Ambient space and integration of the trace anomaly

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Conformal and Weyl symmetries

Classical Weyl symmetry

Take some action $S[\Phi, g_{\mu\nu}]$, metric is the **source** of the energy-momentum tensor

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}$$

Local Weyl rescalings for $\sigma = \sigma(x)$

$$\delta_\sigma^W g_{\mu\nu} = 2\sigma g_{\mu\nu} \qquad \delta_\sigma^W \Phi = w_\Phi \sigma \Phi$$

Noether identities of Diff and Weyl symmetries on-shell

$$\nabla_\mu T^{\mu\nu} = 0 \qquad T^\mu{}_\mu = 0$$

Weyl symmetry $\Longrightarrow$ Conformal symmetry

Conformal (isometries) group

$$0 = \delta^W_\sigma g_{\mu\nu} + \delta^E_\xi g_{\mu\nu} = 2\sigma g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

Solution iff $\sigma = -\frac{1}{d}\nabla_\mu \xi^\mu$

$$\delta^C_\xi = \delta^W_\sigma + \delta^E_\xi$$

Flat space limit $g_{\mu\nu} \to \eta_{\mu\nu}$ has $\frac{(d+1)(d+2)}{2} = \frac{d(d+1)}{2} + d + 1$ generators in $d \neq 2$

$$P_\mu\,,\;\; J_{\mu\nu}\,,\;\; K_\mu\,,\;\; D$$

leading to conserved charges

Weyl symmetry and the trace anomaly

Quantum Weyl symmetry

From the path-integral

$$e^{-\Gamma} = \int [d\Phi]\, e^{-S}$$

The renormalized EMT

$$\langle T^{\mu\nu} \rangle = -\frac{2}{\sqrt{g}} \frac{\delta\Gamma}{\delta g_{\mu\nu}}$$

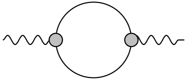
Trace is dimension d operator but it can be nonzero even for Weyl invariant S

Duff, Deser-Schwimmer, Jack-Osborn ...

A bit of history

Compute the 2pf of EMT for classically Weyl theory

$$\Pi_{\mu\nu,\alpha\beta} = \int \mathrm{d}^4 x \langle T_{\mu\nu}(x)\, T_{\alpha\beta}(0)\rangle \mathrm{e}^{ip\cdot x} \quad \propto \quad 2$$



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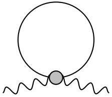


In dimreg Noether ids. are OK for divergences

$$p^\mu \Pi_{\mu\nu,\alpha\beta}|_{\mathrm{div}} = \Pi^\mu{}_{\mu,\alpha\beta}|_{\mathrm{div}} = 0$$

But the finite part is anomalous

$$p^\mu \Pi_{\mu\nu,\alpha\beta}|_{\mathrm{finite}} = 0 \qquad\qquad \Pi^\mu{}_{\mu,\alpha\beta}|_{\mathrm{finite}} \neq 0$$

history summarized by Duff in Class. Quant. Grav. 11 (1994) 1387-1404

What does it mean?

Given that Γ is the generator of $T_{\mu\nu}$ correlators

$$\langle T^{\mu}{}_{\mu} \rangle \equiv g^{\mu\nu} \langle T_{\mu\nu} \rangle \sim g_{\mu\nu} \frac{\delta \Gamma}{\delta g_{\mu\nu}}$$

Then schematically

$$\Pi^{\mu}{}_{\mu,\alpha\beta} \sim \frac{\delta}{\delta g_{\alpha\beta}} \, \langle T^{\mu}{}_{\mu} \rangle|_{\text{flat space}}$$

Which implies

$$\langle T^{\mu}{}_{\mu} \rangle \sim \text{stuff involving } g_{\mu\nu}$$

What is this “stuff”?

Imagine $g_{\mu\nu}$ is the only source and no interactions for simplicity

$$\langle T^{\mu}{}_{\mu}\rangle \sim \delta_\sigma \Gamma[\mathrm{e}^{2\sigma} g_{\mu\nu}]\Big|_{\sigma=0} \qquad \Longrightarrow \qquad A_\sigma = \int 2\sigma g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\Gamma \equiv \Delta_\sigma \Gamma = \int \sigma \langle T^{\mu}{}_{\mu}\rangle$$

Due to the Abelian nature of the rescalings we have Wess-Zumino consistency

$$[\Delta_\sigma, \Delta_{\sigma'}]\Gamma = 0$$

Which implies an “integrability” condition

$$\Delta_\sigma A_{\sigma'} = \Delta_{\sigma'} A_\sigma$$

Plan

By now we understand that the trace anomaly has this structure:

$$\langle T^{\mu}{}_{\mu}\rangle = \text{geometry of sources} + \text{renormalization group} + \text{EOMs}$$

Plan:

- renormalization group $\Longrightarrow$ β-functions
- EOMs $\Longrightarrow$ Flavor current and B-functions
- geometry $\Longrightarrow$ Ambient space and nonlocal actions

Local RG analysis of the anomaly

$$\langle T^{\mu}{}_{\mu} \rangle = \text{geometry} + \boxed{\text{renormalization group} + \text{flavor and EOMs}}$$

Renormalization with local couplings

Consistency requires **local couplings** to source observables $\mu \to \mathrm{e}^{-\sigma(x)}\mu$ **Shore 80s**

$$S \supset \int \mathrm{d}^d x \sqrt{g}\, \lambda^I(x)\, \mathcal{O}_I + \int \mathrm{d}^d x \sqrt{g}\, \mathcal{J}_i(x) \varphi^i$$

Currents source the expectation values

$$\langle T^{\mu\nu} \rangle = -\frac{2}{\sqrt{g}} \frac{\delta \Gamma}{\delta g_{\mu\nu}} \qquad \langle \mathcal{O}_i \rangle = -\frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta \lambda^i}$$

Notice field renormalization from 1PF

$$\langle \varphi^I \rangle = -\frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta \mathcal{J}_I}$$

Local rg interpretation

Local scale transformation on the geometrical sources

$$\Delta_\sigma^W \simeq \int \Big\{ 2\sigma g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \sigma \mathcal{J}^i \big((d - \Delta_\varphi)\delta^i_j - \gamma^i{}_j \big) \frac{\delta}{\delta \mathcal{J}^i} + \cdots \Big\}$$

Local scale transformation caused by the rg beta functions $\beta^i = \frac{\mathrm{d}}{\mathrm{d}\log\mu}\lambda^i$

$$\Delta_\sigma^\beta = -\int \sigma \beta^i \frac{\delta}{\delta \lambda^i}$$

The anomaly of Γ is expressed **Osborn 80s – 90s**

$$\Delta_\sigma^W \Gamma = \Delta_\sigma^\beta \Gamma + A_\sigma \qquad\qquad A_\sigma \supset \{\partial_\mu \lambda^i, R, \mathcal{J}^i, \cdots\}$$

Wess-Zumino consistency

Rewrite as a full transformation

$$\Delta_\sigma \Gamma = (\Delta_\sigma^W - \Delta_\sigma^\beta)\Gamma = A_\sigma$$

For Wess-Zumino's consistency and Abelian transformations

$$[\Delta_\sigma, \Delta_{\sigma'}]\Gamma = 0$$

Consistency condition for the anomaly

$$(\Delta_\sigma^W - \Delta_\sigma^\beta)A_{\sigma'} - (\sigma \leftrightarrow \sigma') = 0$$

Topological charge in $d = 4$

Parametrize the anomaly using Euler density E_4 **Osborn, Jack-Osborn**

$$A_\sigma \supset \int \mathrm{d}^4x\sqrt{g}\,\sigma\Big\{a(\lambda)\,E_4+b(\lambda)\,W^2+\cdots+\tilde{\chi}(\lambda)_{IJ}\Box\lambda^I\Box\lambda^J+\chi(\lambda)_{IJ}R^{\mu\nu}\partial_\mu\lambda^I\partial^\mu\lambda^J+\cdots\Big\}$$

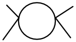
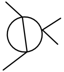
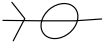
Integrability implies for $B^I = \beta^I + (\gamma_a \cdot \lambda)^I$ **Herren-Thomsen 2021**

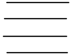
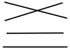
$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}a \sim \chi_{IJ}B^I B^J \qquad \Longleftrightarrow \qquad \partial_I a \sim \chi_{IJ}B^J$$

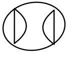
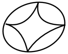
Not obvious to establish positivity/irreversibility of χ_{IJ} because it is 3pf

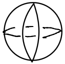
$$\chi_{IJ} \sim \langle \mathcal{O}_I \mathcal{O}_J T\rangle \qquad \text{instead of} \qquad \tilde{\chi}_{IJ} \to |x|^4\langle \mathcal{O}_I(x)\mathcal{O}_J(0)\rangle \geq 0$$

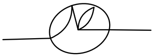
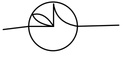
but perturbatively $\chi_{IJ} > 0$

$\beta = \beta_1$  $+ \beta_2$  $+ \gamma_2$  + 3 loops

$G =$  $+ g_2$  + 1 loop

$A = a_{1,1}$  $+ a_{2,1}$  $+ a_{2,2}$ 

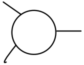
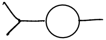
$+ a_{2,3}$  + 4 loops

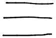
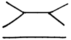
$\gamma^{(a)} = S_{6,1} \Big\{$  $-$  $\Big\}$ + 4 terms

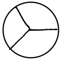
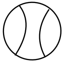
+ 6 loops

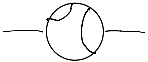
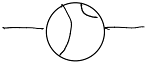
Example: $\lambda_I \mathcal{O}_I = \frac{1}{3!}\lambda_{ijk}\varphi_i\varphi_j\varphi_k$ in $d = 6$

Gracey-Jack-Poole 2015

$\beta = \beta_1$  $+\ \gamma'_1$  + 2 loops

$G =$  $+\ g_1$  + 1 loop

$A = a_{1,1}$  $+\ a_{1,2}$  + 4 loops

$\gamma^{(a)} = S_{3,1} \{$  $-$  $\}$ + 4 loops

$$S_{3,1} \Big|_{\overline{MS}} = -\frac{137}{10368}$$

Conformal geometry and ambient space

$$\langle T^{\mu}{}_{\mu}\rangle = \text{geometry} + \text{renormalization group} + \text{flavor and EOMs}$$

Crucial actors are special conformal tensors

$$A_\sigma \supset \int \mathrm{d}^4x\sqrt{g}\,\sigma\Big\{a\,E_4 + b\,W^2 + \cdots\Big\}$$

Objective: use an "ambient space" to find all integrable geometrical terms

Lightcone embedding in flat space

Move from $\mathbb{R}^d$ to $\mathbb{R}^{d+2}$ on the lightcone $Y^2 = 0$

$$Y^A = (Y^\mu, Y^+, Y^-) \qquad \eta_{AB} Y^A Y^B = 0 \qquad Y^A \sim \lambda Y^A$$

Spacetime x^μ embedding on the lightcone Y^A

$$\begin{aligned} x^\mu &\to Y^A = (Y^\mu, Y^+, Y^-) = Y^+(x^\mu, 1, -x^2) \\ Y^A &\to x^\mu = \frac{Y^\mu}{Y^+} \end{aligned}$$

Embedding Lorentz $Y^A \to Y'^A = \Lambda^A{}_B Y^B$ generates conformal on spacetime

$$(Y'^+)^2 \eta_{\mu\nu} dx'^\mu dx'^\nu = (Y^+)^2 \eta_{\mu\nu} dx^\mu dx^\nu$$

Fefferman-Graham ambient space

Use Cartesian coordinates, $X^2 = 2t^2\rho$, $t = X^+$

$$Y^A \to X^A = (X^\mu, X^{d+1}, X^{d+2}) \stackrel{*}{=} t\Big(x^\mu, \frac{1+2\rho - x^2}{2}, \frac{1-2\rho + x^2}{2}\Big)$$

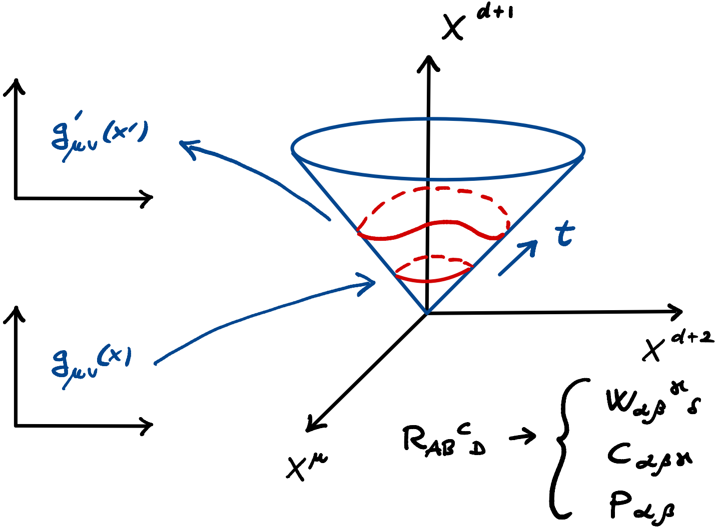
The flat embedding metric

$$\tilde{\eta} = \eta_{AB} dx^A dx^B \stackrel{*}{=} 2\rho dt^2 + 2t dt d\rho + t^2 \eta_{\mu\nu} dx^\mu dx^\nu$$

In curved space: FG metric with $R_{AB} = 0$, $\mathcal{L}_{t\partial_t}\tilde{g} = 2\tilde{g}$ and $h_{\mu\nu}(x, \rho = 0) = g_{\mu\nu}$

$$\tilde{g} = \tilde{g}_{AB} dx^A dx^B \stackrel{*}{=} 2\rho dt^2 + 2t dt d\rho + t^2 h_{\mu\nu}(x,\rho) dx^\mu dx^\nu$$

Ambient Space in a nutshell



Relation with AdS/CFT

Coordinates $\rho = -\frac{r^2}{2}$ and $t = \frac{s}{r}$

$$\tilde{g} = -ds^2 + s^2\Big(\frac{dr^2 + h_{\mu\nu}(x,r)dx^\mu dx^\nu}{r^2}\Big)$$

Asymptotically (in r) local (in s) AdS space **Parisini-Skenderis-Withers**

Fixed s: approaching lightcone with hyperobolas. Geometrical fundation of AdS/CFT

Note: If you are familiar with AdS/CFT you can replace $\rho \sim r$ *otherwise don't worry*

PBH diffeomorphisms

A diffeomorphism of the ambient **Imbimbo et al.**

$$\delta_\zeta \tilde{g}_{AB} = \mathcal{L}_\zeta \tilde{g}_{AB} = \zeta^C \partial_C \tilde{g}_{AB} + \tilde{g}_{AC} \partial_B \zeta^C + \tilde{g}_{BC} \partial_A \zeta^C$$

Preserves the form of the ambient metric (changing $h_{\mu\nu} \to h'_{\mu\nu}$) **Penrose, Brown-Henneaux**

$$\zeta^t = t\,\sigma(x) \qquad \zeta^\rho = -2\rho\,\sigma(x) \qquad \zeta^\mu = \xi^\mu(x) + \cdots$$

It generates Diff $\ltimes$ Weyl on spacetime

$$\delta_\zeta h_{\mu\nu}|_{\rho=0} = \delta_\zeta g_{\mu\nu} = \delta_{\sigma,\xi} g_{\mu\nu} = 2\sigma g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\mu \xi_\nu$$

Ricci-flatness determines $h_{\mu\nu}$

Expand in ρ to solve $\tilde{R}_{AB} = 0$ iteratively

$$h_{\mu\nu}(x,\rho) = g_{\mu\nu}(x) + \rho {h^{(1)}}_{\mu\nu} + \frac{1}{2}\rho^2 {h^{(2)}}_{\mu\nu} + \cdots$$

The coefficients find obstructions in even d

Fefferman-Graham

$$\begin{aligned}
{h^{(1)}}_{\mu\nu} &= +\frac{2}{d-2}\left(R_{\mu\nu} - \frac{R}{2(d-1)} g_{\mu\nu}\right) = 2K_{\mu\nu} \\
{h^{(2)}}_{\mu\nu} &= -\frac{2}{d-4} B_{\mu\nu} + 2K_{\mu\sigma}{K^{\sigma}}_{\nu} \\
{h^{(3)}}_{\mu\nu} &= +\frac{2}{d-6} B'_{\mu\nu} + \cdots
\end{aligned}$$

Circumvent obstructions by "analitically" continuing d.
Poles disappear in computations, however...

Ambient Laplacian(s)

Scalar Laplacian of the embedding

$$-\Box_{\tilde{g}}\Phi = -\frac{1}{t^2}\Box_h\Phi - \frac{2}{t}\partial_t\partial_\rho\Phi - \frac{1}{2t}\partial_t\Phi - \frac{d-2}{t^2}\partial_\rho\Phi + \frac{\rho}{t^2}{h'}_\mu{}^\mu\partial_\rho\Phi$$

Consider a scaling scalar field $\Phi = t^{\Delta_\varphi}\varphi(x)$ and project to Yamabe

$$-\Box_{\tilde{g}}\Phi|_{\rho=0} = t^{\Delta_\varphi-2}\Big(-\Box_g - \frac{d-2}{4(d-1)}R\Big)\varphi$$

We can construct a family of powers of conformal GJMS Laplacians **Graham et al.**

$$P_{2n}\varphi(x) \equiv t^{-\frac{2n+d}{2}}(-\Box_{\tilde{g}})^n(t^{\frac{2n-d}{2}}\varphi)|_{\rho=0}$$

Conformal Laplacians and conformal invariants

There are derivative $\Delta_{2n} \sim \partial^{2n}$ and constant parts (exist in $d \geq 2n$)

$$P_{2n}\varphi(x) = \Delta_{2n} + \frac{d-2n}{2} Q_{2n}$$

Constant part transforms nicely for $g_{\mu\nu} = \mathrm{e}^{2\sigma}\overline{g}_{\mu\nu}$ in $d = 2n$ **Branson et al.**

$$\sqrt{g}\, Q_d = \sqrt{\bar{g}}(\bar{Q}_d + \bar{\Delta}_d \sigma)$$

Conformal invariants are also easy to obtain

$$\tilde{\mathcal{R}}_i^n \longrightarrow \mathcal{W}_i \qquad \text{e.g.} \qquad \tilde{R}^2_{ABCD} \longrightarrow W^2_{\mu\nu\alpha\beta}$$

The integrable terms

We now have all "integrable" geometrical terms such that

$$[\delta_\sigma, \delta_{\sigma'}]\, \mathcal{W}_i = 0 \qquad\qquad [\delta_\sigma, \delta_{\sigma'}]\, Q_d = 0$$

We deduce

$$\langle T^\mu{}_\mu \rangle = a\, Q_d + \sum_i b_i\, \mathcal{W}_i$$

$E_d \sim Q_d$ plus total derivatives (finite renormalizations). Compare Cardy's conjecture:

$$\langle T^\mu{}_\mu \rangle = a\, E_d + \sum_i b_i\, \mathcal{W}_i$$

Integration of the anomaly and nonlocal actions

Example: $d = 4$

Integrability at a RG fixed point implies

$$\langle T \rangle = a\, Q_4 + b\, W^2$$

The Q-curvature is

$$Q_4 = \frac{1}{6} R^2 - \frac{1}{2} R_{\mu\nu} R^{\mu\nu} - \frac{1}{6} \Box R$$

It transforms $Q_4 \rightsquigarrow Q_4 + \Delta_4 \sigma$ with the Paneitz operator

$$\Delta_4 = \Box^2 + 2 \nabla^\mu \left(R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R \right) \nabla^\nu$$

In the real world

Covariant computations (eg. heat-kernel methods)

$$a = -\frac{1}{360}N_\phi - \frac{31}{180}N_A - \frac{11}{360}N_\psi$$
$$b = \frac{1}{120}N_\phi + \frac{1}{10}N_A - \frac{1}{20}N_\psi$$

for N_ϕ, N_A and N_ψ the number of scalars, vectors, Dirac spinors

Physical meaning

$\langle T \rangle \neq 0$ implies that the conformal factor σ in $g_{\mu\nu} \sim \mathrm{e}^{2\sigma}\overline{g}_{\mu\nu}$ is dynamical with effective action

$$\Gamma[\sigma, \overline{g}_{\mu\nu}] \supset \int \mathrm{d}^4x \sqrt{\overline{g}}\sigma \left(a\, Q_4[\overline{g}] + b\, W^2[\overline{g}] + \frac{a}{2}\overline{\Delta}_4\sigma \right)$$

"Integration" means to find an action $\Gamma[g_{\mu\nu}]$ such that

$$\langle T \rangle \sim \frac{\delta}{\delta\sigma}\Gamma[\mathrm{e}^{2\sigma}g]$$

General strategy: think at $\overline{g}_{\mu\nu}$ as "picked" by some gauge fixing **Barvinsky-Wachowsky**

$$\chi[\overline{g}] = 0 \qquad \Longrightarrow \qquad \sigma = \Sigma_\chi[\overline{g}]$$

Nonlocal actions

Simplest choice

$$\chi[\bar{g}] = Q_4[\bar{g}] = 0 \qquad \Longrightarrow \qquad \sigma = \frac{1}{\Delta_4} Q_4$$

Results in

$$\Gamma[g_{\mu\nu}] = \Gamma_{\mathrm{conf}}[g] + \Gamma_{\mathrm{an,loc}}[g] - \int \mathrm{d}^4x \sqrt{g} \left(\frac{a}{2} Q_d + b\, W^2 \right) \frac{1}{\Delta_4} Q_4$$

- Applications: cosmology, black-holes (effective scalar mode coupled at any scale)
- Problems: ambiguities in the choice of χ, higher point structure of correlators

Problems

Ambiguities! E.g. alternative choice

$$Q_2[\overline{g}] = 0 \qquad \Longrightarrow \qquad \sigma = -\log\left(1 + \frac{1}{\Box_g - \frac{R}{6}}\,\frac{R}{6}\right)$$

Results in a different nonlocal action, more akin to RG of curvature terms

$$\Gamma[g_{\mu\nu}] \supset \int \mathrm{d}^4x\,\sqrt{g}\,\mathcal{R}\log(\Box_g)\mathcal{R}$$

But neither choice seems to be consistently reproducing 4PFs **Corianò et al.**

$$\langle TT\mathcal{J}\mathcal{J}\rangle$$

Conclusions

- Rich geometrical structure of conformal anomaly linked to scale
  - Implications for stat. mech. (reversibility, gradient...)
  - Implications for SM (B-functions, RG and flavor...)
- Problems to be solved – have to do with logarithms/obstructions?
- Implications for Quantum Gravity?

Thank you