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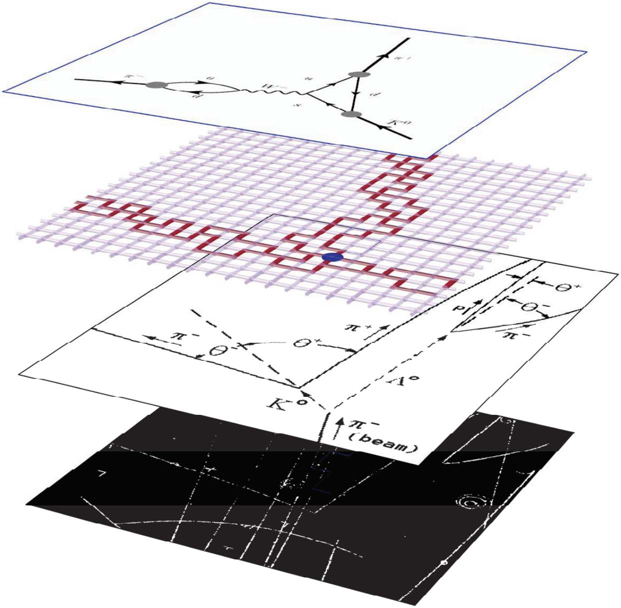

Lattice computation of $K \to \pi\pi$ decay amplitudes

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(RBC-UKQCD Collaboration)

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Outline of Talk

1. Introduction
   - Non-leptonic kaon decays
   - Ensembles
   - Fits
   - Matching of results with different actions
2. $K \to (\pi\pi)_{I=2}$ Decays
   - $K^+ \to \pi^+\pi^0 \Rightarrow K^+ \to \pi^+\pi^+$
   - Raw matrix elements
   - Renormalization of four-quark operators
   - Remaining systematic uncertainties
3. $K \to (\pi\pi)_{I=0}$ Decays
   - Status and prospects
4. Long-distance effects in other processes
   - ΔM_K as an example
5. Summary, Conclusions and Prospects

- $K \to \pi\pi$ decays are a very important class of processes for standard model phenomenology.
- Among the interesting issues are the origin of the $\Delta I = 1/2$ rule and an understanding of the experimental value of ε'/ε, the parameter which was the first experimental evidence of direct CP-violation.
- At lowest order in the SU(3) chiral expansion one can obtain the $K \to \pi\pi$ decay amplitude by calculating $K \to \pi$ and $K \to$ vacuum matrix elements.
  - In 2001, two collaborations, RBC and CP-PACS, published some very interesting (quenched) results, at pion masses greater than about 600 MeV.
  - RBC/(UKQCD) have repeated the calculation with dynamical fermions in the pion-mass range 240-415 MeV.
  - Conclusion - soft-pion theorems are not sufficiently reliable $\Rightarrow$ need to compute $K \to \pi\pi$ matrix elements directly.
- The evaluation of $K \to \pi\pi$ matrix elements requires an extension of the standard computations of $\langle 0\,|\,O(0)\,|\,h\rangle$ and $\langle h_2\,|\,O(0)\,|\,h_1\rangle$ matrix elements with a single hadron in the initial and/or final state.

Parameters of the simulation

We have two datasets of $N_f = 2+1$ DWF with the Iwasaki Gauge Action:

1. $a \simeq 0.114\,$fm:
   $24^3 \times 64 \times 16 \quad (L \simeq 2.74\,\text{fm})$
   $(16^3 \times 32 \times 16 \quad (L \simeq 1.83\,\text{fm})$ – not discussed here.)

   - Four light-quark masses:

     $$ma = 0.03\ (m_\pi \simeq 670\,\text{MeV}); \qquad ma = 0.02\ (m_\pi \simeq 555\,\text{MeV});$$
     $$ma = 0.01\ (m_\pi \simeq 415\,\text{MeV}); \qquad ma = 0.005\ (m_\pi \simeq 330\,\text{MeV}).$$

   - The lightest partially quenched pion has a mass of about 240 MeV.
   - Only data from masses with $m_\pi \lesssim 420\,$MeV are used in the analyses.

2. $a \simeq 0.086\,$fm:
   $32^3 \times 64 \times 16 \quad (L \simeq 2.765\,\text{fm})$

   - Three light-quark masses:

$$ma = 0.008\ (m_\pi \simeq 390\,\text{MeV});\ ma = 0.006\ (m_\pi \simeq 343\,\text{MeV});\ ma = 0.004\ (m_\pi \simeq 290\,\text{MeV}).$$

   - The lightest partially quenched pion has a mass of about 223 MeV.

For $K \to \pi\pi$ decays we require pions with a physical mass and hence a large volume $\Rightarrow$ coarse lattice.

3. $a \simeq 0.14$ fm, (DWF+IDSDR)
$32^3 \times 64 \times 32$ ($L \simeq 4.58$ fm)

C.Kelly, arXiv:1201.0706; RBC-UKQCD, in preparation

- Two light-quark masses:

$$ma = 0.0042\ (m_\pi \simeq 250\,\text{MeV});\ ma = 0.001\ (m_\pi \simeq 170\,\text{MeV})\,.$$

- The lightest partially quenched pion has a mass of about 143 MeV.
- The goal was to have a physical $K \to \pi\pi$ decay, with $|p_\pi| = \sqrt{2}\pi/L$.
- With this coarse lattice, it will not be surprising that lattice artefacts are the largest source of systematic error.
We mitigate against this in a number of ways.

- Our standard global chiral-continuum fitting procedure is to use SU(2) χPT keeping terms of $O(m_\pi^2)$ and $O(a^2)$ but not higher order terms.

  Y.Aoki et al., arXiv:1011.0892 [hep-lat]

- We use m_π, m_K and m_Ω to calibrate the lattice.
- Although we only have performed DWF+IDSDR simulations at a single value of β, it is nevertheless possible to estimate the lattice artefacts for those quantities which have also been calculated on the DWF+Iwasaki lattices.
  - Calculate the quantities on the Iwasaki lattices and extrapolate to the continuum.
  - Compare results on the Iwasaki and IDSDR lattices $\Rightarrow$ IDSDR $O(a^2)$ artefacts.
  - Can use the IDSDR lattices in global fits.

T. Blum, P.A. Boyle, N.H. Christ, N. Garron, E. Goode, T. Izubuchi, C. Jung, C. Kelly, C. Lehner, M. Lightman, Q. Liu, A.T. Lytle, R.D. Mawhinney, C.T. Sachrajda, A. Soni, C. Sturm, arXiv:1111.1699 [hep-lat].

- Of course we would like to evaluate all the $K \to \pi\pi$ matrix elements in lattice simulations and reconstruct A_0 and A_2 and understand the $\Delta I = 1/2$ rule and the value of ε'/ε (see below).
- In the meantime however, we know Re A_0 and Re A_2 from experiment.
  I now attempt to demonstrate that we can also compute Re A_2.
- The experimental value of ε'/ε gives us one relation between Im A_0 and Im A_2, thus if we evaluate Im A_2 then within the standard model we know Im A_0 to some precision. Thanks to Andrzej Buras for stressing this to me.
  I also attempt to demonstrate that we can indeed compute Im A_2.
- I stress again that ultimately of course, we wish to do better than this. See below

- The main theoretical ingredients of the *infrared* problem with two-pions in the s-wave are understood.
- Two-pion quantization condition in a finite Euclidean volume

$$\delta(q^*) + \phi^P(q^*) = n\pi\,,$$

where $E^2 = 4(m_\pi^2 + q^{*\,2})$, δ is the s-wave $\pi\pi$ phase shift and ϕ^P is a known kinematic function. M.Lüscher, 1986, 1991, $\cdots$.

- The relation between the physical $K \to \pi\pi$ amplitude A and the finite-volume matrix element M

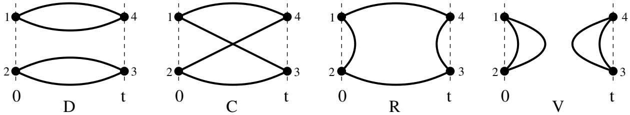
$$|A|^2 = 8\pi V^2 \frac{m_K E^2}{q^{*\,2}} \left\{\delta'(q^*) + \phi^{P\prime}(q^*)\right\} |M|^2\,,$$

where $\prime$ denotes differentiation w.r.t. q^* .

L.Lellouch and M.Lüscher, hep-lat/0003023; C.J.D.Lin, G.Martinelli, CTS, M.Testa, hep-lat/0104006; C.h.Kim, CTS and S.Sharpe, hep-lat/0507006; N.H.Christ, C.h.Kim and T.Yamazaki hep-lat/0507009

  - We understand how to calculate the $\Delta I = 3/2$ $K \to \pi\pi$ matrix elements.
- Our aim is to calculate the matrix elements with as good a precision as we can.

- Computation of $K \to (\pi\pi)_{I=2}$ matrix elements does not require the subtraction of power divergences or the evaluation of disconnected diagrams.
- Consider for example, the two-pion correlation functions, which are an important ingredient in the evaluation of $K \to \pi\pi$ amplitudes.



- For I=2 $\pi\pi$ states the correlation function is proportional to D-C and we do not need to evaluate the disconnected vacuum diagram V.
- Our calculation using today's techniques and ensembles to improve on:
  An exploratory lattice study of $\Delta I = 3/2$ $K \to \pi\pi$ decays at next-to-leading order in the chiral expansion, hep-lat/0412029
  P.Boucaud, V.Gimenez, C.-J.D.Lin, V.Lubicz, G.Martinelli, M.Papinutto, CTS
- This was the latest and 65th paper written with Guido Martinelli.

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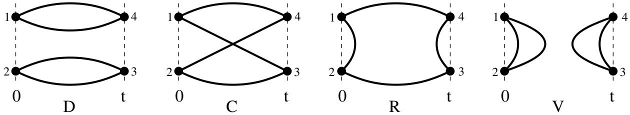

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At the workshop - Guido Martinelli (left) and Chris Sachrajda contemplate power subtractions for non-leptonic kaon decays.

(CERN Courier, reporting on the 2000 Ringberg Workshop on Current Theoretical Problems in Lattice Field Theory)

- The operators whose matrix elements have to be calculated are:

$$
\begin{aligned}
O^{3/2}_{(27,1)} &= (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L \right\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_L \\
O^{3/2}_{7} &= (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_R - (\bar{d}^j d^j)_R \right\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_R \\
O^{3/2}_{8} &= (\bar{s}^i d^j)_L \left\{ (\bar{u}^j u^i)_R - (\bar{d}^j d^i)_R \right\} + (\bar{s}^i u^j)_L (\bar{u}^j d^i)_R
\end{aligned}
$$

- It is convenient to use the Wigner-Eckart Theorem: (Notation - $O^{\Delta I}_{\Delta I_z}$)

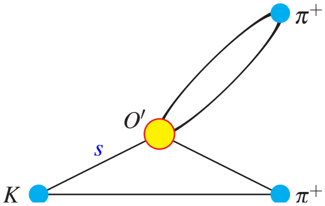
$$
{}_{I=2}\langle \pi^+(p_1)\pi^0(p_2)\,|O^{3/2}_{1/2}|K^+\rangle = \frac{\sqrt{3}}{2}\langle \pi^+(p_1)\pi^+(p_2)\,|O^{3/2}_{3/2}|K^+\rangle\,,
$$

where

  - $O^{3/2}_{3/2}$ has the flavour structure $(\bar{s}d)\,(\bar{u}d)$.
  - $O^{3/2}_{1/2}$ has the flavour structure $(\bar{s}d)\,(\,(\bar{u}u)-(\bar{d}d)\,)+(\bar{s}u)\,(\bar{u}d)$.

- We can then use antiperiodic boundary conditions for the u-quark say, so that the $\pi\pi$ ground-state is $\langle\pi^+(\pi/L)\,\pi^+(-\pi/L)|$. C-h Kim, Ph.D. Thesis
  - Do not have to isolate an excited state.
  - Size (L) needed for physical $K\to\pi\pi$ decay halved (6 fm $\to$ 3 fm).

Computing $\Delta I = 3/2$ Matrix Elements



- The RBC/UKQCD strategy at this stage is to perform the simulations on a large lattice, $L \simeq 4.5$ fm, with light pions $(32^3 \times 64 \times 32)$

$$m_\pi \simeq 143\,\mathrm{MeV} \qquad \mathrm{Unitary}\ m_\pi \simeq 170\,\mathrm{MeV}\,.$$

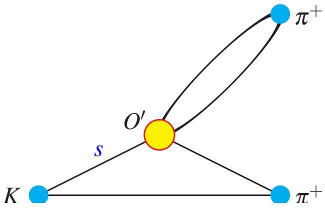
- The price is that the lattice is coarse, $a^{-1} \simeq 1.4$ GeV.
- With DWF, m_{res} increases as the coupling becomes stronger $\Rightarrow$ change the gauge action (from Iwasaki) by multiplying by the *Auxilliary Determinant* .

D.Renfrew, T.Blum, N.Christ, R.Mawhinney and P.Vranas, arXiv:0902.2587

R. Mawhinney, Lattice 2010

- This is tuned to suppress m_{res} but to maintain topology changing.

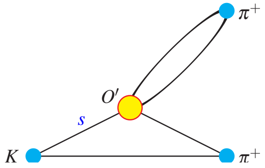
Preliminary $\Delta I = 3/2$ Matrix Elements – Cont.



- The masses and momenta are as follows:

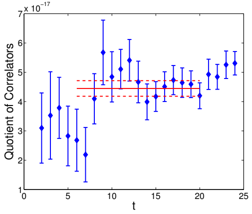
Quantity	This Calculation	Physical
m_π	142.9(1.1) MeV	139.6 MeV
m_K	511(4) MeV	493.7 MeV
$E_{\pi\pi}(p_\pi \simeq \sqrt{2}\pi/L)$	493(6) MeV	m_K
$E_{\pi\pi}(p_\pi \simeq \sqrt{2}\pi/L) - m_K$	-18.7 MeV	0

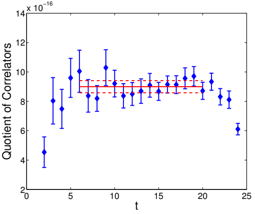
- The results presented here were obtained with 63 configurations .
  (We now have more than twice this number and are continuing to run.)

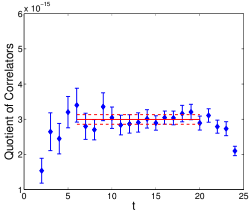


Source	Re(A_2) (10^{-8} GeV)
$t_K = 20$	1.38(10)
$t_K = 24$	1.47(10)
$t_K = 28$	1.58(11)
$t_K = 32$	1.25(15)
Weighted Average	1.44(6)
Experiment	1.5

Stat. error only







$O'^{3/2}_{(27,1)} = (\bar{s}d)_L\,(\bar{u}d)_L$ $\quad O'^{3/2}_7 = (\bar{s}d)_L\,(\bar{u}d)_R$ $\quad O'^{3/2}_8 = (\bar{s}^i d^j)_L\,(\bar{u}^j d^i)_R$

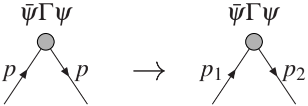
Sample plateaus for the matrix elements at matched kinematics ($p_\pi = \sqrt{2}p_{\min}$).

We follow our standard strategy:

bare lattice operators	$\longrightarrow$ NPR	operators renormalized in operators Intermediate Scheme(s) (RI-MOM, RI-SMOM)
	$\longrightarrow$ Perturbation Theory	renormalized in $\overline{\rm MS}$-NDR scheme.

- This is well understood here in Rome, but I would like to underline one or two points from our procedure.

Renormalization of quark bilinear operators in a MOM-scheme with a non-exceptional subtraction point C.Sturm, Y.Aoki, N.H.Christ, T.Izubuchi, CTS, and A.Soni, arXiv:0901.2599 [hep-ph]



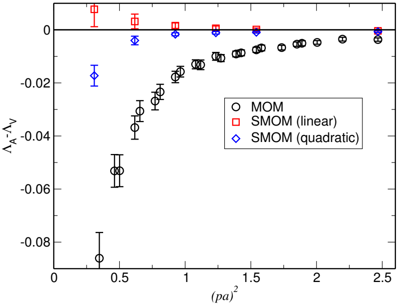
$$p_1^2 = p_2^2 = (p_1 - p_2)^2$$

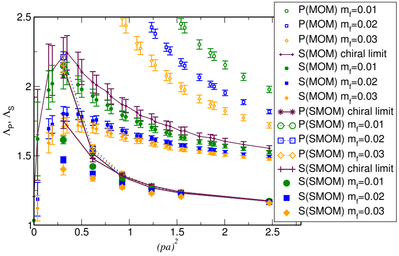
- In this paper we develop the scheme with the non-exceptional subtraction point

$$p_1^2 = p_2^2 = (p_1 - p_2)^2 \,.$$

- We calculate the one-loop conversion factors between this scheme and the $\overline{\text{MS}}$ scheme. This is entirely a continuum exercise.

Evidence for small chiral symmetry breaking

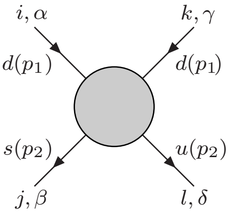




- $\Lambda_A - \Lambda_V$.
- Λ_S and Λ_P.

Y.Aoki arXiv:0901.2595 [hep-lat]

SMOM Kinematics for Four-Quark Operators



- For four-quark operators there is no such natural choice for the kinematics.
- We still choose $p_1^2 = p_2^2 = (p_1 - p_2)^2 \equiv p^2$.
- By using momentum sources for the Green functions, the statistical errors are tiny and $O(4)$-breaking lattice artefacts are seen in the renormalization constants relating the bare lattice operators and the renormalized ones in the SMom scheme.
  - In other words, after the running due to the anomalous dimension is divided out, the remaining artefacts are not simply $O(a^2p^2)$ but contain terms such as $O(a^2 \sum_\mu p_\mu^4/p^2)$.

See M.Constantinou et al., arXiv:1011.6059

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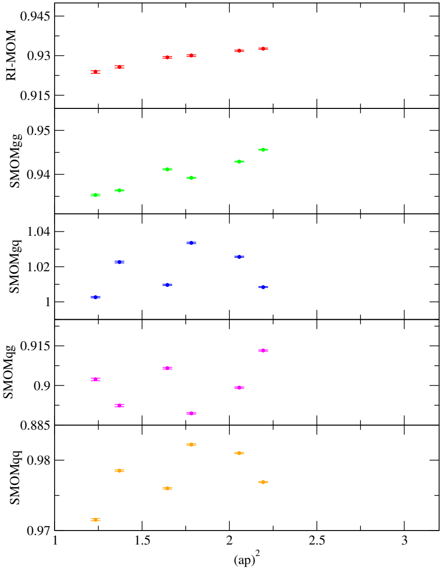

- The panels show $Z_{B_K}^{\rm NDR}$ at a renormalization scale of 2 GeV as a function of $(ap)^2$ for 5 intermediate renormalization schemes (RI-Mom and 4 different SMom schemes).
- We interpret the scatter as being due to $O(4)$-breaking effects.
- With (partially) twisted boundary conditions, we can simply scale the momenta so that the effects should only depend on p^2 making it possible to model and remove them.

Y.Aoki et al., arXiv:1012.4178

NPR with twisted boundary conditions (cont.)

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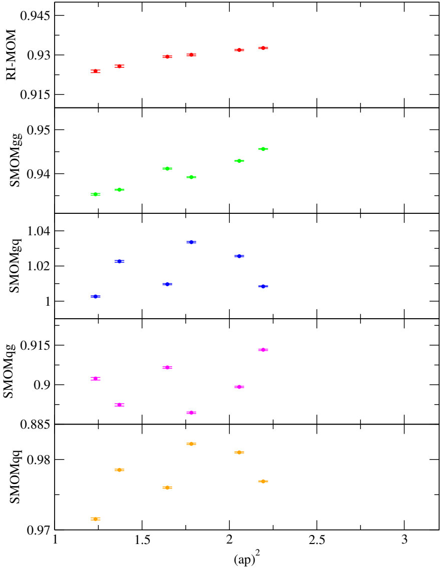

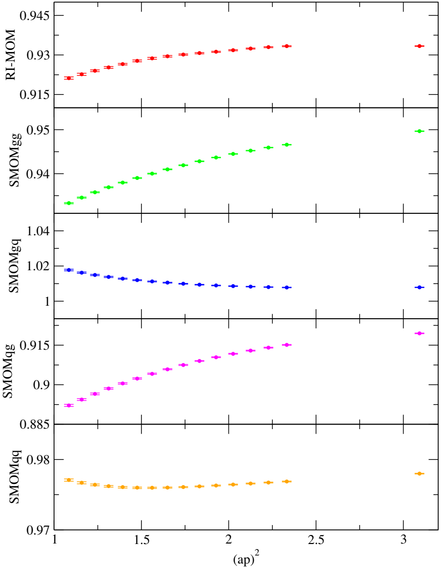

Procedure for Renormalizing the $\Delta I = 3/2\ \Delta S = 1$ Operators

1. Perform the renormalization into the intermediate schemes on the IDSDR lattices ($a^{-1} \simeq 1.4$ GeV) at $\mu = 1.145$ GeV.
   At such a low scale we cannot use perturbation theory reliably to match onto the Wilson Coefficient functions which are calculated in the $\overline{\text{MS}}$-NDR scheme.
2. Match the result onto the finer Iwasaki lattices at $\mu = 1.145$ GeV.
3. Perform step-scaling on the Iwasaki lattices and the continuum extrapolation to obtain the renormalization constants at $\mu = 3$ GeV.
4. Convert the results to the $\overline{\text{MS}}$-NDR scheme at 3 GeV using perturbation theory.

$$Z^{\overline{\text{MS}},(\text{latt})}_{(\gamma^\mu,\gamma^\mu)}(3\,\text{GeV}) = \begin{pmatrix} 0.421\,(02)(00) & 0 & 0 \\ 0 & 0.479\,(03)(07) & -0.024\,(04)(17) \\ 0 & -0.045\,(11)(11) & 0.543\,(18)(23) \end{pmatrix}$$

$$Z^{\overline{\text{MS}},(\text{latt})}_{(\not{q},\not{q})}(3\,\text{GeV}) = \begin{pmatrix} 0.427\,(03)(03) & 0 & 0 \\ 0 & 0.473\,(05)(06) & -0.026\,(05)(17) \\ 0 & -0.070\,(23)(25) & 0.564\,(27)(13) \end{pmatrix}.$$

	ReA_2	ImA_2
lattice artefacts	15%	15%
finite-volume corrections	6.2%	6.8%
partial quenching	3.5%	1.7%
renormalization	1.7%	4.7%
unphysical kinematics	3.0%	0.22%
derivative of the phase shift	0.32%	0.32%
Wilson coefficients	7.1%	8.1%
Total	18%	19%

- Since the lattice is so coarse and the results are proportional to a^{-3}, the systematic errors are dominated by the lattice aretefacts. We estimate these in two different ways:
  1. from the variation in the value of a obtained using m_Ω, f_π, f_K and r_0 to set the scale;
  2. from the a^2 terms in global chiral continuum fits of B_K, performed using both IDSDR and Iwasaki lattices.

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Our results for the $M_i = \langle \pi^+\pi^+ \,|\, Q_i \,|\, K^+ \rangle$ are:

$$M_{(27,1)} = (3.20 \pm 0.13_{\rm stat} \pm 0.58_{\rm syst})\, 10^{-2}\, {\rm GeV}^3\,,$$
$$M_{(8,8)} = (5.85 \pm 0.89_{\rm stat} \pm 1.11_{\rm syst})\, 10^{-1}\, {\rm GeV}^3\,,$$
$$M_{(8,8){\rm mix}} = (2.75 \pm 0.12_{\rm stat} \pm 0.52_{\rm syst})\, {\rm GeV}^3\,.$$

In terms of the amplitude A_2 these results imply:

$$\mathrm{Re}A_2 = (1.436 \pm 0.062_{\rm stat} \pm 0.258_{\rm syst})\, 10^{-8}\, {\rm GeV}$$
$$\mathrm{Im}A_2 = -(6.83 \pm 0.51_{\rm stat} \pm 1.30_{\rm syst})\, 10^{-13}\, {\rm GeV}.$$

- The result for Re A_2 agrees well with the experimental value of $1.479(4) \times 10^{-8}$ GeV obtained from K^+ decays and $1.573(57) \times 10^{-8}$ GeV obtained from K_S decays .
- Im A_2 is unknown so that our result provides its first direct determination.
- For the phase of A_2 we find Im A_2/Re$A_2 = -4.76(37)_{\rm stat}(81)_{\rm syst}\, 10^{-5}$.
- Combining our result for Im A_2 with the experimental results for Re A_2, Re $A_0 = 3.3201(18)\dot{1}0^{-7}$ GeV and ε'/ε we obtain:

$$\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} = -1.63(19)_{\rm stat}(20)_{\rm syst} \times 10^{-4}\,.$$

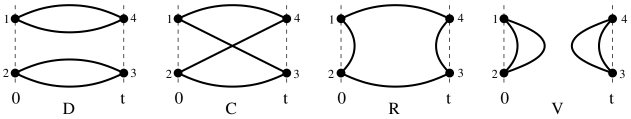
Of course, we wish to confirm this directly.

Conclusions on the $\Delta I = 3/2$ Project

- The *ab initio* calculation of A_2 described above builds upon substantial theoretical advances, achieved over many years.
- The agreement we find for Re A_2 with the experimental result is very satisfying.
- We are also able to determine Im A_2 for the first time.
- It will be important to repeat this calculation using a second lattice spacing so that a continuum extrapolation can be performed thus eliminating the dominant contribution to the error, reducing the total uncertainty to about 5%.
- We expect that the dominant remaining errors in A_2 will then come from the omission of electromagnetic and other isospin breaking mixing between the large amplitude A_0 and A_2.
- Much more challenging but of even greater interest is the application of these methods to the evaluation of A_0 allowing for a calculation of ε'/ε and an understanding of the $\Delta I = 1/2$ rule.

T. Blum, P.A. Boyle, N.H. Christ, N. Garron, E. Goode, T. Izubuchi, C. Lehner, Q. Liu, R.D. Mawhinney, C.T. Sachrajda, A. Soni, C. Sturm, H. Yin, R. Zhou arXiv:1106.2714.

- The $I = 0$ final state has vacuum quantum numbers.
- Vacuum contribution must be subtracted; disconnected diagrams require statistical cancelations to obtain the $e^{-2m_\pi t}$ behaviour.
- Consider first the two-pion correlation functions, which are an important ingredient in the evaluation of $K \to \pi\pi$ amplitudes.

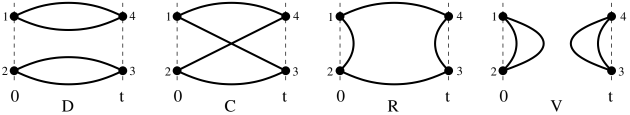


- For I=2 $\pi\pi$ states the correlation function is proportional to D-C.
- For I=0 $\pi\pi$ states the correlation function is proportional to 2D+C-6R+3V.

The major practical difficulty is to subtract the vacuum contribution with sufficient precision.

- In the paper we report on high-statistics experiments on a $16^3 \times 32$ lattice, $a^{-1} = 1.73$ GeV, $m_\pi = 420$ MeV, with the propagators evaluated from each time-slice.

Diagrams contributing to two-pion correlation functions



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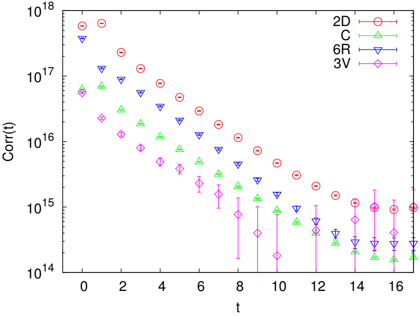
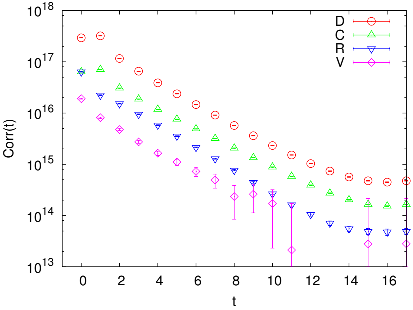
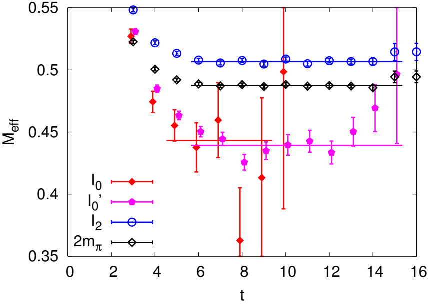


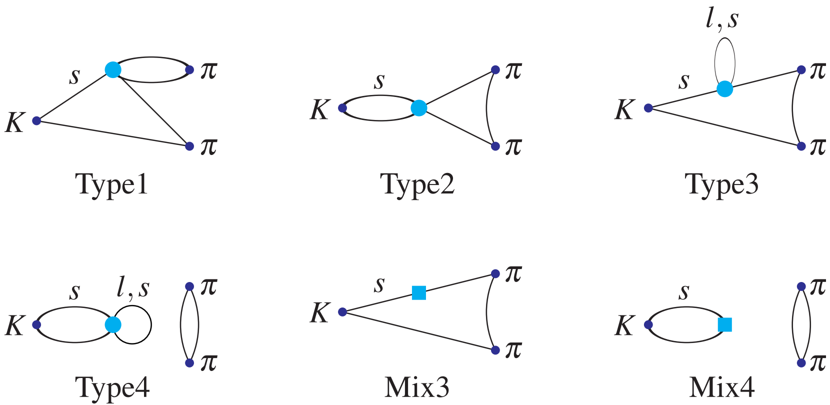
RBC/UKQCD, Qi Liu – Lattice 2010

Two-pion Correlation Functions

RBC/UKQCD, arXiv:1106.2714







- There are 48 different contractions and we classify the contributions into the 6 different types illustrated above.
- Mix3 and Mix4 are needed to subtract the power divergences which are proportional to matrix elements of $\bar{s}\gamma_5 d$.

RBC/UKQCD arXiv:1106.2714

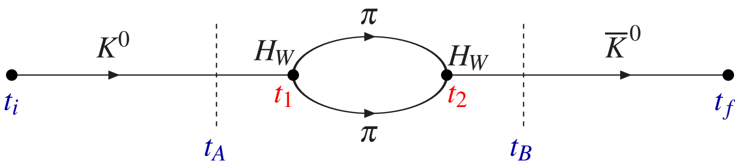
- These results are for the $K \to \pi\pi$ (almost) on-shell amplitudes with 420 MeV pions at rest:

$$\begin{array}{ll} \text{Re}\, A_0 & (3.80 \pm 0.82)\, 10^{-7}\ \text{GeV} \\ \text{Im}\, A_0 & -(2.5 \pm 2.2)\, 10^{-11}\ \text{GeV} \\ \text{Re}\, A_2 & (4.911 \pm 0.031)\, 10^{-8}\ \text{GeV} \\ \text{Im}\, A_2 & -(5.502 \pm 0.0040)\, 10^{-13}\ \text{GeV} \end{array}$$

- This is an exploratory exercise in which we are learning how to do the calculation.
- Since this work was finished we have been developing techniques which seem to enhance the signal considerably.
- The exploratory results for $K \to (\pi\pi)_{I=0}$ decays encourage us to proceed to physical kinematics.

$\Rightarrow$ an understanding of the $\Delta I = 1/2$ rule and the value of ε'/ε.

- The evaluation of disconnected diagram has allowed us to study the η and η' mesons and their mixing. RBC-UKQCD – arXiV:1002.2999



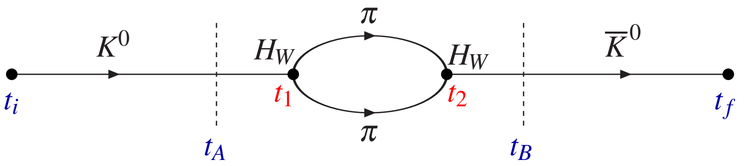
- We have in mind to calculate the amplitude

$$\mathscr{A} = \frac{1}{2}\int_{-\infty}^{\infty} dt_1\, dt_2\; T\,\langle \bar{K}^0\,|\,H_W(t_2)\,H_W(t_1)\,|\,K^0\rangle$$

  and to determine the K_L-K_S mass difference:

$$\Delta M_K = 2\mathscr{P}\sum_\alpha \frac{\langle \bar{K}^0\,|\,H_W\,|\alpha\rangle\,\langle\alpha\,|H_W\,|\,K^0\rangle}{m_K - E_\alpha}$$

  where the sum over $|\alpha\rangle$ includes an energy-momentum integral.
- In a finite-volume calculation we have to ensure that the K^0 is created first and the $\overline{K}^0$ is annihilated last $\Rightarrow$ integrals over t_1 and t_2 are over a sub-interval, $t_A \le t_{1,2} \le t_B$.



- In a finite volume the correlator is given by

$$\mathscr{A} = |Z_K|^2 e^{-M_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle}{(m_K - E_n)^2} \left\{ e^{(m_K-E_n)T} - (m_K - E_n)T - 1 \right\}$$

where $T \equiv t_B - t_A + 1$.

- By studying the time dependence to identify the coefficient of T, one can determine

$$\Delta M_K^{\mathrm{FV}} = 2 \sum_n \frac{\langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle}{m_K - E_n} .$$

- What is the relation between the finite-volume sum and the infinite-volume integral?

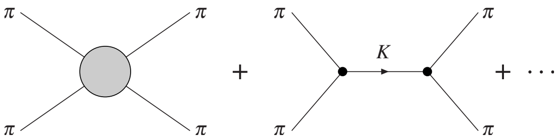
1 **Quantization Condition:** Lüscher quantisation condition for two-pions in the centre-of-mass frame with s-wave interactions

$$\delta(q)+\phi(q)=n\pi\,,$$

where $E^2=4(m_\pi^2+q^2)$, δ is the s-wave $\pi\pi$ phase shift and ϕ is a known kinematic function:

$$\tan\phi(q)=\frac{q}{4\pi}\,\frac{1}{c(q^2)}\quad\text{where}\quad c(q^2)=\frac{1}{L^3}\sum_{\vec{k}}\frac{1}{q^2-k^2}\,.$$

M.Lüscher, 1986, 1991, $\cdots$.



2 Lellouch and Lüscher derive the relation between the physical $K \to \pi\pi$ amplitudes and finite-volume matrix elements using degenerate perturbation theory, starting with $m_K = E_{\pi\pi}$:

$$|A^2| = 8\pi V^2 \frac{E^3}{q^2} \left[\delta'(q) + \phi'(q)\right] |M^2| .$$

L.Lellouch & M.Lüscher, hep-lat/0003023

- We interpret the LL factor as being the density of states (+ trivial normalization factors)

$$\frac{dn}{dE} = \frac{1}{\pi} \frac{d(\delta+\phi)}{dE} \qquad (\text{recall that } \delta + \phi = n\pi),$$

with no need for the m_K and $E_{2\pi}$ to be degenerate.

C.-J.D.Lin, G.Martinelli, CTS & M.Testa, hep-lat/0104006

3 ΔM_K: At Lattice 2010, Norman Christ presented the evaluation of the long-distance effects in ΔM_K using second-order degenerate perturbation theory:

$$\Delta M_K = 2\sum_{n\neq n_0}\frac{|\langle n\,|\,H_W\,|\,K_S\rangle|^2}{m_K-E_n}+\left[\frac{\partial(\delta+\phi)}{\partial E}\right]^{-1}\left[\frac{1}{2}\,\frac{\partial^2(\delta+\phi)}{\partial E^2}\,|\langle n_0\,|\,H_W\,|\,K_S\rangle|^2 - \frac{\partial}{\partial E_{n_0}}\left\{\left.\frac{\partial(\delta+\phi)}{\partial E}\right|_{E_{n_0}}|\langle n_0\,|\,H_W\,|\,K_S\rangle|^2\right\}_{E_{n_0}=m_K}\right]$$

N.H.Christ, PoS LATTICE2010 (2010) 300

Guido Martinelli and I have studied the relation between the finite-volume sum and infinite-volume integral and found the relation preliminary

$$\sum_{E_n}\frac{f(E_n)}{m_K^2-E_n^2}=\mathscr{P}\int\rho(E)\,dE\,\frac{f(E)}{m_K^2-E^2}+\frac{f(m_K)}{2m_K}\left[\cot(\delta+\phi)\,\frac{d(\delta+\phi)}{dE}\right]_{m_K},$$

where ρ is the density of states, $\rho = dn/dE$.

- Thus again we do not need $m_K = E_{n_0}$.
- If $m_K = E_{n_0}$ then the result reduces to that above.
- Indeed, it may be best to work with $\cot(\delta+\phi)=0$.

- We have performed the first direct calculation of the $K \to (\pi\pi)_{I=2}$ decay amplitude A_2.
  We believe that this will serve as an important benchmark for future improved calculations.
- Although significant technical problems remain, we are well on our way towards calculating A_0.
  (I did not talk about our exploratory studies refining all-to-all propagators or using G-parity.)
- We are beginning to tackle the calculation of long-distance effects for a variety of processes.