i \sigma^{\mu \Delta}}{m} A_5 + \frac{P^\mu i \sigma^{z \Delta}}{m} A_6 + \frac{z^\mu i \sigma^{z \Delta}}{m} A_7 + \frac{\Delta^\mu i \sigma^{z \Delta}}{m} A_8 \right] u(p, \lambda).$$

Special case of purely longitudinal momentum transfer: M. Chu et al., arXiv:2508.17998, accepted in PRD

$$F^{\mu, L}(z, P, \Delta_L) = \bar{u}(P_f, \lambda') \left[\frac{P^\mu}{m} A_1^L + m z^\mu A_2^L + i m \sigma^{\mu z} A_4^L \right] u(P_i, \lambda).$$

Redefined amplitudes A_i^L are linear combinations of the general amplitudes A_i , e.g.:

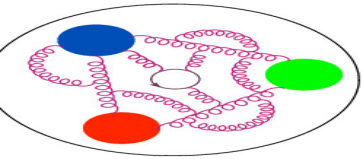
$$A_1^L \equiv A_1 + \alpha A_3 - \frac{\alpha^2}{2} A_5 - \frac{\alpha}{2} (z \cdot \Delta_L) A_6 - \frac{\alpha^2}{2} (z \cdot \Delta_L) A_8.$$

GPDs in terms of amplitudes:

$$H_{\text{LI}} = A_1 - 2\xi A_3,$$

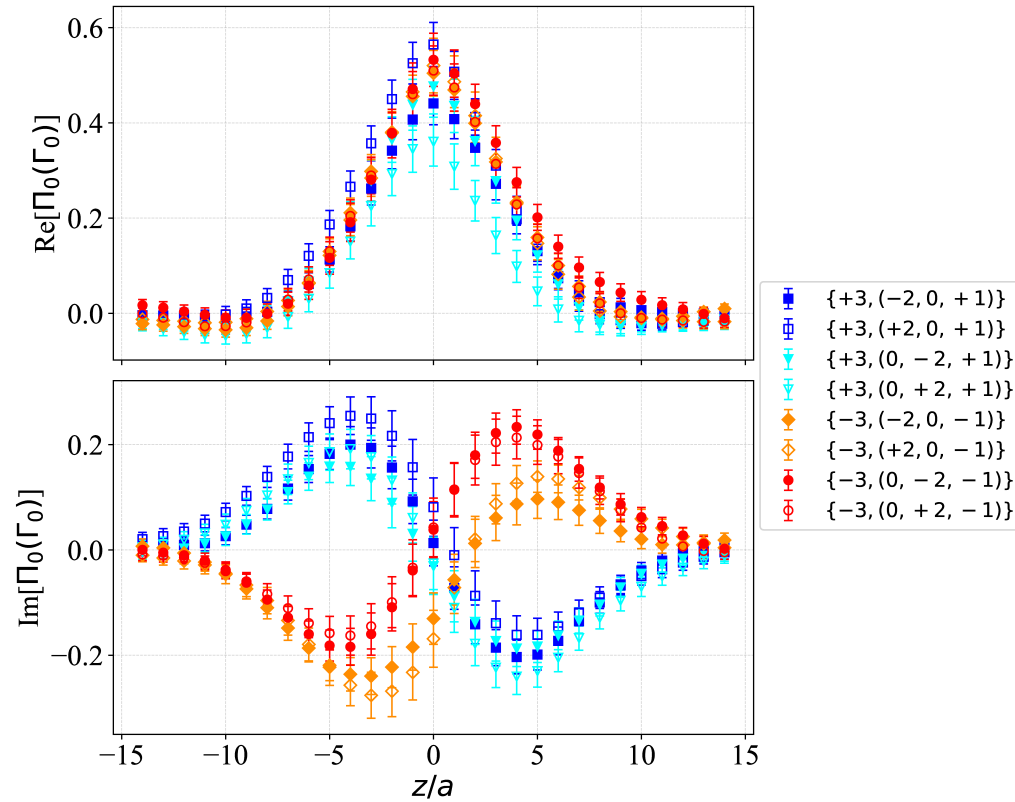
$$E_{\text{LI}} = -A_1 + 2\xi A_3 + 2A_5 + 2P_3 z A_6 - 4\xi P_3 z A_8,$$

$$H_{\text{LI}}^L \equiv \left(H + \frac{1}{2} \alpha \xi (H + E) \right)_{\text{LI}} = A_1^L, \quad \text{with} \quad \alpha = \frac{-(z \cdot \Delta)(z \cdot P)}{z^2 P^2 - (z \cdot P)^2}.$$

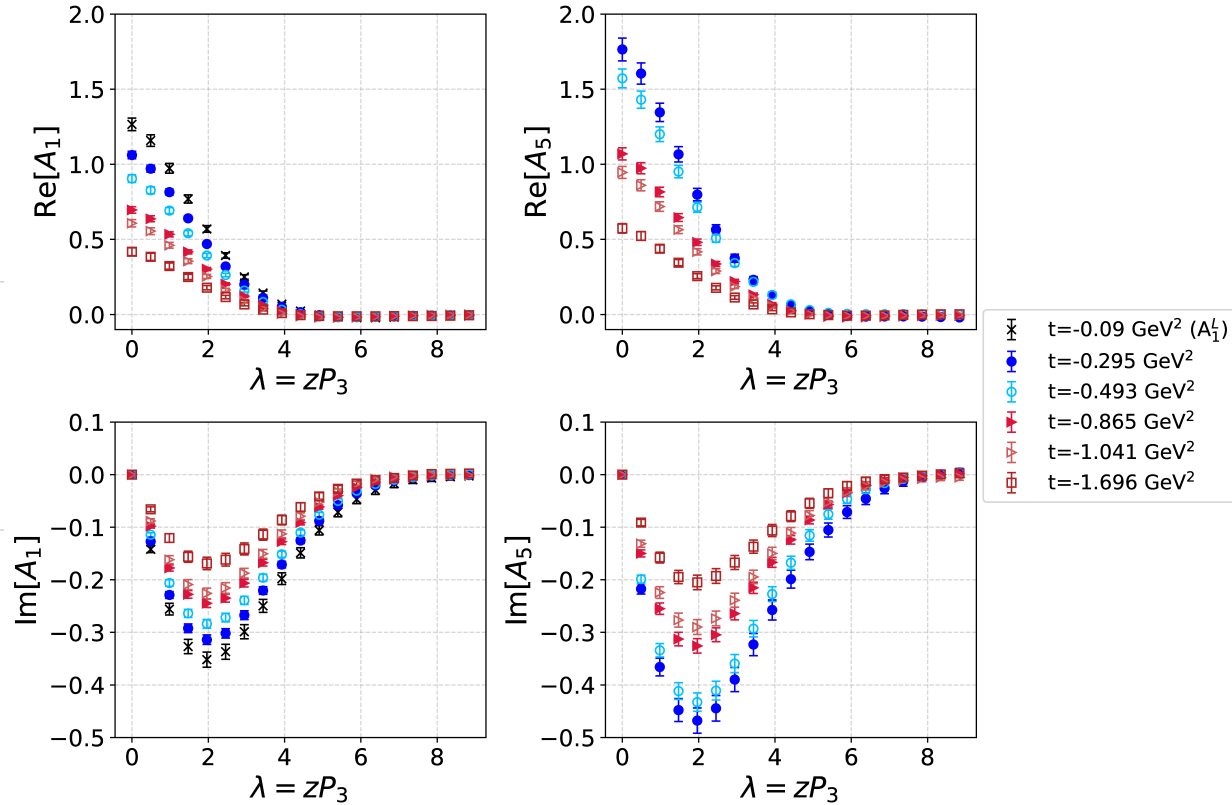


Nonzero skewness GPDs – MEs and amplitudes

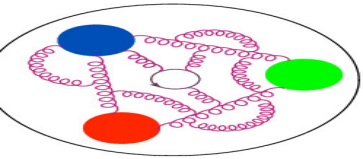
Example bare MEs:
 γ_0 , unpolarized projector



Example amplitudes:
 A_1 and A_5 (leading ones)

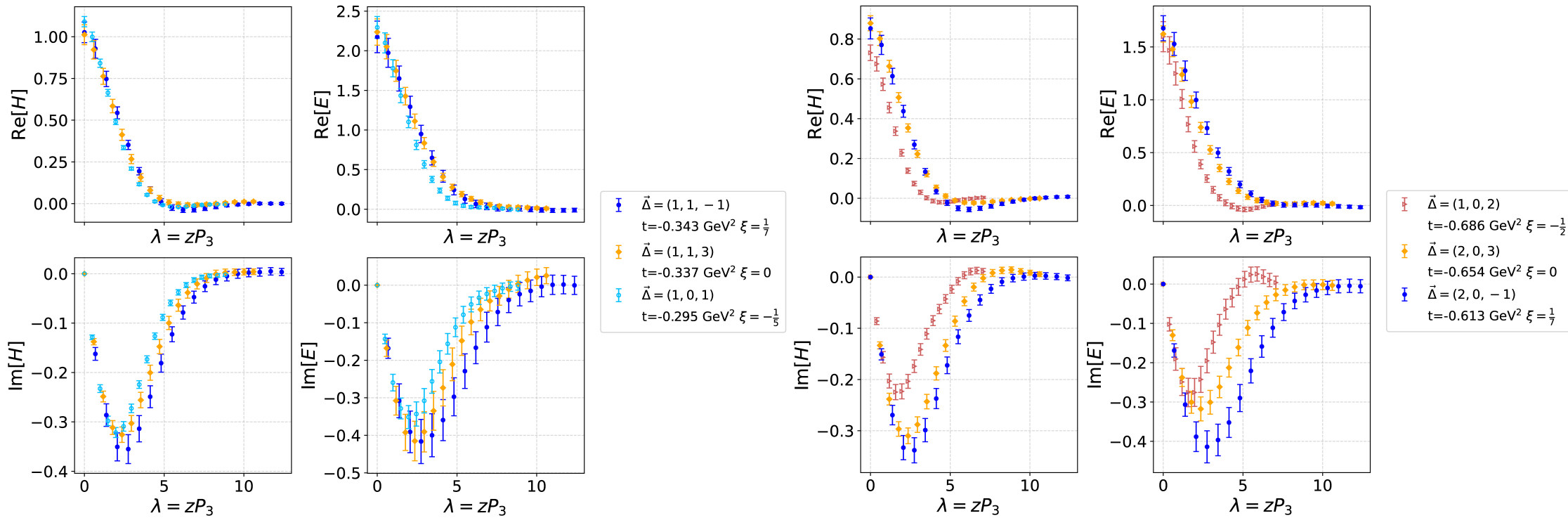


M. Chu, M. Colaço, S. Bhattacharya, KC, M. Constantinou, A. Metz, F. Steffens, arXiv:2508.17998, accepted in PRD

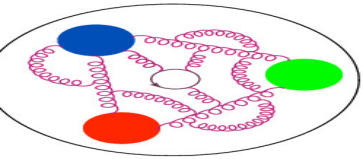


Nonzero skewness GPDs – H and E

Skewness dependence of H and E GPDs at approx. equal $-t$



M. Chu, M. Colaço, S. Bhattacharya, KC, M. Constantinou, A. Metz, F. Steffens, arXiv:2508.17998, accepted in PRD



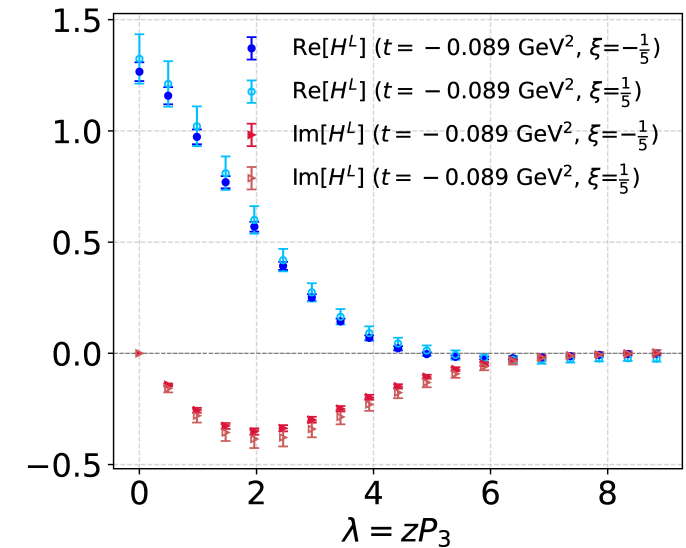
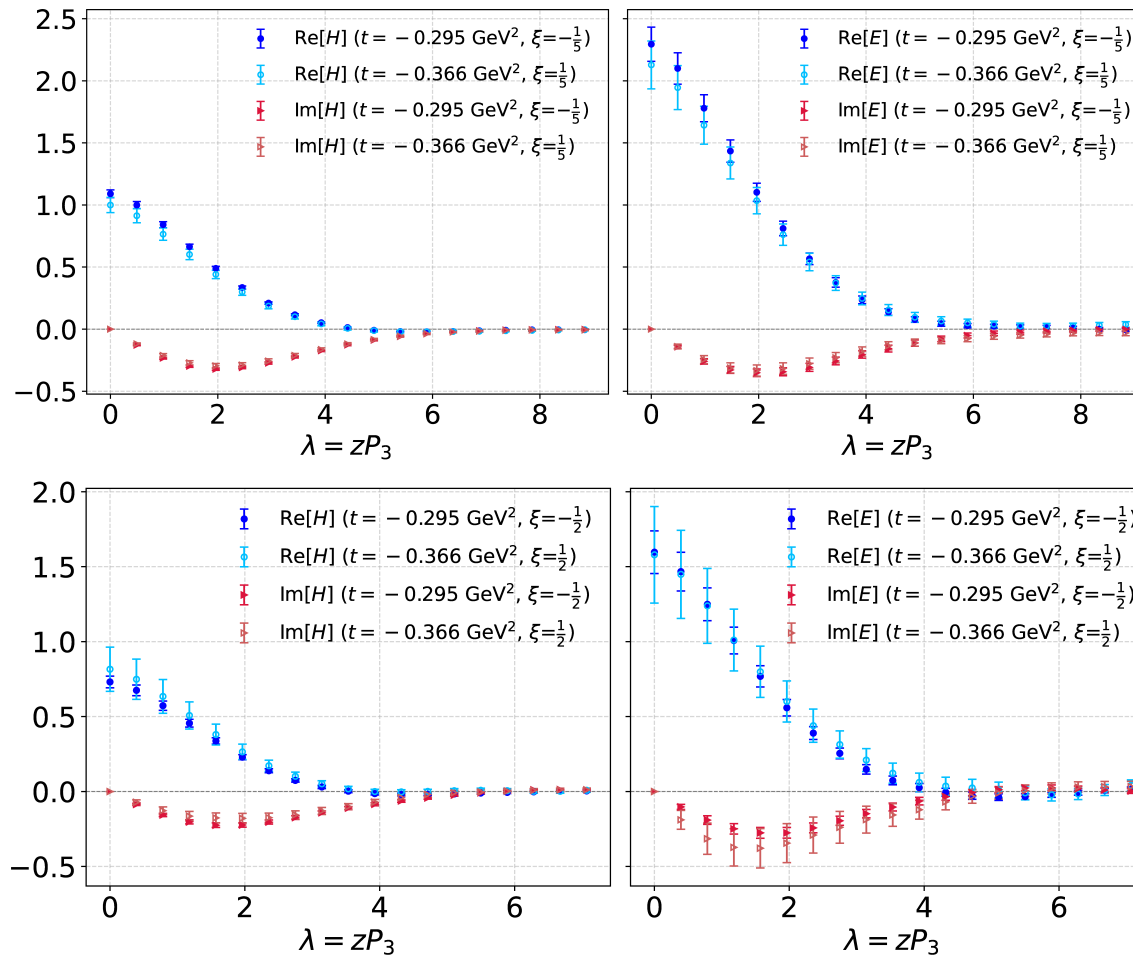
Nonzero skewness GPDs – skewness symmetry

opposite skewness
approx. equal $-t$

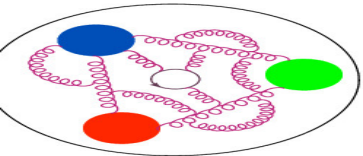
$$\vec{\Delta} = (\Delta_1, \Delta_2, \Delta_3)$$

opposite skewness
precisely equal $-t$

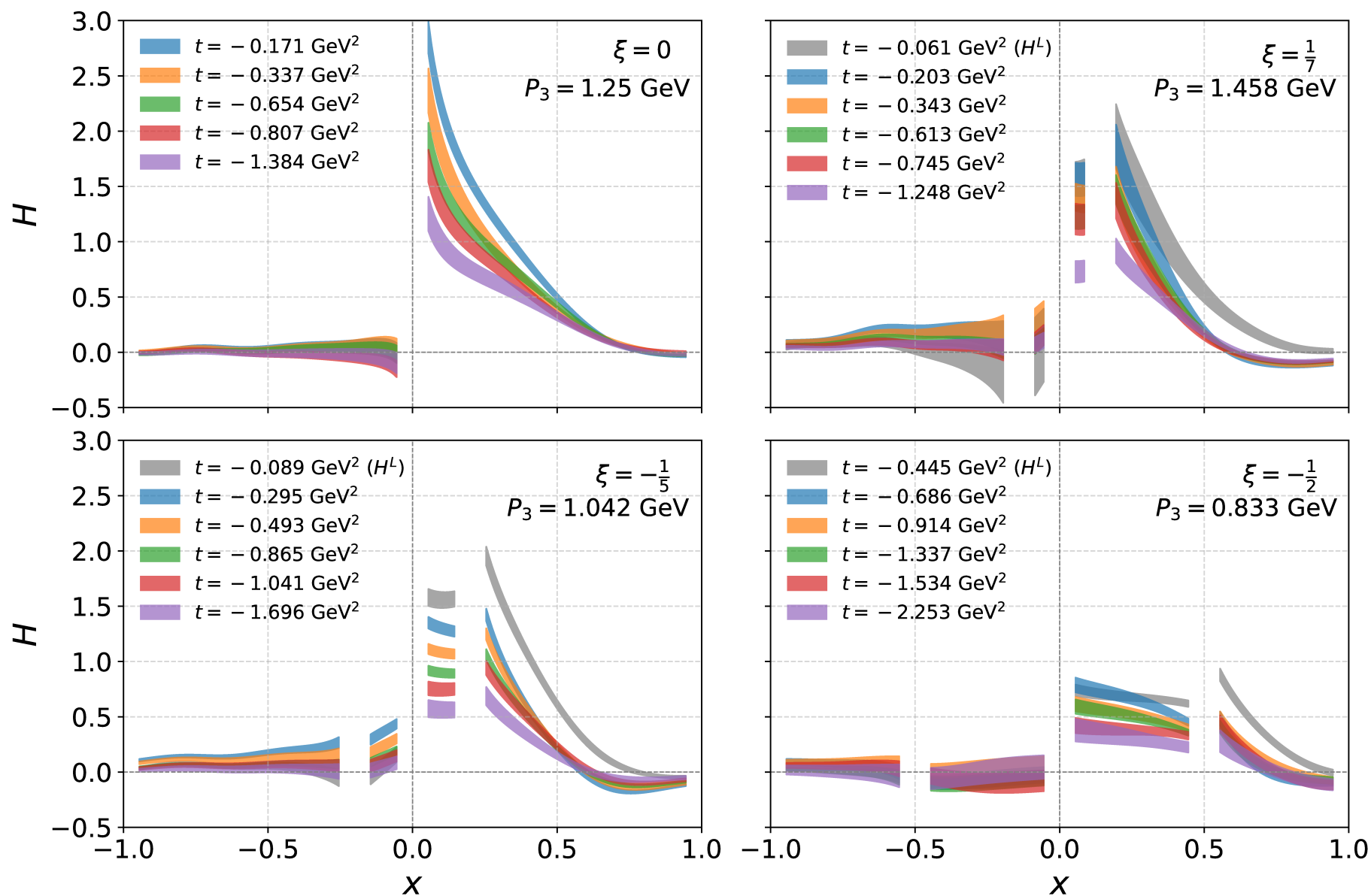
$$\vec{\Delta} = (0, 0, \Delta_3)$$



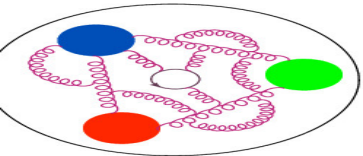
M. Chu, M. Colaço, S. Bhattacharya, KC, M. Constantinou, A. Metz, F. Steffens, arXiv:2508.17998, accepted in PRD



Nonzero skewness GPDs – momentum space



M. Chu, M. Colaço, S. Bhattacharya, KC, M. Constantinou, A. Metz, F. Steffens, arXiv:2508.17998, accepted in PRD



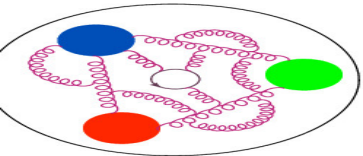
Conclusions and prospects



Nucleon structure
Lattice QCD
Quasi and pseudo
First extraction
GPDs program
ANNs
 $\xi \neq 0$

Summary

- Main message: **probing nucleon's 3D structure with LQCD becomes feasible and reliable!**
- This talk: overview of progress for GPDs + new development improving lattice extraction of PDFs/GPDs by:
 - ★ (both) combining the two most popular approaches, quasi- and pseudo-distributions, and allowing them to work in synergy,
 - ★ (both) employing a modern tool, artificial neural networks, to reconstruct the x dependence and mitigate the inverse problem,
 - ★ (GPDs) reconstructing the t dependence in a model-independent way, also with artificial neural networks.
- **With this, we have a very flexible and robust framework (still improvable!) for optimal use of expensive lattice data.**
- Still, a lot of work to be done: quantify all systematics → laborious, but crucial.
- **Consistent progress will ensure complementary role of lattice to phenomenology and experiment!**



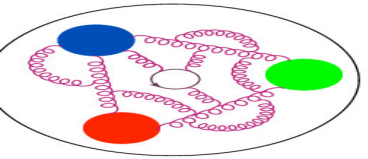
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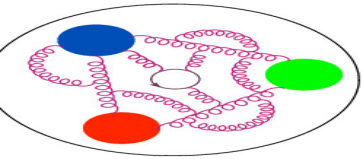
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 $\xi \neq 0$
Summary

Backup slides

Backup slides



Lorentz-covariant parametrization

Main theoretical tool: Lorentz-covariant parametrization of matrix elements:

unpolarized: S. Bhattacharya et al., PRD106(2022)114512

$$F^\mu(z, P, \Delta) = \bar{u}(p', \lambda') \left[\frac{P^\mu}{m} A_1 + m z^\mu A_2 + \frac{\Delta^\mu}{m} A_3 + i m \sigma^{\mu z} A_4 + \frac{i \sigma^{\mu \Delta}}{m} A_5 + \frac{P^\mu i \sigma^{z \Delta}}{m} A_6 + \frac{z^\mu i \sigma^{z \Delta}}{m} A_7 + \frac{\Delta^\mu i \sigma^{z \Delta}}{m} A_8 \right] u(p, \lambda),$$

helicity: S. Bhattacharya et al., PRD109(2024)034508

$$F[\gamma^\mu \gamma_5] = \bar{u}(p', \lambda') \left[\frac{i \epsilon^{\mu P z \Delta}}{m} \widetilde{A}_1 + \gamma^\mu \gamma_5 \widetilde{A}_2 + \gamma_5 \left(\frac{P^\mu}{m} \widetilde{A}_3 + m z^\mu \widetilde{A}_4 + \frac{\Delta^\mu}{m} \widetilde{A}_5 \right) + m \not{z} \gamma_5 \left(\frac{P^\mu}{m} \widetilde{A}_6 + m z^\mu \widetilde{A}_7 + \frac{\Delta^\mu}{m} \widetilde{A}_8 \right) \right] u(p, \lambda)$$

transversity: S. Bhattacharya et al., arXiv:2505.11288 $F^{[i \sigma^{\mu \nu} \gamma_5]} = \bar{u}(p', \lambda') \left[\sum_{i=1}^{12} \Gamma_i^{\mu \nu} A_{Ti} \right] u(p, \lambda)$

- most general parametrization in terms of 8 or 12 linearly-independent Lorentz structures,
- 8/12 Lorentz-invariant amplitudes $A_i / \widetilde{A}_i / A_{Ti}(z \cdot P, z \cdot \Delta, \Delta^2, z^2)$.

Example: (γ_0 insertion, unpolarized projector)

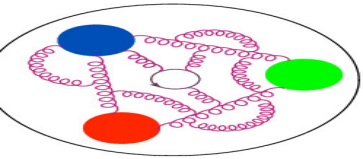
symmetric frame:

$$\Pi_0^s(\Gamma_0) = C \left(\frac{E(E+m) - P_3^2}{2m^3} A_1 + \frac{(E+m)(-E^2 + m^2 + P_3^2)}{m^3} A_5 + \frac{EP_3(-E^2 + m^2 + P_3^2)z}{m^3} A_6 \right),$$

asymmetric frame:

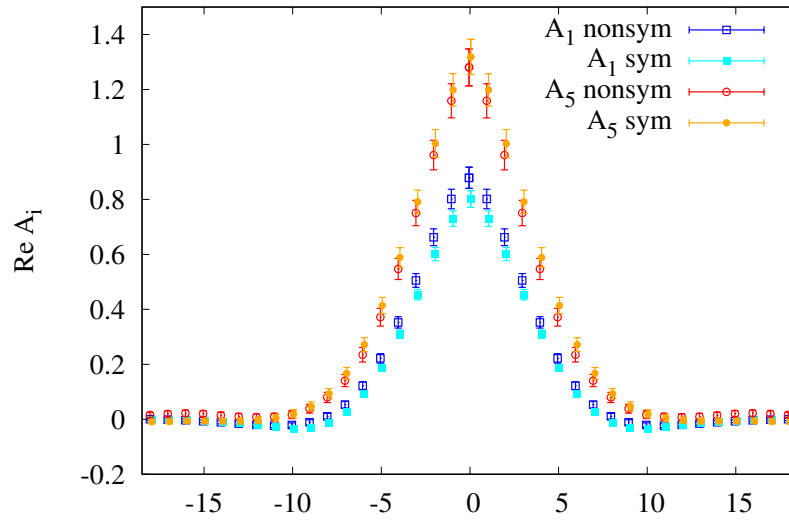
$$\begin{aligned} \Pi_0^a(\Gamma_0) = C \left(- \frac{(E_f + E_i)(E_f - E_i - 2m)(E_f + m)}{8m^3} A_1 - \frac{(E_f - E_i - 2m)(E_f + m)(E_f - E_i)}{4m^3} A_3 + \frac{(E_i - E_f)P_3 z}{4m} A_4 \right. \\ \left. + \frac{(E_f + E_i)(E_f + m)(E_f - E_i)}{4m^3} A_5 + \frac{E_f(E_f + E_i)P_3(E_f - E_i)z}{4m^3} A_6 + \frac{E_f P_3(E_f - E_i)^2 z}{2m^3} A_8 \right). \end{aligned}$$

- matrix elements $\Pi_\mu(\Gamma_\nu)$ or $\Pi_{\mu 5}(\Gamma_\nu)$ are **frame-dependent**
- but the amplitudes $A_i, \widetilde{A}_i$ or A_{Ti} are **frame-invariant**.



Proof of concept (comparison between frames)

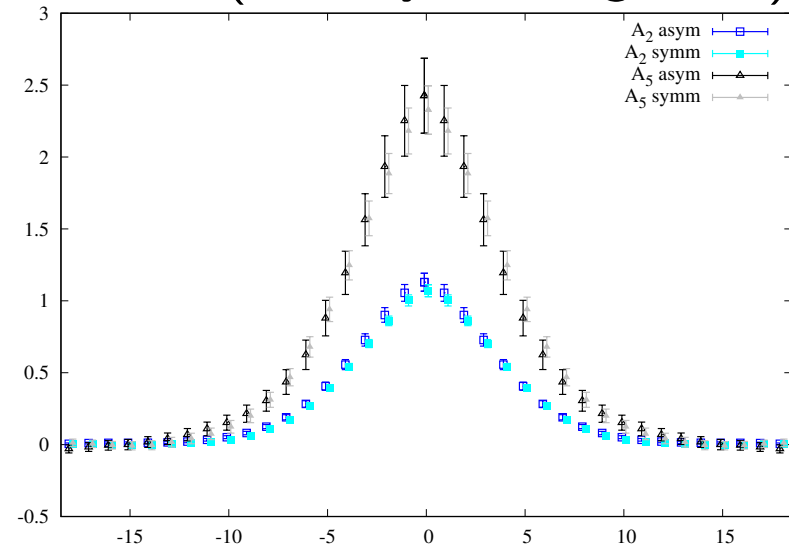
A_1, A_5 (unpolarized leading ones)



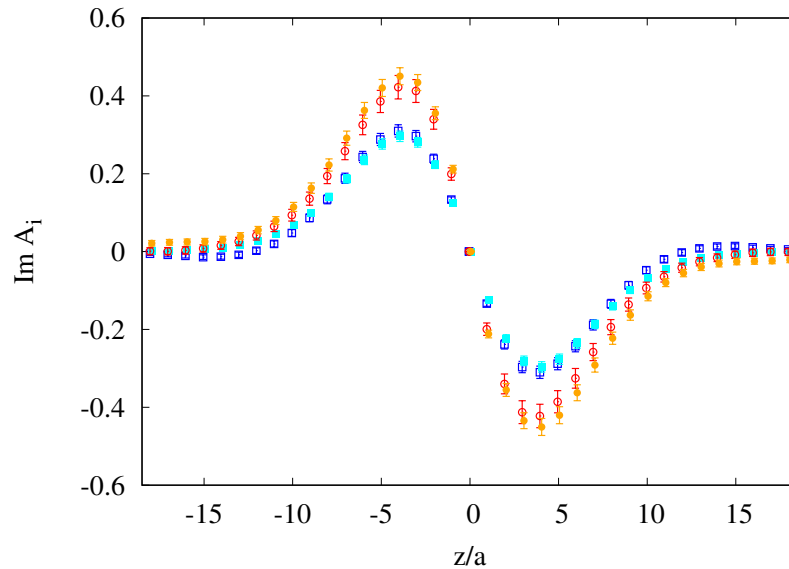
PRD106(2022)114512

S. Bhattacharya et al.

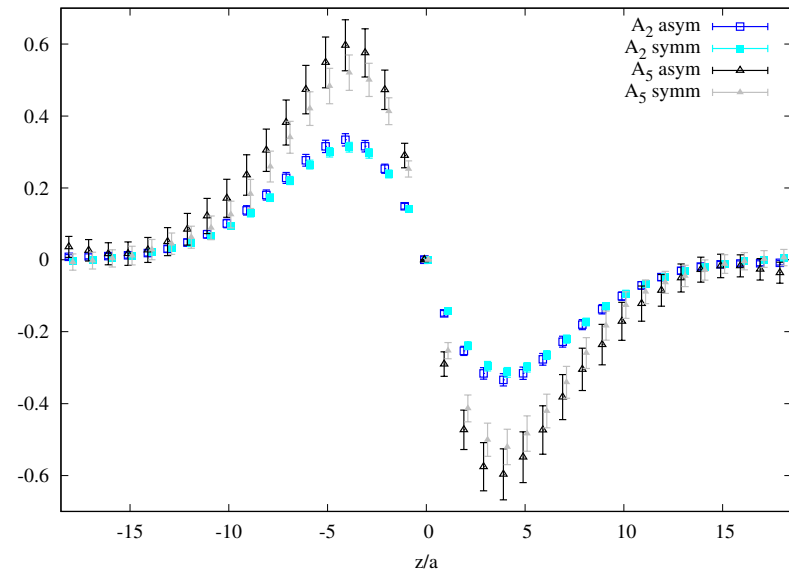
$\widetilde{A}_2, \widetilde{A}_5$ (helicity leading ones)

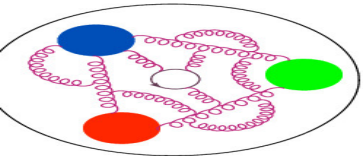


PRD109(2024)034508



Im





GPDs – possible definitions



Defining H and E GPDs in the standard way, expressions are frame-dependent:

SYMMETRIC frame:

$$F_{H(0)} = A_1 + \frac{z(\Delta_1^2 + \Delta_2^2)}{2P_3} A_6, \quad F_{E(0)} = -A_1 + 2A_5 + \frac{z(4E^2 - \Delta_1^2 - \Delta_2^2)}{2P_3} A_6.$$

ASYMMETRIC frame:

$$F_{H(0)} = A_1 + \frac{\Delta_0}{P_0} A_3 + \frac{m^2 z \Delta_0}{2P_0 P_3} A_4 + \frac{z(\Delta_0^2 + \Delta_\perp^2)}{2P_3} A_6 + \frac{z(\Delta_0^3 + \Delta_0 \Delta_\perp^2)}{2P_0 P_3} A_8,$$

$$F_{E(0)} = -A_1 - \frac{\Delta_0}{P_0} A_3 - \frac{m^2 z(\Delta_0 + 2P_0)}{2P_0 P_3} A_4 + 2A_5 - \frac{z(\Delta_0^2 + 2P_0 \Delta_0 + 4P_0^2 + \Delta_\perp^2)}{2P_3} A_6 - \frac{z\Delta_0(\Delta_0^2 + 2\Delta_0 P_0 + 4P_0^2 + \Delta_\perp^2)}{2P_0 P_3} A_8.$$

One can also modify the definition to make it Lorentz-invariant and arrive at:

ANY frame:

$$F_H = A_1, \quad F_E = -A_1 + 2A_5 + 2zP_3 A_6.$$

With respect to the standard definition, removed/reduced contribution from A_3, A_4, A_6, A_8 .

In terms of matrix elements: standard definition – only $\Pi_0(\Gamma_0), \Pi_0(\Gamma_{1/2})$,

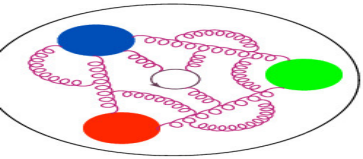
LI definition – additionally: $\Pi_{1/2}(\Gamma_3)$ (both frames), $\Pi_{1/2}(\Gamma_3), \Pi_{1/2}(\Gamma_0), \Pi_1(\Gamma_2), \Pi_2(\Gamma_1)$ (asym.).

Two definitions of $\tilde{H}$:

standard ($\gamma_5 \gamma_3$ operator): $F_{\tilde{H}} = \widetilde{A}_2 + zP_3 \widetilde{A}_6 - m^2 z^2 \widetilde{A}_7,$

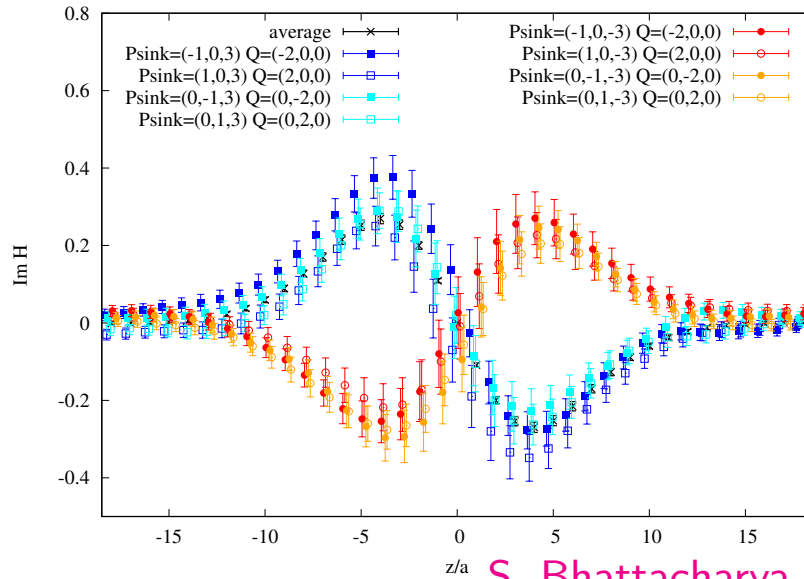
another ($\gamma_5 \gamma_i$ operators, $i = 0, 1, 2$): $F_{\tilde{H}} = \widetilde{A}_2 + zP_3 \widetilde{A}_6.$

$\tilde{E}$ impossible to extract at zero skewness: $F_{\tilde{E}} = 2 \frac{P \cdot z}{\Delta \cdot z} \widetilde{A}_3 + 2\widetilde{A}_5.$

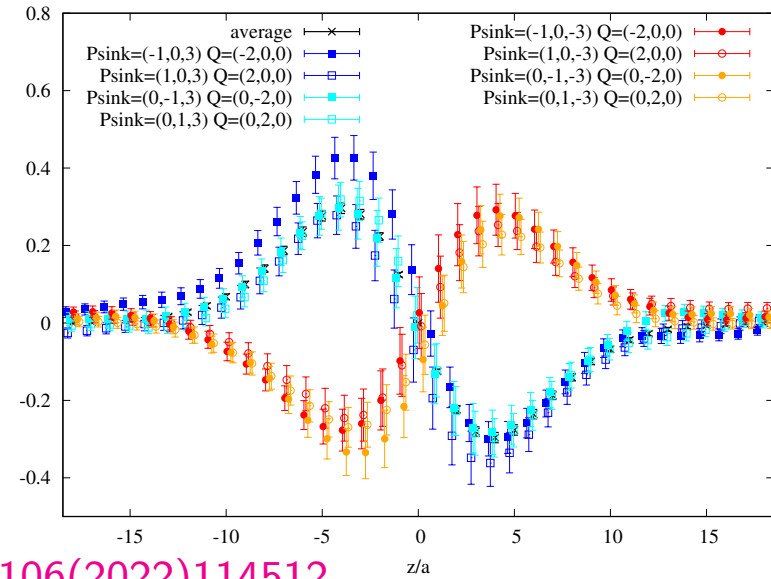


Signal comparison for different definitions

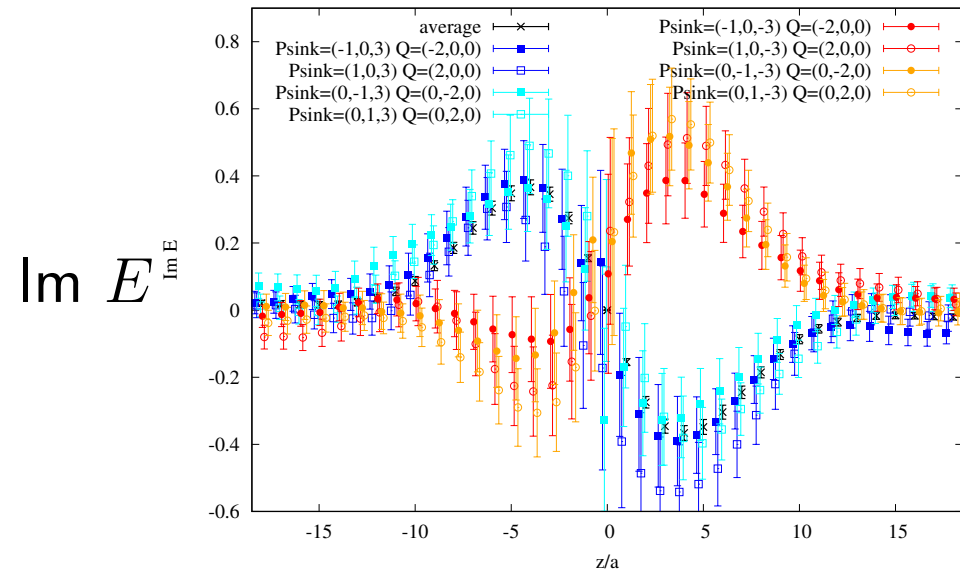
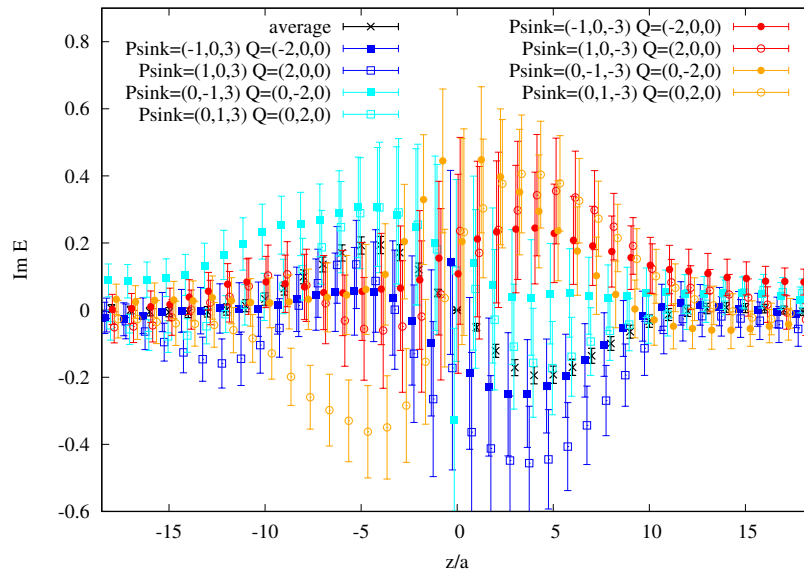
standard

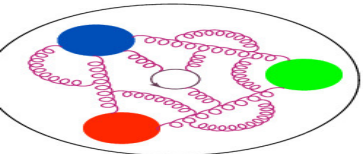


Lorentz-invariant



S. Bhattacharya et al., PRD106(2022)114512



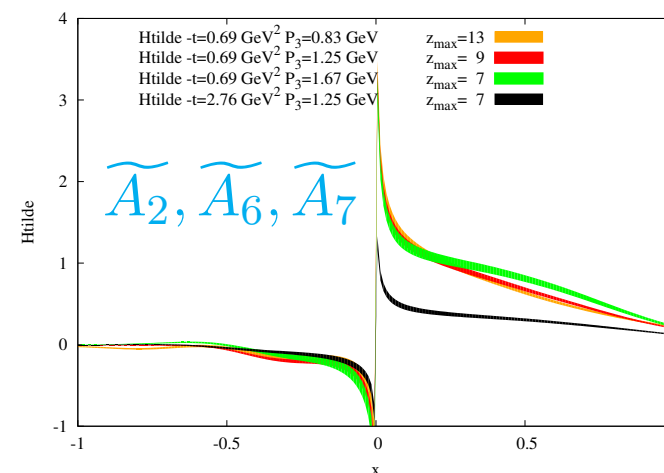
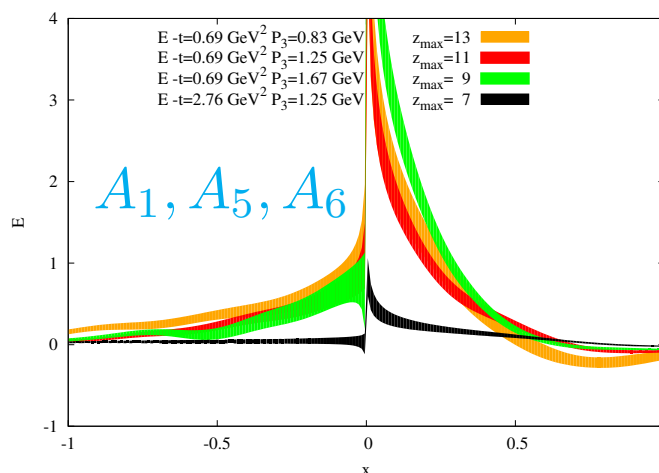
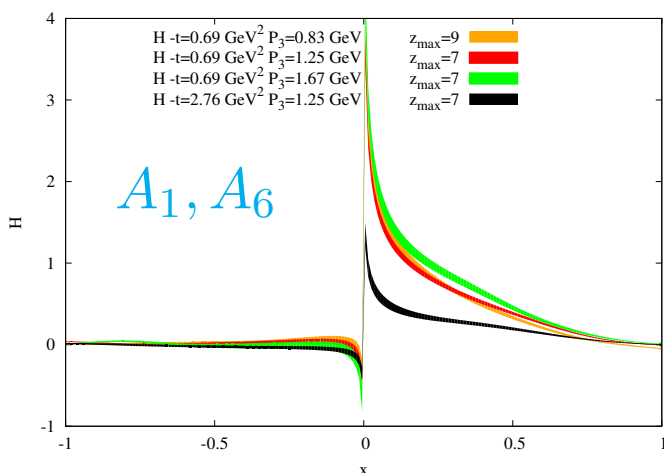


Convergence of different definitions of $\tilde{H}/H/E$

STANDARD

UNPOLARIZED

HELICITY



γ_0 operator (non-LI)

$\gamma_5 \gamma_3$ operator (LI)

H -GPD

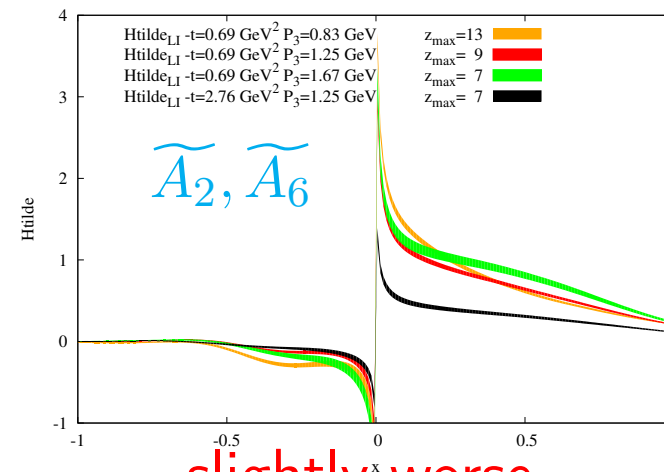
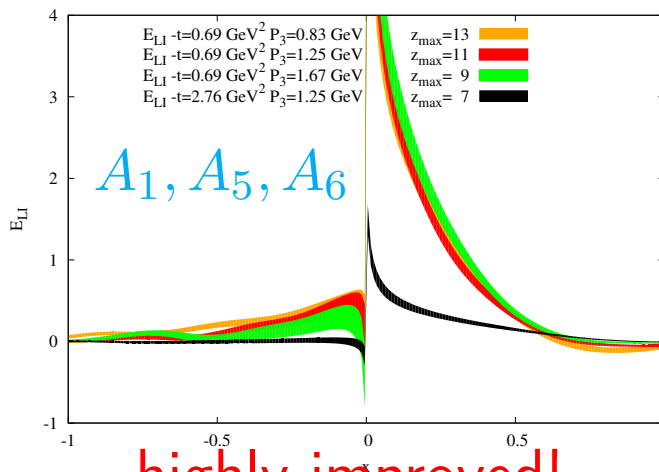
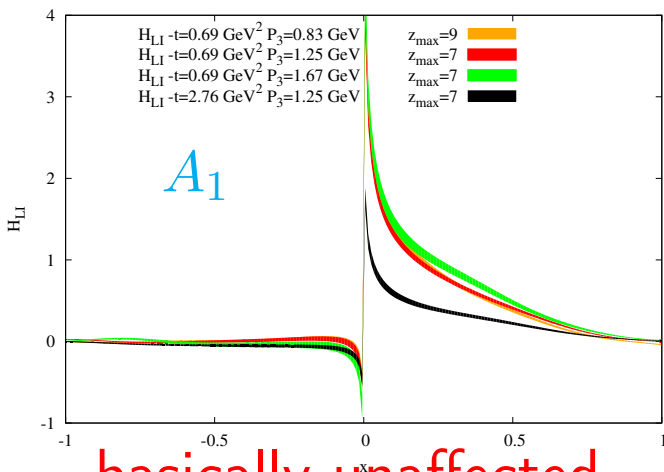
E -GPD

$\tilde{H}$ -GPD

γ_0, γ_T operators (LI)

$\gamma_5 \gamma_0, \gamma_5 \gamma_T$ operators (LI)

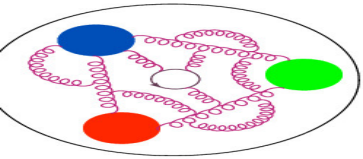
LORENTZ-INV.



basically unaffected

highly-improved!

slightly worse



GPDs moments from OPE of non-local operators



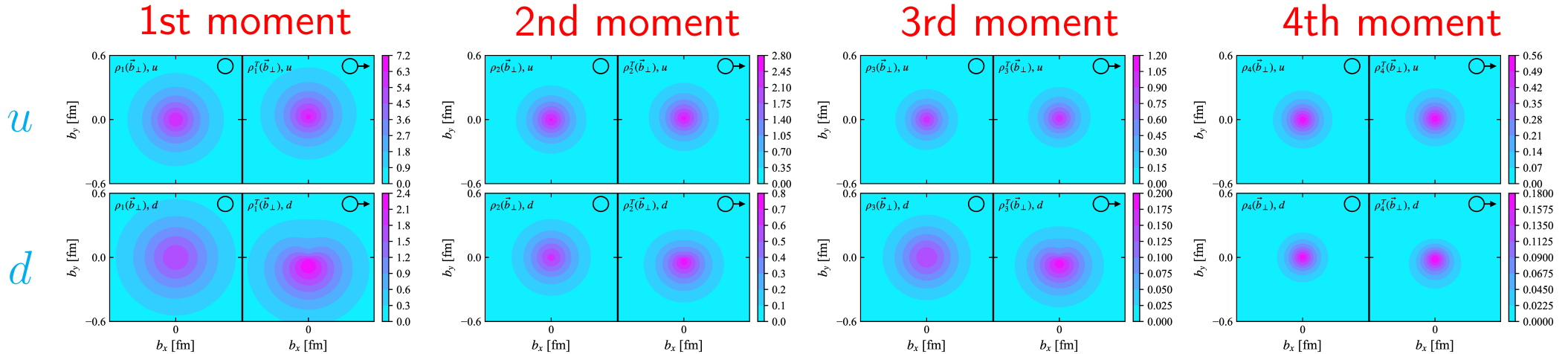
Short-distance factorization (SDF) can also be used to extract moments of GPDs.

For ratio-renormalized H/E : $\mathcal{F}^{\overline{\text{MS}}}(z, P, \Delta) = \sum_{n=0} \frac{(-izP)^n}{n!} C_n^{\overline{\text{MS}}}(\mu^2 z^2) \langle x^n \rangle + \mathcal{O}(\Lambda_{\text{QCD}}^2 z^2)$,
 $C_n^{\overline{\text{MS}}}(\mu^2 z^2)$ – Wilson coefficients (NNLO for $u - d$, NLO for $u + d$)

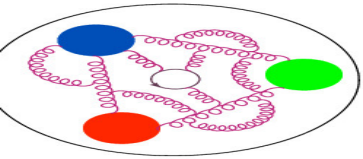
Moments of impact parameter parton distributions in the transverse plane:

$$\rho_{n+1}(\vec{b}_\perp) = \int \frac{d^2 \vec{\Delta}_\perp}{(2\pi)^2} A_{n+1,0}(-\vec{\Delta}_\perp^2) e^{-i\vec{b}_\perp \cdot \vec{\Delta}_\perp},$$

$$\rho_{n+1}^T(\vec{b}_\perp) = \int \frac{d^2 \vec{\Delta}_\perp}{(2\pi)^2} [A_{n+1,0}(-\vec{\Delta}_\perp^2) + i \frac{\Delta_y}{2M} B_{n+1,0}(-\vec{\Delta}_\perp^2)] e^{-i\vec{b}_\perp \cdot \vec{\Delta}_\perp}.$$



S. Bhattacharya, KC, M. Constantinou, X. Gao, A. Metz, J. Miller, S. Mukherjee, P. Petreczky, F. Steffens, Y. Zhao, PRD108(2023)014507

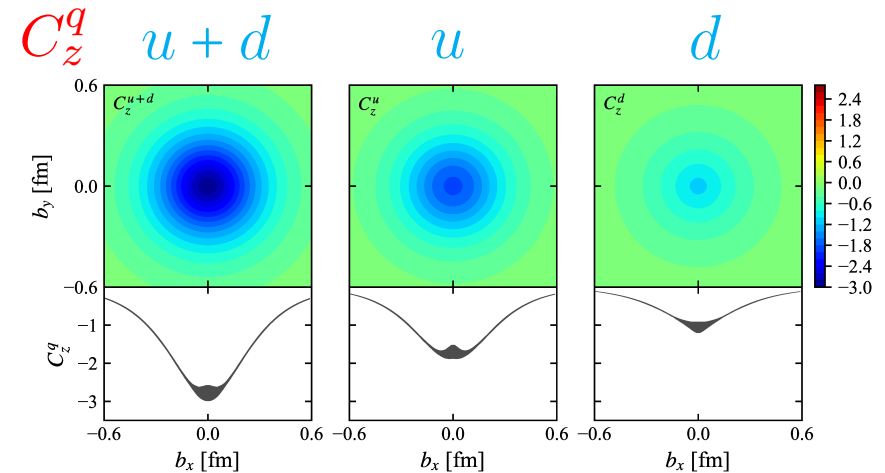
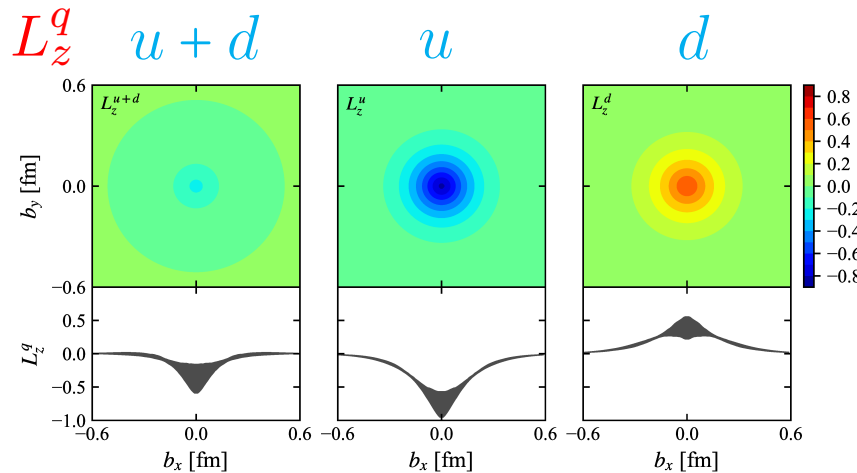
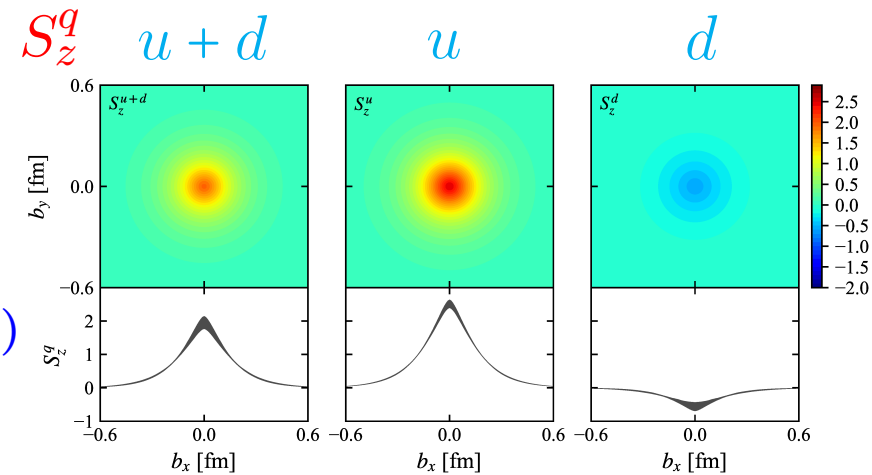


GPDs moments (axial vector)

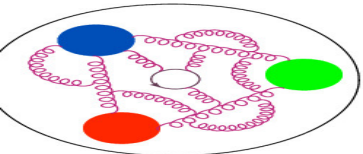
Similar extraction of moments $\tilde{A}_{n,0}^{u\pm d}$ using SDF can be done in the axial vector sector.

In particular, one can extract moments related to:

- quark helicity: $S_z^q = \frac{1}{2} \int_{-1}^1 dx \tilde{H}^q(x, 0, 0) = \frac{1}{2} \tilde{A}_{1,0}^q(0)$
- quark OAM: $L_z^q = J_z^q - S_z^q = \frac{1}{2} (A_{2,0}^q(0) + B_{2,0}^q(0)) - \frac{1}{2} \tilde{A}_{1,0}^q(0)$
- spin-orbit correlation: $C_z^q = \frac{1}{2} (\tilde{A}_{2,0}^q(0) - A_{1,0}^q(0))$
(ignoring term suppressed by $m_q/2m_N$ with $E_T + 2\tilde{H}_T$)



S. Bhattacharya, KC, M. Constantinou, X. Gao, A. Metz, J. Miller, S. Mukherjee, P. Petreczky, F. Steffens, Y. Zhao, JHEP01(2025)146



Twist-3 GPDs

PDFs/GPDs can be classified according to their twist, which describes the order in $1/Q$ at which they appear in the factorization of structure functions.

Twist-2 – probability densities for finding partons carrying fraction x of the hadron momentum.

Twist-3 – no density interpretation, contain important information about qgq correlations, appear in QCD factorization theorems for a variety hard scattering processes, interesting connections with TMDs, important for JLab12 and EIC, but difficult to measure.

Exploratory studies:

S. Bhattacharya, KC, M. Constantinou
A. Metz, A. Scapellato, F. Steffens

- matching for twist-3 PDFs: g_T, h_L, e

S. Bhattacharya et al., PRD102(2020)034005, PRD102(2020)114025

BC-type sum rules S. Bhattacharya, A. Metz, PRD105(2022)054027

Note: neglected qgq correlations

see also: V. Braun, Y. Ji, A. Vladimirov, JHEP 05(2021)086, 11(2021)087

- lattice extraction of $g_T^{u-d}(x)$ and $h_L^{u-d}(x)$
+ test of Wandzura-Wilczek approximation

S. Bhattacharya et al., PRD102(2020)111501(R), PRD104(2021)114510

- first exploration of twist-3 axial GPDs: $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{G}_4$

S. Bhattacharya et al., PRD108(2023)054501

$$\mathcal{F}[\gamma_j \gamma_5] = -i \frac{\Delta_j \gamma_5}{2m} F_{\tilde{E}+\tilde{G}_1} + \gamma_j \gamma_5 F_{\tilde{H}+\tilde{G}_2} + \frac{\Delta_j \gamma_3 \gamma_5}{P_3} F_{\tilde{G}_3} - \frac{\text{sign}[P_3] \varepsilon_{\perp}^j \rho \Delta_{\rho} \gamma_3}{P_3} F_{\tilde{G}_4}$$

