Università degli Studi di Padova

Istituto Nazionale di Fisica Nucleare
Sezione di Padova

Solving the Strong CP Problem with Modular Invariance

Arsenii Titov

Dipartimento di Fisica "Galileo Galilei", Università di Padova
INFN, Sezione di Padova

F. Feruglio, A. Strumia, AT, *JHEP* **07** (2023) 027 [2305.08908]
F. Feruglio, M. Parriciatu, A. Strumia, AT, *JHEP* **08** (2024) 214 [2406.01689]

Theory Group Seminar, Genova, 19 February 2025

Finanziato dall'Unione europea
NextGenerationEU

Ministero dell'Università e della Ricerca

Italiadomani
PIANO NAZIONALE DI RIPRESA E RESILIENZA

Outline

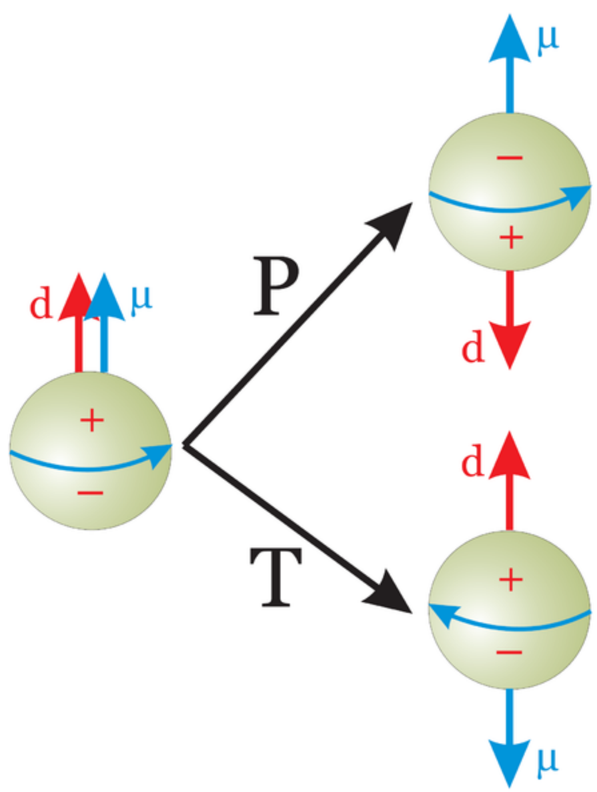
1. Strong CP problem
2. Existing solutions
3. Spontaneous CP violation
   - 3.1. Toy model with 2 quark generations
   - 3.2. Realistic scenario with 3 quark generations
   - 3.3. General conditions for $\bar{\theta} = 0$ and $\delta_{\mathrm{CKM}} \neq 0$
4. QFT realisation with supersymmetry
   - 4.1. U(1) flavour symmetry
   - 4.2. Modular invariance
   - 4.3. Flavour-SM anomalies
5. Modular-invariant models
   - 5.1. Models with SM quarks only
   - 5.2. Models with heavy vector-like quarks
6. Corrections to $\bar{\theta} = 0$
7. Conclusions

The strong CP problem

$$\mathscr{L}_{\text{QCD}} = \bar{q}\left(i\not{D} - M_q\right)q - \frac{1}{4g_3^2}G^a_{\mu\nu}G^{a,\mu\nu} + \frac{\theta_{\text{QCD}}}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}$$

$$\bar{\theta} = \theta_{\text{QCD}} + \arg\det M_q$$ CPV parameter

Neutron EDM d



$d = 2.4 \times 10^{-16}\ \bar{\theta}\ e \cdot$ cm Pospelov, Ritz, hep-ph/9908508v4

$|d| \leq 1.8 \times 10^{-26}\ e \cdot$ cm (90% C.L.) Abel et al., 2001.11966

$$|\bar{\theta}| \lesssim 10^{-10}$$

Why so small???

… and the CPV phase in the CKM matrix $\delta_{\text{CKM}} \approx 1.2$

Solution 1: the Axion

Promote $\bar{\theta}$ to a dynamical scalar field a, the axion, which washes out CP violation in QCD

$$\mathscr{L}_a = \frac{1}{2}(\partial_\mu a)^2 + \frac{1}{32\pi^2}\frac{a}{f_a}G\tilde{G} + \ldots$$

$$\bar{\theta} = \frac{\langle a \rangle}{f_a} \qquad \text{with} \qquad \langle a \rangle = 0$$

New global $\mathrm{U(1)_{PQ}}$ — Peccei, Quinn, PRL **38** (1977) 1440; PRD **16** (1977) 1791

- spontaneously broken => the axion is a NGB
- anomalous under QCD $(\partial_\mu J^\mu_{\mathrm{PQ}} \propto G\tilde{G})$ => the axion is a pNGB

Quality problem

- Corrections of order $(f_a/M_{\mathrm{Pl}})^\#$ from higher-dimensional operators
- $\mathrm{U(1)_{PQ}}$ should be an accidental symmetry in a complete model

Solution 2: P or CP is symmetry of UV

$\theta_{\rm QCD} G\tilde{G}$ is P-odd and C-even => CP-odd

- P or CP is a symmetry of the UV theory
- Challenge: break P/CP spontaneously such that $\bar{\theta} = 0$ and $\delta_{\rm CKM} = \mathcal{O}(1)$
- In the SM both P and CP are broken explicitly => BSM extensions

Solution 2.1: P is symmetry of UV

P-invariant theory Beg, Tsao, PRL **41** (1978) 278; Mohapatra, Senjanovic, PLB **79** (1978) 283

- Extended gauge group (e.g. left-right symmetry)
- Mirror fermions

Example: Babu-Mohapatra model Babu, Mohapatra, PRD **41** (1990) 1286

$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ with $L \leftrightarrow R$ and $H_L \sim (\mathbf{2}, \mathbf{1})$ and $H_R \sim (\mathbf{1}, \mathbf{2})$

$q_{L,R}$ are $\mathrm{SU}(2)_{L,R}$ doublets and $Q' = Q_L \oplus Q_R$ is vector-like singlet quark

$$(\overline{q_L}\ \overline{Q'_L})\, M_q \begin{pmatrix} q_R \\ Q'_R \end{pmatrix} = (\overline{q_L}\ \overline{Q'_L}) \begin{pmatrix} 0 & y_q \mathrm{v}_L \\ y_q^\dagger \mathrm{v}_R & M_{Q'} \end{pmatrix} \begin{pmatrix} q_R \\ Q'_R \end{pmatrix}$$

- $\det M_q = - y_q y_q^\dagger \mathrm{v}_L \mathrm{v}_R$ is real if v_L and v_R are real => $\arg\det M_q = 0$

Solution 2.2: CP is symmetry of UV

CP-invariant theory

- CP is broken by the Yukawa interactions
- Promote Yukawa couplings to dynamical variables
- Break CP spontaneously

Example: Nelson-Barr models Nelson, PLB **136** (1984) 387; Barr, PRL **53** (1984) 329

New heavy vector-like quarks Q' and scalars z with CPV complex VEVs $\langle z \rangle$

$$\left(\overline{q_L}\ \overline{Q'_L}\right) M_q \begin{pmatrix} q_R \\ Q'_R \end{pmatrix} = \left(\overline{q_L}\ \overline{Q'_L}\right) \begin{pmatrix} y \mathrm{v}_H & 0 \\ y' \langle z \rangle & M_{Q'} \end{pmatrix} \begin{pmatrix} q_R \\ Q'_R \end{pmatrix}$$

- CP is a symmetry => the couplings y, y' and the mass $M_{Q'}$ are real
- $\det M_q = y \mathrm{v}_H M_{Q'}$ is real (and positive) => $\arg \det M_q = 0$
- Light quark mass matrix depends on (multiple) $\langle z \rangle$ => $\delta_{\mathrm{CKM}} \neq 0$

Additional matter, tuning, loop corrections… Dine, Draper, 1506.05433

Toy model with 2 quark generations

$$\mathscr{L}_{\text{Yuk}} = U_i^c \, Y_{ij}^u(z) \, Q_j \, H_u + D_i^c \, Y_{ij}^d(z) \, Q_j \, H_d \qquad i, j = 1, 2$$

1. Give each field a charge: $k_{U_i^c}$, $k_{D_i^c}$, k_{Q_i} , k_{H_u} , k_{H_d} and $k_z = +1$
2. Assume $Y_{ij}^u = z^{k_{U_i^c} + k_{Q_j} + k_{H_u}}$ and $Y_{ij}^d = z^{k_{D_i^c} + k_{Q_j} + k_{H_d}}$

Toy model with 2 quark generations

$$\mathscr{L}_{\text{Yuk}} = U_i^c\, Y_{ij}^u(z)\, Q_j\, H_u + D_i^c\, Y_{ij}^d(z)\, Q_j\, H_d \qquad i, j = 1, 2$$

1. Give each field a charge: $k_{U_i^c}$, $k_{D_i^c}$, k_{Q_i}, k_{H_u}, k_{H_d} and $k_z = +1$
2. Assume $Y_{ij}^u = z^{k_{U_i^c} + k_{Q_j} + k_{H_u}}$ and $Y_{ij}^d = z^{k_{D_i^c} + k_{Q_j} + k_{H_d}}$

$$m_d = \mathrm{v}_d Y^d = \begin{pmatrix} c_{11}^d\, z^{k_{D_1^c} + k_{Q_1} + k_{H_d}} & c_{12}^d\, z^{k_{D_1^c} + k_{Q_2} + k_{H_d}} \\ c_{21}^d\, z^{k_{D_2^c} + k_{Q_1} + k_{H_d}} & c_{22}^d\, z^{k_{D_2^c} + k_{Q_2} + k_{H_d}} \end{pmatrix}$$

$c_{ij}^d \in \mathbb{R}$ by CP

$$\det m_d = \left(c_{11}^d c_{22}^d - c_{12}^d c_{21}^d\right) z^{d_d} \quad \text{with} \quad d_d = \sum_{i=1}^{2} \left(k_{D_i^c} + k_{Q_i}\right) + 2k_{H_d}$$

$$\det M_q = \det\left(m_u \oplus m_d\right) = \det\left(c^u c^d\right) z^d \quad \text{with} \quad d \equiv d_u + d_d$$

$d = 0 \quad \Rightarrow \quad \det M_q$ is a z-independent real constant

Example

$$k_{D^c} = (-1, +1) \qquad k_Q = (-1, +1) \qquad k_{H_d} = 0$$

$$m_d = \begin{pmatrix} c^d_{11}\, z^{-2} & c^d_{12} \\ c^d_{21} & c^d_{22}\, z^2 \end{pmatrix} \quad \text{with} \quad \det m_d = \det c^d$$

In an EFT, singularities occur when would-be heavy states integrated out from the full theory accidentally become massless (EFT breaks down)

Assuming no singularities leads to $c^d_{11} = 0$ and

$$m_d = \begin{pmatrix} 0 & c^d_{12} \\ c^d_{21} & c^d_{22}\, z^2 \end{pmatrix} \quad \text{with} \quad \det m_d = -\, c^d_{12}\, c^d_{21} \in \mathbb{R}$$

Model with 3 quark generations

$$\mathscr{L}_{\rm Yuk} = U_i^c\, Y_{ij}^u\left(z_1, \ldots, z_N\right)\, Q_j\, H_u + D_i^c\, Y_{ij}^d\left(z_1, \ldots, z_N\right)\, Q_j\, H_d \qquad i, j = 1, 2, 3$$

- No physical $\delta_{\rm CKM}$ in the case of 2 quark generations
- No physical $\delta_{\rm CKM}$ in the case of 3 quark generations and 1 complex scalar z
- At least 2 complex scalars z_1 and z_2 with a relative phase between their VEVs are needed to have $\delta_{\rm CKM} \neq 0$ see e.g. Kanemura et al., 0704.0697

Model with 3 quark generations

$$\mathscr{L}_{\text{Yuk}} = U_i^c\, Y_{ij}^u \left(z_1, \ldots, z_N\right)\; Q_j H_u + D_i^c\, Y_{ij}^d \left(z_1, \ldots, z_N\right)\; Q_j H_d \qquad i, j = 1, 2, 3$$

- No physical δ_{CKM} in the case of 2 quark generations
- No physical δ_{CKM} in the case of 3 quark generations and 1 complex scalar z
- At least 2 complex scalars z_1 and z_2 with a relative phase between their VEVs are needed to have $\delta_{\text{CKM}} \neq 0$ see e.g. Kanemura et al., 0704.0697

Example (assuming no singularities)
with 2 scalars z_+ and z_{++} with charges $k_{z_+} = +1$ and $k_{z_{++}} = +2$

$$k_{D^c} = (-1, 0, +1) \qquad k_Q = (-1, 0, +1) \qquad k_{H_d} = 0$$

$$m_d = \begin{pmatrix} 0 & 0 & c_{13}^d \\ 0 & c_{22}^d & c_{23}^d\, z_+ \\ c_{31}^d & c_{32}^d\, z_+ & c_{33}^d\, z_+^2 + c'^d_{33}\, z_{++} \end{pmatrix}$$

$\delta_{\text{CKM}} \neq 0$ if $\arg z_{++} \neq \arg z_+^2$

General conditions for $\bar{\theta} = 0$ and $\delta_{\text{CKM}} \neq 0$

$$\mathscr{L}_{\text{Yuk}} = U^c_i \, Y^u_{ij}\left(z_1, \ldots, z_N\right) \, Q_j \, H_u + D^c_i \, Y^d_{ij}\left(z_1, \ldots, z_N\right) \, Q_j \, H_d$$

1. $Y^q_{ij} = \sum_{\alpha_1 \ldots \alpha_N} c^q_{ij, \alpha_1 \ldots \alpha_N} \, z_1^{\alpha_1} \ldots z_N^{\alpha_N}$ with z_a being 'scalars' with complex VEVs

   $\alpha_a \geq 0$ (no singularities)

2. $Y^q_{ij}\left(\lambda^{k_1} z_1, \ldots, \lambda^{k_N} z_N\right) = \lambda^{k^q_{ij}} \, Y^q_{ij}\left(z_1, \ldots, z_N\right) \quad \forall \lambda \in \mathbb{C}$ with $k_a \geq 0$

3. $\det\left[Y^q_{ij}\left(\lambda^{k_1} z_1, \ldots, \lambda^{k_N} z_N\right)\right] = \lambda^{d_q} \det\left[Y^q_{ij}(z_1, \ldots, z_N)\right]$ with $d_q \equiv \sum_i k^q_{ii}$

4. $d \equiv d_u + d_d = 0$

General conditions for $\bar{\theta} = 0$ and $\delta_{\text{CKM}} \neq 0$

$$\mathscr{L}_{\text{Yuk}} = U_i^c \, Y_{ij}^u \left(z_1, \ldots, z_N \right) \, Q_j \, H_u + D_i^c \, Y_{ij}^d \left(z_1, \ldots, z_N \right) \, Q_j \, H_d$$

$$k_{ij}^u = k_{U_i^c} + k_{Q_j} + k_{H_u} \qquad\qquad k_{ij}^d = k_{D_i^c} + k_{Q_j} + k_{H_d}$$

$$\det \left[Y_{ij}^u \left(\lambda^{k_1} z_1, \ldots, \lambda^{k_N} z_N \right) \right] = \det \left[\lambda^{k_{ij}^u} \, Y_{ij}^u (z_1, \ldots, z_N) \right] = \lambda^{d_u} \det \left[Y_{ij}^u (z_1, \ldots, z_N) \right]$$

$$d_u = \sum_{i=1}^{3} \left(k_{U_i^c} + k_{Q_i} \right) + 3 k_{H_u} \qquad\qquad d_d = \sum_{i=1}^{3} \left(k_{D_i^c} + k_{Q_i} \right) + 3 k_{H_d}$$

$$d = \sum_{i=1}^{3} \left(2 k_{Q_i} + k_{U_i^c} + k_{D_i^c} \right) + 3 \left(k_{H_u} + k_{H_d} \right) = 0$$

Patterns of Yukawas for $\bar{\theta} = 0$ and $\delta_{\rm CKM} \neq 0$

$$Y = \begin{pmatrix} Y_{11}^{(0)} & Y_{12}^{(p_1-p_2)} & Y_{13}^{(p_1-p_3)} \\ Y_{21}^{(p_2-p_1)} & Y_{22}^{(0)} & Y_{23}^{(p_2-p_3)} \\ Y_{31}^{(p_3-p_1)} & Y_{32}^{(p_3-p_2)} & Y_{33}^{(0)} \end{pmatrix} \quad \text{with} \quad \det Y = \text{const}$$

- $p_1 = p_2 = p_3 \quad \Rightarrow \quad Y = \begin{pmatrix} Y_{11}^{(0)} & Y_{12}^{(0)} & Y_{13}^{(0)} \\ Y_{21}^{(0)} & Y_{22}^{(0)} & Y_{23}^{(0)} \\ Y_{31}^{(0)} & Y_{32}^{(0)} & Y_{33}^{(0)} \end{pmatrix} = \text{const}$
- $p_1 = p_2 \neq p_3 \quad \Rightarrow \quad Y = \begin{pmatrix} Y_{11}^{(0)} & Y_{12}^{(0)} & Y_{13}^{(p_1-p_3)}(z) \\ Y_{21}^{(0)} & Y_{22}^{(0)} & Y_{23}^{(p_1-p_3)}(z) \\ 0 & 0 & Y_{33}^{(0)} \end{pmatrix}$
- $p_1 \neq p_2 \neq p_3 \quad \Rightarrow \quad Y = \begin{pmatrix} Y_{11}^{(0)} & Y_{12}^{(p_1-p_2)}(z) & Y_{13}^{(p_1-p_3)}(z) \\ 0 & Y_{22}^{(0)} & Y_{23}^{(p_2-p_3)}(z) \\ 0 & 0 & Y_{33}^{(0)} \end{pmatrix}$

Feruglio, Parriciatu, Strumia, **AT**, 2406.01689 (see also Penedo, Petcov, 2404.08032)

QFT realisation with SUSY

Y^q depend on z_a and not on $\bar{z}_a$ => SUSY

$\mathcal{N} = 1$ global SUSY action

$$\mathcal{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\; K\left(e^{2V}\Phi, \Phi^\dagger\right) + \left[\int \mathrm{d}^2\theta\; W(\Phi) + \frac{1}{16}\int \mathrm{d}^2\theta\, f(\Phi)\; \mathcal{G}\,\mathcal{G} + \text{h.c.}\right]$$

Kähler potential K
(kinetic terms,
gauge interactions)

Superpotential W
(Yukawa interactions)

Gauge kinetic function f

$$f_3 = \frac{1}{g_3^2} - i\frac{\theta_{\mathrm{QCD}}}{8\pi^2}$$

$\Phi = \{M, Z\}$

$Z = \{Z_1, \ldots, Z_N\}$ dimensionless gauge-invariant chiral superfields

$M = \{U_i^c, D_i^c, E_i^c, Q_i, L_i, H_{u,d}\}$ matter and Higgs chiral super fields

QFT realisation with SUSY

The general conditions 1–4 are realised assuming invariance under

$$Q_i \to \Lambda^{-k_{Q_i}} Q_i \qquad U_i^c \to \Lambda^{-k_{U_i^c}} U_i^c \qquad D_i^c \to \Lambda^{-k_{D_i^c}} D_i^c \qquad Z_a \to \Lambda^{k_{Z_a}} Z_a$$

- The transformations Λ can be either global or local
- If local, they form a gauge group Γ, either continuous or discrete
- Γ can be realised either linearly or non-linearly

QFT realisation with SUSY

The general conditions 1–4 are realised assuming invariance under

$$Q_i \to \Lambda^{-k_{Q_i}} Q_i \qquad U_i^c \to \Lambda^{-k_{U_i^c}} U_i^c \qquad D_i^c \to \Lambda^{-k_{D_i^c}} D_i^c \qquad Z_a \to \Lambda^{k_{Z_a}} Z_a$$

- The transformations Λ can be either global or local
- If local, they form a gauge group Γ, either continuous or discrete
- Γ can be realised either linearly or non-linearly

The invariance of the superpotential

$$W(\Phi) = U_i^c\, Y_{ij}^u(Z)\, Q_j\, H_u + D_i^c\, Y_{ij}^d(Z)\, Q_j\, H_d$$

under these transformations requires

$$Y_{ij}^u\left(\Lambda^{k_{Z_1}} Z_1, \ldots, \Lambda^{k_{Z_N}} Z_N\right) = \Lambda^{k_{U_i^c}+k_{Q_j}+k_{H_u}}\, Y_{ij}^u\left(Z_1, \ldots, Z_N\right)$$

$$Y_{ij}^d\left(\Lambda^{k_{Z_1}} Z_1, \ldots, \Lambda^{k_{Z_N}} Z_N\right) = \Lambda^{k_{D_i^c}+k_{Q_j}+k_{H_d}}\, Y_{ij}^d\left(Z_1, \ldots, Z_N\right)$$

Linear vs non-linear realisation

Linearly realised Γ

$$\Lambda = \Lambda(\gamma) \qquad \gamma \in \Gamma$$

Z are elementary

$$\begin{cases} M_i \to \Lambda(\gamma)^{-k_{M_i}} M_i \\ Z_a \to \Lambda(\gamma)^{+k_{Z_a}} Z_a \end{cases}$$

The invariance of the Kähler potential

$$K\left(e^{2V}\Phi, \Phi^\dagger\right) = \Phi^\dagger e^{2V}\Phi$$

$$V = V_{\rm SM} + kV_{\rm BSM} \quad k = {\rm diag}\left(-k_{M_i}, k_{Z_a}\right)$$

under Γ implies

$$\begin{cases} V_{\rm SM} \to V_{\rm SM} \\ 2V_{\rm BSM} \to 2V_{\rm BSM} - \ln\Lambda(\gamma) - \ln\bar{\Lambda}(\gamma) \end{cases}$$

Linear vs non-linear realisation

Linearly realised Γ

$$\Lambda = \Lambda(\gamma) \qquad \gamma \in \Gamma$$

Z are elementary

$$\begin{cases} M_i \to \Lambda(\gamma)^{-k_{M_i}} M_i \\ Z_a \to \Lambda(\gamma)^{+k_{Z_a}} Z_a \end{cases}$$

The invariance of the Kähler potential

$$K\left(e^{2V}\Phi, \Phi^\dagger\right) = \Phi^\dagger e^{2V}\Phi$$

$$V = V_{\mathrm{SM}} + kV_{\mathrm{BSM}} \quad k = \mathrm{diag}\left(-k_{M_i}, k_{Z_a}\right)$$

under Γ implies

$$\begin{cases} V_{\mathrm{SM}} \to V_{\mathrm{SM}} \\ 2V_{\mathrm{BSM}} \to 2V_{\mathrm{BSM}} - \ln\Lambda(\gamma) - \ln\bar{\Lambda}(\gamma) \end{cases}$$

Non-linearly realised Γ

$$\Lambda = \Lambda(\gamma, \tau) \qquad \gamma \in \Gamma$$

$Z = Z(\tau)$ are composite

$$\begin{cases} \tau \to \gamma\tau \quad \text{non-linear transformation} \\ M_i \to \Lambda(\gamma, \tau)^{-k_{M_i}} M_i \\ Z_a(\tau) \to Z_a(\gamma\tau) = \Lambda(\gamma, \tau)^{+k_{Z_a}} Z_a(\tau) \end{cases}$$

$$K = M^\dagger e^{2V} M + K\left(\tau, \tau^\dagger\right)$$

$$V = V_{\mathrm{SM}} + kV_{\mathrm{BSM}}\left(\tau, \tau^\dagger\right)$$

$$K\left(\tau, \tau^\dagger\right) \to K\left(\tau, \tau^\dagger\right) + f_K(\gamma, \tau) + \bar{f}_K\left(\gamma, \tau^\dagger\right)$$

(invariant up to Kähler transformation)

U(1) vs modular symmetry

Linearly realised $\Gamma = \mathrm{U}(1)$

$$\Lambda = e^{i\,\alpha}$$

Z are elementary

$$\begin{cases} M_i \to e^{-i\,k_{M_i}\alpha}\, M_i \\ Z_a \to e^{+i\,k_{Z_a}\alpha}\, Z_a \end{cases}$$

The VEVs of Z_a break spontaneously U(1) and CP

One needs to build U(1) and CP invariant (super)potential realising the desired vacuum alignment

U(1) vs modular symmetry

Linearly realised $\Gamma = \mathrm{U}(1)$

$$\Lambda = e^{i\,\alpha}$$

Z are elementary

$$\begin{cases} M_i \to e^{-i\,k_{M_i}\alpha}\,M_i \\ Z_a \to e^{+i\,k_{Z_a}\alpha}\,Z_a \end{cases}$$

The VEVs of Z_a break spontaneously U(1) and CP

One needs to build U(1) and CP invariant (super)potential realising the desired vacuum alignment

Non-linearly realised modular symmetry

What is this symmetry?

What is τ ?

What are $\Lambda(\gamma, \tau)$ and $Z_a(\tau)$ in this case?

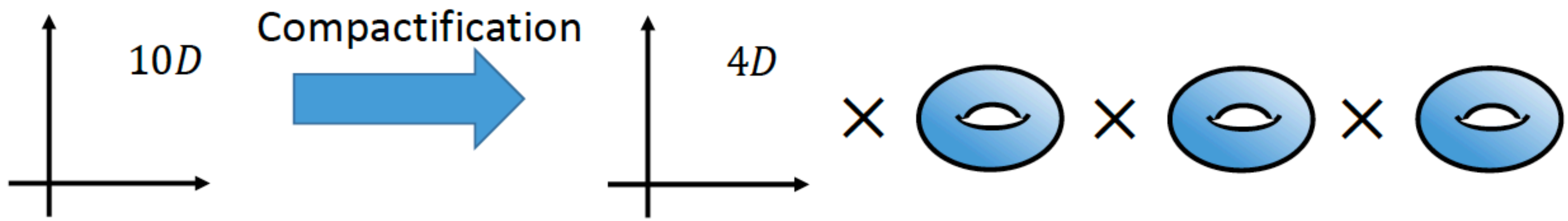
Modular invariance and CP are broken by the VEV of a single field τ

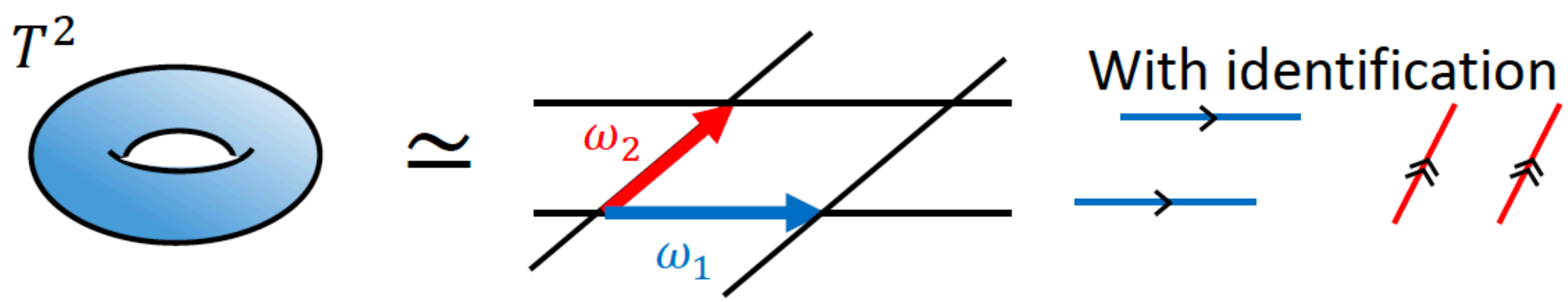
Presence of multiple 'scalars' $Z_a(\tau)$ with CP-violating VEVs is the result of the modular mathematics

Modular invariance

String theory requires extra dimensions

Images: Takuya H. Tatsuishi





$$\tau = \frac{\omega_2}{\omega_1}, \quad \mathrm{Im}\,(\tau) > 0$$

modulus

Lattice left invariant by modular transformations

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d} \qquad a, b, c, d \in \mathbb{Z} \qquad ad - bc = 1$$

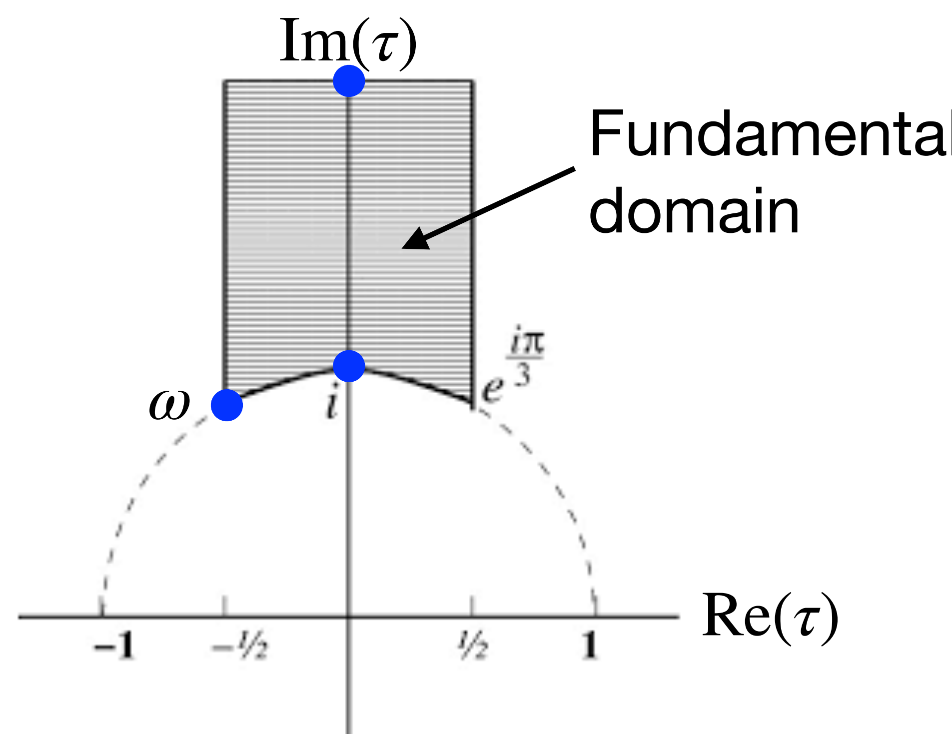
non-linear transformation

These transformations form an infinite discrete group

Modular group

Homogeneous modular group

$$\Gamma = \langle S,\, T \mid S^4 = (ST)^3 = I \rangle \cong \mathrm{SL}(2,\mathbb{Z})$$



$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad \tau \xrightarrow{T} \tau + 1$$

duality | discrete shift symmetry

Special points

- $\tau = i: \quad i \xrightarrow{S} -\frac{1}{i} = i \quad \Rightarrow \quad Z_4^S$
- $\tau = \omega \equiv e^{\frac{2\pi i}{3}}: \quad \omega \xrightarrow{ST} -\frac{1}{\omega+1} = \omega \quad \Rightarrow \quad Z_3^{ST} \times Z_2^{S^2}$
- $\tau = i\infty: \quad i\infty \xrightarrow{T} i\infty + 1 = i\infty \quad \Rightarrow \quad Z^T \times Z_2^{S^2}$

Modular forms

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under Γ as

$$Z\left(\gamma\tau\right) = \left(c\tau + d\right)^k Z(\tau)\,, \qquad \gamma \in \Gamma$$

k is weight, a non-negative even integer

Normalised Eisenstein series

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k}$$

Each modular form of weight k can be written as a polynomial in E_4 and E_6

$$Z(\tau) = \sum_{a,b\geq 0} c_{ab}\, E_4^a(\tau)\, E_6^b(\tau) \qquad \text{with} \qquad 4a + 6b = k$$

Modular weight k	0	2	4	6	8	10	12	14
Modular forms	1	–	E_4	E_6	$E_8 = E_4^2$	$E_{10} = E_4 E_6$	E_4^3, E_6^2	$E_{14} = E_4^2 E_6$

Modular-invariant SUSY theories

$\mathcal{N} = 1$ global SUSY action

$$\mathcal{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; K\left(\tau, e^{2V}M, \tau^\dagger, M^\dagger\right) + \left[\int \mathrm{d}^2\theta\; W(\tau, M) + \frac{1}{16}\int \mathrm{d}^2\theta\; f(\tau)\,\mathcal{G}\,\mathcal{G} + \text{h.c.}\right]$$

Kähler potential K
(kinetic terms,
gauge interactions)

Superpotential W
(Yukawa interactions)

Gauge kinetic function f

$$f_3 = \frac{1}{g_3^2} - i\frac{\theta_{\mathrm{QCD}}}{8\pi^2}$$

Under modular transformations $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ M \to \left(c\tau + d\right)^{-k_M} M \\ V \to V - k\ln|c\tau + d| \end{cases}$$

τ is promoted to a (dimensionless) superfield

matter supermultiplets $\quad \Lambda(\gamma, \tau) = c\tau + d$

vector supermultiplets $\quad 2V_{\mathrm{BSM}} = \ln\left(-i\tau + i\tau^\dagger\right)$

Modular symmetry acts non-linearly

Ferrara et al., PLB **225** (1989) 363; PLB **233** (1989) 147; Feruglio, 1706.08749

Modular-invariant SUSY theories

$\mathcal{N} = 1$ global SUSY action

$$\mathscr{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; K\left(\tau, e^{2V}M, \tau^\dagger, M^\dagger\right) + \left[\int \mathrm{d}^2\theta\; W(\tau, M) + \frac{1}{16}\int \mathrm{d}^2\theta\, f(\tau)\, \mathscr{G}\mathscr{G} + \text{h.c.}\right]$$

Kähler potential K
(kinetic terms,
gauge interactions)

Superpotential W
(Yukawa interactions)

Gauge kinetic function f

$$f_3 = \frac{1}{g_3^2} - i\frac{\theta_{\text{QCD}}}{8\pi^2}$$

Modular invariance of the action requires

$$\begin{cases} K(\tau, e^{2V}M, \tau^\dagger, M^\dagger) \to K(\tau, e^{2V}M, \tau^\dagger, M^\dagger) + f_K(\tau, M) + \overline{f_K}\left(\tau^\dagger, M^\dagger\right) \\ W(\tau, M) \to W(\tau, M) \\ f(\tau) \to f(\tau) \end{cases}$$

Ferrara et al., PLB **225** (1989) 363; PLB **233** (1989) 147; Feruglio, 1706.08749

Modular-invariant SUSY theories

Minimal Kähler potential

$$K = -h^2 \ln\left(-i\tau + i\tau^\dagger\right) + \sum_M \frac{M^\dagger e^{2V} M}{\left(-i\tau + i\tau^\dagger\right)^{k_M}}$$

Superpotential

$$W = Y^u_{ij}(\tau)\, U^c_i\, Q_j\, H_u + Y^d_{ij}(\tau)\, D^c_i\, Q_j\, H_d$$

τ-dependent Yukawa couplings

$$Y^q_{ij}(\tau) \to (c\tau + d)^{k^q_{ij}}\, Y^q_{ij}(\tau) \qquad \text{with} \qquad k^{u(d)}_{ij} = k_{U^c_i(D^c_i)} + k_{Q_j} + k_{H_{u(d)}}$$

are modular forms!

$$Y^q_{ij}(\tau) = c^q_{ij}\, Z_{k^q_{ij}}(\tau) \qquad \text{with} \qquad c^q_{ij} \in \mathbb{R} \qquad \text{because of CP}$$

Gauge kinetic function

$$f = \frac{1}{g_3^2} \qquad \theta_{\rm QCD} = 0 \quad \text{because of CP}$$

Modular invariance and CP

Fields

$$\tau \xrightarrow{\text{CP}} -\tau^\dagger \qquad \text{and} \qquad M \xrightarrow{\text{CP}} M^\dagger$$

Modular forms

$$Z(\tau) \xrightarrow{\text{CP}} Z(-\tau^*) = Z(\tau)^*$$

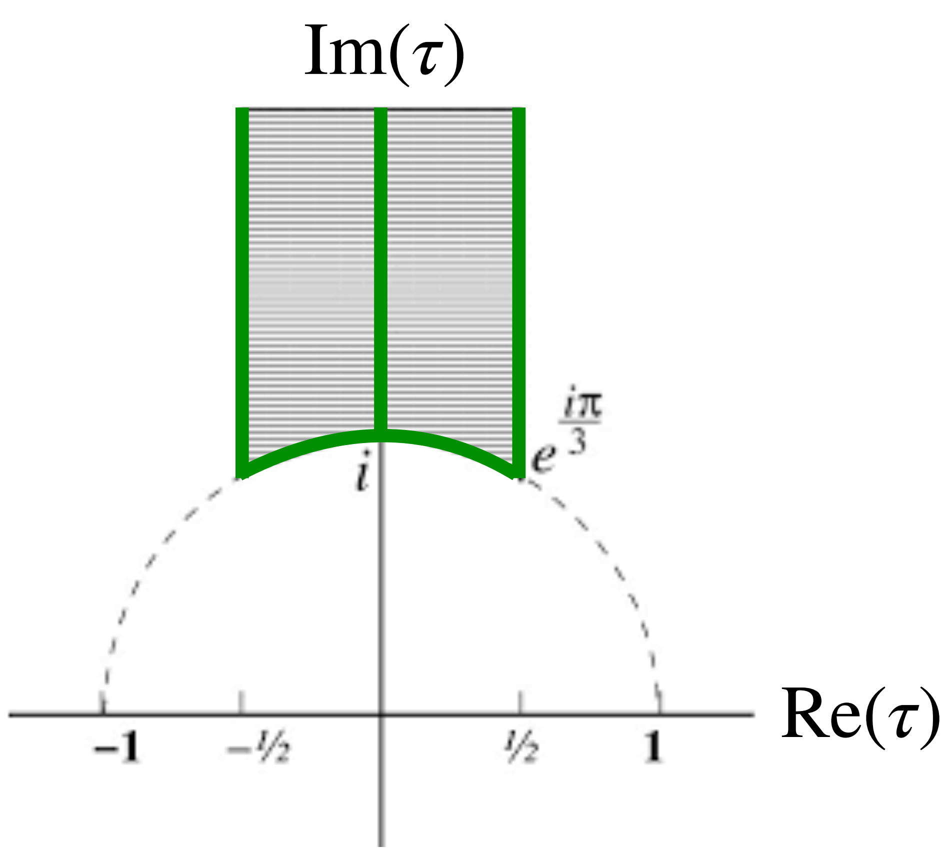
CP-conserving values of τ

$$\tau \xrightarrow{\text{CP}} -\tau^* = \gamma\tau$$ (goes to itself up to γ)

1. $\tau = iy \xrightarrow{\text{CP}} iy$

2. $\tau = -\frac{1}{2} + iy \xrightarrow{\text{CP}} \frac{1}{2} + iy = T\tau$

3. $\tau = e^{i\varphi} \xrightarrow{\text{CP}} -e^{-i\varphi} = S\tau$



Novichkov, Penedo, Petcov, **AT**, 1905.11970; Baur, Nilles, Trautner, Vaudrevange, 1901.03251

Determinant of quark mass matrix

$$M_u = \mathrm{v}_u Y^u \qquad M_d = \mathrm{v}_d Y^d$$

$$\det M_q = \det M_u \det M_d \propto \det Y^u \det Y^d$$

$$Y^q(\tau) = \begin{pmatrix} Z_{k^q_{11}} & Z_{k^q_{12}} & Z_{k^q_{13}} \\ Z_{k^q_{21}} & Z_{k^q_{22}} & Z_{k^q_{23}} \\ Z_{k^q_{31}} & Z_{k^q_{32}} & Z_{k^q_{33}} \end{pmatrix} \quad \Rightarrow$$

$\det Y^q(\tau)$ is a modular form of weight k^q_{det}

$$k^u_{\text{det}} = k^u_{11} + k^u_{22} + k^u_{33} = \ldots = \sum_{i=1}^{3} \left(k_{U^c_i} + k_{Q_i} \right) + 3k_{H_u}$$

And $\det Y^u(\tau) \det Y^d(\tau)$ is a modular form of weight k_{det}

$$k_{\text{det}} = k^u_{\text{det}} + k^d_{\text{det}} = \sum_{i=1}^{3} \left(2k_{Q_i} + k_{U^c_i} + k_{D^c_i} \right) + 3\left(k_{H_u} + k_{H_d} \right)$$

$k_{\text{det}} = 0 \quad \Rightarrow \quad \det Y^u(\tau) \det Y^d(\tau) =$ (real) constant

Matter fields and canonical normalisation

Gauge quantum numbers

	Q	U^c	D^c	L	E^c	H_u	H_d
$\mathrm{SU}(3)_c$	$\mathbf{3}$	$\overline{\mathbf{3}}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathrm{SU}(2)_L$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$
$\mathrm{U}(1)_Y$	$\frac{1}{6}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$

Canonical normalisation

$$K \supset \frac{M^\dagger M}{(-i\tau + i\tau^\dagger)^{k_M}} = M^\dagger_{\mathrm{can}} M_{\mathrm{can}} \qquad M_{\mathrm{can}} = \{\phi_{\mathrm{can}}, \psi_{\mathrm{can}}\}$$

$$\psi_{\mathrm{can}} \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{-\frac{k_M}{2}} \psi_{\mathrm{can}} = e^{-ik_M \alpha(\tau)} \psi_{\mathrm{can}} \qquad \alpha(\tau) = \arg(c\tau + d)$$

Modular symmetry acts on canonically normalised fields
as a τ-dependent phase rotation (with $\tau = \tau(x)$)

Modular-SM anomalies

Conditions for modular-gauge anomaly cancellation

$$\mathrm{SU}(3)_c: \quad A \equiv \sum_{i=1}^{3} \left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) = 0$$

$$\mathrm{SU}(2)_L: \quad \sum_{i=1}^{3} \left(3k_{Q_i} + k_{L_i}\right) + k_{H_u} + k_{H_d} = 0$$

$$\mathrm{U}(1)_Y: \quad \sum_{i=1}^{3} \left(k_{Q_i} + 8k_{U_i^c} + 2k_{D_i^c} + 3k_{L_i} + 6k_{E_i^c}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

Simplest solution

$$k_Q = k_{U^c} = k_{D^c} = k_L = k_{E^c} = (-k, 0, k) \qquad \text{and} \qquad k_{H_u} + k_{H_d} = 0$$

Cancellation of modular-QCD anomaly along with $k_{H_u} + k_{H_d} = 0$ implies

$$k_{\text{det}} = \sum_{i=1}^{3} \left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

Simplest example: quarks

Simplest non-trivial example giving $k_{\rm det} = 0$ and $A = 0$

$$k_Q = k_{U^c} = k_{D^c} = (-6, 0, 6) \quad \text{and} \quad k_{H_u} = k_{H_d} = 0$$

Yukawa matrices

$$Y^q = \begin{pmatrix} 0 & 0 & c^q_{13} \\ 0 & c^q_{22} & c^q_{23} E_6 \\ c^q_{31} & c^q_{32} E_6 & c^q_{33} E_4^3 + c'^q_{33} E_6^2 \end{pmatrix} \quad \Rightarrow \quad Y^q|_{\rm can} = \begin{pmatrix} 0 & 0 & c^q_{13} \\ 0 & c^q_{22} & c^q_{23}\,(2{\rm Im}\tau)^3\, E_6 \\ c^q_{31} & c^q_{32}\,(2{\rm Im}\tau)^3\, E_6 & (2{\rm Im}\tau)^6 \left[c^q_{33} E_4^3 + c'^q_{33} E_6^2\right] \end{pmatrix}$$

$$\det Y^q|_{\rm can} = -\, c^q_{13} c^q_{22} c^q_{31} \in \mathbb{R}$$

Fixing $\tau = 1/8 + i$ and $\tan\beta = 10$

$$c^u_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.56 \\ 0 & -1.86 & 0.87 \\ 1.29 & 4.14 & 3.51,\ 1.40 \end{pmatrix} \qquad c^d_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.55 \\ 0 & -2.59 & 4.59 \\ 0.378 & 0.710 & 0.734,\ 1.76 \end{pmatrix}$$

reproduce the quark masses, mixing angles and $\delta_{\rm CKM}$ at the GUT scale of 2×10^{16} GeV

Simplest example: leptons

$k_L = k_{E^c} = (-6, 0, 6)$

Weinberg operator $\mathscr{C}^\nu_{ij}(L_i H_u)(L_j H_u)$ for neutrino masses

Charged lepton Yukawa matrix and coefficient of the Weinberg operator

$$Y^e = \begin{pmatrix} 0 & 0 & c^e_{13} \\ 0 & c^e_{22} & c^e_{23} E_6 \\ c^e_{31} & c^e_{32} E_6 & c^e_{33} E_4^3 + c'^e_{33} E_6^2 \end{pmatrix} \qquad \mathscr{C}^\nu = \begin{pmatrix} 0 & 0 & c^\nu_{13} \\ 0 & c^\nu_{22} & c^\nu_{23} E_6 \\ c^\nu_{13} & c^\nu_{23} E_6 & c^\nu_{33} E_4^3 + c'^\nu_{33} E_6^2 \end{pmatrix}$$

Fixing $\tau = 1/8 + i$ and $\tan\beta = 10$

$$c^e_{ij} = 10^{-3} \begin{pmatrix} 0 & 0 & 1.29 \\ 0 & 5.95 & 0.35 \\ -2.56 & 1.47 & 1.01,\ 1.32 \end{pmatrix} \qquad c^\nu_{ij} = \frac{1}{10^{16}\ \mathrm{GeV}} \begin{pmatrix} 0 & 0 & 3.4 \\ 0 & 7.1 & 1.2 \\ 3.4 & 1.2 & 0.19,\ 0.95 \end{pmatrix}$$

reproduce the lepton masses and mixings, including δ_{PMNS}

Models with larger modular weights

Yukawa matrices	Modular weights			Alternative bigger weights		
$Y_{u,d}$	$(u_L, d_L)_{1,2,3}$	$u_{R1,2,3}$	$d_{R1,2,3}$	$(u_L, d_L)_{1,2,3}$	$u_{R1,2,3}$	$d_{R1,2,3}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_6 \\ 1 & E_6 & E_4^3 + E_6^2 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ -2 \\ 4 \end{pmatrix}$	$\begin{pmatrix} -8 \\ -2 \\ 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^2 \\ 1 & E_4 & E_4^3 + E_6^2 \end{pmatrix}$	$\begin{pmatrix} -6 \\ -2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^2 \\ 1 & E_4^2 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4 E_6 \\ 1 & E_6 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ -2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 2 \\ 8 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^3 + E_6^2 \\ 1 & E_4 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ -4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix}$

Finite modular groups

Principle congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ of level $N = 2\,,\,3\,,\,4\,,\ldots$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Finite modular groups

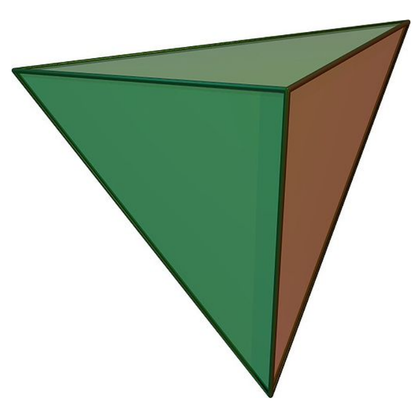
$$\Gamma_N \equiv \Gamma/\Gamma(N)$$

$$\Gamma_2 = \left\langle S,\, T \mid S^2 = (ST)^3 = T^2 = I \right\rangle$$

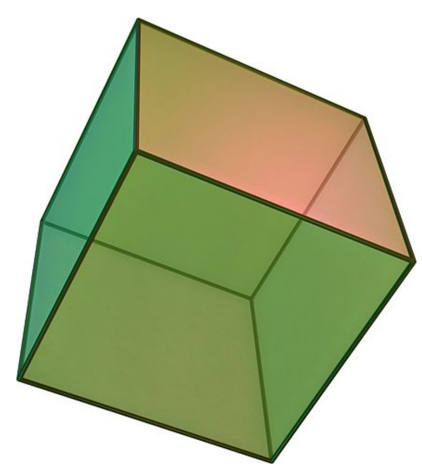
$$\Gamma_N = \left\langle S,\, T \mid S^4 = (ST)^3 = T^N = I\,, \quad S^2T = TS^2 \right\rangle\,, \quad N = 3\,,\,4\,,\,5$$

$\Gamma_2 \cong S_3$

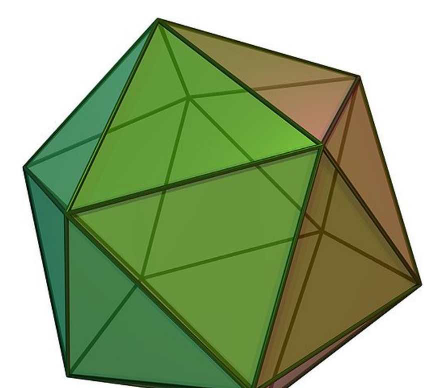
$\Gamma_3 \cong A_4' = T'$



$\Gamma_4 \cong S_4'$



$\Gamma_5 \cong A_5'$



Models with finite modular symmetries

Feruglio, 1706.08749

$$\begin{cases} M \to \left(c\tau + d\right)^{-k_M} \rho_M(\gamma)\, M \\ Z \to \left(c\tau + d\right)^{k_Z} \rho_Z(\gamma)\, Z \end{cases}$$

ρ is a unitary representation of the finite modular group Γ_N

Invariance of the superpotential $W \sim Z_{ijk}(\tau)\, M_i M_j M_k$ under Γ and Γ_N requires

$$\begin{cases} k_{Z_{ijk}} = k_{M_i} + k_{M_j} + k_{M_k} \\ \rho_Z \otimes \rho_{M_i} \otimes \rho_{M_j} \otimes \rho_{M_k} \supset \mathbf{1}_0 \end{cases}$$

Models with finite modular symmetries

$$\begin{cases} M \to (c\tau + d)^{-k_M}\, \rho_M(\gamma)\, M \\ Z \to (c\tau + d)^{k_Z}\, \rho_Z(\gamma)\, Z \end{cases}$$

Feruglio, 1706.08749

ρ is a unitary representation of the finite modular group Γ_N

Invariance of the superpotential $W \sim Z_{ijk}(\tau)\, M_i M_j M_k$ under Γ and Γ_N requires

$$\begin{cases} k_{Z_{ijk}} = k_{M_i} + k_{M_j} + k_{M_k} \\ \rho_Z \otimes \rho_{M_i} \otimes \rho_{M_j} \otimes \rho_{M_k} \supset \mathbf{1}_0 \end{cases}$$

Penedo, Petcov, 2404.08032

Multi-dim irreps $\rho \sim \mathbf{r}\,,\ \ \mathbf{r} > 1$ for SM quarks do not allow to simultaneously

- realise the proposed mechanism for $\bar{\theta} = 0$
- successfully describe quark masses and mixings

For the mechanism to work, SM quarks must furnish 1D irreps $\rho \sim \mathbf{1}_0,\ \mathbf{1}_1,\ \mathbf{1}_2, \dots$

Models with finite modular symmetries

Minimal models (6 Lagrangian parameters per sector) Penedo, Petcov, 2404.08032

$$Y^q = \begin{pmatrix} c^q_{11} & 0 & c^q_{13}\, Z^{(k')}_{\mathbf{1}^*} + c'^q_{13}\, Z'^{(k')}_{\mathbf{1}^*} \\ 0 & c^q_{22} & c^q_{23}\, Z^{(k)}_{\mathbf{1}^*} \\ 0 & 0 & c^q_{33} \end{pmatrix}$$

(up to weak basis transformations)

Minimal models (I and II)	
All Γ'_N	$(k, k') = (10, 12),\ (12, 14),\ (14, 16)$
S_3 only	$(10', 12),\ (10, 18'),\ (10', 18),\ (12, 12'),$ $(12, 14'),\ (12, 16'),\ (12', 16),\ (12, 20'),$ $(12', 20),\ (14', 16),\ (14, 18'),\ (14', 18),$ $(14, 22'),\ (14', 22),\ (16, 16'),\ (16', 18),$ $(16', 18'),\ (16, 20'),\ (16', 20),\ (18, 20'),$ $(18', 20'),\ (20, 20'),\ (20', 22),\ (20', 22')$
A'_4 only	$(8', 12),\ (8', 18),\ (10', 12),\ (10, 16'),$ $(10', 16),\ (10, 20''),\ (10', 20),\ (12, 12'),$ $(12, 12''),\ (12, 14'),\ (12, 14''),\ (12, 16''),$ $(12', 16),\ (12'', 16'),\ (12, 18'),\ (12, 18''),$ $(12', 18),\ (12'', 18),\ (12, 22''),\ (12', 22),$ $(12'', 22'),\ (14, 16'),\ (14', 16),\ (14', 16'),$ $(14'', 16),\ (14'', 16'),\ (14', 18),\ (14, 20'),$ $(14, 20''),\ (14', 20),\ (14', 20''),\ (14'', 20),$ $(14'', 20'),\ (14, 24''),\ (14'', 24'),\ (16, 16''),$ $(16', 16''),\ (16, 18'),\ (16, 18''),\ (16', 18'),$ $(16', 18''),\ (16'', 18),\ (16'', 20'),\ (16, 22''),$ $(16', 22''),\ (16'', 22),\ (16'', 22'),$ $(16'', 26'),\ (18, 18'),\ (18, 18''),\ (18', 20),$ $(18', 20'),\ (18', 20''),\ (18'', 20),\ (18'', 20'),$ $(18'', 20''),\ (18, 22''),\ (18', 22),\ (18'', 22'),$ $(18', 24''),\ (18'', 24'),\ (20, 22''),\ (20', 22''),$ $(20'', 22''),\ (22, 22''),\ (22', 22''),$ $(22'', 24'),\ (22'', 24''),\ (22'', 26')$

Modular weights are large

String compactifications
favour smaller weights

(For models based on modular A_4
see also Petcov, Tanimoto, 2404.00858)

Heavy quarks and singularities

- Heavy quarks are not needed for the mechanism to work, but assume they exist

$$k_q = (-6, -2, 0, +2, +6) \quad \text{and} \quad k_{H_u} = k_{H_d} = 0$$

Light chiral quarks Heavy vector-like quarks

- In the full theory $f_{\text{UV}} \in \mathbb{R}$ and $\det M_{\text{all}} \in \mathbb{R}$ => $\bar{\theta}_{\text{UV}} = 0$

$$M_{\text{all}} = \begin{pmatrix} M_{LL} & M_{LH} \\ M_{HL} & M_{HH} \end{pmatrix}$$

$$M_{\text{light}} \approx M_{LL} - M_{LH} M_{HH}^{-1} M_{HL} \qquad M_{\text{heavy}} \approx M_{HH}$$

Singularities where $\det M_{\text{heavy}}(\tau) = 0$
(breakdown of EFT)

$$\det M_{\text{all}} = \det M_{\text{light}} \det M_{\text{heavy}}$$

$$\det M_{\text{light}} \to (c\tau + d)^{k_{\text{light}}} \det M_{\text{light}} \qquad \det M_{\text{heavy}} \to (c\tau + d)^{k_{\text{heavy}}} \det M_{\text{heavy}}$$

$$k_{\text{all}} = k_{\text{light}} + k_{\text{heavy}} = 0$$

EFT of light quarks

In the EFT of light quarks

$$\bar{\theta}_{\mathrm{IR}} = \theta_{\mathrm{QCD}} + \arg\det M_{\mathrm{light}} = -8\pi^2\,\mathrm{Im}\,f_{\mathrm{IR}} + \arg\det M_{\mathrm{light}}$$

The EFT has anomalous field content with $k_q = (-6, -2, 0)$

Anomaly is cancelled by a new contribution to the gauge kinetic function arising from the integration over the heavy quarks

$$f_{\mathrm{IR}} = f_{\mathrm{UV}} - \frac{1}{8\pi^2}\ln\det M_{\mathrm{heavy}}$$

Thus

$$\bar{\theta}_{\mathrm{IR}} = \arg\det M_{\mathrm{heavy}} + \arg\det M_{\mathrm{light}} = \arg\det M_{\mathrm{all}} = 0$$

Extensions of Nelson-Barr models

Heavy vector-like $\mathrm{SU}(2)_L$ singlet quarks — for review see Alves et al., 2304.10561

$$D'_\alpha \sim (\mathbf{3}, \mathbf{1}, -1/3) \qquad D'^c_\alpha \sim (\bar{\mathbf{3}}, \mathbf{1}, +1/3) \qquad \alpha = \{1, \ldots, P\}$$

$$W_{\rm UV} = y^d_{ij}\, Q_i\, D^c_j\, H_d + y'^d_{i\beta}\, Q_i\, D'^c_\beta\, H_d + N^d_{\alpha i}\, D'_\alpha\, D^c_i + M^d_{\alpha\beta}\, D'_\alpha\, D'^c_\beta$$

$$\mathscr{M}_d = \begin{matrix} \\ Q \\ D' \end{matrix} \begin{matrix} D^c \quad D'^c \\ \begin{pmatrix} m^d & n^d \\ N^d & M^d \end{pmatrix} \end{matrix} \qquad m^d = \mathrm{v}_d\, y^d \qquad n^d = \mathrm{v}_d\, y'^d$$

$$m^q_{\rm IR} = K_Q^{-1/2\,T} \left[m^q - n^q \left(M^q\right)^{-1} N^q \right] K_{q^c}^{-1/2}$$

$$K_Q^T = \mathbf{1} + \left(m^q\, N^{q\dagger} + n^q\, M^{q\dagger}\right) \left(M^q M^{q\dagger} + N^q N^{q\dagger}\right)^{-2} \left(N^q\, m^{q\dagger} + M^q\, n^{q\dagger}\right)$$

$$K_{q^c} = \mathbf{1} + N^{q\dagger} \left(M^q M^{q\dagger}\right)^{-1} N^q$$

Nelson-Barr limit: $\mathscr{M}_q = \begin{pmatrix} m^q & 0 \\ N^q(Z) & M^q \end{pmatrix}$ and $m^q_{\rm IR} = K_Q^{-1/2\,T}\, m^q\, K_{q^c}$

In particular, $\delta_{\rm CKM}$ originates from wave-function renormalisation factors K_Q and K_{q^c}

Model with VLQs based on $\Gamma_2 \cong S_3$

Feruglio, Parriciatu, Strumia, **AT**, 2406.01689

	SM quarks			Extra vector-like quarks			
	Q	D^c	U^c	D'^c	D'	U'^c	U'
$\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$	$2_{1/6}$	$1_{1/3}$	$1_{-2/3}$	$1_{1/3}$	$1_{-1/3}$	$1_{-2/3}$	$1_{2/3}$
Flavor symmetry Γ_2	$\mathbf{2}\oplus\mathbf{1}_0$	$\mathbf{2}\oplus\mathbf{1}_1$	$\mathbf{2}\oplus\mathbf{1}_0$	$\mathbf{2}\oplus\mathbf{1}_0$	$\mathbf{2}\oplus\mathbf{1}_1$	$\mathbf{2}\oplus\mathbf{1}_0$	$\mathbf{2}\oplus\mathbf{1}_0$
Modular weights k_Φ	-2	-2	-2	$+2$	$+2$	$+2$	$+2$

$$m^d = 0_{3\times 3} \qquad n^d = n_d \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha_d \end{pmatrix} \qquad N^d = N_d \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta_d \end{pmatrix} \qquad M^d = M_d \begin{pmatrix} -Z_1^{(4)} + \gamma_{d1} Z_3^{(4)} & Z_2^{(4)} & \gamma_{d2} Z_1^{(4)} \\ Z_2^{(4)} & Z_1^{(4)} + \gamma_{d1} Z_3^{(4)} & \gamma_{d2} Z_2^{(4)} \\ \gamma_{d3} Z_2^{(4)} & -\gamma_{d3} Z_1^{(4)} & 0 \end{pmatrix}$$

$(Z_1^{(4)}(\tau), Z_2^{(4)}(\tau))^T \sim \mathbf{2}$ $\qquad Z_3^{(4)}(\tau) \sim \mathbf{1}_0$ $\qquad$ level $N = 2$ modular forms of weight $k = 4$

$$\det \mathscr{M}_d = \det\left[m^d - n^d \left(M^d\right)^{-1} N_d\right] \det M^d = -\det\left[n_d N_d\right] = -n_d^3 N_d^3 \alpha_d \beta_d \in \mathbb{R}$$

In the full theory: $\bar{\theta}_{\mathrm{UV}} = \arg\det\left(\mathscr{M}_u \mathscr{M}_d\right) = 0$

In the EFT: $\bar{\theta}_{\mathrm{IR}} = -8\pi^2 \,\mathrm{Im}\, f_{\mathrm{IR}} + \arg\det\left(m^u_{\mathrm{IR}} m^d_{\mathrm{IR}}\right) = \arg\det\left(\mathscr{M}_u \mathscr{M}_d\right) = 0$

$$f_{\mathrm{IR}} = f_{\mathrm{UV}} - \frac{1}{8\pi^2} \ln\det\left(M^u M^d\right)$$

Model with VLQs based on $\Gamma_2 \cong S_3$

Feruglio, Parriciatu, Strumia, **AT**, 2406.01689

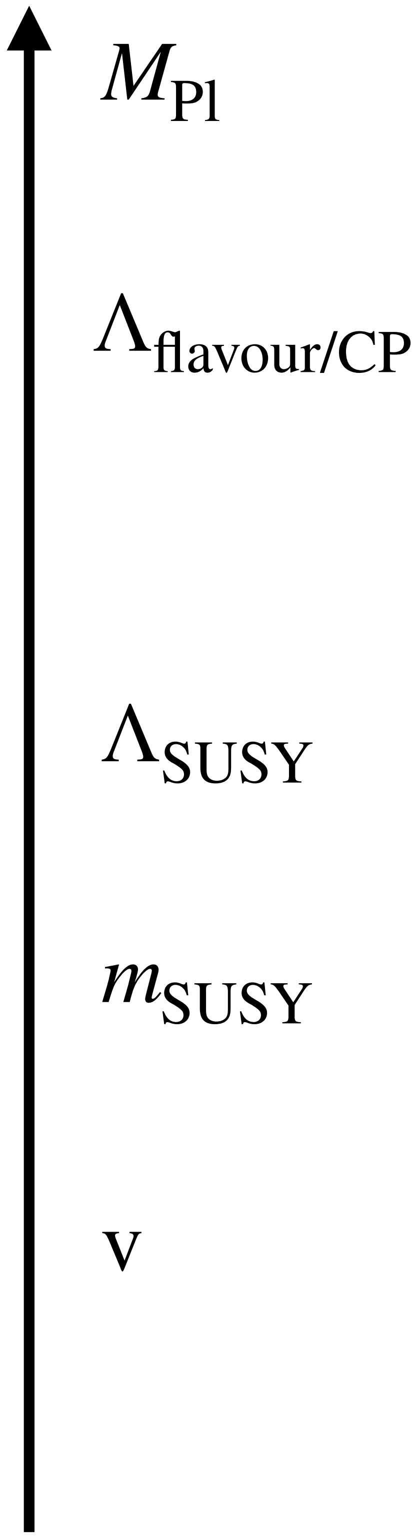
	SM quarks			Extra vector-like quarks			
	Q	D^c	U^c	D'^c	D'	U'^c	U'
$\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$	$2_{1/6}$	$1_{1/3}$	$1_{-2/3}$	$1_{1/3}$	$1_{-1/3}$	$1_{-2/3}$	$1_{2/3}$
Flavor symmetry Γ_2	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_1$	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_1$	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_0$
Modular weights k_Φ	-2	-2	-2	$+2$	$+2$	$+2$	$+2$

Input parameters					
Modulus		Up sector		Down sector	
$\mathrm{Re}\,\tau$	-0.4347	α_u	13.71	α_d	13.30
$\mathrm{Im}\,\tau$	1.646	β_u	-0.0723	β_d	-4.681
		γ_{u0}	0.1087		
		γ_{u1}	-0.2442	γ_{d1}	1.105
		γ_{u2}	-0.0216	γ_{d2}	-0.2165
		γ_{u3}	-62.38	γ_{d3}	-0.0460
		N_u/M_u	1.515	N_d/M_d	0.6220
		n_u [GeV]	24.82	n_d [GeV]	0.3411

Output values	
m_u/m_c	1.46×10^{-3}
m_c/m_t	2.68×10^{-3}
m_d/m_s	4.99×10^{-2}
m_s/m_b	1.42×10^{-2}
$\sin^2\theta_{12}$	5.06×10^{-2}
$\sin^2\theta_{13}$	1.03×10^{-5}
$\sin^2\theta_{23}$	1.25×10^{-3}
δ/π	0.391
χ^2	0.47

- Low weights $|k_M| \leq 2$ (favoured by string theory compactifications) 😊
- 15 dimensionless parameters to describe 8 dimensionless observables 😢

Corrections to $\bar{\theta} = 0$



Corrections to $\bar{\theta} = 0$

$M_{\rm Pl}$

$\Lambda_{\rm flavour/CP}$

$\Lambda_{\rm SUSY}$

$m_{\rm SUSY}$

v

SUSY unbroken

- Modular invariance determines completely (up to real couplings) the functional dependence $W(\tau)$
- It is not the case for K, but $\bar{\theta}$ is insensitive to K
- No-renormalisation theorems Ellis, Ferrara, Nanopoulos, PLB **114** (1982) 231

Corrections to $\bar{\theta} = 0$

$M_{\rm Pl}$

SUSY unbroken

$\Lambda_{\rm flavour/CP}$

- Modular invariance determines completely (up to real couplings) the functional dependence $W(\tau)$
- It is not the case for K, but $\bar{\theta}$ is insensitive to K
- No-renormalisation theorems Ellis, Ferrara, Nanopoulos, PLB **114** (1982) 231

$\Lambda_{\rm SUSY}$

SUSY breaking corrections

- In general, can be large

$m_{\rm SUSY}$

- Small if $\Lambda_{\rm flavour/CP} \gg \Lambda_{\rm SUSY}$ (as e.g. in gauge mediation) and soft ~~SUSY~~ terms respect the flavour structure of the SM

v

$$\bar{\theta} \lesssim \frac{M_t^4 M_b^4 M_c^2 M_s^2}{\mathrm{v}^{12}} J_{\rm CP} \tan^6\beta \sim 10^{-28} \tan^6\beta$$

Corrections to $\bar{\theta} = 0$

M_{Pl}

SUSY unbroken

- Modular invariance determines completely (up to real couplings) the functional dependence $W(\tau)$

$\Lambda_{\text{flavour/CP}}$

- It is not the case for K, but $\bar{\theta}$ is insensitive to K
- No-renormalisation theorems Ellis, Ferrara, Nanopoulos, PLB **114** (1982) 231

Λ_{SUSY}

SUSY breaking corrections

- In general, can be large
- Small if $\Lambda_{\text{flavour/CP}} \gg \Lambda_{\text{SUSY}}$ (as e.g. in gauge mediation) and soft ~~SUSY~~ terms respect the flavour structure of the SM

m_{SUSY}

$$\bar{\theta} \lesssim \frac{M_t^4 M_b^4 M_c^2 M_s^2}{\text{v}^{12}} J_{\text{CP}} \tan^6\beta \sim 10^{-28} \tan^6\beta$$

v

SM corrections are negligible

- $\bar{\theta} \sim 10^{-18}$ at four loops Khriplovich, PLB **173** (1986) 193
Ellis, Gaillard, NPB **150** (1979) 141

Conclusions

- General conditions for $\bar{\theta} = 0$ and $\delta_{\rm CKM} \neq 0$ in theories with spontaneous $\not{CP}$
- Can be implemented with an anomaly-free local flavour symmetry (and SUSY)

Conclusions

- General conditions for $\bar{\theta} = 0$ and $\delta_{\text{CKM}} \neq 0$ in theories with spontaneous $\not{CP}$
- Can be implemented with an anomaly-free local flavour symmetry (and SUSY)

- Modular invariance is inherent to toroidal compactifications in string theory
- It can be consistently implemented in a SUSY QFT
- The VEV of the modulus τ is the only source of spontaneous $\not{CP}$

Conclusions

- General conditions for $\bar{\theta} = 0$ and $\delta_{\text{CKM}} \neq 0$ in theories with spontaneous $\not{CP}$
- Can be implemented with an anomaly-free local flavour symmetry (and SUSY)

- Modular invariance is inherent to toroidal compactifications in string theory
- It can be consistently implemented in a SUSY QFT
- The VEV of the modulus τ is the only source of spontaneous $\not{CP}$

$$\bar{\theta} = \theta_{\text{QCD}} + \arg\det M_q$$

- $\theta_{\text{QCD}} = 0$ because the UV theory is CP-conserving
- $\arg\det M_q = 0$ because of anomaly-free modular symmetry
- Heavy VLQs are not needed, but can help to lower modular weights
- Corrections to $\bar{\theta} = 0$ are small under certain assumptions on SUSY breaking

Back-up slides

Our solution: CP + modular invariance

1. CP is a symmetry => $\theta_{\text{QCD}} = 0$ (and real Lagrangian couplings)

2. Modular invariance/anomaly cancellation => $\arg\det M_q = 0$

3. CP is broken spontaneously by the VEV of a single complex scalar field, the modulus τ => $\delta_{\text{CKM}} = \mathcal{O}(1)$

4. Quark mass hierarchies and mixing angles are reproduced by $\mathcal{O}(1)$ parameters

5. Corrections to $\bar{\theta} = 0$ are small under certain assumptions on SUSY breaking

Phenomenology and cosmology

- Couplings to matter are suppressed by $1/h$ ($1/M_{\rm Pl}$ in SUGRA)
- No couplings to gauge bosons in the exact SUSY limit
- $m_\tau \gtrsim 10$ TeV not to spoil BBN
- Fermionic component of τ could be LSP and maybe DM
- Scalar potential $V(\tau) = V(-\tau^*)$ => CP-conjugated minima
  (domain walls are inflated away if CP breaking occurs before inflation)

CPon dark matter

Feruglio, Ziegler, 2411.08101

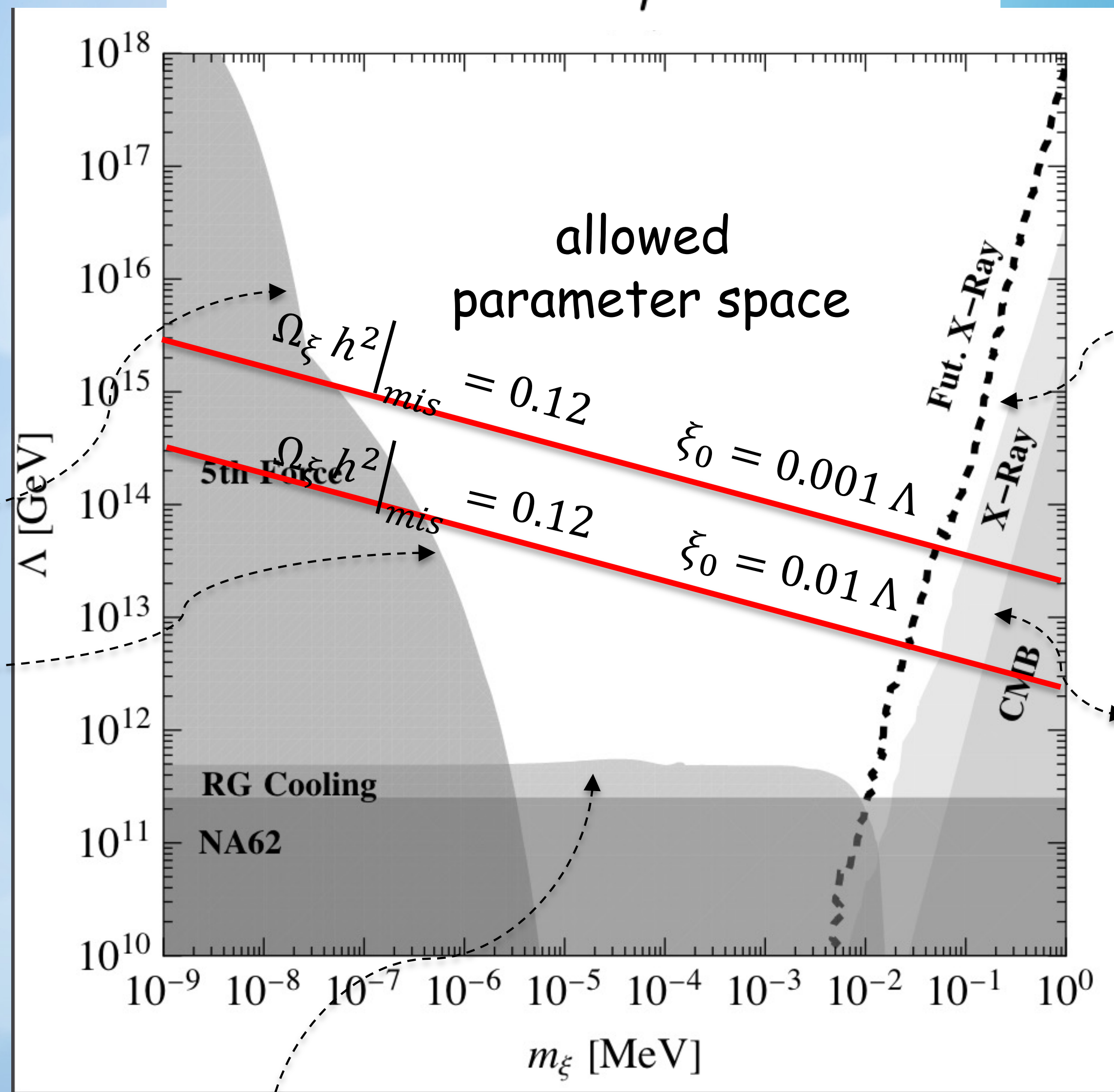
$\alpha \propto (g_{\xi NN})^2$

$\approx \left(\frac{2}{27}\frac{m_N}{\Lambda}\right)^2$

$r = 1/M$

$$\delta V_{ISL}(r) = -\frac{G m_1 m_2}{r}\alpha e^{-r/\lambda}.$$

$$\Gamma(\xi \to \gamma\gamma) \propto \frac{\alpha^2\, m_\xi^{\,3}}{\Lambda^2}\frac{m_\xi^{\,4}}{m_e^4}$$



limits from Inverse Square Law of gravity

X-rays diffuse emissions from DM decay in galaxy clusters

CMB distortion from energy of DM decay or annihilation

stellar energy loss in Red Giants and White Dwarfs

$\frac{Y_{See}}{\Lambda} < 7\times 10^{-16}$

[here $Y_{See} \approx m_e$]

[From Feruglio's talk at DISCRETE 2024]

Toy model with 2 quark generations

$$\mathscr{L}_{\text{Yuk}} = U_i^c\, Y_{ij}^u(z)\, Q_j\, H_u + D_i^c\, Y_{ij}^d(z)\, Q_j\, H_d \qquad i, j = 1, 2$$

Assume

$$m_d = \mathrm{v}_d\, Y^d = \begin{pmatrix} d_{11}\, z^{k_{11}^d} & d_{12}\, z^{k_{12}^d} \\ d_{21}\, z^{k_{21}^d} & d_{22}\, z^{k_{22}^d} \end{pmatrix} \qquad d_{ij} \in \mathbb{R} \text{ by CP invariance}$$

Full quark mass matrix $M_q = m_d \oplus m_u$ has

$$\det M_q = \left(d_{11} d_{22} z^{k_{11}^d + k_{22}^d} - d_{12} d_{21} z^{k_{12}^d + k_{21}^d}\right) \left(u_{11} u_{22} z^{k_{11}^u + k_{22}^u} - u_{12} u_{21} z^{k_{12}^u + k_{21}^u}\right)$$

We want $\det M_q$ to be a positive constant

$\det M_q \propto z^d \quad \forall \quad d_{ij}\,, u_{ij}$ if

$$k_{11}^d + k_{22}^d = k_{12}^d + k_{21}^d \qquad\qquad k_{11}^u + k_{22}^u = k_{12}^u + k_{21}^u$$

$$\det M_q = \left(d_{11} d_{22} - d_{12} d_{21}\right) \left(u_{11} u_{22} - u_{12} u_{21}\right) z^d \qquad d = \sum_{i=1}^{2} \left(k_{ii}^d + k_{ii}^u\right)$$

$d = 0 \quad \Rightarrow \quad \det M_q$ is a z-independent real constant

Toy model with 2 quark generations

Interesting class of solutions to

$$k^d_{11} + k^d_{22} = k^d_{12} + k^d_{21} \qquad\qquad k^u_{11} + k^u_{22} = k^u_{12} + k^u_{21}$$

is given by

$$k^d_{ij} = k_{D^c_i} + k_{Q_j} + k_{H_d} \qquad\qquad k^u_{ij} = k_{U^c_i} + k_{Q_j} + k_{H_u}$$

k_Φ are determined up to additive constants Δ_Φ, $\Phi = \{U^c, D^c, Q, H_{u/d}\}$:

$$k_{H_q} \to k_{H_q} + \Delta_{H_q} \qquad k_{Q_i} \to k_{Q_i} + \Delta_Q \qquad k_{D^c_i(U^c_i)} \to k_{D^c_i(U^c_i)} - \Delta_{H_{d(u)}} - \Delta_Q$$

k_Φ can be interpreted as charges (modular weights) in models with U(1) (modular) flavour symmetry. For example, in the case of U(1)

$$\Phi_i \to e^{-ik_{\Phi_i}\alpha}\,\Phi_i \qquad\qquad z \to e^{+ik_z\alpha}\,z$$

$$\mathscr{L}_{\rm Yuk} \supset D^c_i\, Q_j\, H_d\, z^{k^d_{ij}} \qquad\qquad \left(k_z = +1\right)$$

$$d = \sum_{i=1}^{2} \left(k^d_{ii} + k^u_{ii}\right) = \sum_{i=1}^{2} \left(k_{D^c_i} + k_{U^c_i} + 2k_{Q_i}\right) + 2\left(k_{H_d} + k_{H_u}\right) = 0$$

Toy model with 2 quark generations

Example

$$k_{D_1^c} = k_{U_1^c} = -k_{Q_1} > 0 \qquad k_{D_2^c} = k_{U_2^c} = -k_{Q_2} < 0 \qquad k_{H_{u,d}} = 0$$

$$m_d = \begin{pmatrix} d_{11} & d_{12}\, z^k \\ d_{21}\, z^{-k} & d_{22} \end{pmatrix} \qquad m_u = \begin{pmatrix} u_{11} & u_{12}\, z^k \\ u_{21}\, z^{-k} & u_{22} \end{pmatrix} \qquad k = k_{D_1^c} + k_{Q_2} > 0$$

The elements $m_{q\,21}$ are singular in the limit $z \to 0$

In an EFT, singularities occur when would-be heavy states integrated out from the full theory accidentally become massless (EFT breaks down)

If no such states exist in the full theory, $d_{21} = u_{21} = 0$ and

$$m_d = \begin{pmatrix} d_{11} & d_{12}\, z^k \\ 0 & d_{22} \end{pmatrix} \qquad m_u = \begin{pmatrix} u_{11} & u_{12}\, z^k \\ 0 & u_{22} \end{pmatrix} \qquad k = k_{D_1^c} + k_{Q_2} > 0$$

Modular group

Homogeneous modular group

$$\Gamma = \langle S,\, T \mid S^4 = (ST)^3 = I \rangle \cong \mathrm{SL}(2,\mathbb{Z})$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad\qquad \tau \xrightarrow{T} \tau + 1$$

duality — discrete shift symmetry

Inhomogeneous modular group

$$\overline{\Gamma} = \langle S,\, T \mid S^2 = (ST)^3 = I \rangle \cong \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{I, -I\}$$

In other words, $\mathrm{SL}(2,\mathbb{Z})$ matrices γ and $-\gamma$ are identified

$$\tau \xrightarrow{\gamma} \gamma\tau = \frac{a\tau + b}{c\tau + d} \qquad\qquad \tau \xrightarrow{-\gamma} (-\gamma)\,\tau = \frac{-a\tau - b}{-c\tau - d} = \gamma\tau$$

Modular forms

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under Γ as

$$f\left(\gamma\tau\right) = \left(c\tau + d\right)^k f(\tau)\,, \qquad \gamma \in \Gamma$$

k is weight, a non-negative even integer

$$\gamma = -I \quad \Rightarrow \quad f(\tau) = (-1)^k f(\tau) \quad \Rightarrow \quad k \text{ is even}$$

Modular forms are periodic and admit q-expansions

$$\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad f(\tau + 1) = f(\tau) \quad \Rightarrow \quad f(\tau) = \sum_{n=0}^{\infty} a_n q^n\,, \quad q = e^{2\pi i\tau}$$

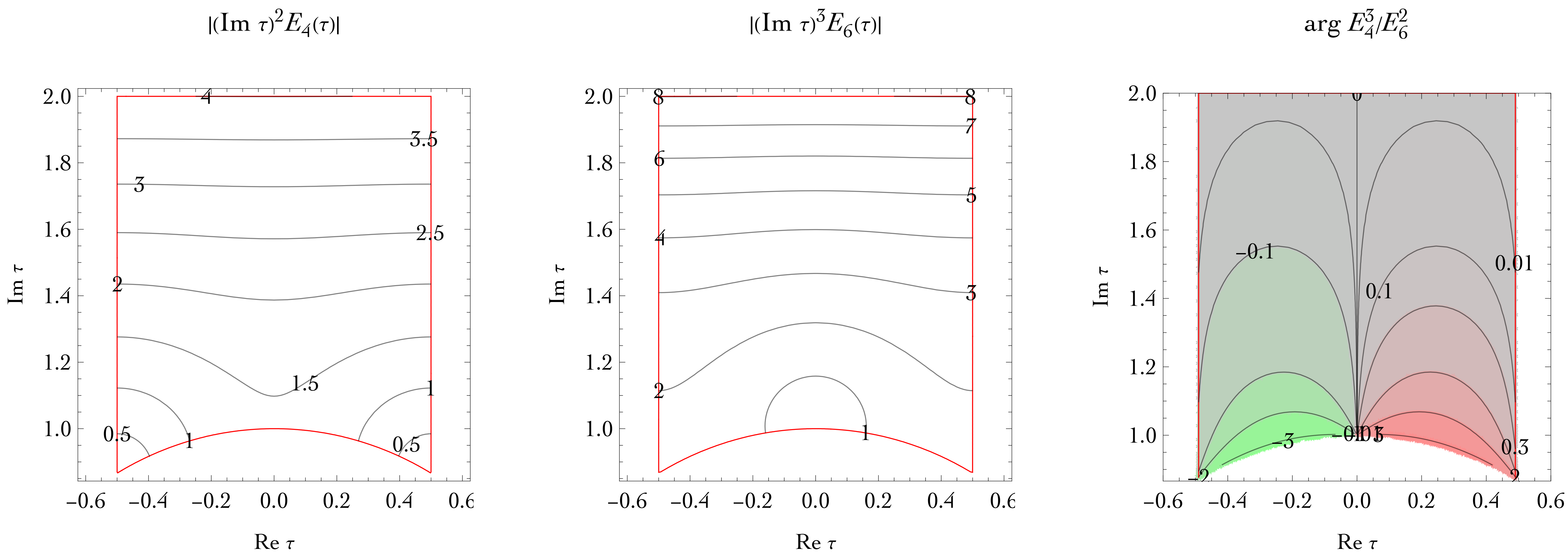
Modular forms of weight k form a linear space $\mathscr{M}_k$ of finite dimension

$$\dim \mathscr{M}_k = \begin{cases} 0 & \text{if } k \text{ is negative or odd} \\ \lfloor k/12 \rfloor & \text{if } k = 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1 & \text{if } k \neq 2 \pmod{12} \end{cases}$$

Modular forms of level 1: E4 and E6

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \mathcal{O}(q^5)$$

$$E_6(\tau) = 1 - 540 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \mathcal{O}(q^5)$$



Modular forms of level 2

Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - q^n\right) \qquad q \equiv e^{2\pi i \tau}$$

$$Z_1^{(2)} = \frac{2i}{\pi} \left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - 8\frac{\eta'(2\tau)}{\eta(2\tau)} \right] = 1 + 24q + 24q^2 + 96q^3 + 24q^4 + \mathcal{O}\left(q^5\right)$$

$$Z_2^{(2)} = \frac{2\sqrt{3}i}{\pi} \left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} \right] = 8\sqrt{3}q^{1/2}\left(1 + 4q + 6q^2 + 8q^3 + \mathcal{O}\left(q^4\right)\right)$$

$$\begin{pmatrix} Z_1^{(2)} \\ Z_2^{(2)} \end{pmatrix} \sim \mathbf{2} \quad \text{of} \quad \Gamma_2 \cong S_3$$

$$\left\{Z_1^{(4)}, Z_2^{(4)}, Z_3^{(4)}\right\} = \left\{Z_2^{(2)2} - Z_1^{(2)2}, 2Z_1^{(2)}Z_2^{(2)}, Z_1^{(2)2} + Z_2^{(2)2}\right\}$$

$$\begin{pmatrix} Z_1^{(4)} \\ Z_2^{(4)} \end{pmatrix} \sim \mathbf{2} \qquad Z_3^{(4)} \sim \mathbf{1}_0 \quad \text{of} \quad \Gamma_2 \cong S_3$$

Group properties of $\Gamma_2 \cong S_3$

$$\Gamma_2 = \left\langle S,\, T \mid S^2 = (ST)^3 = T^2 = I \right\rangle$$

$$\mathscr{S}_{\mathbf{2}} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \qquad \mathscr{T}_{\mathbf{2}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

S_3	$\mathbf{1}_0$	$\mathbf{1}_1$	$\mathbf{2}$
$\mathscr{S}$	1	−1	$\mathscr{S}_{\mathbf{2}}$
$\mathscr{T}$	1	−1	$\mathscr{T}_{\mathbf{2}}$

Tensor products

$$\mathbf{1}_1 \otimes \mathbf{1}_1 = \mathbf{1}_0 \qquad \mathbf{1}_1 \otimes \mathbf{2} = \mathbf{2} \qquad \mathbf{2} \otimes \mathbf{2} = \mathbf{1_0} \oplus \mathbf{1_1} \oplus \mathbf{2}$$

Clebsch-Gordan coefficients

$$\left(\gamma_{\mathbf{1}_1} \otimes \beta_{\mathbf{2}}\right)_{\mathbf{2}} = \left(-\gamma\beta_2\,,\, \gamma\beta_1\right)^T$$

$$\left(\alpha_{\mathbf{2}} \otimes \beta_{\mathbf{2}}\right)_{\mathbf{1}_0} = \alpha_1\beta_1 + \alpha_2\beta_2$$

$$\left(\alpha_{\mathbf{2}} \otimes \beta_{\mathbf{2}}\right)_{\mathbf{1}_1} = \alpha_1\beta_2 - \alpha_2\beta_1$$

$$\left(\alpha_{\mathbf{2}} \otimes \beta_{\mathbf{2}}\right)_{\mathbf{2}} = \left(\alpha_2\beta_2 - \alpha_1\beta_1\,,\, \alpha_1\beta_2 + \alpha_2\beta_1\right)^T$$

More on modular-gauge anomalies

$$M \to M' = \Lambda(\gamma,\tau)^{-k_M} M$$

Jacobian J: $\mathscr{D}M' = J\,\mathscr{D}M$

Arkani-Hamed, Murayama, hep-th/9707133

$$\log J = -\frac{i}{64\pi^2}\int d^4x\, d^2\theta \left[\sum_M T(M)\, k_M\right] W^a W^a \ln\Lambda$$

$T(M)$ is the Dynkin index of the rep of M: $\mathrm{tr}\left(t_a t_b\right) = T(M)\,\delta_{ab}$

$$\sum_M T(M)\, k_M = 0$$

$$\mathrm{SU}(3)_c: \quad \sum_i \left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) = 0$$

$$\mathrm{SU}(2)_L: \quad \sum_i \left(3k_{Q_i} + k_{L_i}\right) + k_{H_u} + k_{H_d} = 0$$

$$\mathrm{U}(1)_Y: \quad \sum_i \left(k_{Q_i} + 8k_{U_i^c} + 2k_{D_i^c} + 3k_{L_i} + 6k_{E_i^c}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

Modular invariance and SUGRA

$\mathcal{N} = 1$ SUGRA action depends on

$$G = \frac{K}{M_{\rm Pl}^2} + \log\left|\frac{W}{M_{\rm Pl}^3}\right|^2$$

For G to be invariant, both K and W have to transform

$$K \to K + M_{\rm Pl}^2\left(F + F^\dagger\right) \qquad \text{and} \qquad W \to e^{-F}W$$

In the case of modular transformations

$$F = \frac{h^2}{M_{\rm Pl}^2}\log(c\tau + d)$$

$$W \to (c\tau + d)^{-k_W}\,W \qquad \text{with} \qquad k_W = \frac{h^2}{M_{\rm Pl}^2} > 0$$

The superpotential is a modular function, having singularities at some values of τ

$k_W \to 0$ rigid SUSY limit

Modular invariance and SUGRA

$$W = Y^u_{ij}(\tau)\, U^c_i\, Q_j\, H_u + Y^d_{ij}(\tau)\, D^c_i\, Q_j\, H_d$$

$$Y^q_{ij}(\tau) \to (c\tau + d)^{k^q_{ij}}\, Y^q_{ij}(\tau) \qquad \text{with} \qquad k^{u(d)}_{ij} = k_{U^c_i(D^c_i)} + k_{Q_j} + k_{H_{u(d)}} - k_W$$

Furthermore, the Kähler transformation must be accompanied by a U(1) rotation

$$\psi \to e^{\frac{F - F^\dagger}{4}}\, \psi \qquad\qquad \lambda \to e^{-\frac{F - F^\dagger}{4}}\, \lambda$$

how gaugino enters the game

$$\psi_{\text{can}} \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{\frac{k_W}{4} - \frac{k_\Phi}{2}} \psi_{\text{can}} \qquad\qquad \lambda \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{-\frac{k_W}{4}} \lambda$$

Modular-QCD anomaly modifies as

$$A = \sum_{i=1}^{3} \left(2k_{Q_i} + k_{U^c_i} + k_{D^c_i} - 2k_W\right) + C k_W$$

$C = 3$ is quadratic Casimir of $\mathbf{8}$ of $\mathrm{SU}(3)_C$

Gluino mass

$$\bar{\theta} = \theta_{\mathrm{QCD}} + \arg\det M_q + C \arg M_3$$

Assume $k_{\mathrm{det}} = 0$ and the quark contribution to A vanishes. Then

$$\bar{\theta} = \theta_{\mathrm{QCD}} + C \arg M_3$$

Gluino mass requires SUSY breaking

$$M_3 = \frac{g_3^2}{2} e^{K/2M_{\mathrm{Pl}}^2} K^{i\bar{j}} D_{\bar{j}} W^\dagger f_i$$

Assuming $D_\tau W = 0$ and no additional phases from SUSY breaking

$$\arg M_3 = -\arg W$$

$$W = \ldots + \frac{c_0 M_{\mathrm{Pl}}^3}{\eta(\tau)^{2k_W}} \qquad \text{and} \qquad f = \ldots + \frac{C k_W}{4\pi^2} \log \eta(\tau)$$

$$\bar{\theta} = -8\pi^2 \mathrm{Im} f - C \arg W = 0$$

More on modular invariance in SUGRA

$$\det M_q \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{\frac{k_{\det}}{2}} \det M_q \qquad k_{\det} = \sum_{i=1}^{3}\left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c} - 2k_W\right) + 3\left(k_{H_u} + k_{H_d}\right)$$

$$M_3 \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{\frac{k_W}{2}} M_3$$

(gluino mass arises only if SUSY is broken)

Theories based on finite modular groups

$\mathcal{N} = 1$ rigid SUSY matter action

$$\mathcal{S} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\ K(\tau,\overline{\tau},\psi,\overline{\psi}) + \int \mathrm{d}^4x\, \mathrm{d}^2\theta\ W(\tau,\psi) + \int \mathrm{d}^4x\, \mathrm{d}^2\overline{\theta}\ \overline{W}(\overline{\tau},\overline{\psi})$$

Ferrara, Lust, Shapere, Theisen, PLB **225** (1989) 363

Ferrara, Lust, Theisen, PLB **233** (1989) 147

$$\begin{cases} \tau \to \dfrac{a\tau+b}{c\tau+d} \\ \psi_i \to \left(c\tau+d\right)^{-k_i} \rho_i(\tilde{\gamma})\, \psi_i \end{cases} \Rightarrow \begin{cases} W(\tau,\psi) \to W(\tau,\psi) \\ K(\tau,\overline{\tau},\psi,\overline{\psi}) \to K(\tau,\overline{\tau},\psi,\overline{\psi}) + f_K(\tau,\psi) + \overline{f_K}(\overline{\tau},\overline{\psi}) \end{cases}$$

unitary representation of Γ_N

Feruglio, 1706.08749

$$W(\tau,\psi) = \sum_n \sum_{\{i_1,\dots,i_n\}} \sum_s g_{i_1\,\dots\,i_n,s} \left(Y_{i_1\,\dots\,i_n,s}(\tau)\, \psi_{i_1}\cdots\psi_{i_n} \right)_{\mathbf{1},s}$$

$$Y(\tau) \xrightarrow{\gamma} (c\tau+d)^{k_Y} \rho_Y(\tilde{\gamma})\, Y(\tau)$$

Yukawa couplings are modular forms!

$$k_Y = k_{i_1} + \ldots + k_{i_n}$$

$$\rho_Y \otimes \rho_{i_1} \otimes \ldots \otimes \rho_{i_n} \supset \mathbf{1}$$

Quark masses and mixings

At the GUT scale of 2×10^{16} GeV,
assuming MSSM with $\tan\beta = 10$ and SUSY breaking scale of 10 TeV

m_u/m_c	$(1.93 \pm 0.60) \times 10^{-3}$
m_c/m_t	$(2.82 \pm 0.12) \times 10^{-3}$
m_d/m_s	$(5.05 \pm 0.62) \times 10^{-2}$
m_s/m_b	$(1.82 \pm 0.10) \times 10^{-2}$
$\sin^2\theta_{12}$	$(5.08 \pm 0.03) \times 10^{-2}$
$\sin^2\theta_{13}$	$(1.22 \pm 0.09) \times 10^{-5}$
$\sin^2\theta_{23}$	$(1.61 \pm 0.05) \times 10^{-3}$
δ/π	0.385 ± 0.017

$$m_t = 87.46 \text{ GeV}$$
$$m_b = 0.9682 \text{ GeV}$$

Antusch, Maurer, 1306.6879
Yao, Lu, Ding, 2012.13390

Lepton masses and mixings

NuFIT 5.2 (2022)

		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 6.4$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
with SK atmospheric data	$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \to 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
	$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
	$\sin^2\theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \to 0.603$	$0.569^{+0.016}_{-0.021}$	$0.412 \to 0.613$
	$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \to 51.0$	$49.0^{+1.0}_{-1.2}$	$39.9 \to 51.5$
	$\sin^2\theta_{13}$	$0.02225^{+0.00056}_{-0.00059}$	$0.02052 \to 0.02398$	$0.02223^{+0.00058}_{-0.00058}$	$0.02048 \to 0.02416$
	$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.94$
	$\delta_{CP}/^\circ$	232^{+36}_{-26}	$144 \to 350$	276^{+22}_{-29}	$194 \to 344$
	$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
	$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.507^{+0.026}_{-0.027}$	$+2.427 \to +2.590$	$-2.486^{+0.025}_{-0.028}$	$-2.570 \to -2.406$

Esteban et al., 2007.14792 and www.nu-fit.org

$$m_e/m_\mu = 0.0048 \pm 0.0002$$

$$m_\mu/m_\tau = 0.0565 \pm 0.0045$$