From data to the analytic S-matrix: A Bootstrap Fit of the $\pi\pi \to \pi\pi$ amplitude

Based on arXiv: 2410.23333

with K. Häring, N. Su

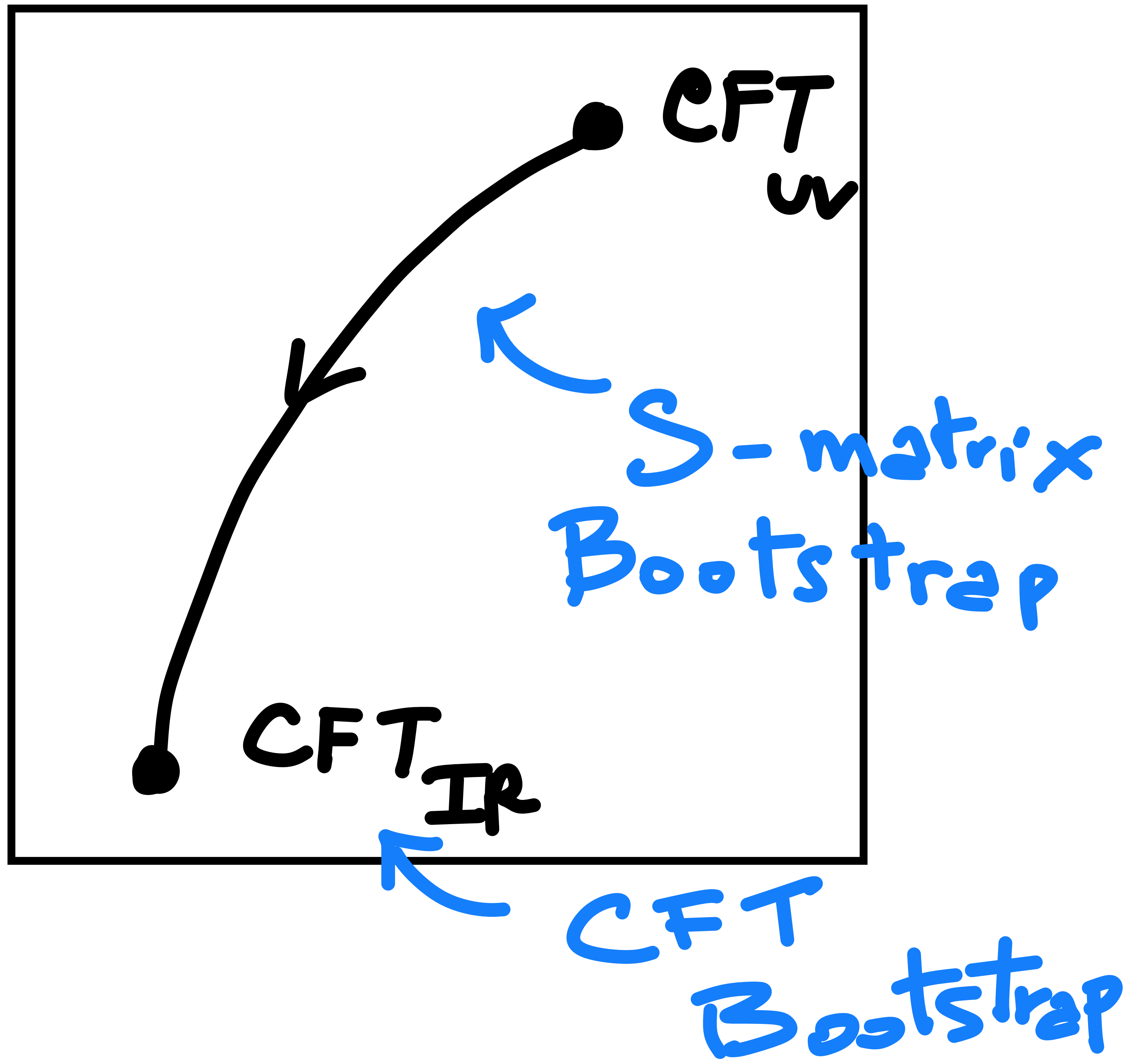
23/04/25

CITY
ST GEORGE'S
UNIVERSITY OF LONDON

Bootstrap

Map out the space of QFTs

↳ Map out the space of Observables



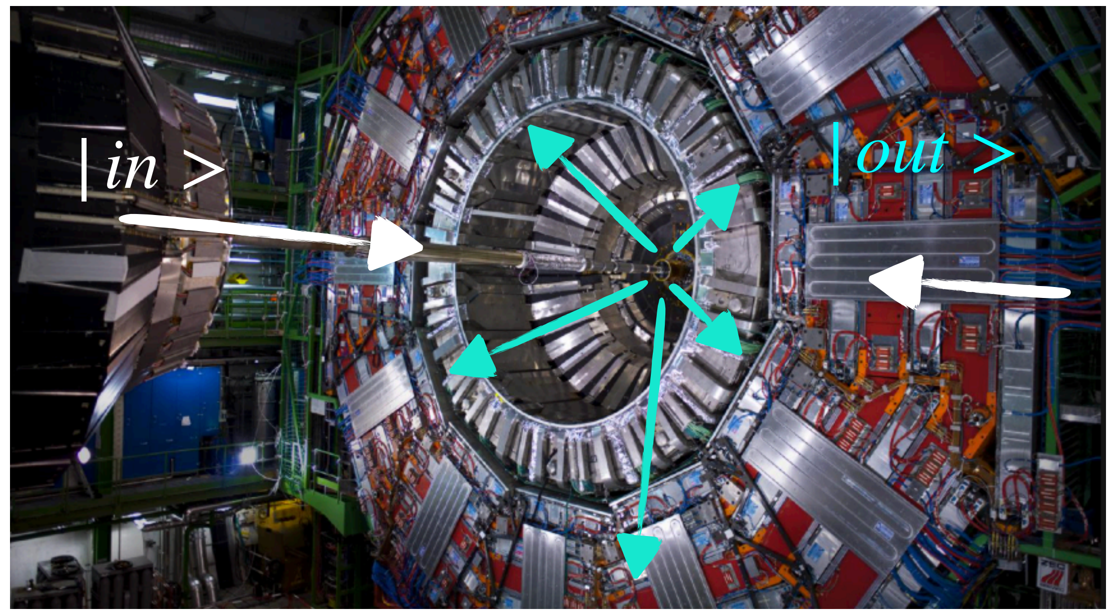
CFT correlators

↳ critical exponents, OPE coefficients, ...

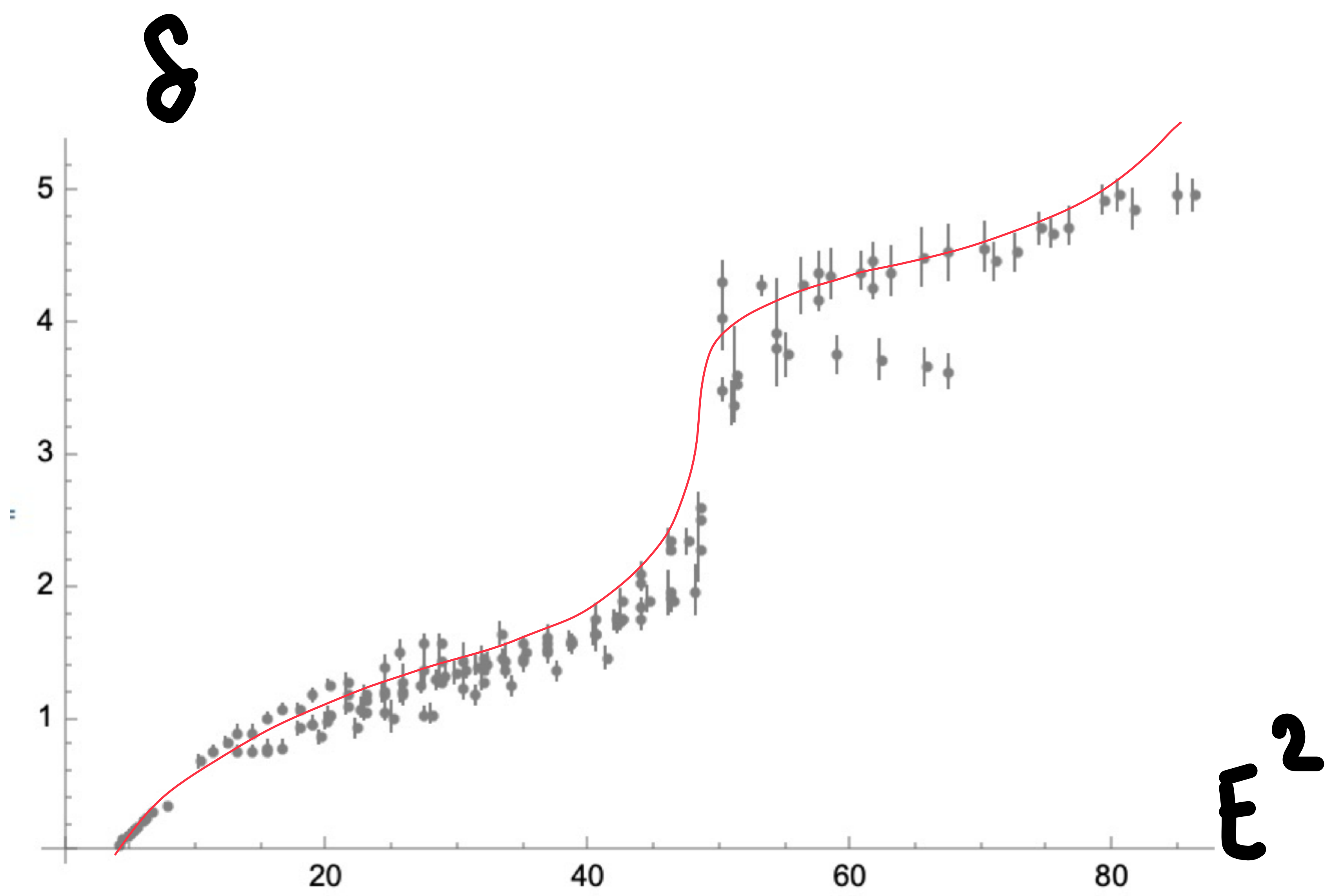
QFT S-matrix/correlators

↳ spectrum, couplings, ...

this talk



General Idea (I)



MODEL

Spectrum

Couplings

low Energy constants

...

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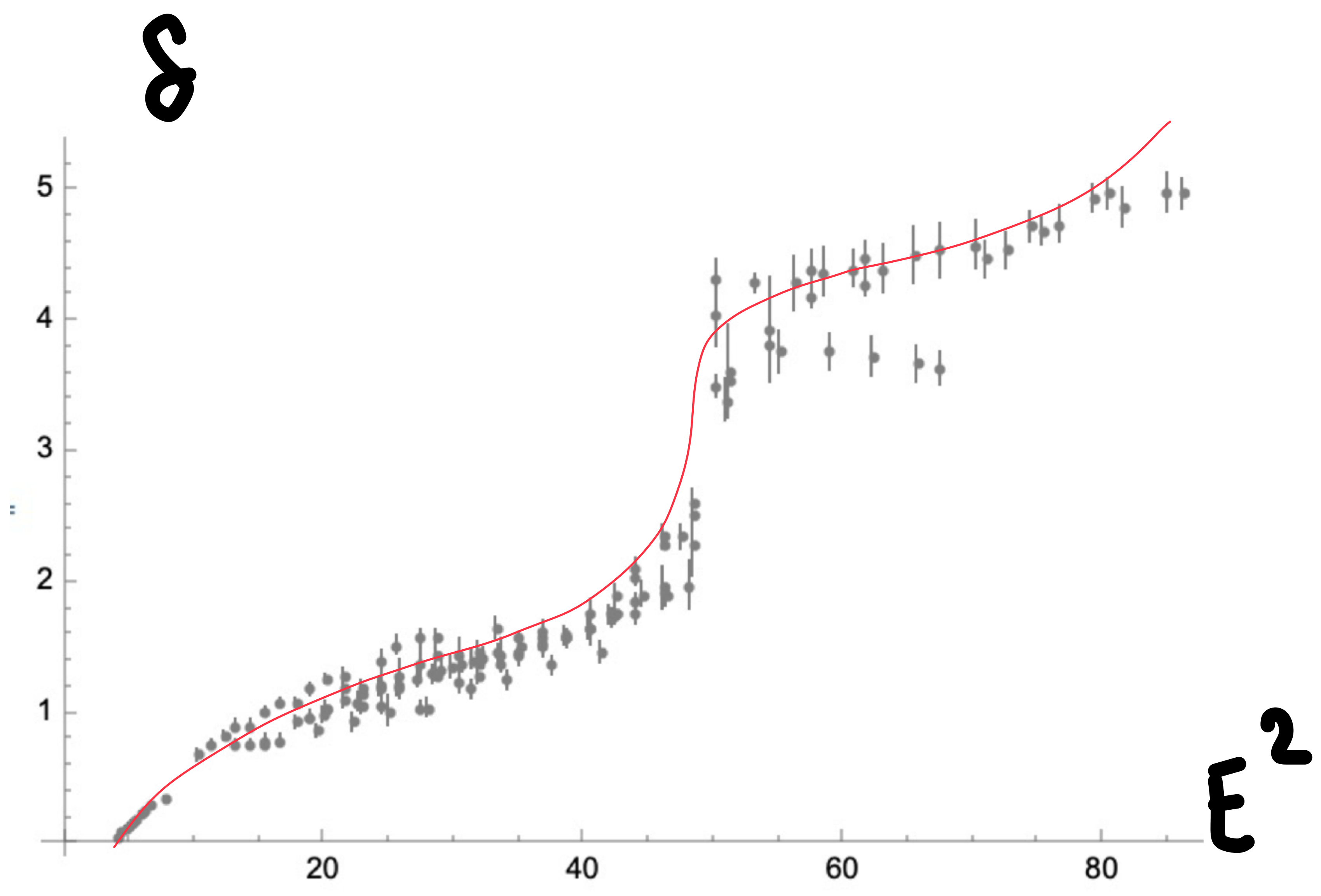

MODEL →

Spectrum
Couplings
low Energy constants
...

1) Isobar Models
(background + resonance)

2) Unitarized χ-PT approaches

3) Roy Equations [Caprini, Colangelo, Gasser, Leutwiler]

4) Can we use the Bootstrap to construct a better Model?

General Idea (II)

The ideal model is the QFT scattering amplitude

1) Causality $\longleftrightarrow$ Analyticity
2) Crossing Symmetry
3) Unitarity

Plan of the talk is to explain how to use the Bootstrap to construct such a model.

(Roy Equations close to satisfy all properties, but limitations: low energy, low spins)

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1) lightest particles in the QCD spectrum $\qquad m_\pi \simeq 140\ \text{MeV}$

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(3 pions)

moderately weakly coupled @ low energy

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4) Approximate Isospin Symmetry

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$$\tau_{\pi^0} \simeq \cancel{8.4 \times 10^{-17}\ \mathrm{sec}} \longrightarrow \infty$$

4) ~~Approximate~~ Isospin Symmetry

$$\frac{m_{\pi^{+-}} - m_{\pi^0}}{m_\pi} \simeq \cancel{0.03} = 0$$

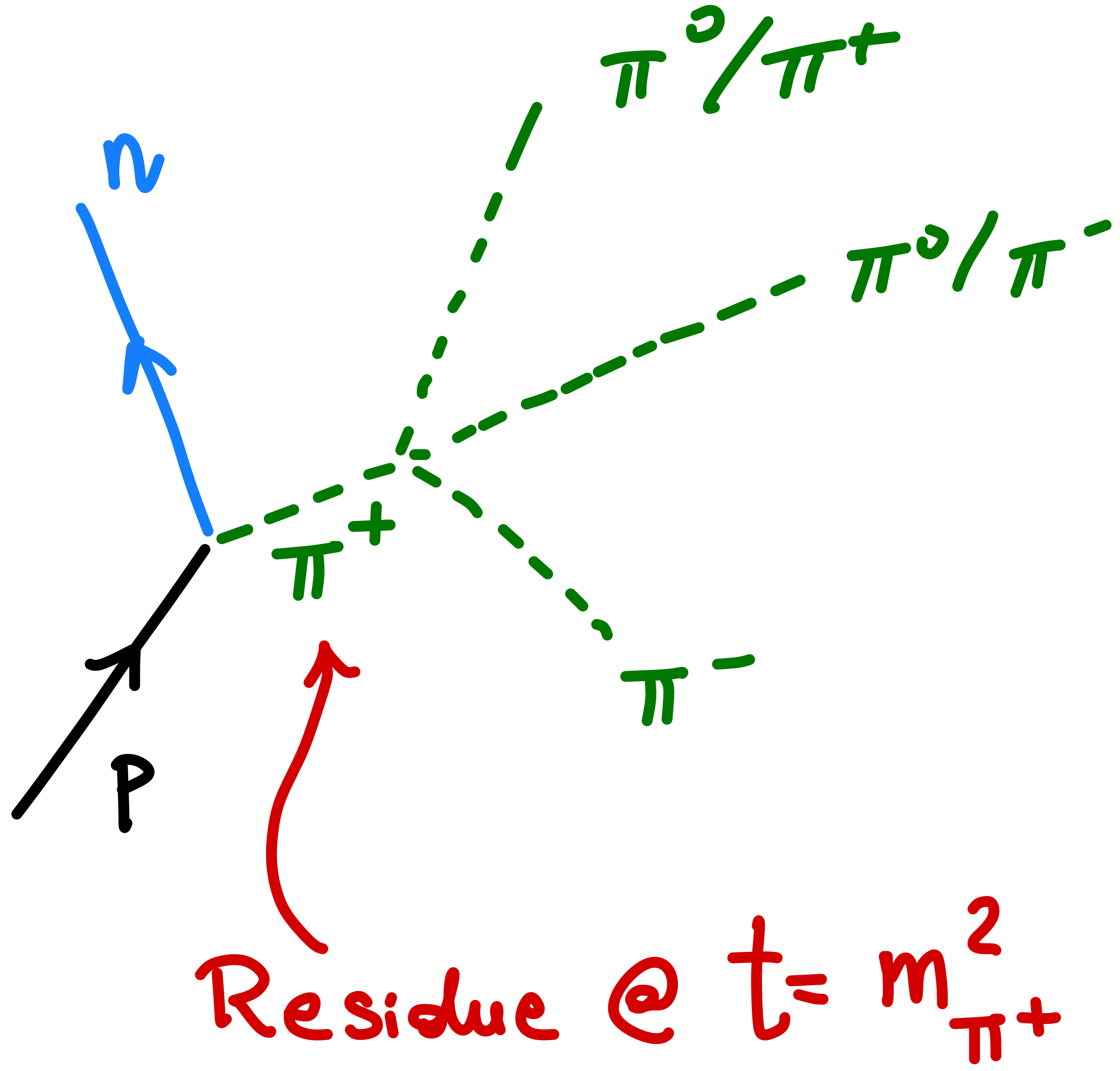
Pion S-matrix $\simeq$ S-matrix of SO(3) scalars

$\pi\pi$ scattering Trivia

In Experiments we have to deal with 3), 4)

- CERN-MUNICH Coll. 60's - 70's

  $\pi N \to \pi\pi N$



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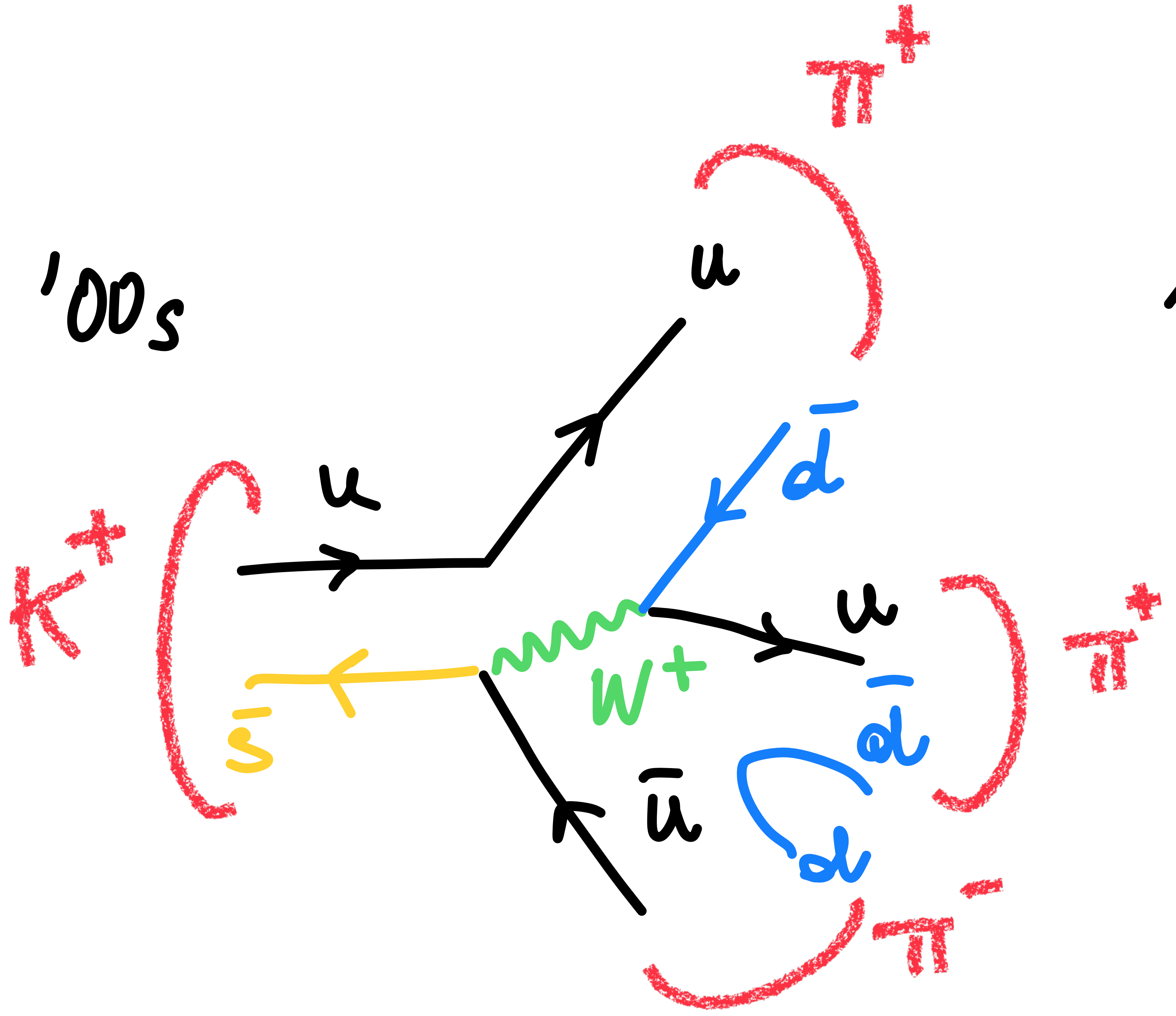
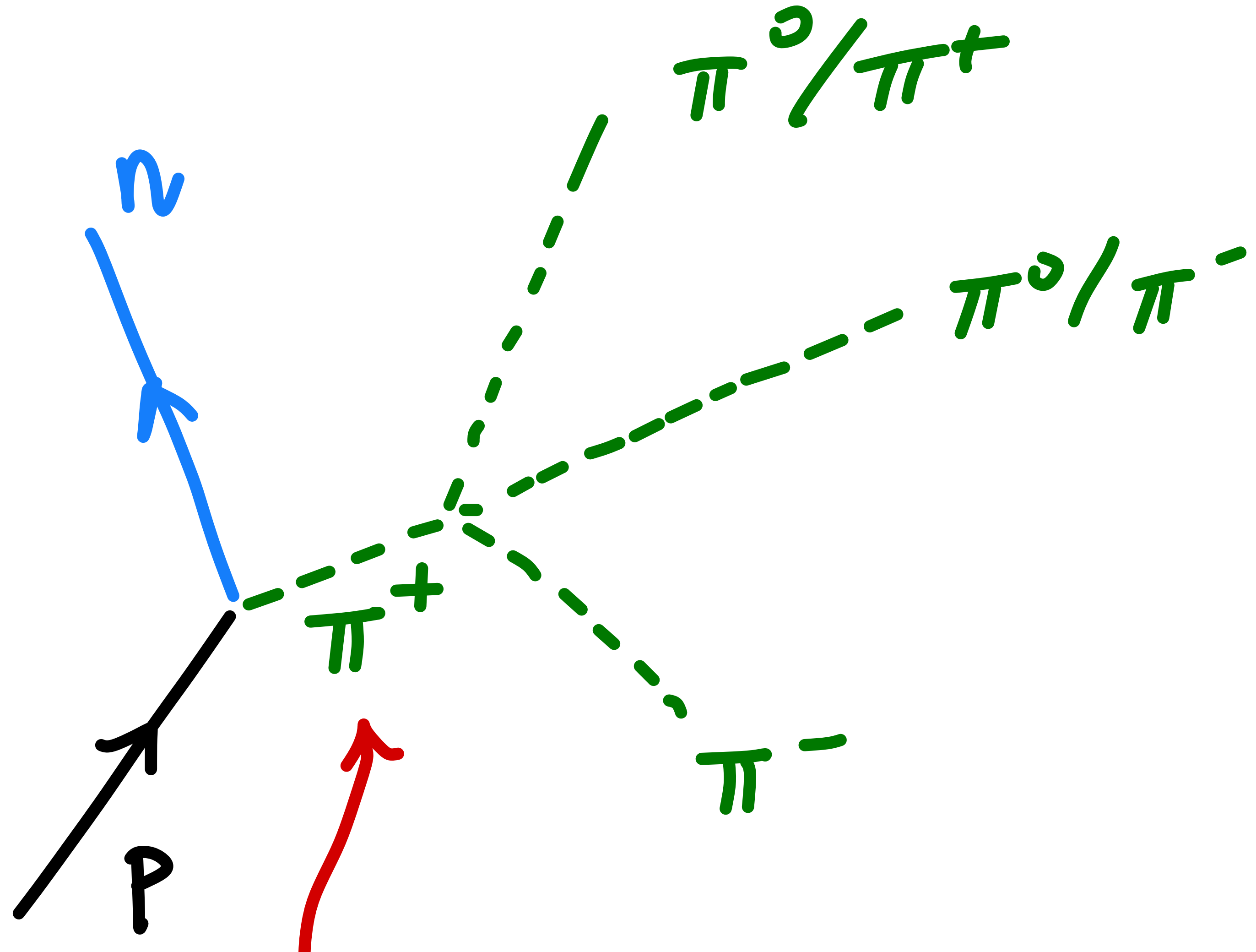
  $K \to \pi\pi\pi$

  (80 MeV)





Residue @ $t = m_{\pi^+}^2$

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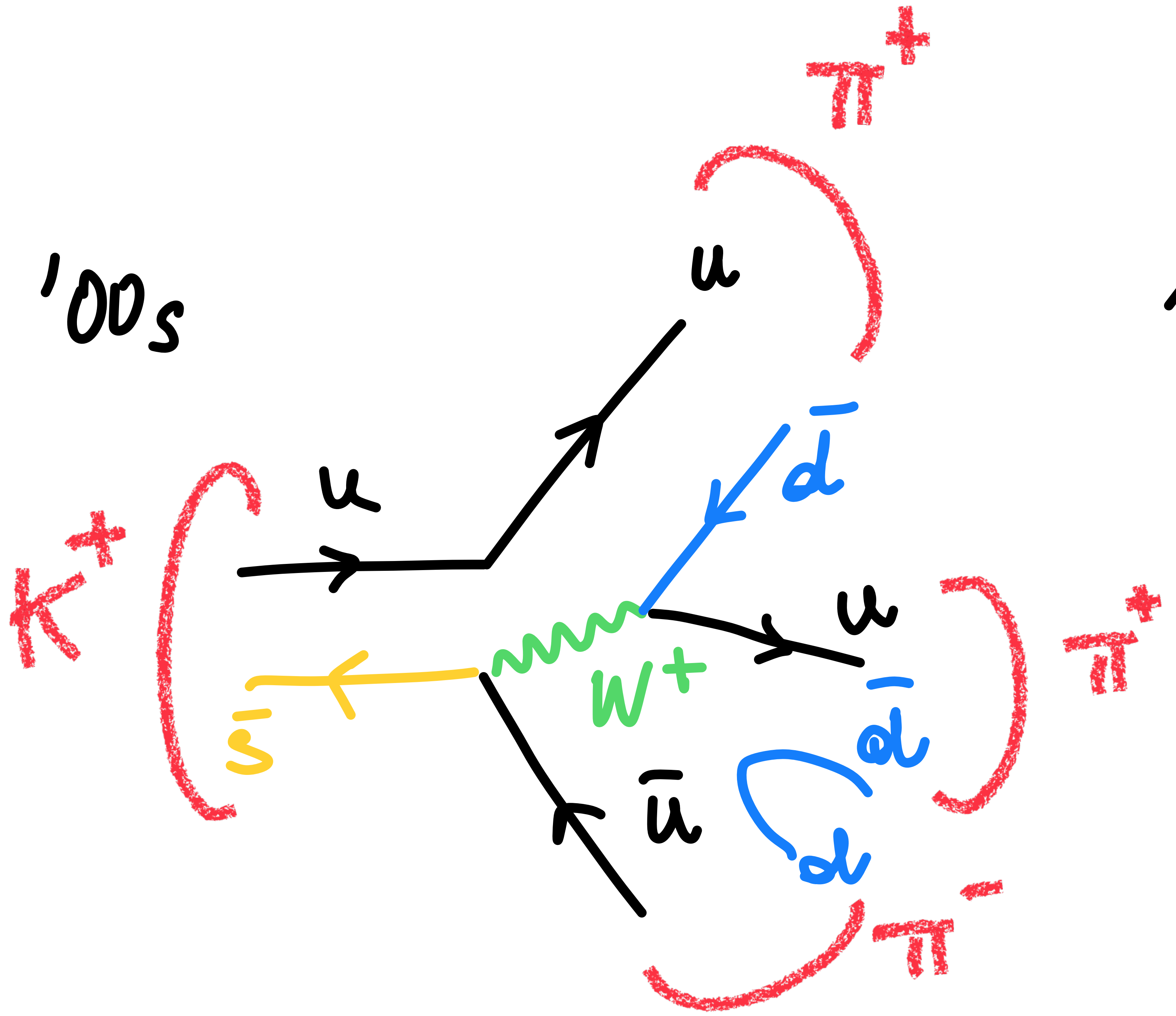
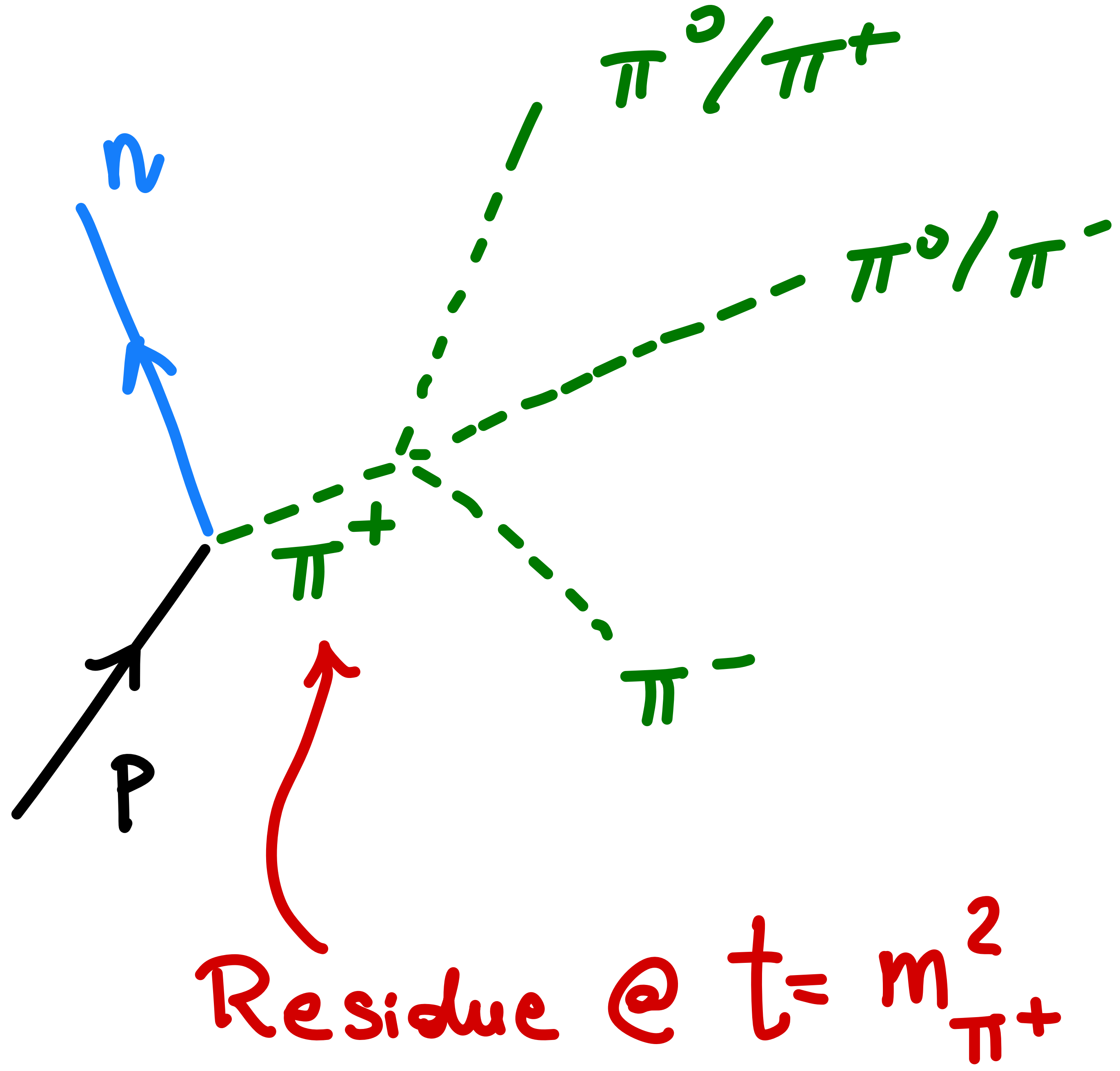
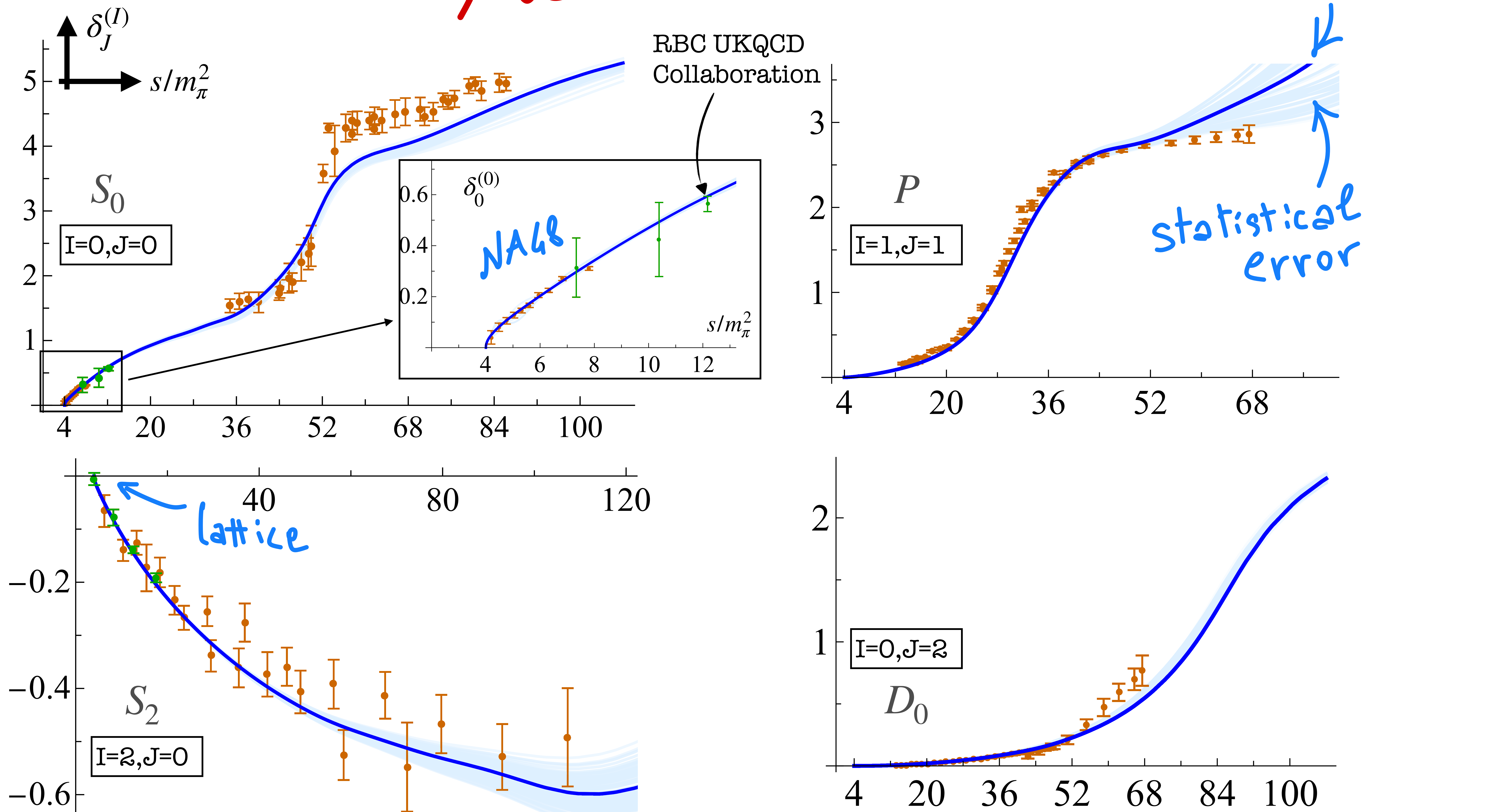




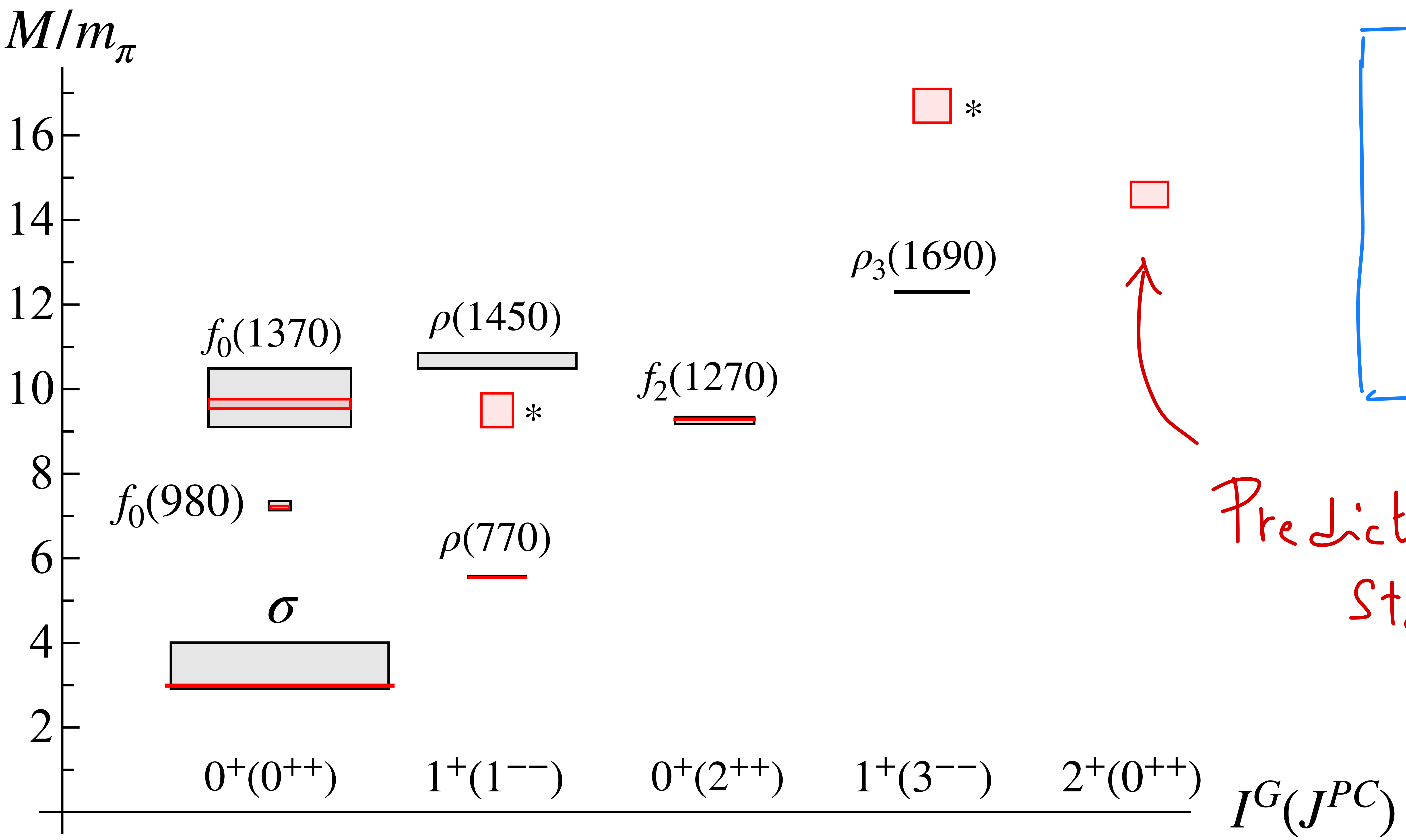
Lattice QCD (the cleanest): few data points, but important!

$\left(S_J^{(I)} = \eta_J^{(I)} e^{2i\delta_J^{(I)}} \right)$

Main Result



Spectrum with G=+1

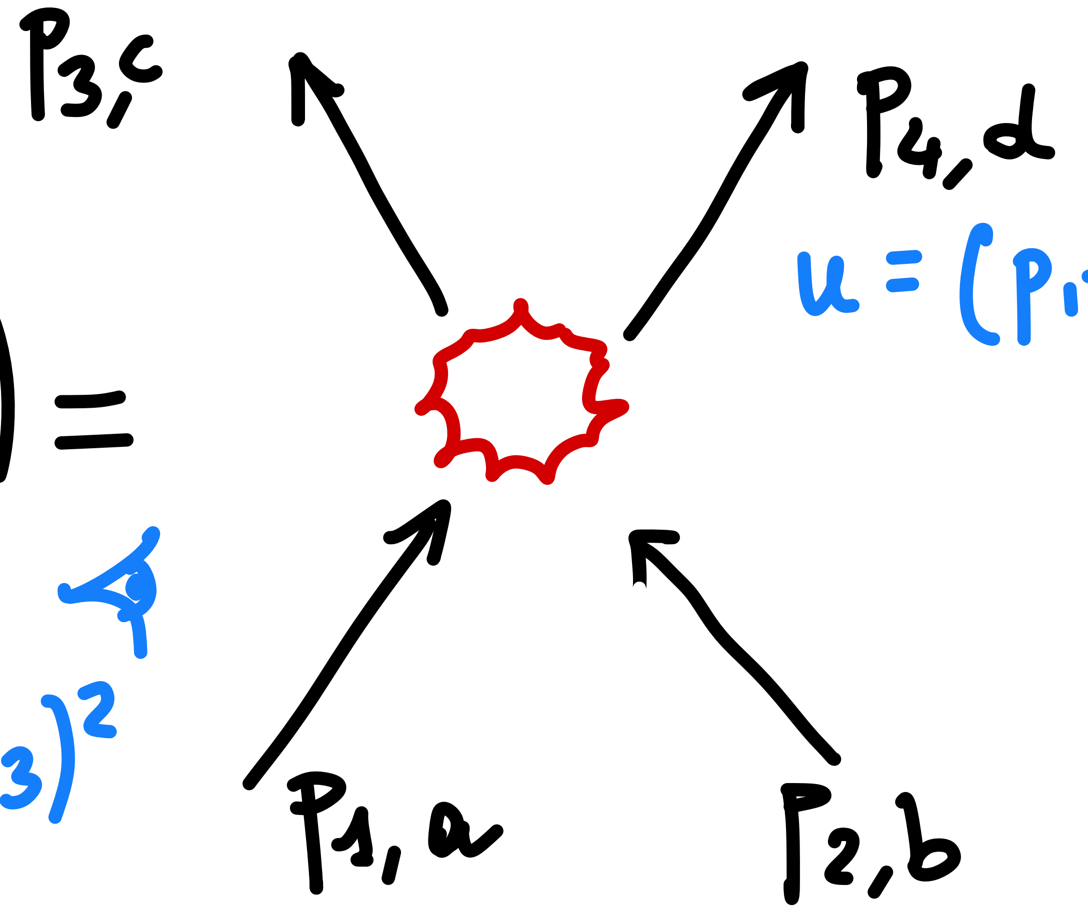


$$G = C(-1)^I$$
$$G(\pi) = -1$$
$$G(\pi\pi) = +1$$

Predict a new State !

The ρ- Expansion

$$\mathcal{M}_{ab}^{cd}(s,t,u) =$$



$$u = (p_1 - p_4)^2 = 4m_\pi^2 - s - t$$

$$=$$

$$t = (p_1 - p_3)^2$$

$$s = (p_1 + p_2)^2$$

<u>O(N) Bootstrap:</u>

Cordova, Vieira, He, Kruczenski (2 D) '18

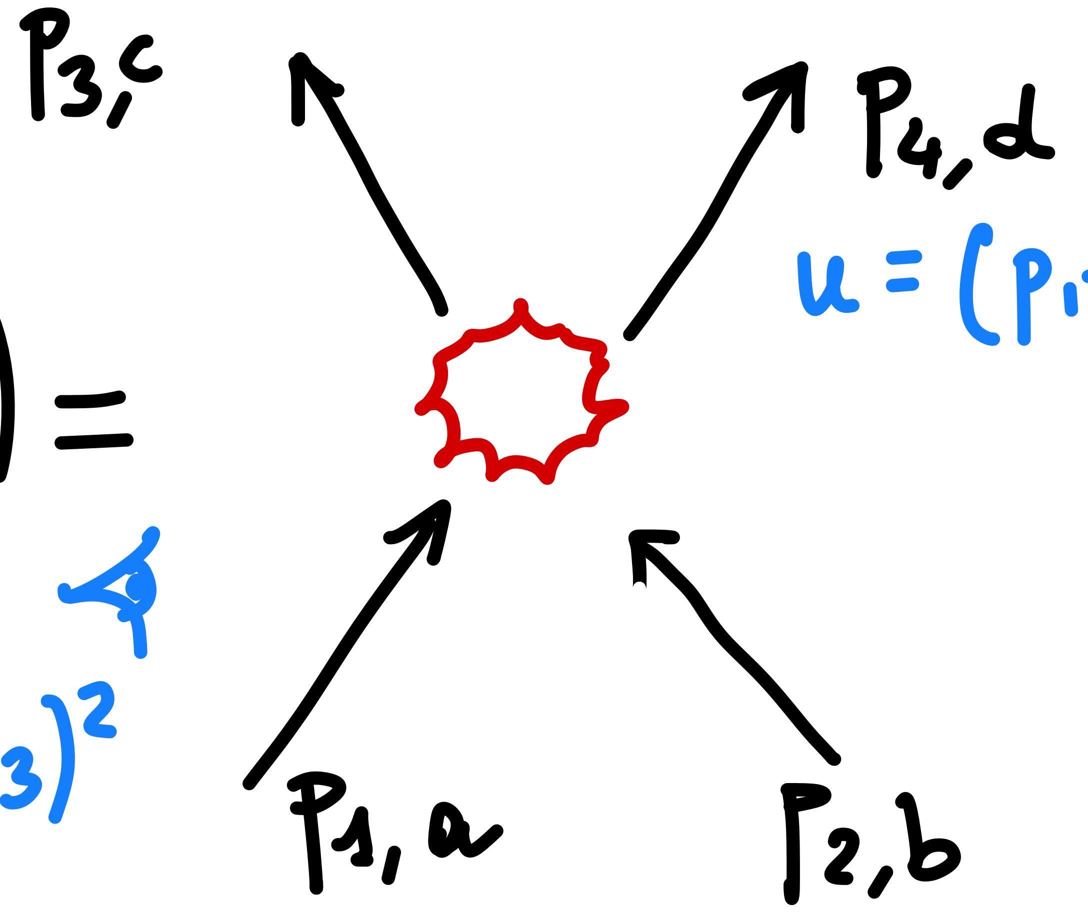
AG, Vieira, Penedones (4D + pions) '18

Sinha et al (pions) '19, '20

Elias-Miró, AG, Gumus (O(2) + Higgs) '23

He, Kruczenski (pions + form factors) '23, '24

The ρ- Expansion



$$\mathcal{M}_{ab}^{cd}(s,t,u) = \ldots = \delta_{ab}\delta^{cd} A_s + \delta_a^c \delta_b^d A_t + \delta_a^d \delta_b^c A_u$$

$t = (p_1 - p_3)^2$

$s = (p_1 + p_2)^2$

$u = (p_1 - p_4)^2 = 4m_\pi^2 - s - t$

where $A_s = A(s|t,u)$

Building blocks are t-u symmetric

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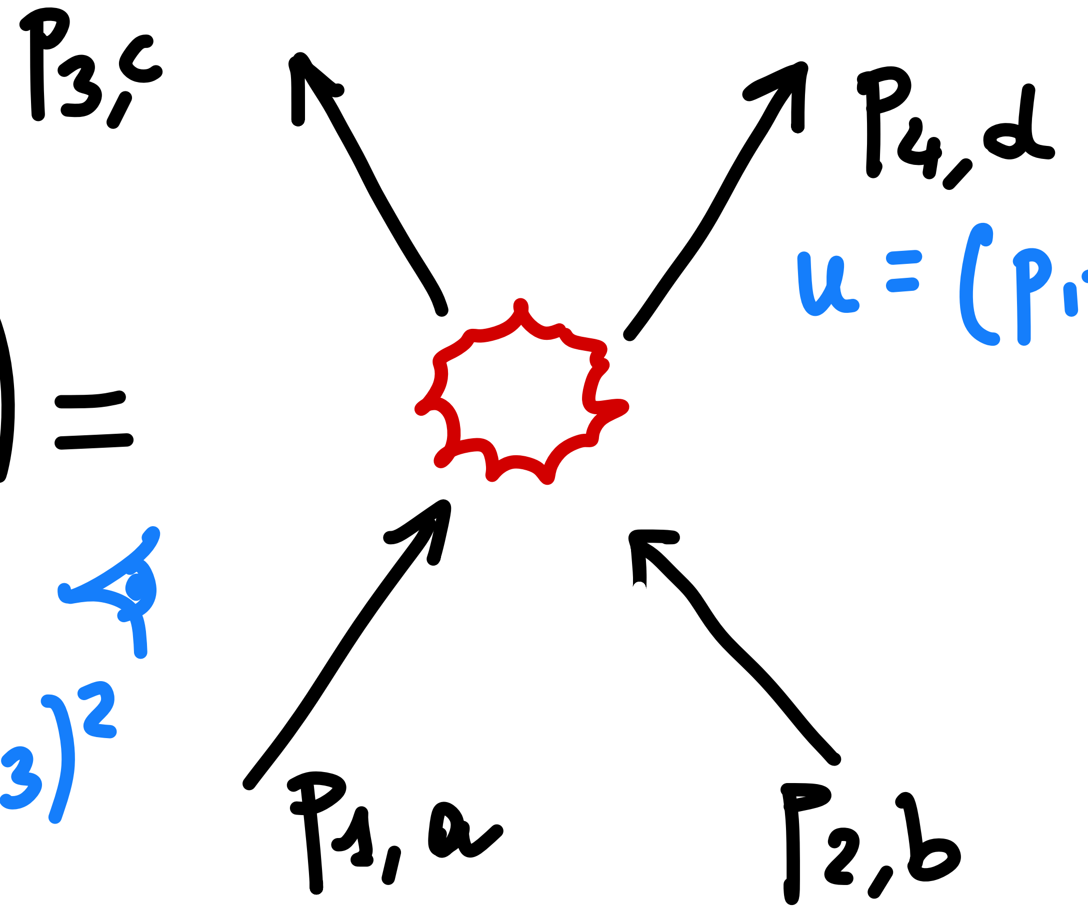

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$(m_\pi = 1)$

$$= \delta_{ab}\delta^{cd} A_s + \delta_a^c \delta_b^d A_t + \delta_a^d \delta_b^c A_u$$

where $A_s = A(s|t,u)$

Building blocks are t-u symmetric

$$A_s = \sum_{n+m \leq N} \alpha_{n,m}\, \rho_s^n \left(\rho_t^m + \rho_u^m\right) + \sum_{n+m \leq N} \beta_{(n,m)}\, \rho_t^n \rho_u^m$$

Crossing Symmetry ✓

Paulos, Penedones, van Rees, Toledo, Vieira '17

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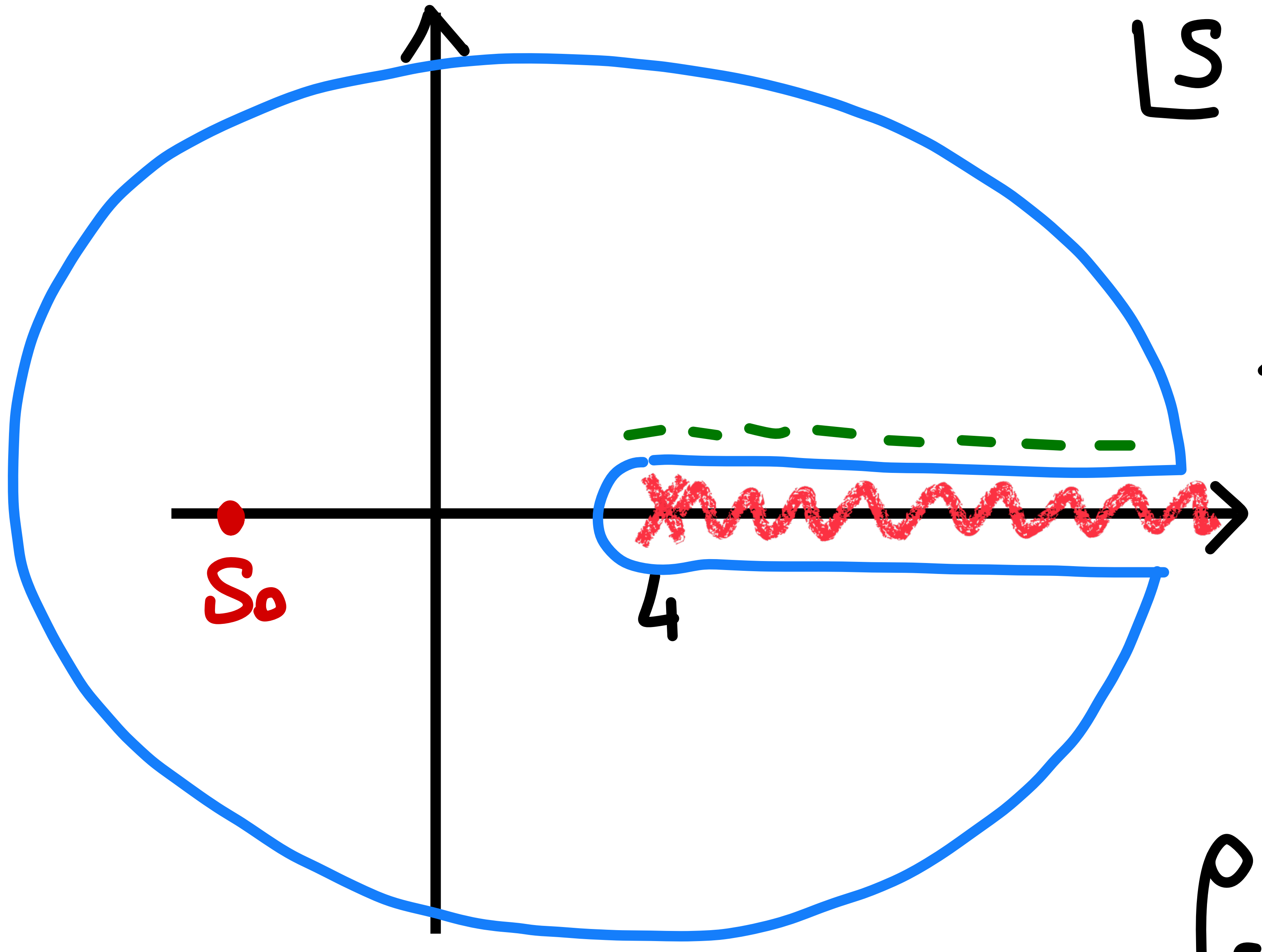

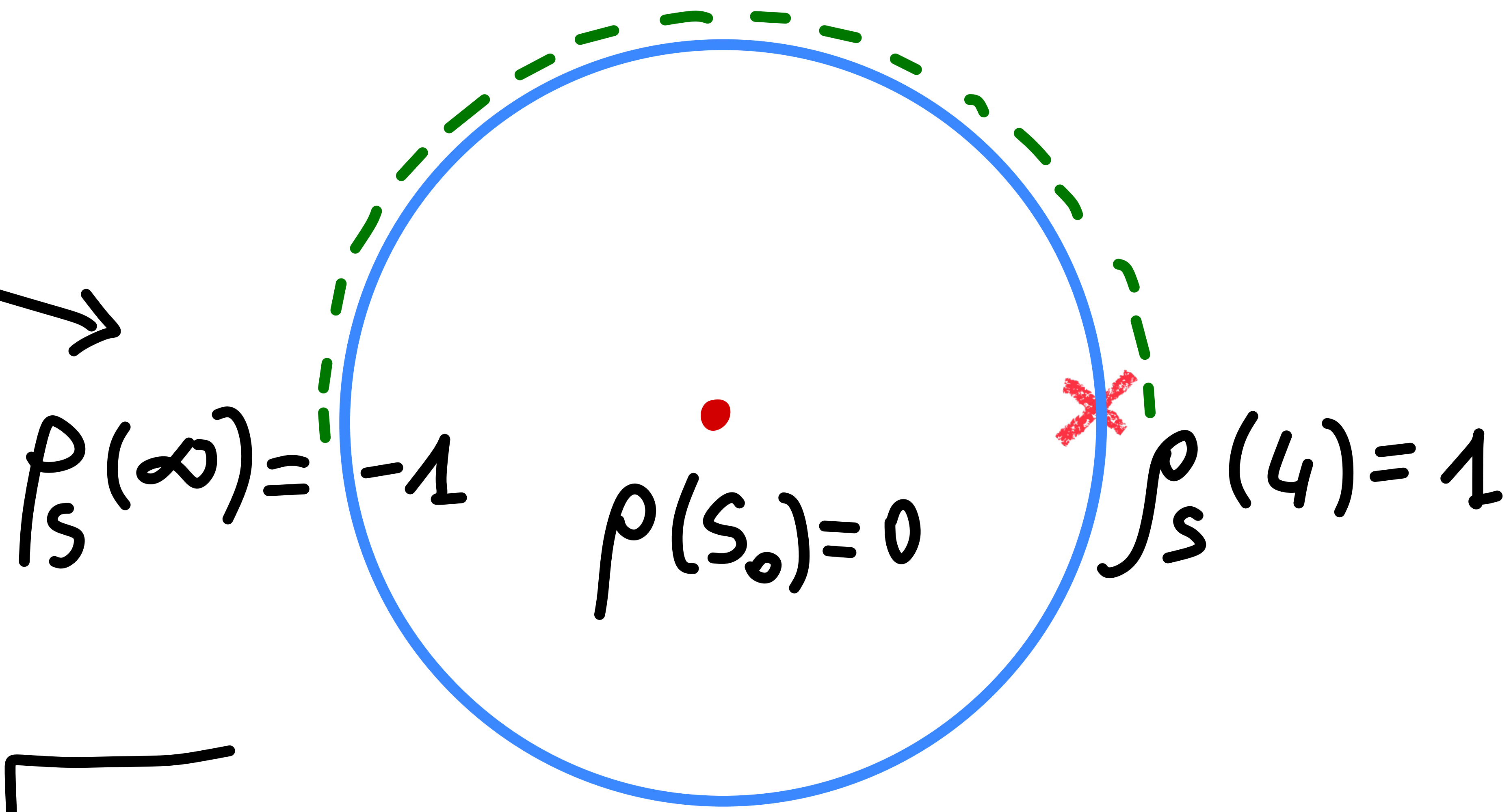

$$\rho_s = \frac{\sqrt{4-s_0}-\sqrt{4-s}}{\sqrt{4-s_0}+\sqrt{4-s}}$$

Analyticity ✓

Unitarity

1) Change of basis to flavour Irreps

$$(I=1) \times (I=1) = \underset{\text{Singlet}}{0} + \underset{\text{Antisymmetric}}{1} + \underset{\text{Symmetric traceless}}{2}$$

$$\left\{ A^{(0)} = \overbrace{N_f}^{=3} A_s + A_t + A_u,\ A^{(1)} = A_t - A_u,\ A^{(2)} = A_t + A_u \right\}$$

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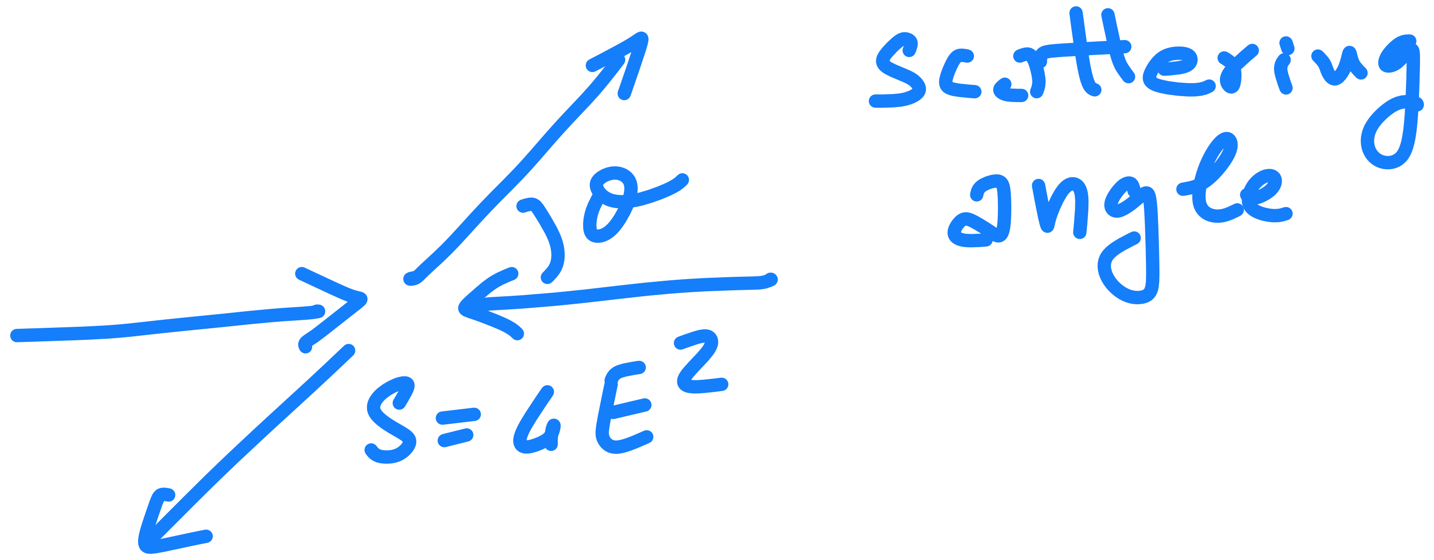
2) Change of basis to Angular Momentum Irreps

$$f_\ell^{(I)}(s) = \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta \, P_\ell(\cos\theta)\, A^{(I)}\!\left(s,\ t = \frac{4-s}{2}(1+\cos\theta),\ u = \frac{4-s}{2}(1+\cos\theta)\right)$$



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linear functions of $\{\alpha_{n,m};\ \beta_{(n,m)}\}$

Unitarity is diagonal

$$S_\ell^{(I)}(s) = 1 + i\sqrt{\frac{s-4}{s}}\, f_\ell^{(I)}(s)$$

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$$\boxed{\text{SDP}}$$

$$|S_\ell(s)|^2 \leq 1 \iff \begin{pmatrix} 1 - \operatorname{Im} S_\ell^{(I)} & \operatorname{Re} S_\ell^{(I)} \\ \operatorname{Re} S_\ell^{(I)} & 1 + \operatorname{Im} S_\ell^{(I)} \end{pmatrix} \succeq 0$$

$\ell = 0, 1, 2, \dots, \infty \; ; \; s \geq 4$

Pion soft theorem

If $m_u = m_d = 0$, then $m_\pi^2 = 0$ and pions are Goldstones

Soft-theorem $\quad \mathcal{M}_{ab}^{cd}(s,t,u) \xrightarrow[p_i \to 0]{} 0$

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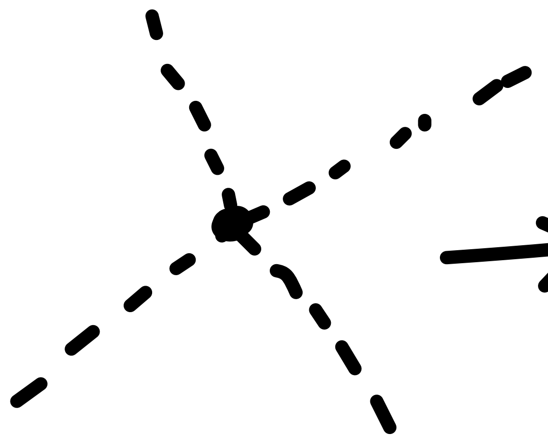
Adler zero

Since $m_\pi^2 \neq 0$, we use χ-EFT (Weinberg)

$$\mathcal{L} = \frac{1}{2}(D_\mu \vec{\pi})^2 - \frac{1}{2} m_\pi^2 \vec{\pi}^2 + \dots$$



$\pi^2 (\partial\pi)^2$

$$\rightarrow \quad A_s = \frac{s - m_\pi^2}{f_\pi^2}$$

pion decay constant

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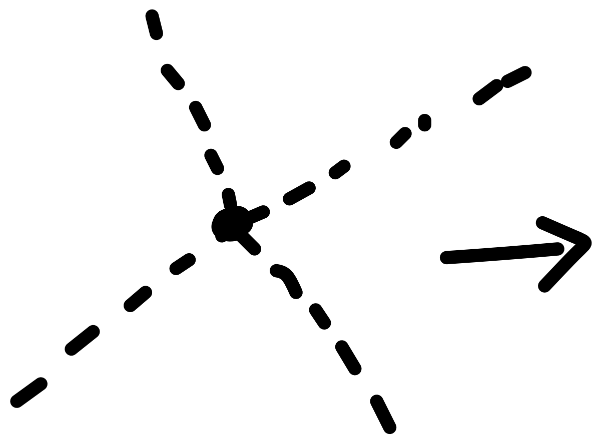

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$$f_0^{(0)} = \frac{2s - m_\pi^2}{32\pi f_\pi^2} \rightarrow f_0^{(0)}\left(\frac{m_\pi^2}{2}\right) = 0$$

$$f_1^{(1)} = \frac{s - 4m_\pi^2}{96\pi f_\pi^2} \rightarrow f_1^{(1)}(4m_\pi^2) = 0$$

$$f_0^{(2)} = \frac{2m_\pi^2 - s}{16\pi f_\pi^2} \rightarrow f_0^{(2)}(2m_\pi^2) = 0$$

Non-perturbatively

$$f_0^{(0)}(z_0) = 0 \;, \quad f_0^{(2)}(z_2) = 0$$

$$0 \leq z_0, z_2 \leq 4 m_\pi^2$$

Spectrum Assumptions

QCD contains many resonances that couple to $\pi\pi$

(Not necessary) We can help the model imposing the existence of some particles

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$$S_\ell^{(I)}(m_R^2) = 0 \quad \text{(pole in the II sheet)}$$

$m_R^2 \in \mathbb{C}$

n	ℓ	I
2	0	0
1	1	1
1	2	0

$m_R^2 = 4 \times 2$ additional parameters

Summary

Crossing + Analyticity

$$A_s = \sum_{n+m\leq N} \alpha_{n,m}\, \rho_s^n \left(\rho_t^m + \rho_u^m\right) + \sum_{n+m\leq N} \beta_{(n,m)}\, \rho_t^n \rho_u^m$$

$N \to \infty$ any analytic function

Unitarity

$$\left|S_\ell^{(I)}(s)\right|^2 \leq 1 \qquad \ell = 0, 1, 2, \ldots, \cancel{\infty}$$

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$L\max \to \infty$

grid $\to \infty$

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Triple extrapolation

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Triple extrapolation Chebyschev grid

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Triple extrapolation Chebyschev grid

$\text{Im}\, A^{(I)}(s, 0 \le t < 4) \ge 0$

positive sum rules for $\ell > L_{max}$

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Unitarity $\left|S_\ell^{(I)}(s)\right|^2 \leq 1$ $\quad \ell = 0, 1, 2, \ldots, \cancel{\infty}$

$\text{Lmax} \to \infty$

$s \geq 4$

grid $\to \infty$

$$\lim_{\text{grid}\to\infty}\ \lim_{\text{Lmax}\to\infty}\ \lim_{N\to\infty}$$

$$\left(\rho_s = \frac{\sqrt{4-s_0} - \sqrt{4-s}}{\sqrt{4-s_0} + \sqrt{4-s}}\right)$$

Triple extrapolation

Chebyschev grid

$\text{Im}\, A^{(I)}(s, 0 \leq t < 4) \geq 0$

positive sum rules for $\ell > \text{Lmax}$

moving s_0

(Wavelets, multi foliation)

The Model (I)

ρ- expansion with parameters $\{\alpha_{n,m};\ \beta_{(n,m)}\}$ – 8 of them

Chiral zeroes $\quad f_0^{(0)}(z_0)=0,\ f_0^{(2)}(z_2)=0$

Spectrum Assumptions $\quad S_e^{(I)}(m_R^2)=0$

In principle, we want to tune α's, β's, z_0, z_2, m_R^2 to fit the data

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ρ-expansion typically $\mathcal{O}(1000)$ parameters $\to$ overfitting

Procedure to choose few of them and fix the rest

EFT

Bootstrap

The Model (II)

$\{\alpha_{n,m}\,;\,\beta_{(n,m)}\}$ have no direct physical interpretation

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$\{ \alpha_{n,m} ; \beta_{(n,m)} \}$ have no direct physical interpretation

Better consider low-energy constants

Universal expansion

$$\mathrm{Re}\, f_\ell^{(I)}(s) = (s-4)^\ell \left[a_\ell^{(I)} + b_\ell^{(I)} (s-4) + \ldots \right]$$

Scattering lengths

effective range

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Scattering lengths

effective range

Choose e.g. $\{a_0^{(0)}, a_1^{(1)}, a_0^{(2)}, \dots\}$, and to fix the rest ...

"Go to the boundary of the allowed region"

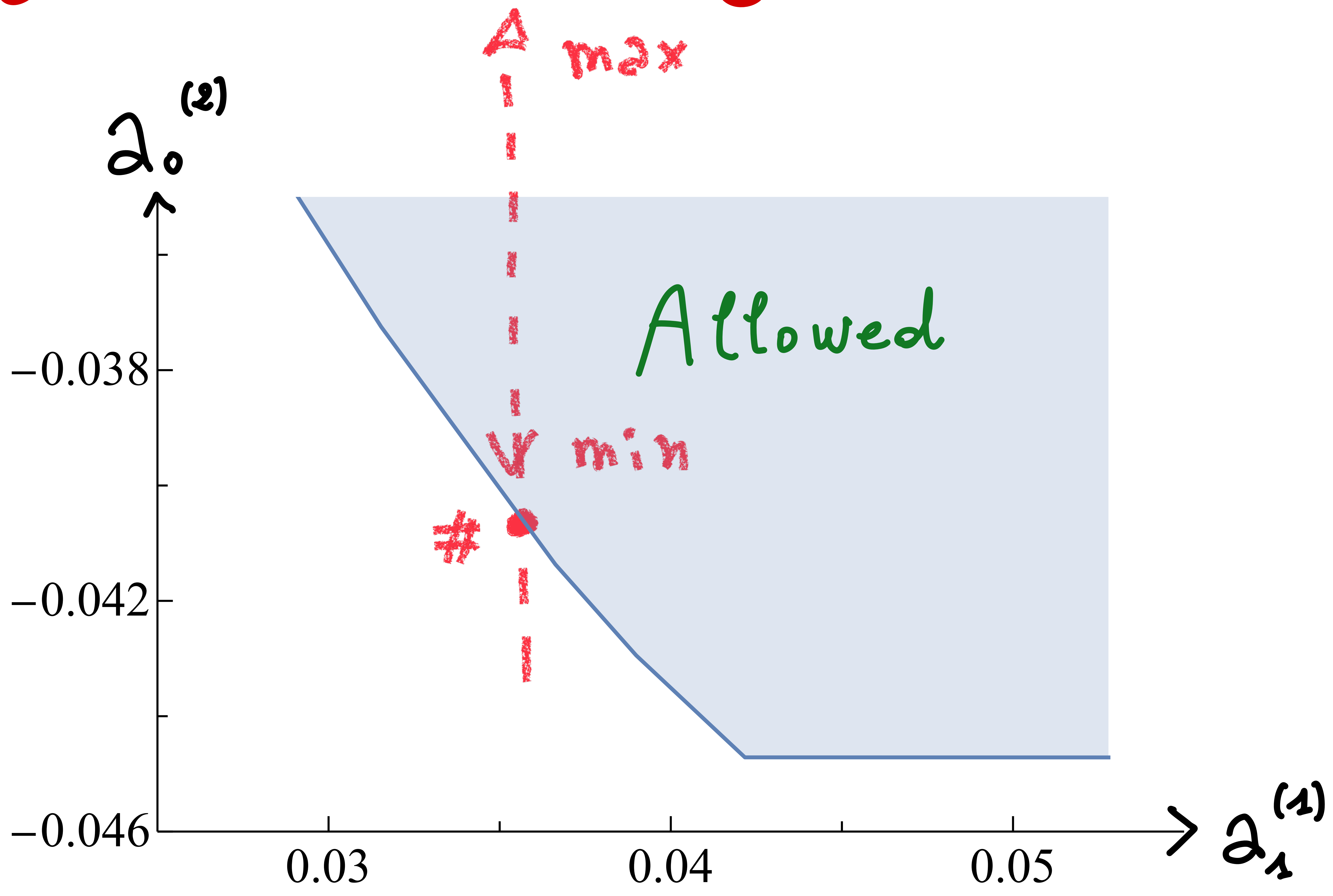
Why the boundary

ρ- expansion
$+ z_0, z_2, m_R^2$
$+$ unitarity

Fix $a_0^{(0)}, a_1^{(1)}$

$\max a_0^{(2)} = \infty$

$\min a_0^{(2)} = \#$



At the max/min all remaining $\{\alpha_{n,m} ; \beta_{(n,m)}\}$ are fixed $\leftarrow$ The amplitude @ the boundary is unique!

Why the boundary

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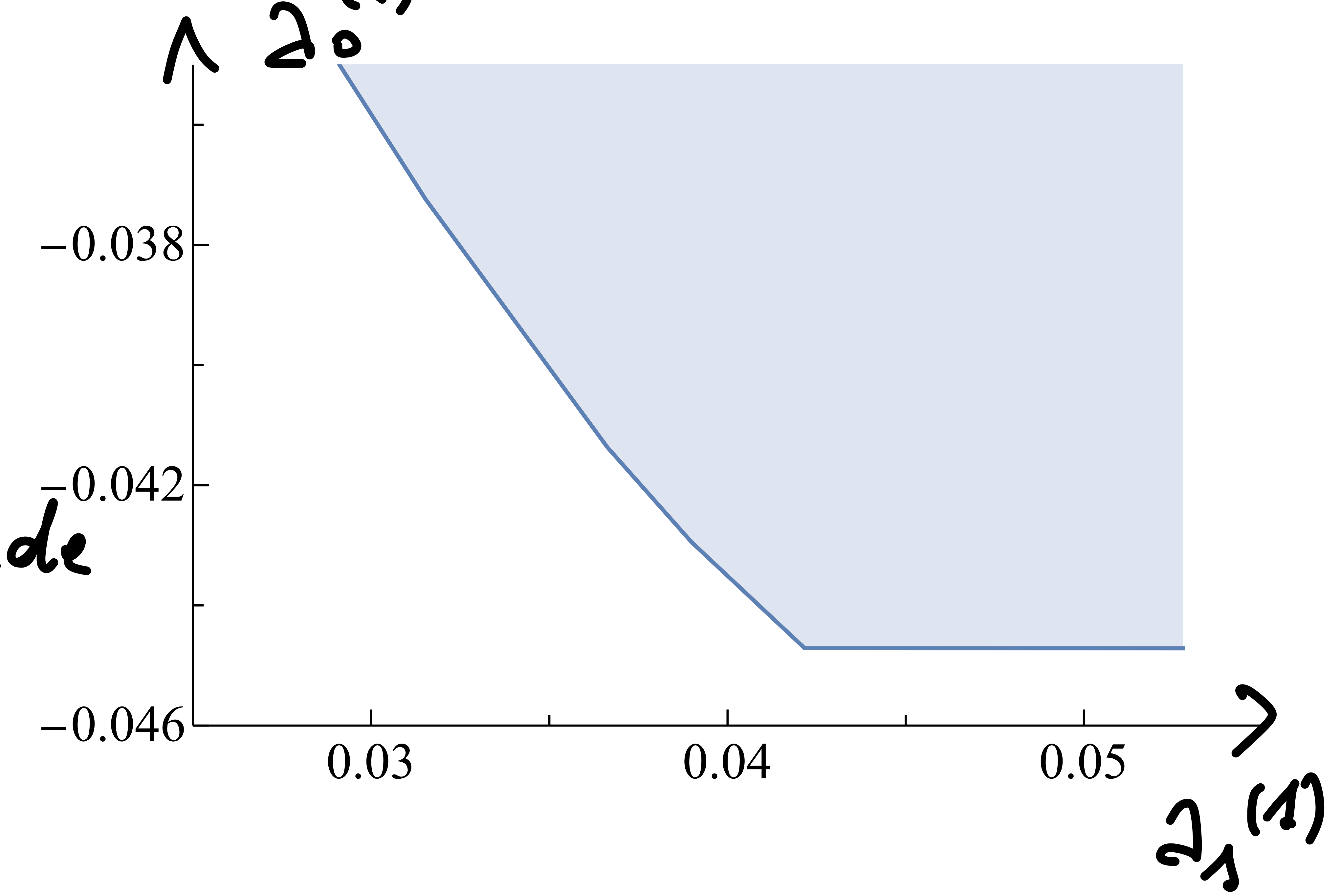
boundary defined by the constraints

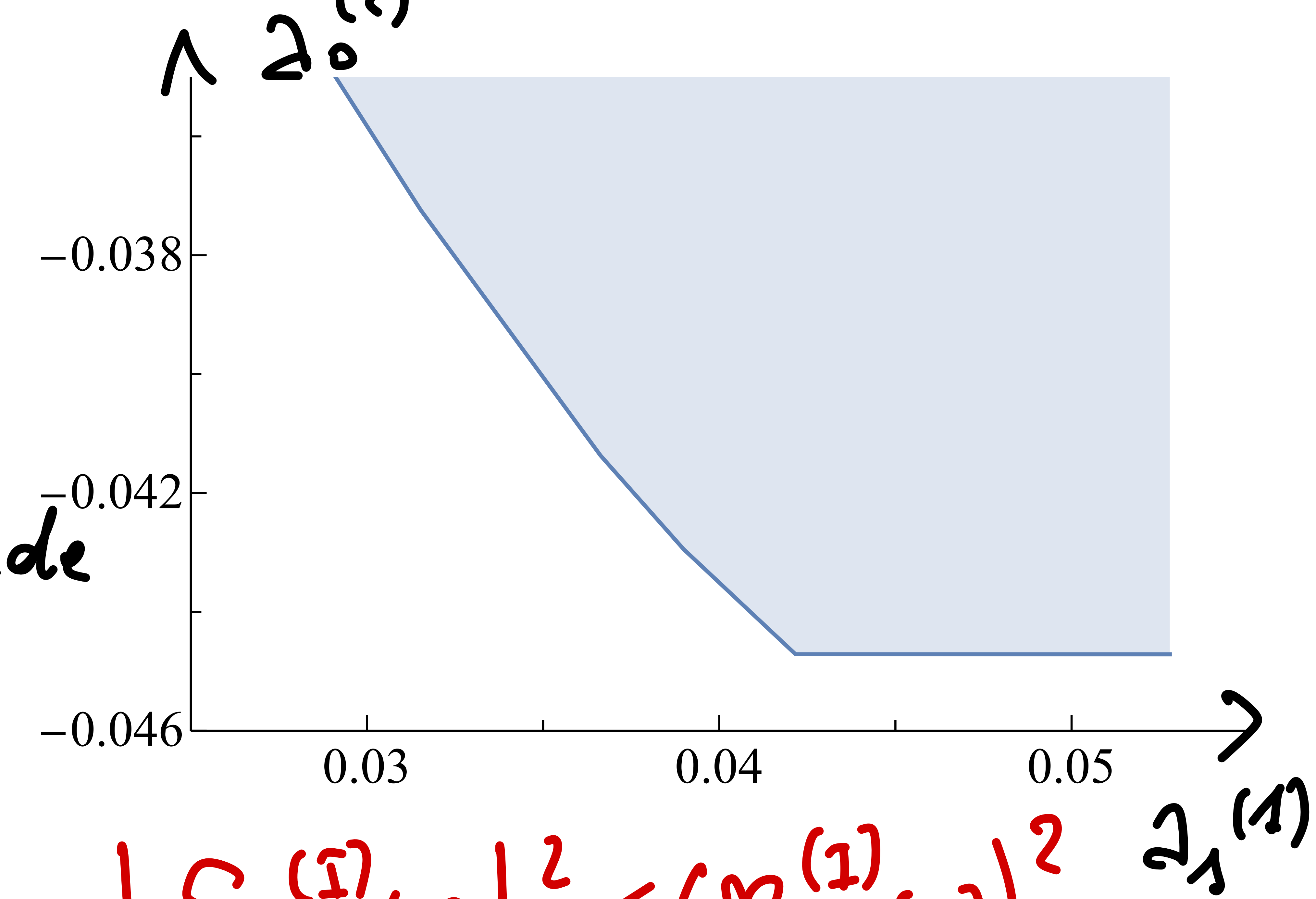


If $|S_\ell^{(I)}(s)|^2 \leq 1 \longrightarrow$ boundary amplitude is elastic

Why the boundary

ρ- expansion
+ z_0, z_2, m_R^2
+ unitarity

} ← boundary defined by the constraints



If $|S_\ell^{(I)}(s)|^2 \leq 1$ ⟶ boundary amplitude is elastic

- We impose the experimental inelasticity $|S_\ell^{(I)}(s)|^2 \leq (\eta_\ell^{(I)}(s))^2$

↪ bndry amplitude has correct production

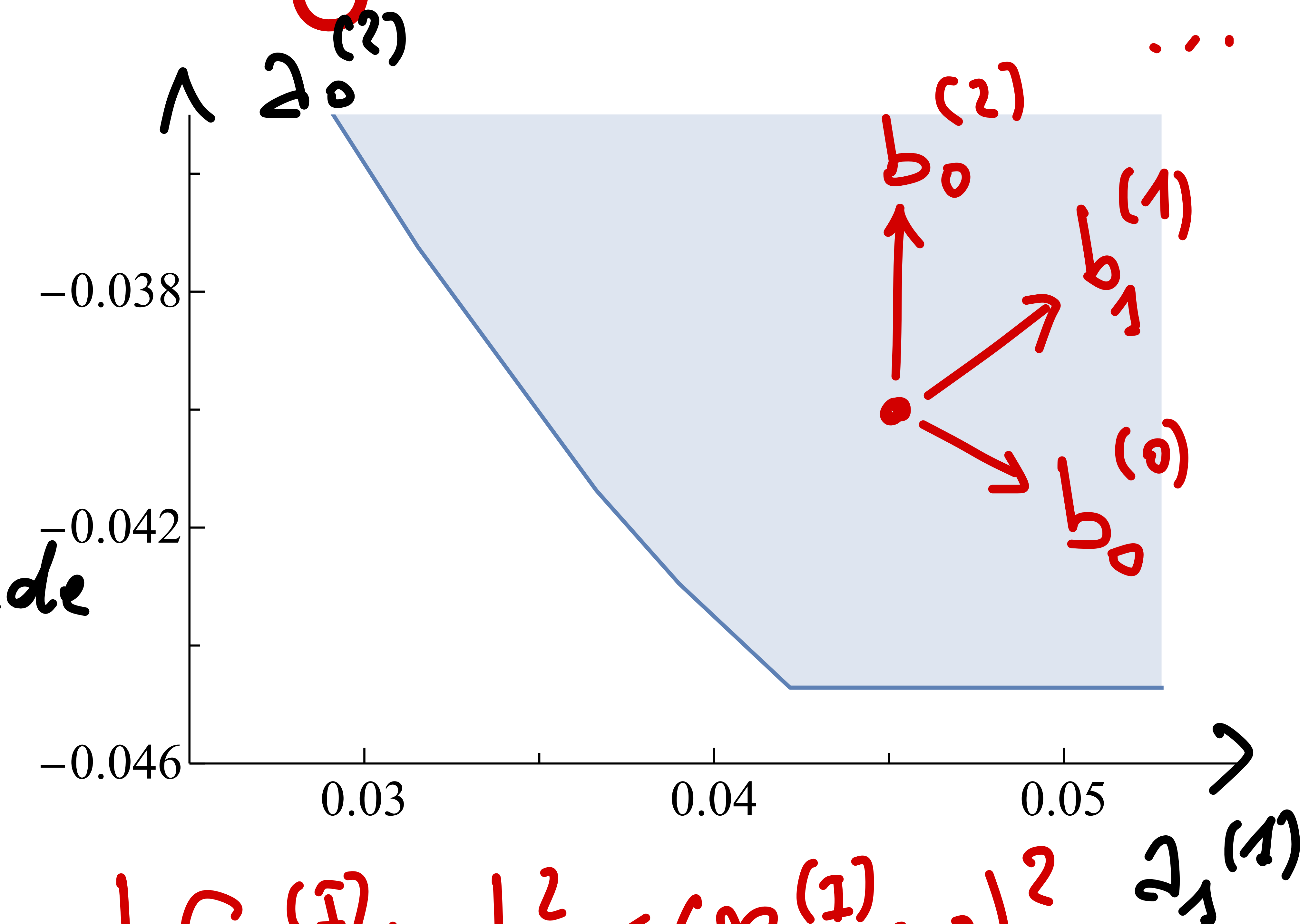
↑ taken from Pelaez et al

Why the boundary

ρ-expansion
$+\ z_0, z_2, m_R^2$
$+$ unitarity

$\Big\}$ ↖ boundary defined by the constraints

If $|S_\ell^{(I)}(s)|^2 \leq 1 \longrightarrow$ boundary amplitude is elastic



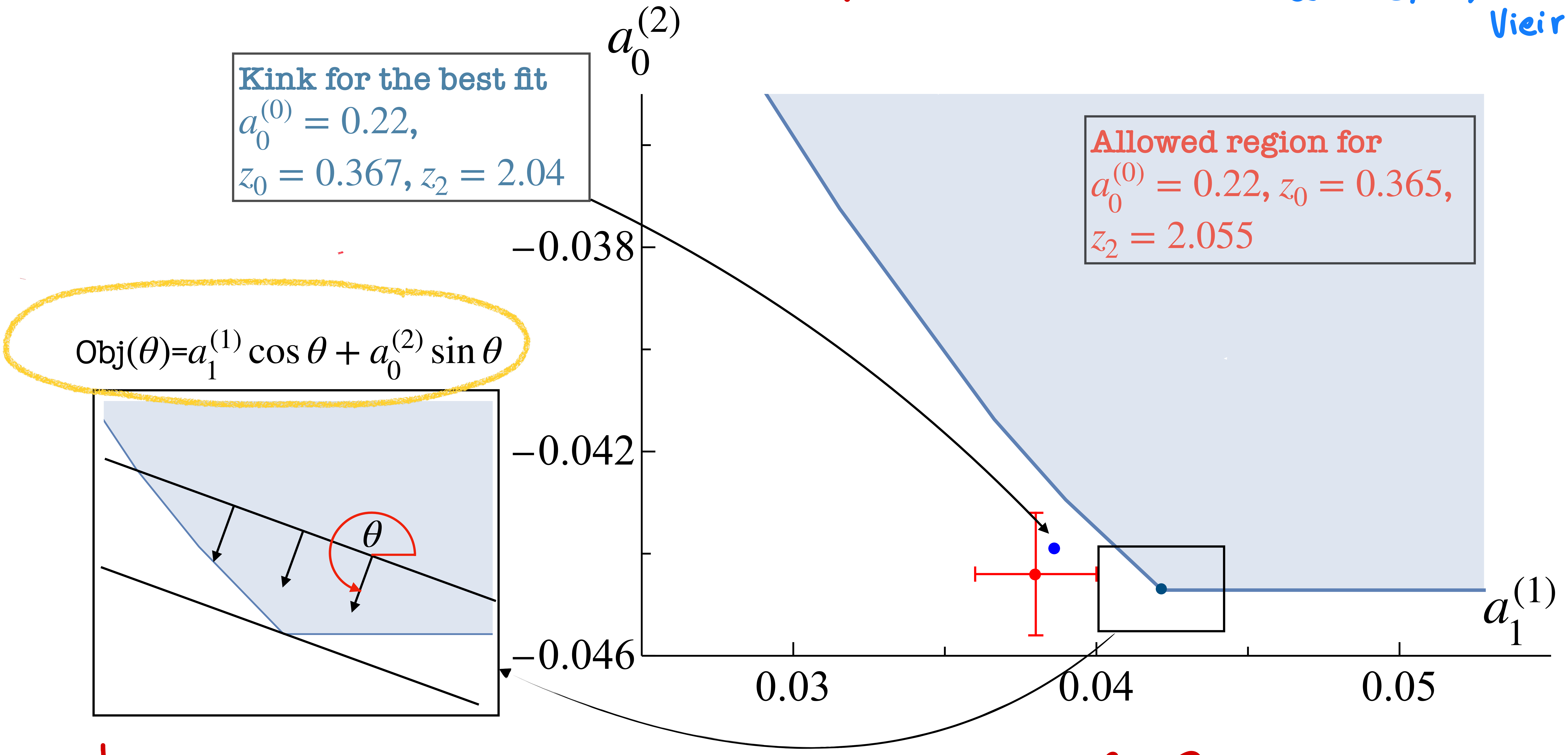
- We impose the experimental inelasticity $|S_\ell^{(I)}(s)|^2 \leq \left(\eta_\ell^{(I)}(s)\right)^2$

  ↳ bndry amplitude has correct production

- We tune more coefficients, e.g. fix $\{a_1^{(1)}, a_0^{(2)}\}$

Normal Optimization

← Cordova, He, Kruczenski, Vieira



We parametrize the bndry using the angle θ

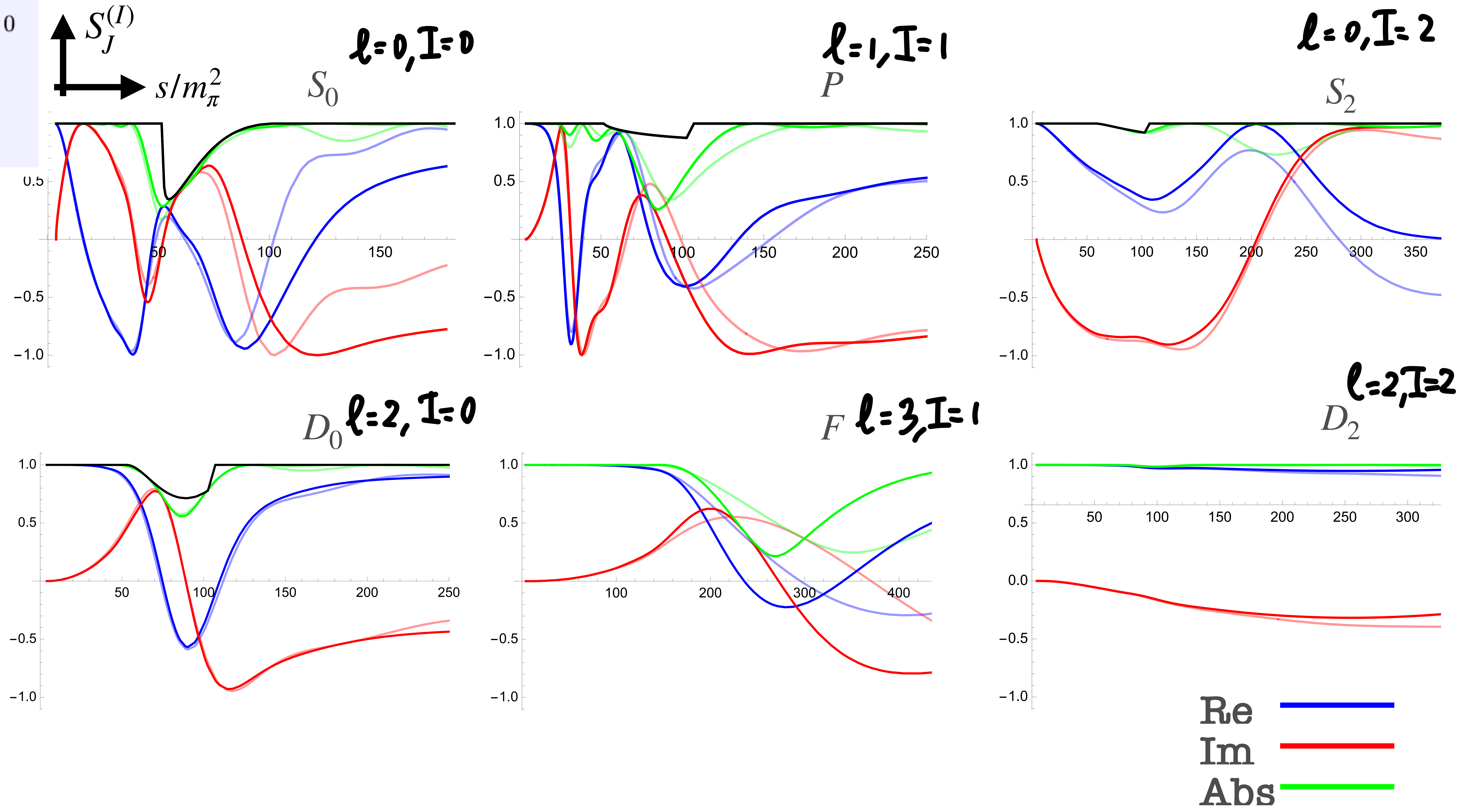
Bootstrap Solution

Fit Ansatz

Given	$\Theta = \{\theta, a_0^{(0)}, z_0, z_2, m_\rho^2, m_{f_0}^2, m_{f_0'}^2, m_{f_2}^2\}$
Maximize in $\mathcal{A}^{\text{ansatz}}(s,t,u)$	$\text{Obj}(\theta)$
constr. by	$t_0^{(0)}(4) = 2a_0^{(0)}, t_0^{(0)}(z_0) = 0, t_0^{(2)}(z_2) = 0$
	$S_1^{(1)}(m_\rho^2)=0, S_0^{(0)}(m_{f_0}^2)=0$
	$S_0^{(0)}(m_{f_0'}^2)=0, S_2^{(0)}(m_{f_2}^2)=0$
$s \geq 4$	$\mathcal{U}_\ell^{(I)} \succeq 0$ for $\ell \in \mathbb{N}$, $I = 0,1,2$ (9)

$A^{\Theta}(s,t,u)$

denotes collectively z_0, z_2, m_R^2, θ, $a_0^{(0)}$



A non-linear / non-convex problem

To find the best fit we must tune Θ to minimize

$$\chi^2_\Theta = \sum_{i=1}^{\#\text{data points}} \left(\left(\delta^{I_i}_{\ell_i}(s_i)\right)^{\text{exp}} - \left(\delta^{I_i}_{\ell_i}(s_i)\right)^{\text{Bootstrap}} \right)^2 \Big/ \left(\sigma^{I_i}_{\ell_i}(s_i)\right)^2$$

$$S^{(I)}_\ell(s) = \eta^{(I)}_\ell(s)\, e^{2i\,\delta^{(I)}_\ell(s)}$$

phase shift

Bottleneck:

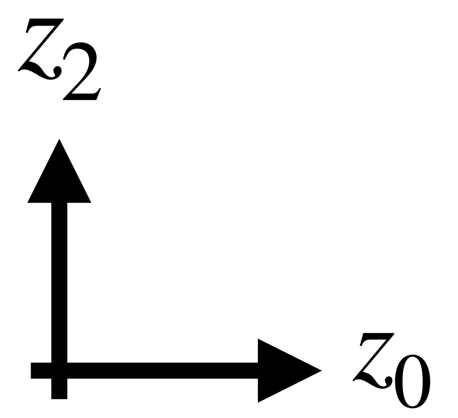
Evaluate $\chi^2_\Theta \leftrightarrow$ Solve a full Bootstrap problem ($\sim 2/3$ h)

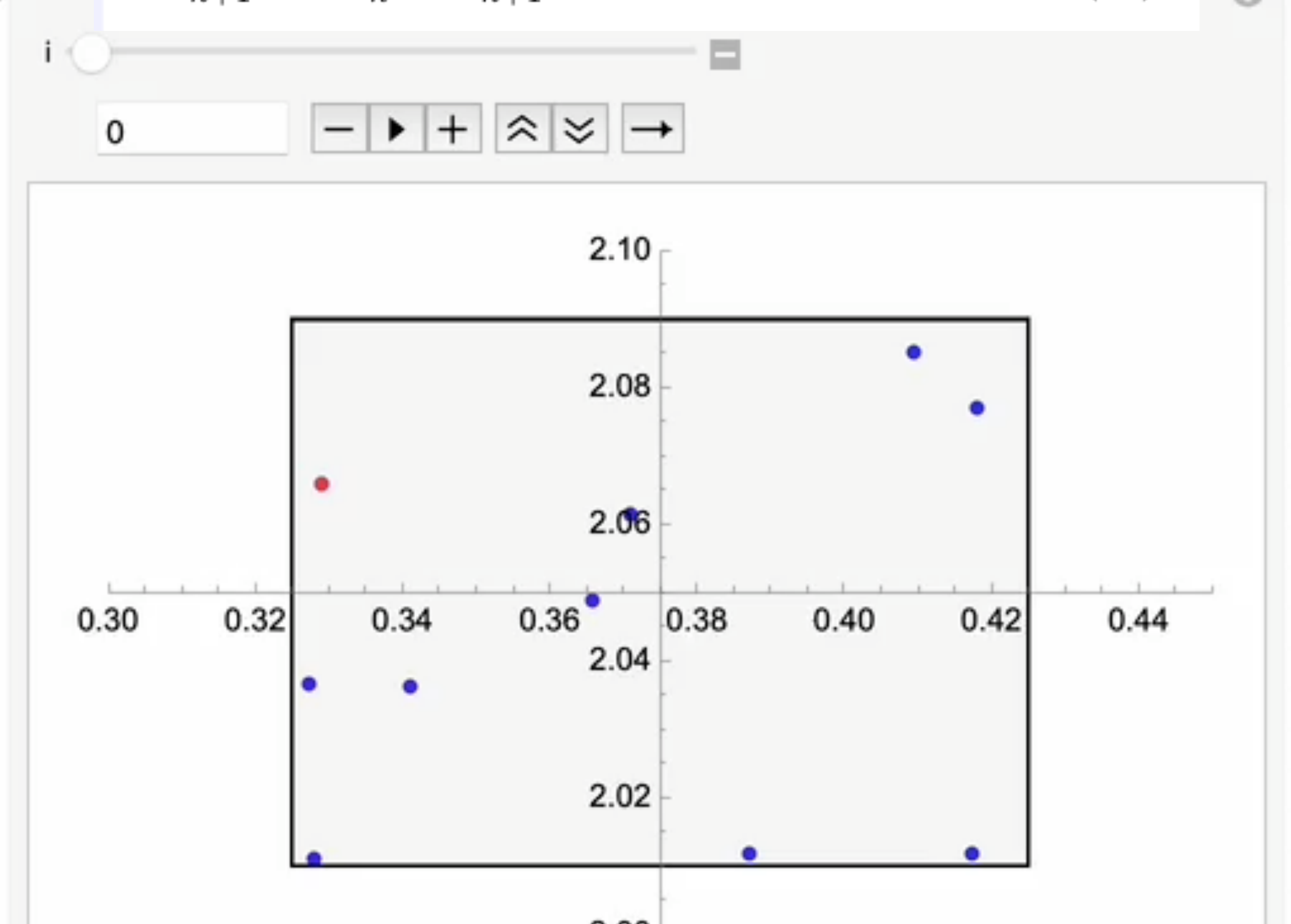
$\partial / \partial\Theta\, \chi^2_\Theta$ even longer $\leftarrow$ Navigator Skydiving Algorithm

Particle Swarm Optimization

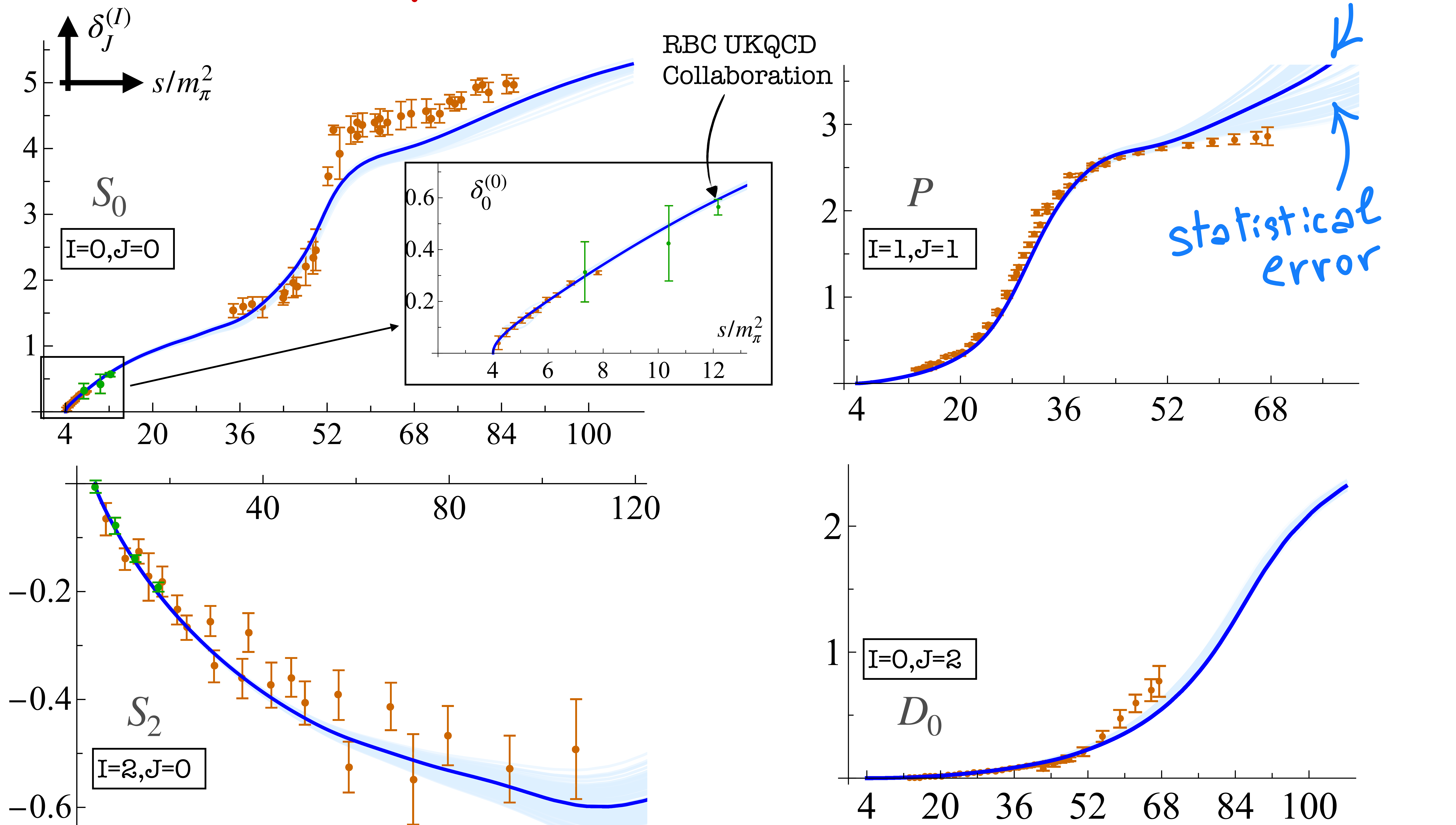
Basic Model
(swarm of birds dynamics)

$$
\begin{aligned}
v_{n+1}^{(i)} &= \omega v_n^{(i)} + c_1 r_1 (\Theta_n^{(i)} - X_n^{(i)}) + c_2 r_2 (\Theta_n^{(i)} - Y_n), \\
\Theta_{n+1}^{(i)} &= \Theta_n^{(i)} + v_{n+1}^{(i)}. \qquad (11)
\end{aligned}
$$





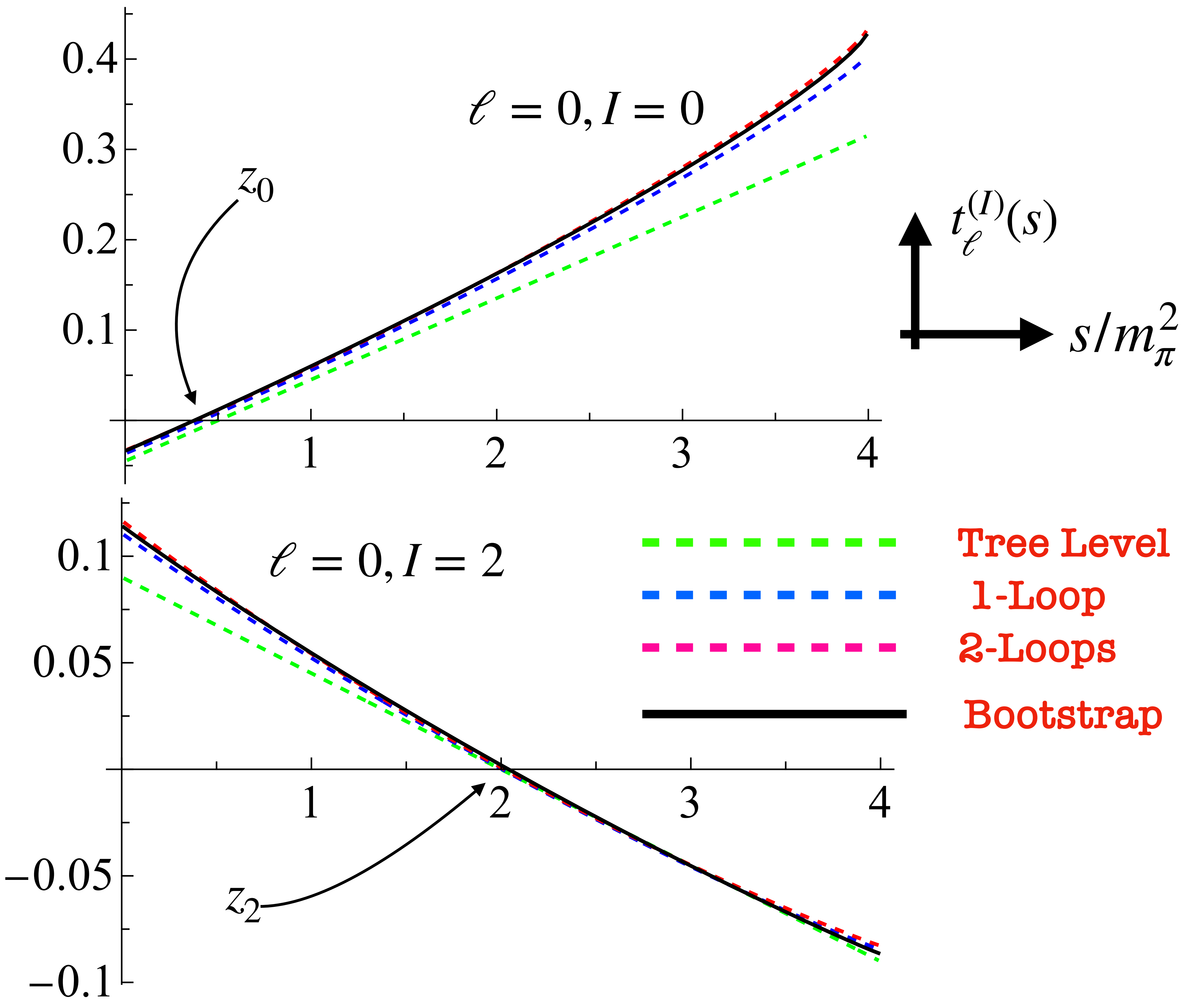
Main Result

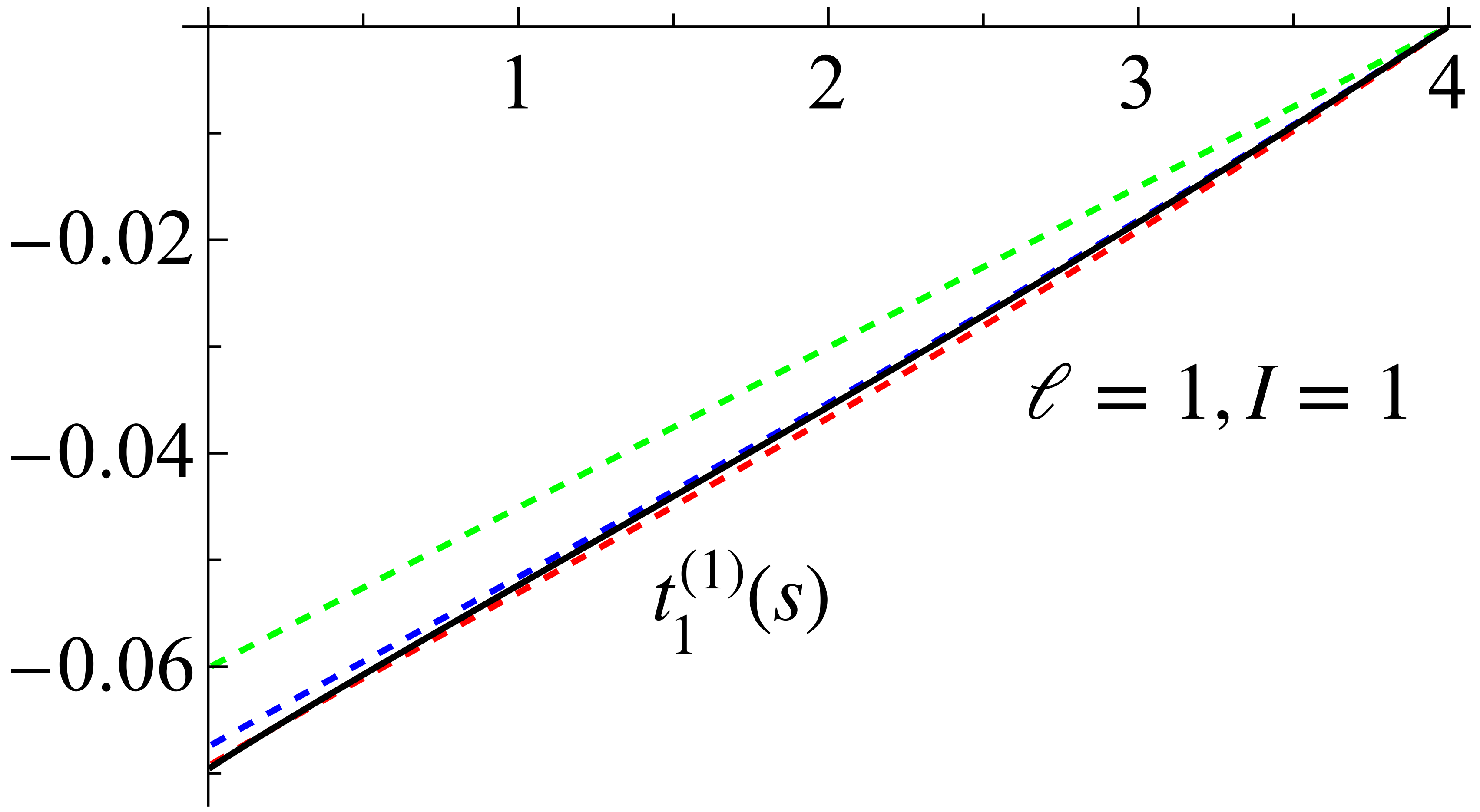


$\delta_J^{(I)}$
s/m_π^2
S_0
I=0,J=0
$\delta_0^{(0)}$
RBC UKQCD Collaboration
best fit
statistical error
P
I=1,J=1
S_2
I=2,J=0
D_0
I=0,J=2

Checks @ low energy

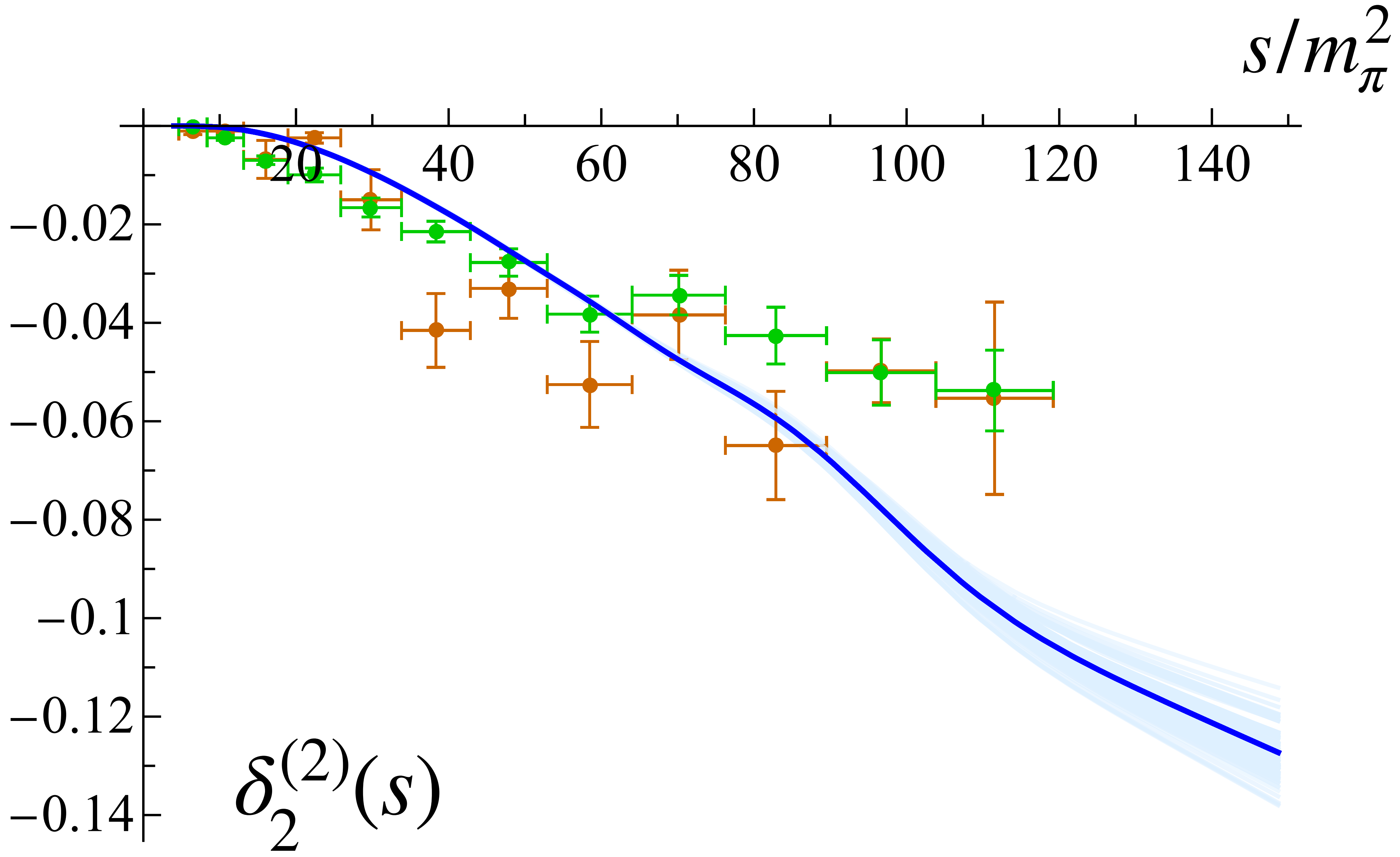
2-loops from Colangelo, Gasser, Leutwyler

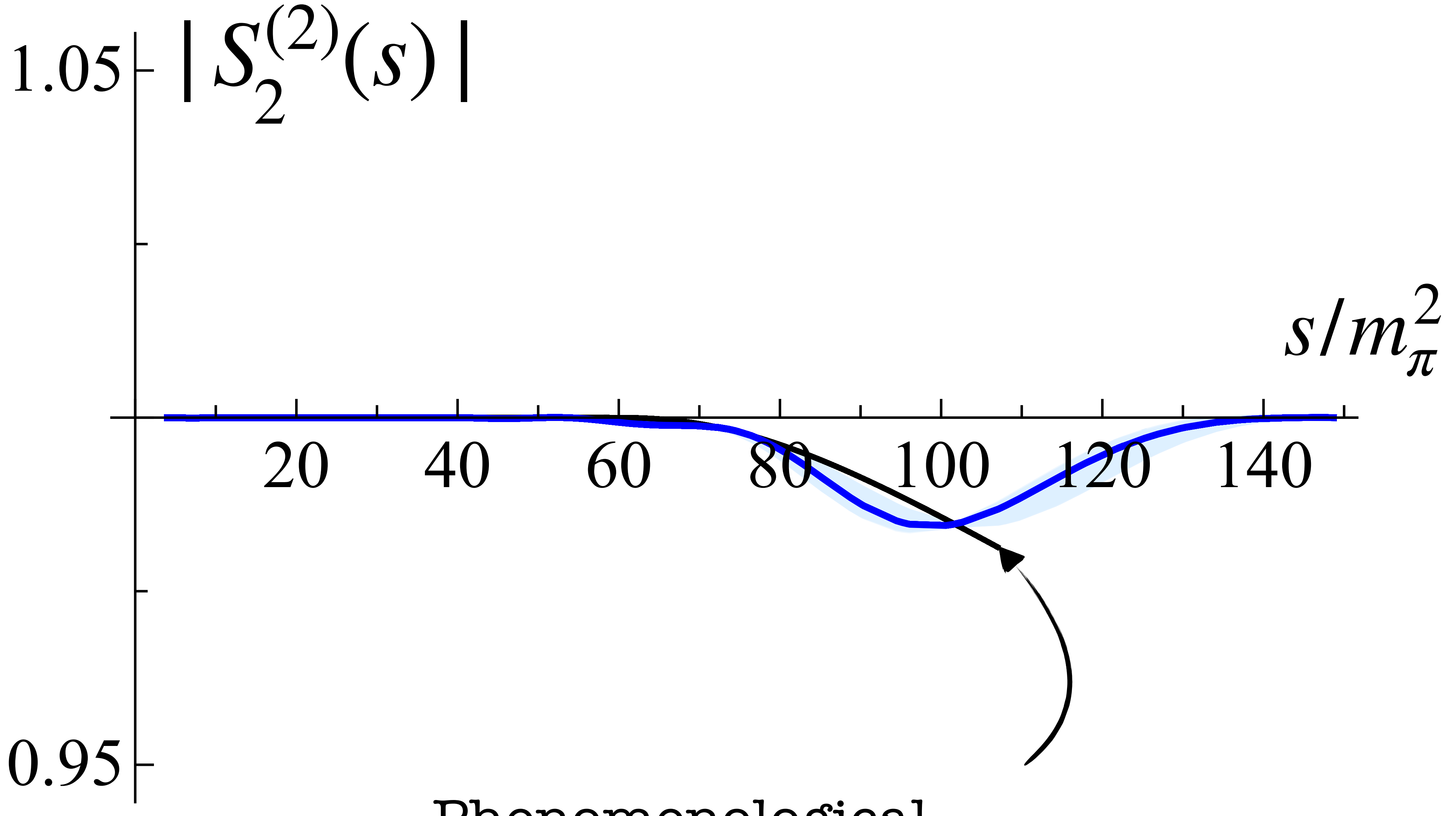




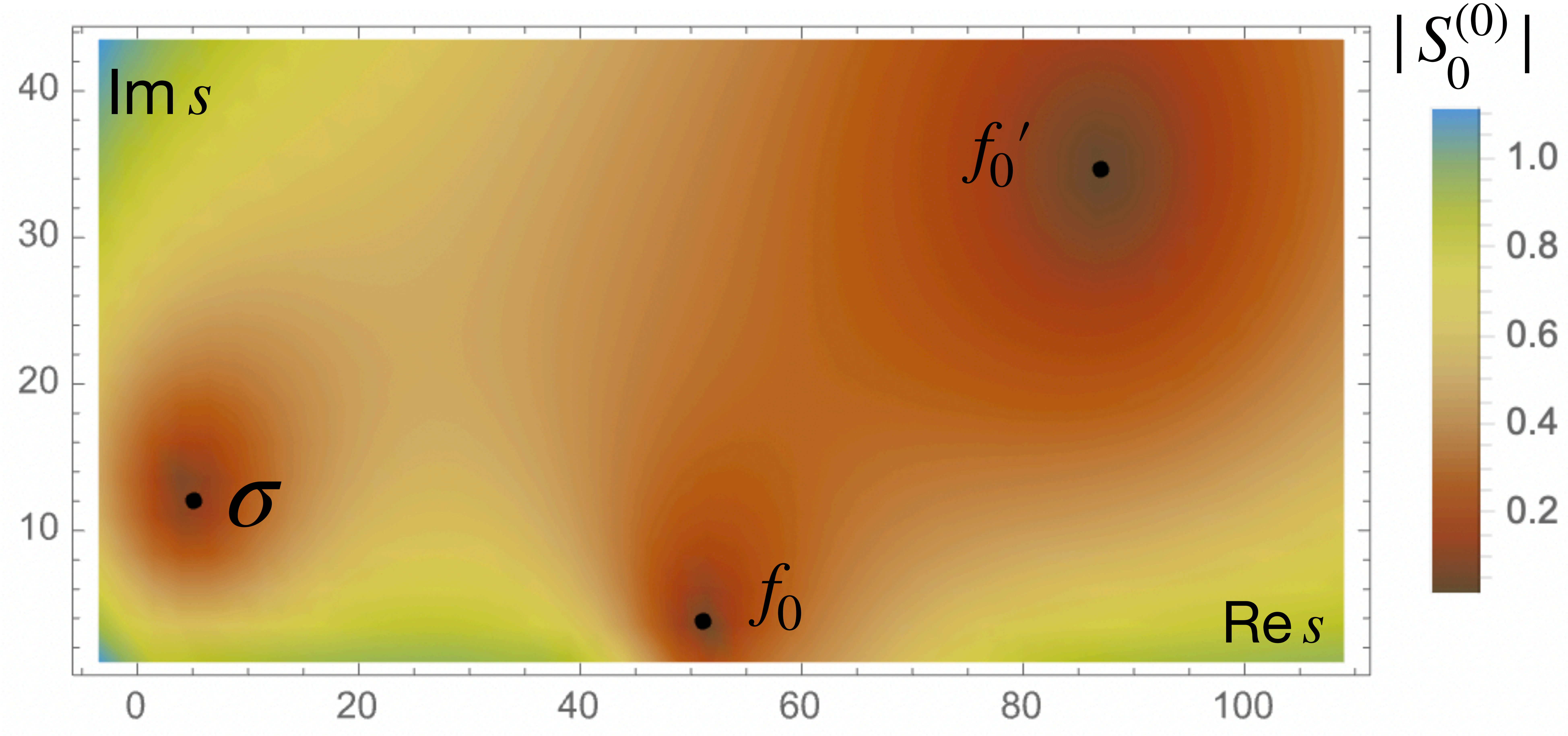
	Bootstrap Fit	Literature
$a_0^{(2)}$	$(-0.432 \pm 0.001) \times 10^{-1}$	$(-0.444 \pm 0.012) \times 10^{-1}$
$a_1^{(1)}$	$(0.380 \pm 0.002) \times 10^{-1}$	$(0.379 \pm 0.05) \times 10^{-1}$
$b_0^{(0)}$	0.265 ± 0.030	0.276 ± 0.006
$b_0^{(2)}$	$(-0.797 \pm 0.002) \times 10^{-1}$	$(-0.803 \pm 0.012) \times 10^{-1}$
$b_1^{(1)}$	$(0.61 \pm 0.02) \times 10^{-2}$	$(0.57 \pm 0.01) \times 10^{-2}$
$a_2^{(0)}$	$(0.53 \pm 0.11) \times 10^{-2}$	$(0.175 \pm 0.003) \times 10^{-2}$
$a_2^{(2)}$	$(0.51 \pm 0.18) \times 10^{-3}$	$(0.170 \pm 0.013) \times 10^{-3}$
$a_1^{(3)}$	$(1.5 \pm 0.4) \times 10^{-4}$	$(0.56 \pm 0.02) \times 10^{-4}$

Predictions for I=2, ℓ=2

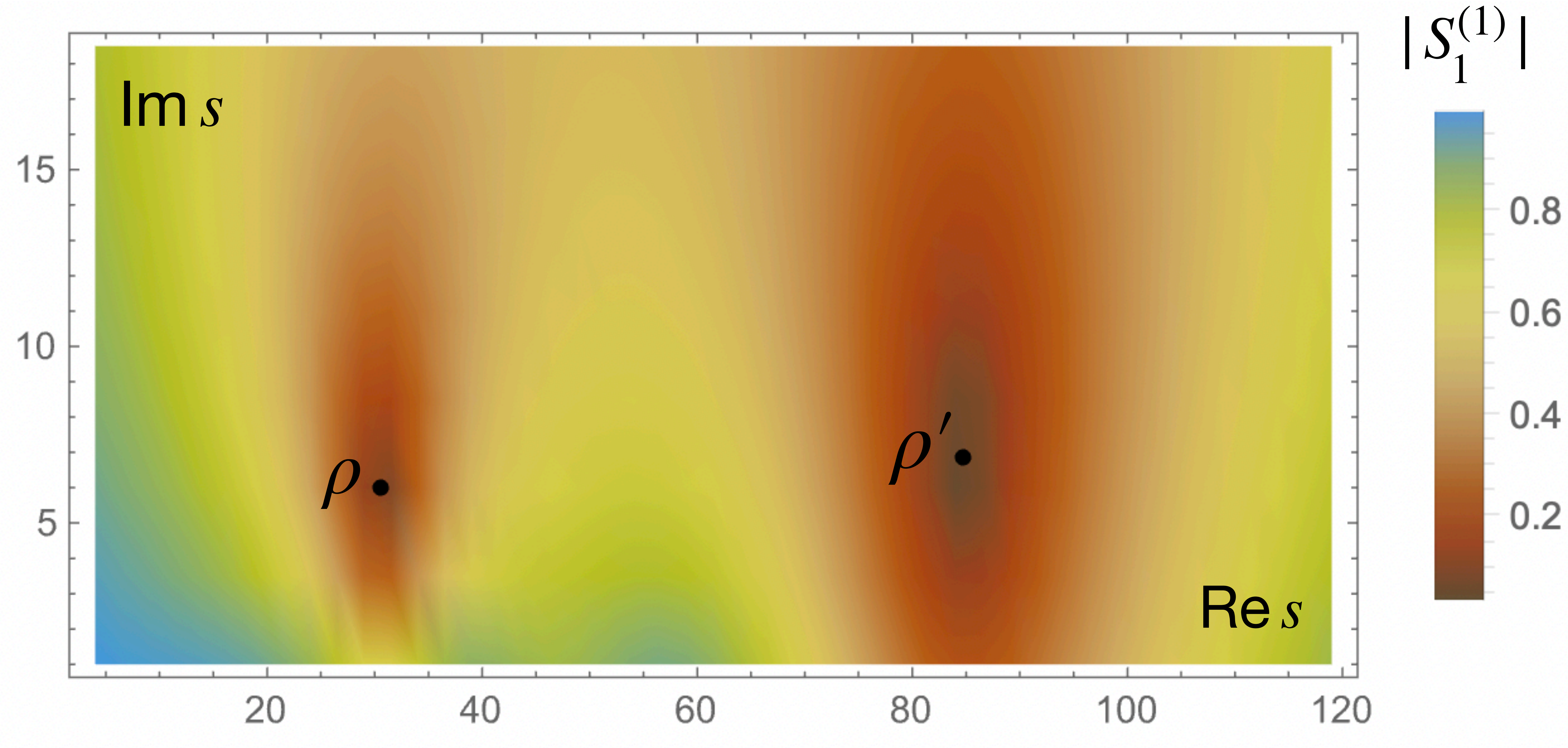




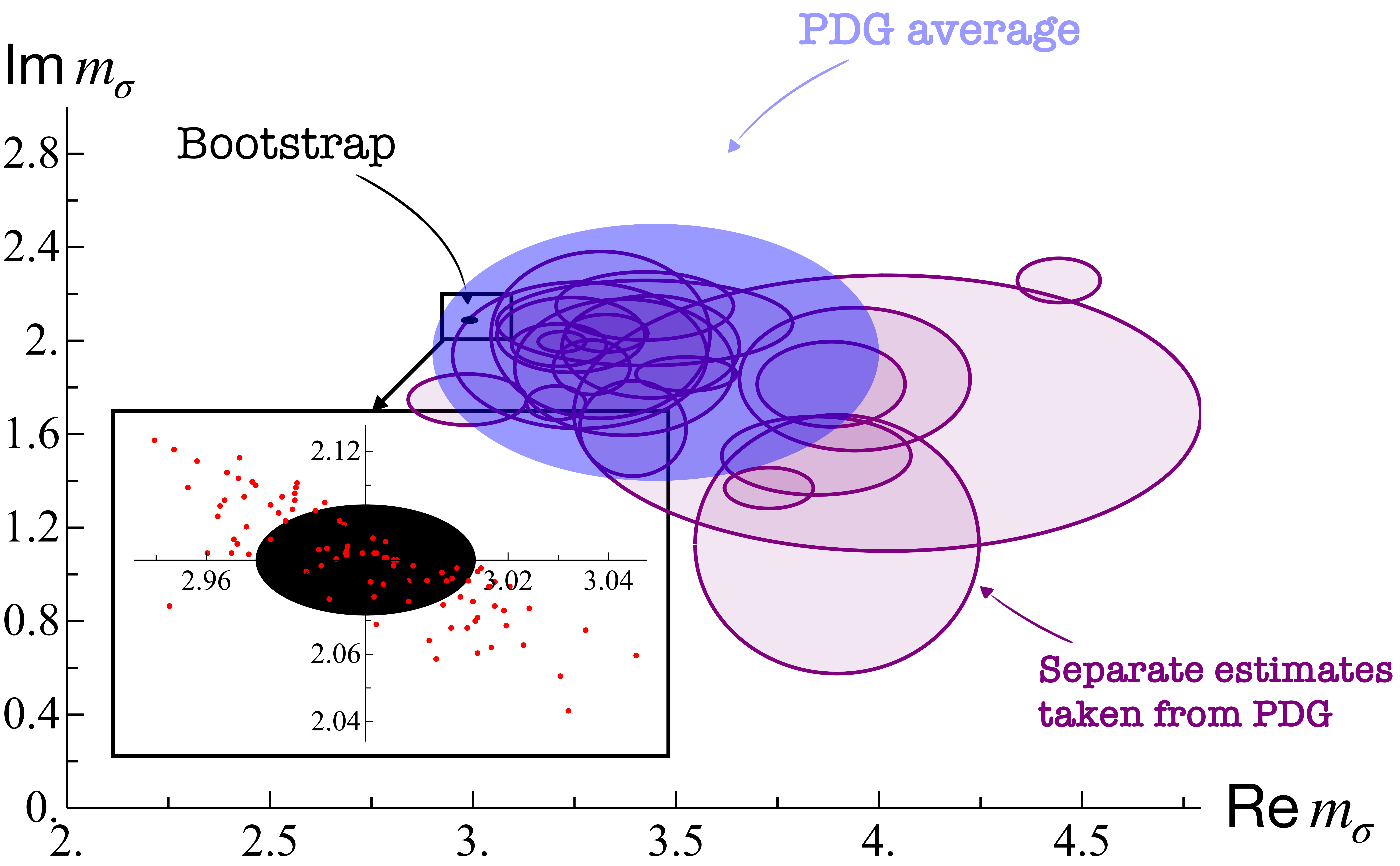
Spectrum for I=0, J=0



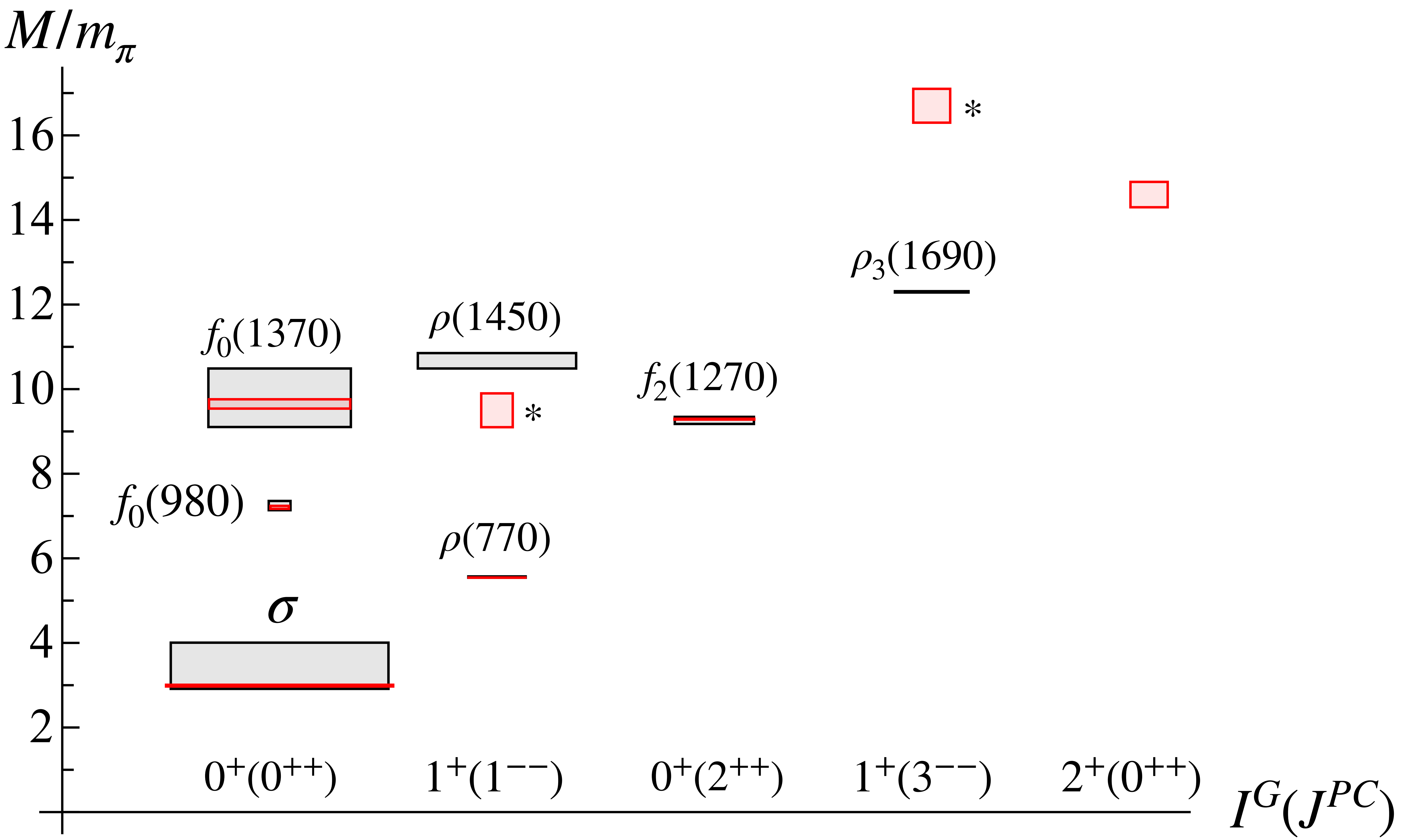
Spectrum for I=1, J=1



Sigma parameters determination

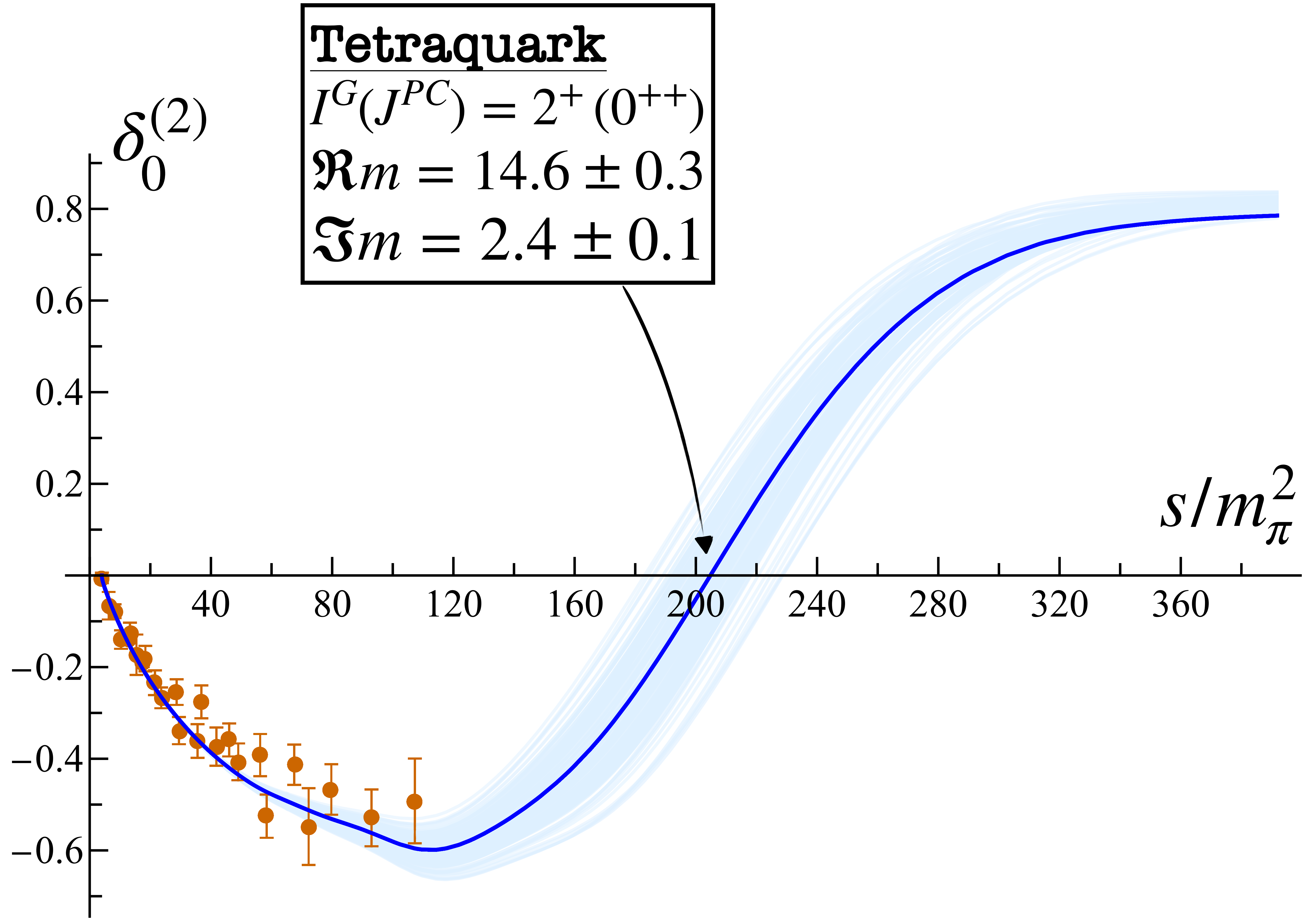


Full Spectrum < 1.4 GeV, with G-parity +1



The Tetraquark

- Each quark carries $I = 1/2$
- 4 quarks to make $I = 2$
- Charge 2 state



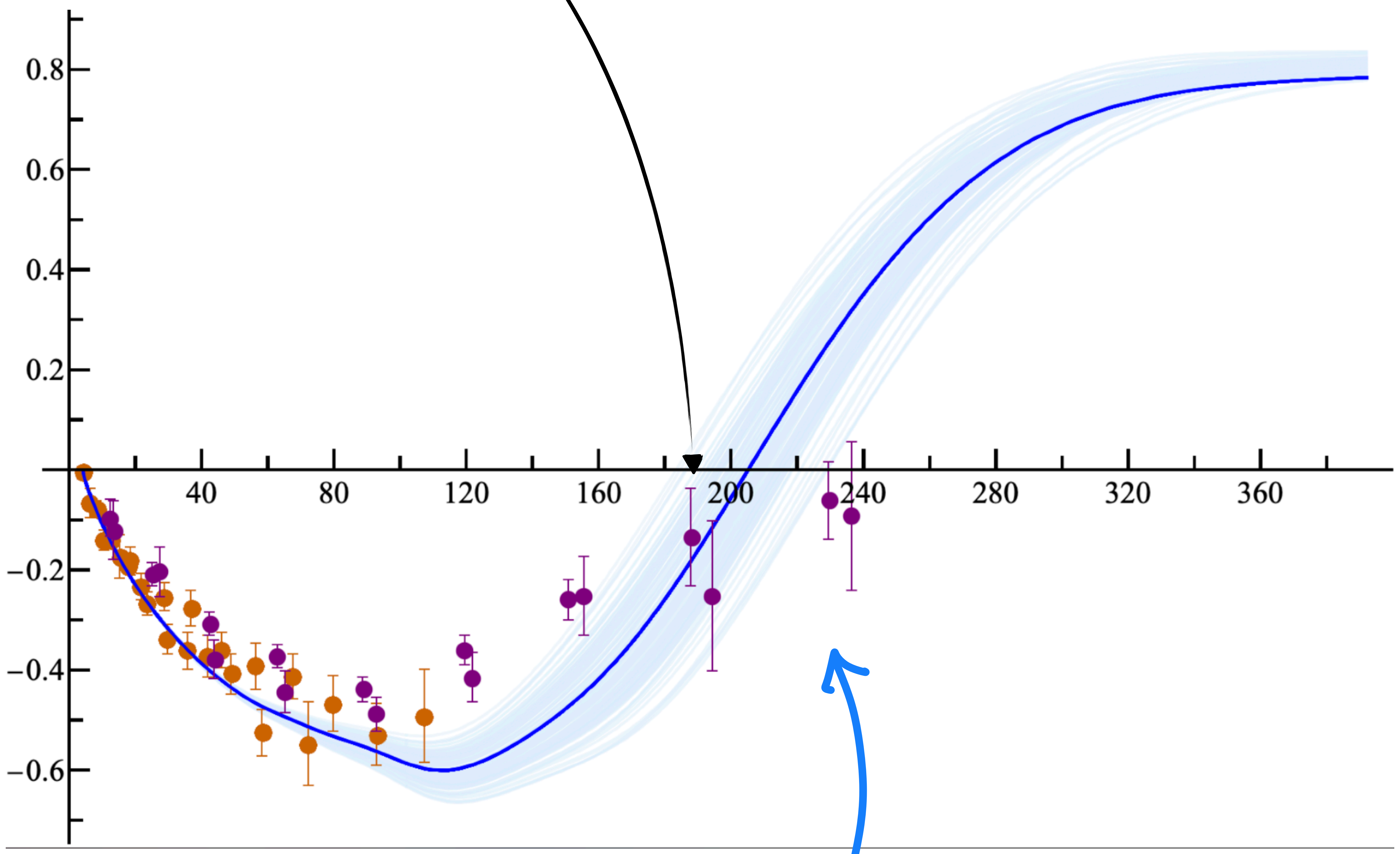
$M \sim 2GeV,$
$\Gamma \sim 600MeV$

LHCb looking into $B^+ \to \pi^-\pi^+\pi^+$!

(Cannot show results yet!)

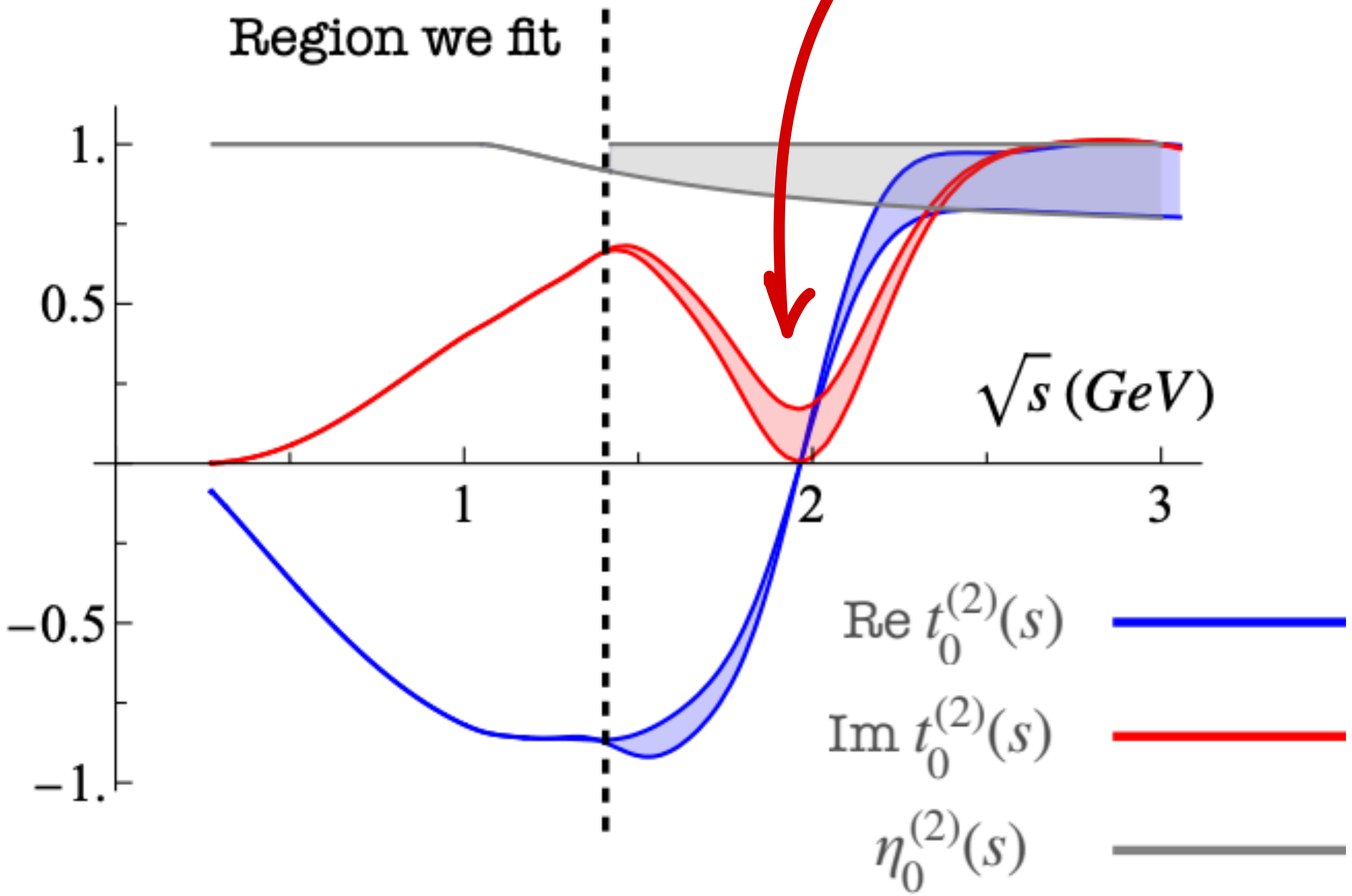
The Tetraquark (news)

Data we had no idea about

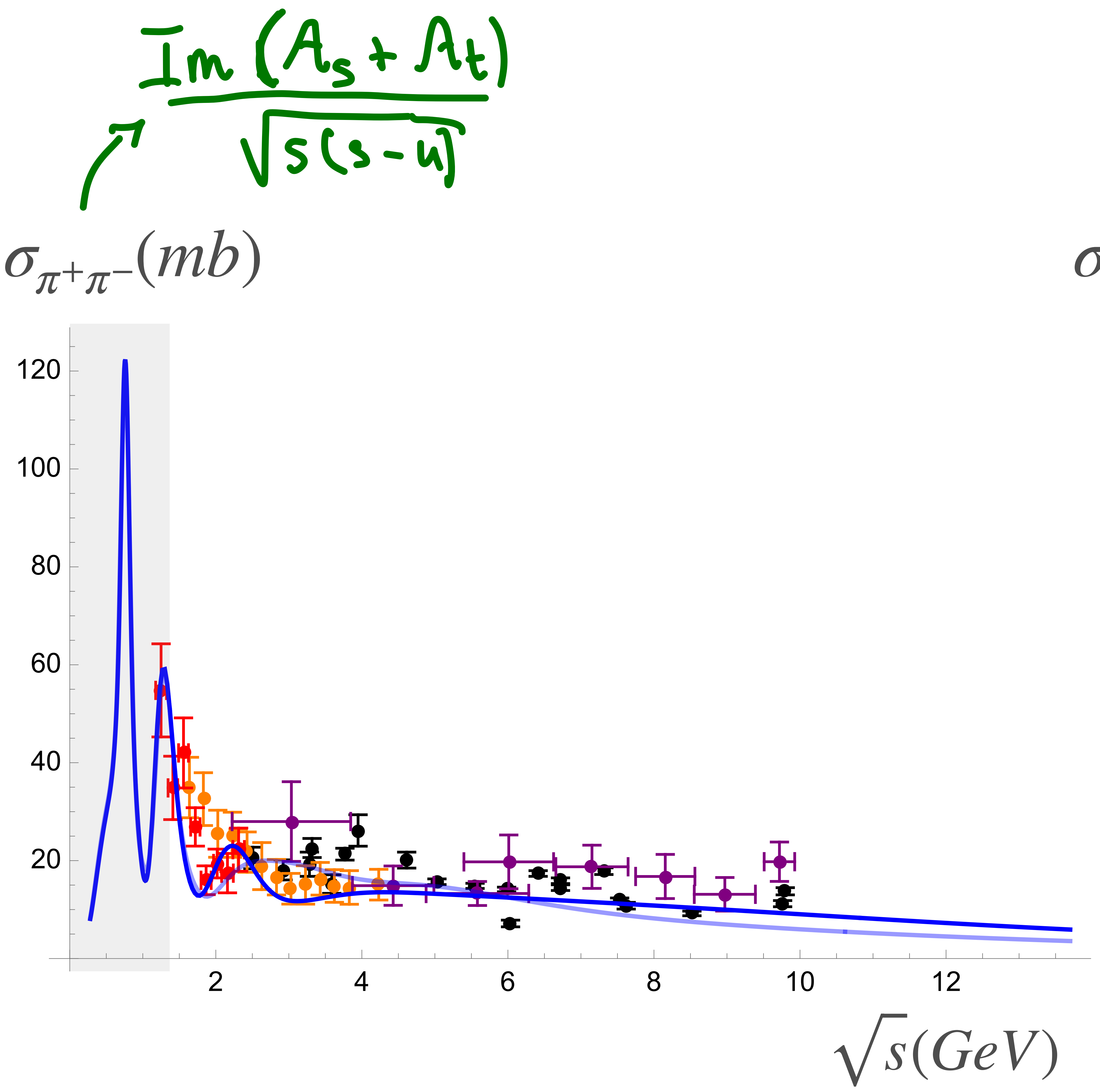


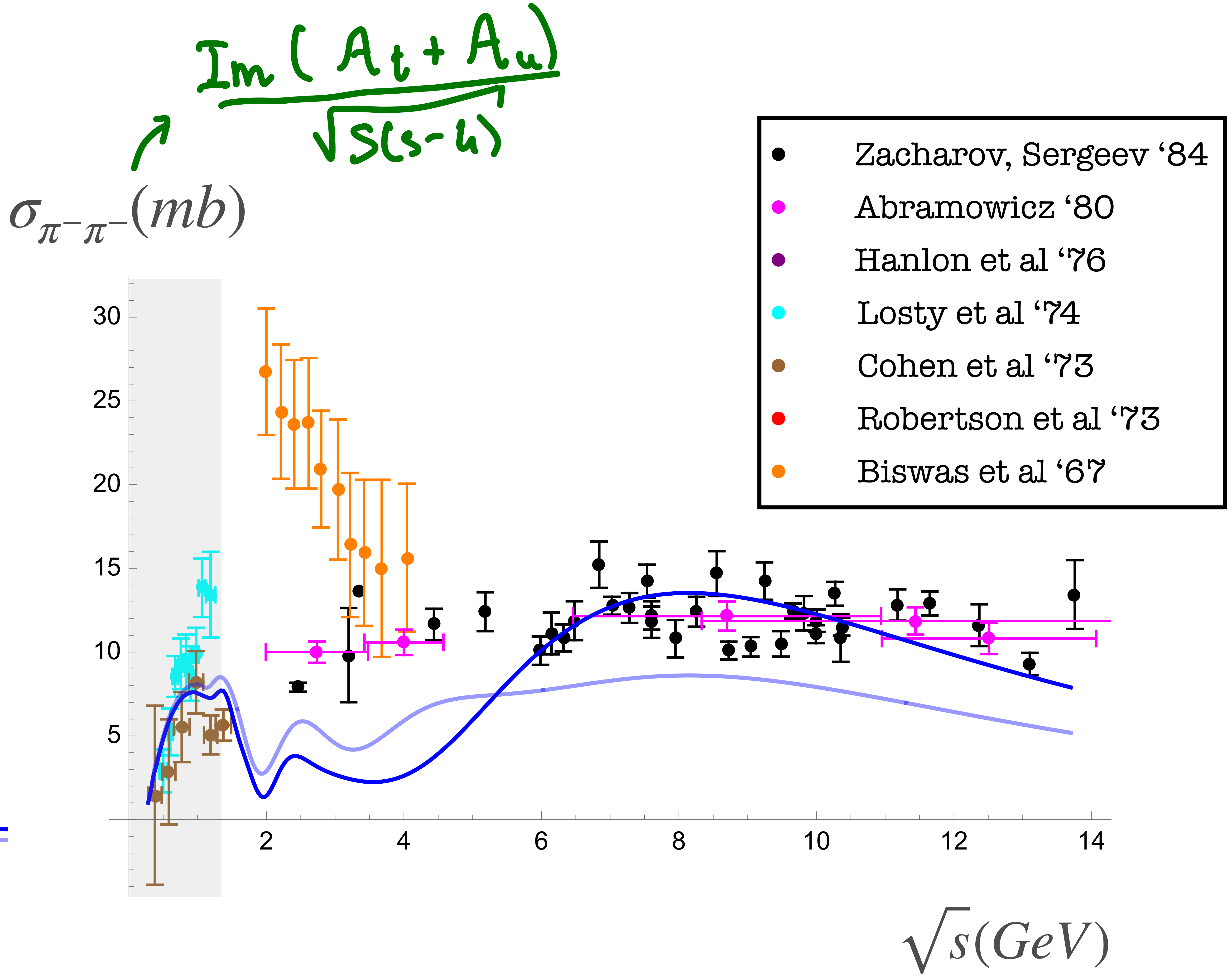
Thanks to Luiz Vale Silva

Experimental signature



Check @ High Energy





Overview

Input

★ Experimental phase shifts data for S_0, S_2, P, D_0 waves

★ Inelasticity model for S_0, S_2, P, D_0 waves

★ $\exists$ chiral zeros

★ $\exists$ resonances $\rho(770)$, $f_0(980)$, $f_0(1370)$, $f_2(1270)$

Output

✤ Fit for the phase shifts, chiral zeros, resonances positions

✤ Scattering lengths and effective ranges for any isospin and spin $\ell < 2$

✤ S_0, S_2, P waves for $0 < s < 4$ compatible with χPT

✤ D_2 phase shift and inelasticity compatible with experiments

✤ Dynamical generation of σ, $\rho(1450)$, ρ_3 resonances, plus a tetra quark

✤ $\sigma_{\pi^+\pi^-}$ and $\sigma_{\pi^-\pi^-}$ cross sections

Outlook

1) Improve numerics, go to higher energy, extract more resonances

2) $\pi\pi \to KK$, $KK \to KK$ into the mix to fit inelasticity and study resonances with $S=1$.

3) Add form factors to fit data and constrain the vacuum hadronic polarization contribution.

To appear soon

1) Maximizing the interaction strength: Cross-section Bootstrap
with A. Georgoudis, and M. Correia

NEW APPROACH TO THE FROISSART BOUND
"BEST TOY MODEL FOR p-p SCATTERING"
(HAVE PLOTS IF YOU WANT)

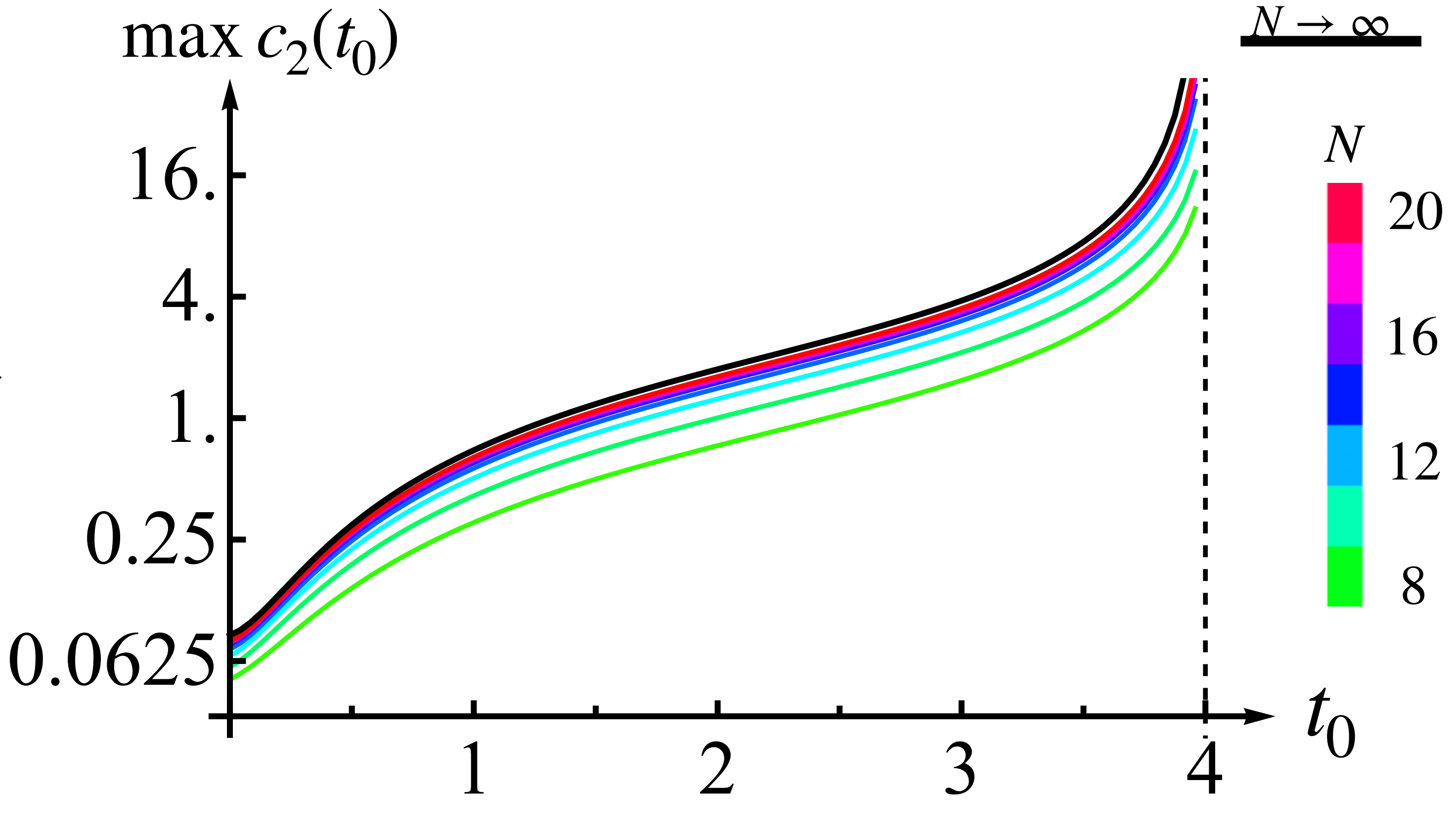
2) Geometry of crossing symmetric dispersion relations
with J. Elias-Miro', M. Gumus, A. Zahed

Connection Algebraic Geometry and Bootstrap

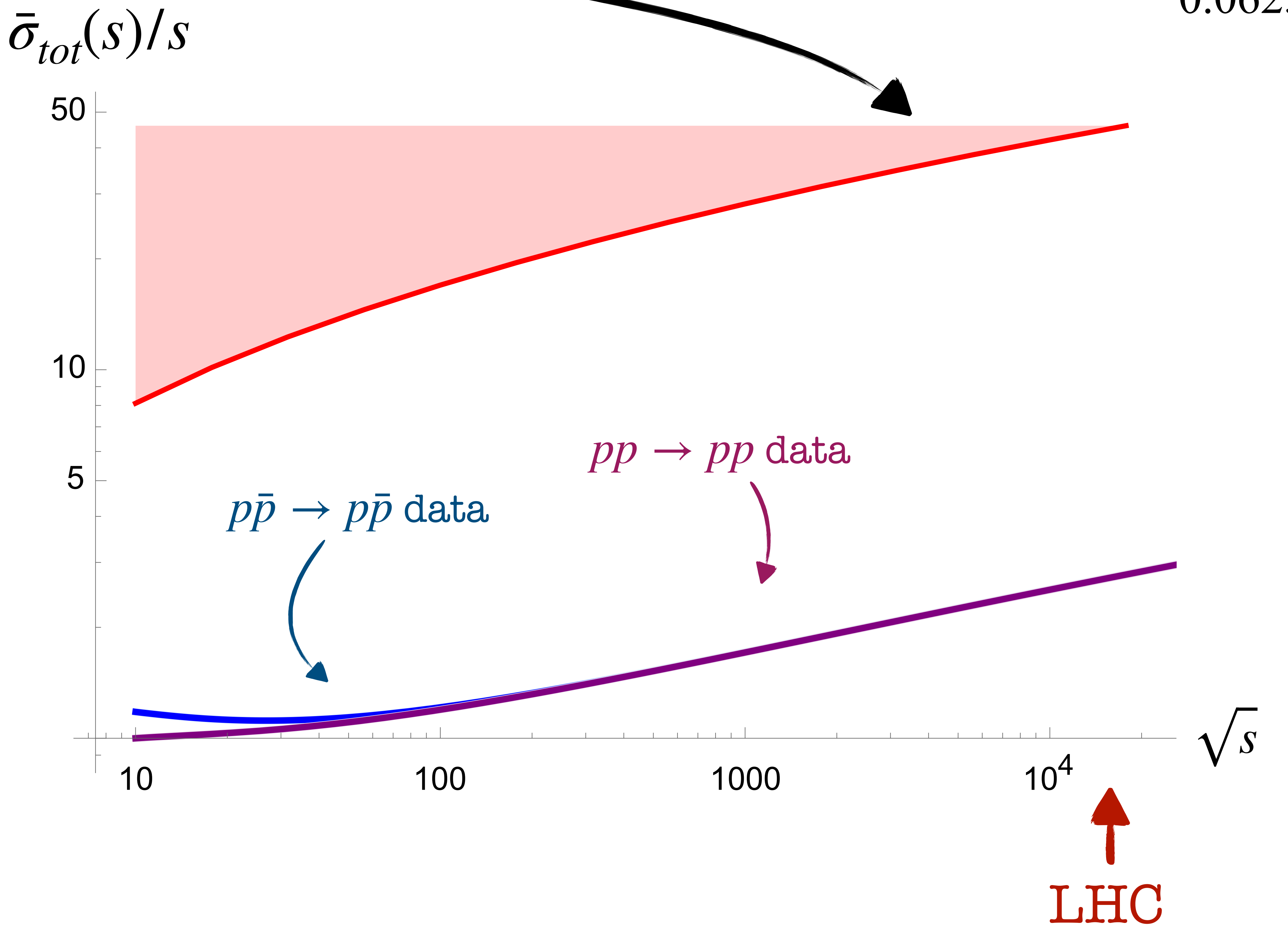
Backup

$$\bar{\sigma}_{\rm tot}(s) \equiv \frac{1}{16\pi}\int_{4m^2}^{s} \frac{s'-4m^2}{s-4m^2}\,\sigma_{\rm tot}(s')\,ds'$$

Plug Numerics into the analytic bound

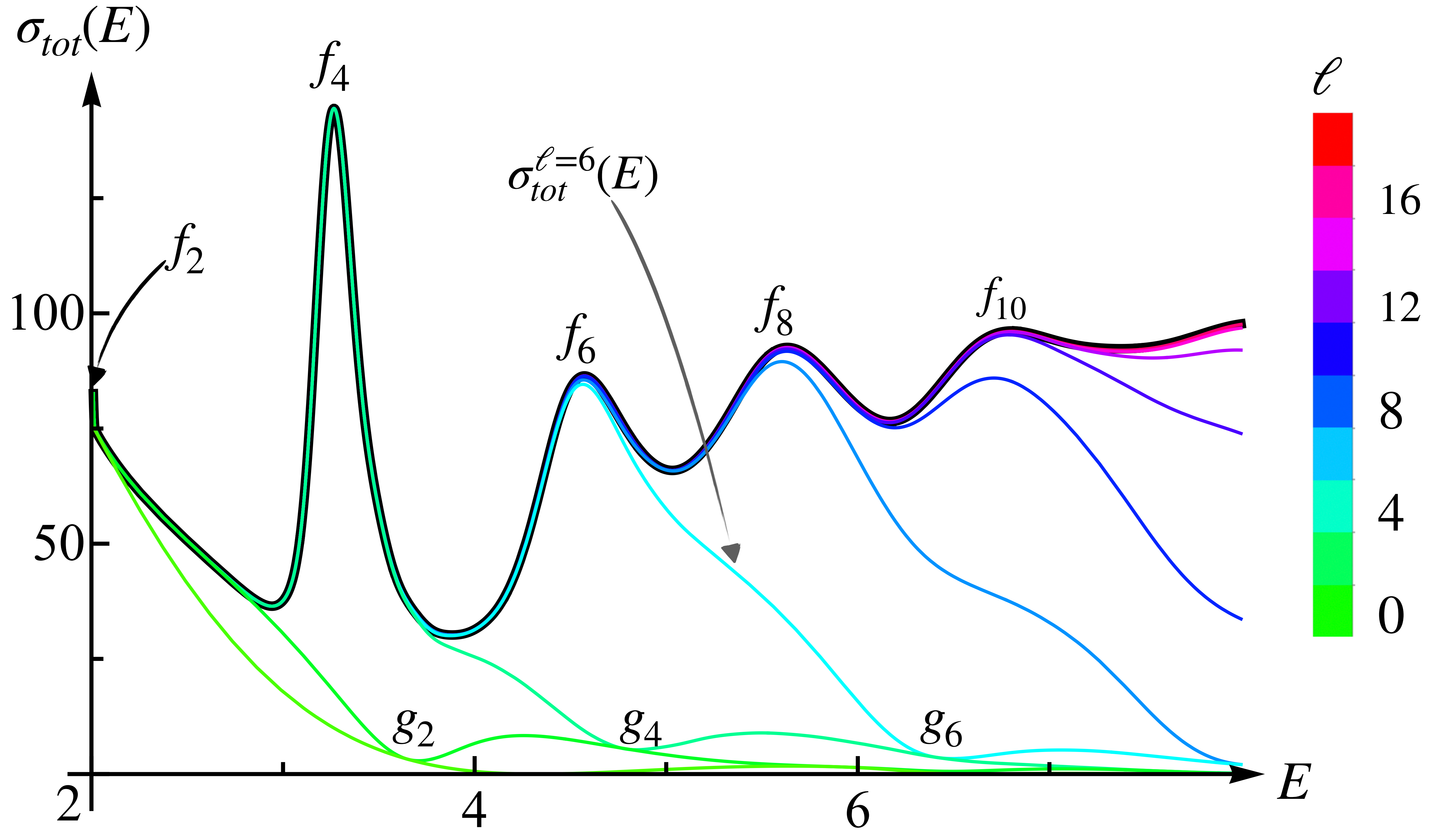


$$\bar{\sigma}_{\rm tot}(s) \le L(L-1) + \frac{2\pi\, c_2(t_0)}{(s-4)\,\mathcal{K}_L(s,t_0)}$$

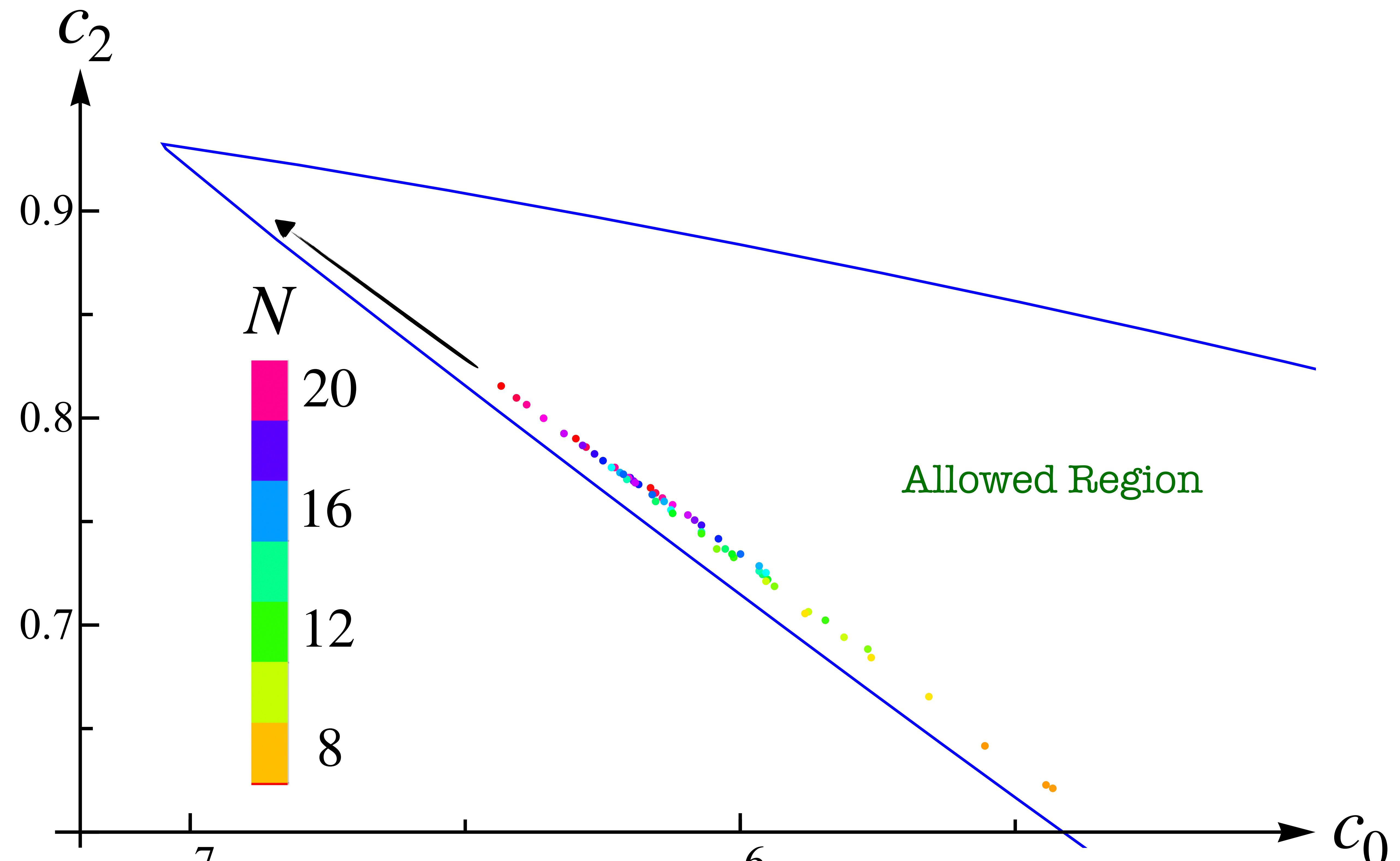


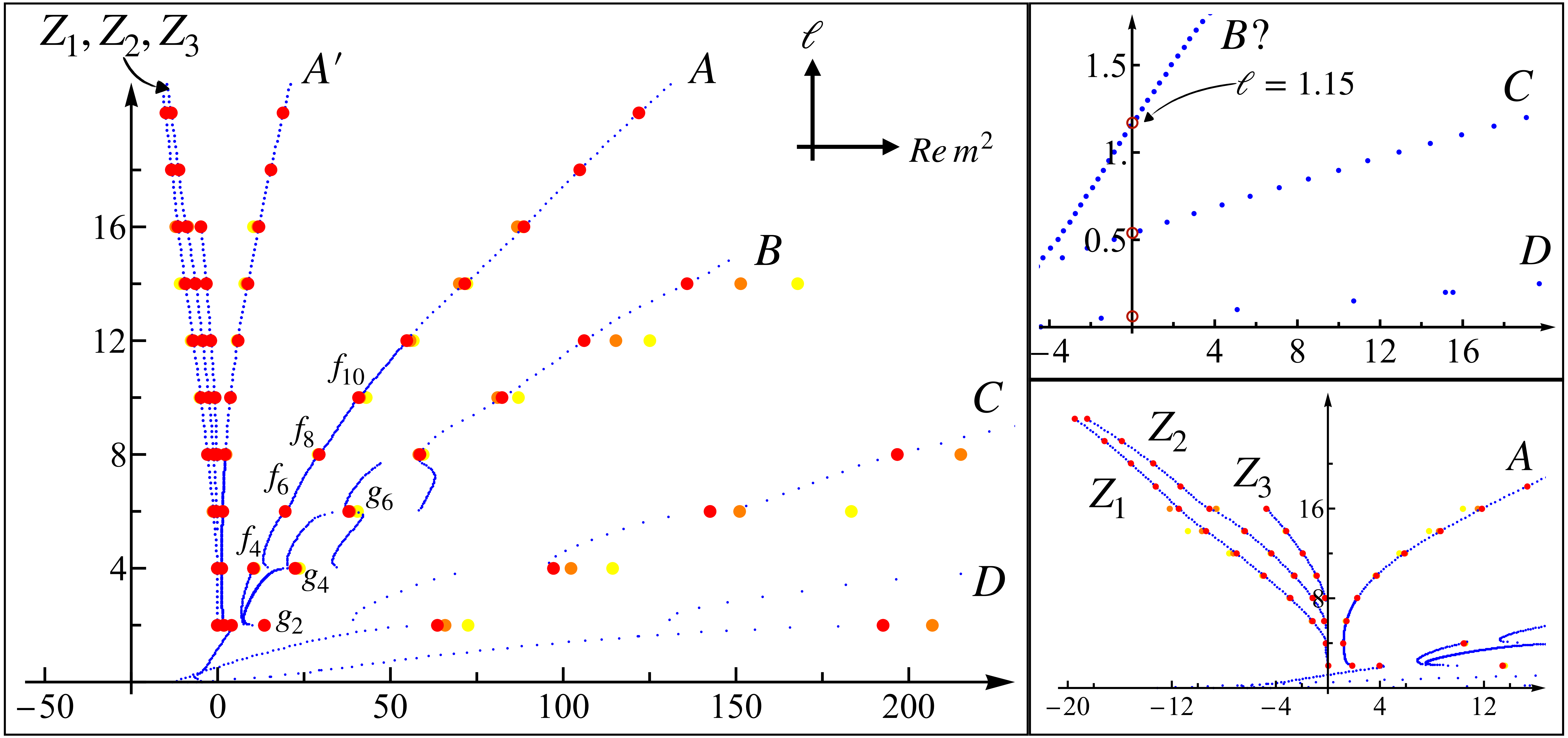
$$\lim_{s\to\infty} \max \bar{\sigma}_{tot} = \infty$$

The amplitude saturating the bound has a finite limit: Froissart Amplitude



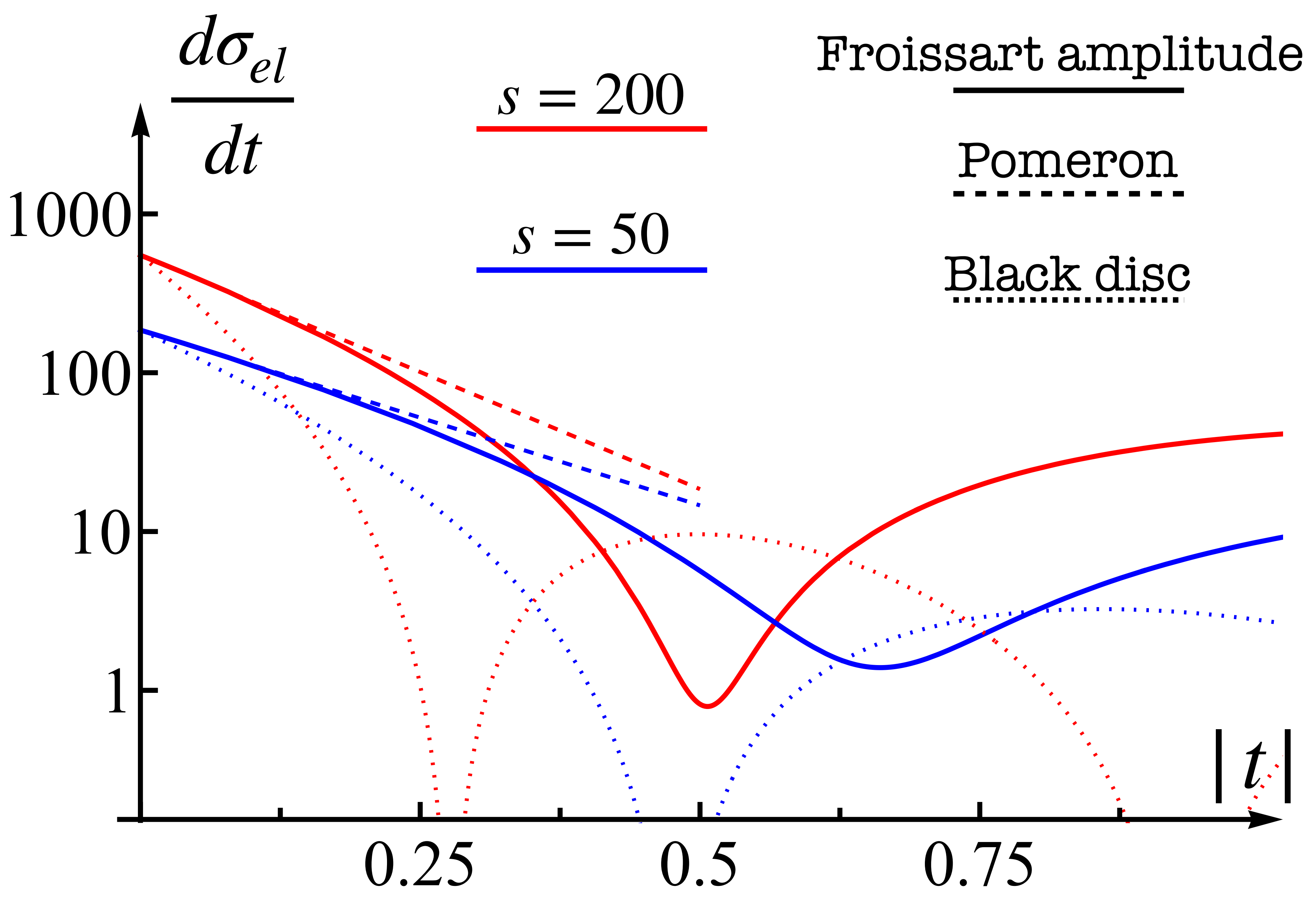
The Froissart Amplitude lives at an infinite dimensional kink

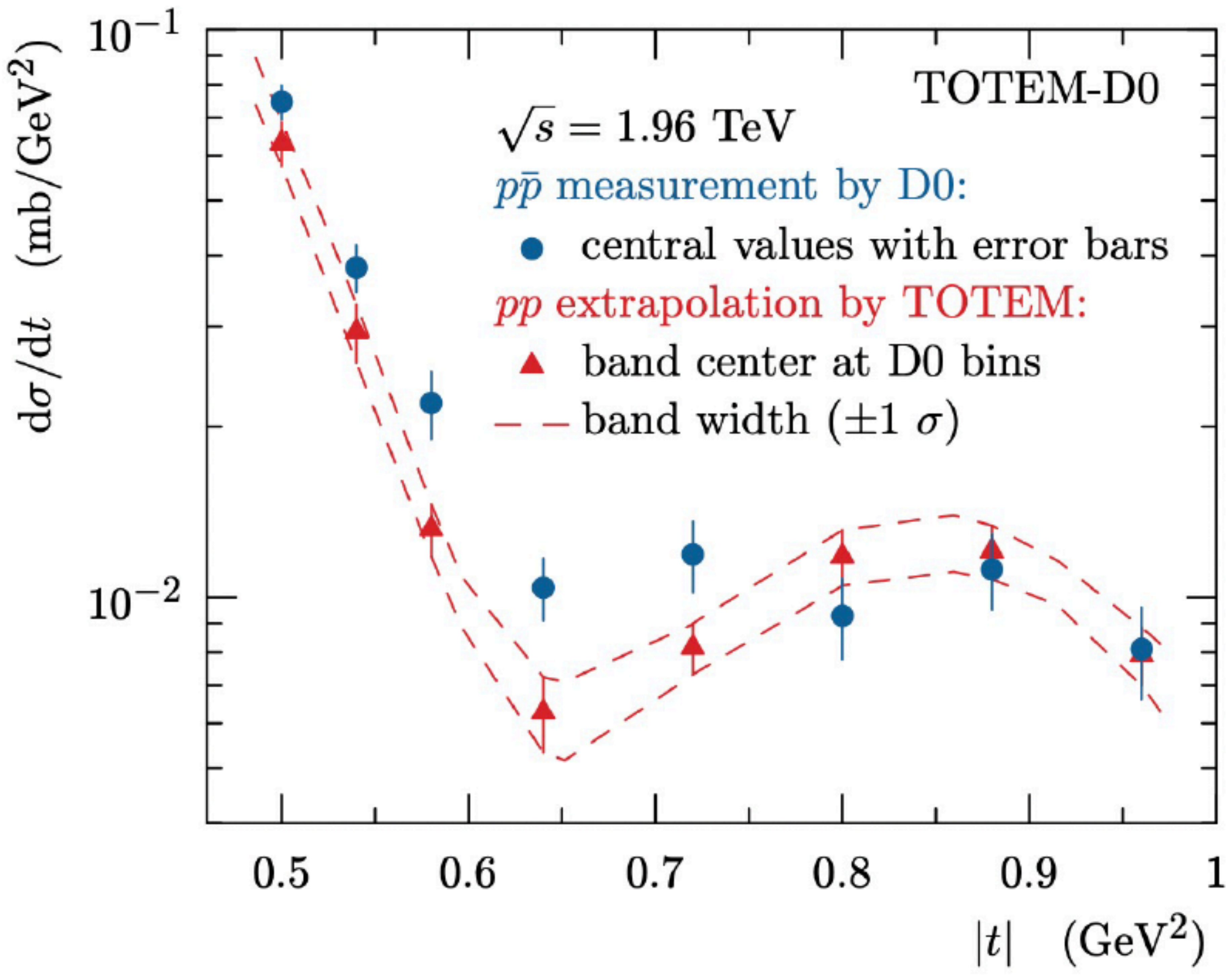




Z_1, Z_2, Z_3
A'
A
B
C
D
ℓ
$Re\, m^2$
f_{10}
f_8
f_6
f_4
g_6
g_4
g_2
16
12
8
4
−50
0
50
100
150
200
$B?$
$\ell = 1.15$
1.5
1.
0.5
−4
4
8
12
16
Z_1
Z_2
Z_3
−20
−12
−4
4
12

Similar Pattern for the diffractive cone





For $s \to \infty$, and fixed-t, Regge theory implies $T(s,t) \sim f(t) s^{\alpha(t)}$

Leading intercept should be $s^{1.15}$, but the ρ ansatz goes to a constant!