Lattice Studies of Non-Leptonic Decays

Chris Sachrajda

School of Physics and Astronomy
University of Southampton
Southampton
UK

Kaon 2007
Frascati
May 21st – 25th 2007

Outline of Talk

- Introduction
- $K \to \pi\pi$ Decays at Lowest Order in Chiral Expansion
  - Reminder of the results of the RBC & CP-PACS Collaborations
- Finite Volume Effects in $K \to \pi\pi$ Decays
- $K \to \pi\pi$ Decays at NLO in Chiral Expansion
- Miscellany
  - Twisted Boundary Conditions and the Evaluation of $\delta'(q^*)$.
  - Rôle of the Charm Quark in the $\Delta I = 1/2$ Rule.
- Summary and Conclusions.

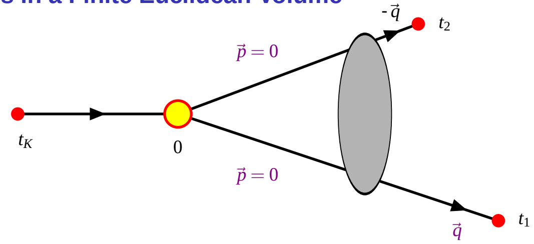
See also Bob Mawhinney's Talk

The $\Delta S = 1$ Weak Hamiltonian

$$\mathcal{H}_{eff}\,(\Delta S = 1) = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} C_i(\mu)\, O_i(\mu)$$

- Non-perturbative QCD effects are contained in the matrix elements of the operators $O_i(\mu)$.
- The challenge for the lattice community is to calculate the matrix elements
$$\langle \pi\pi \,|\, O_i(\mu) \,|\, K \rangle \,.$$
- O_1, O_2 – Current-Current Operators
e.g. $O_2 = (\bar{s}_L \gamma^\mu u_L)\,(\bar{u}_L \gamma_\mu d_L)$ - charm
- $O_3 - O_6$ – QCD Penguin Operators
e.g. $O_6 = (\bar{s}_L^i \gamma^\mu d_L^j)\, \sum_q (\bar{q}_R^j \gamma_\mu q_R^i)$
- $O_7 - O_{10}$ – Electroweak Penguin Operators
e.g. $O_8 = \frac{3}{2}\,(\bar{s}_L^i \gamma^\mu d_L^j)\, \sum_q e_q\,(\bar{q}_R^j \gamma_\mu q_R^i)$.

Two-Particle States in a Finite Euclidean Volume



L.Maiani & M.Testa (1990) made the following two points about the computation of $K \to \pi\pi$ decays in Euclidean Space (in the CoM Frame):

- At large times the correlator is dominated by the unphysical matrix element with the two-pions at threshold;
- In Euclidean space one obtains real quantities, such as

$$\frac{1}{2}\left\{ {}_{\text{out}}\langle \pi\pi | \mathcal{H}_W | K \rangle + {}_{\text{in}}\langle \pi\pi | \mathcal{H}_W | K \rangle \right\}.$$

Following the Maiani-Testa paper there was a break in the calculation of matrix elements between multi-hadron states.

Two-Particle States in a Finite Euclidean Volume – Cont.

Renewed interest was stimulated by L.Lellouch and M.Lüscher (2000) who:

- argued that by tuning the volume, one is in principle able to extract the matrix element corresponding to the physical kinematics for $K \to \pi\pi$ decays.
  - The correlation function will still be dominated by the matrix element with the two pions in the ground state (unphysical kinematics), so one has to determine the coefficient of a non-leading exponential.
  - For a physical $K \to \pi\pi$ decay with the kaon at rest and the energy of the two-pions corresponding to $n = 1$ (the first excited state) for periodic boundary conditions one needs a lattice of about 6 fm.
- derived a formula relating the matrix elements in a finite volume to the modulus of the physical decay amplitudes, up to exponential corrections in the volume.
- This is described below.

LO Chiral Perturbation Theory

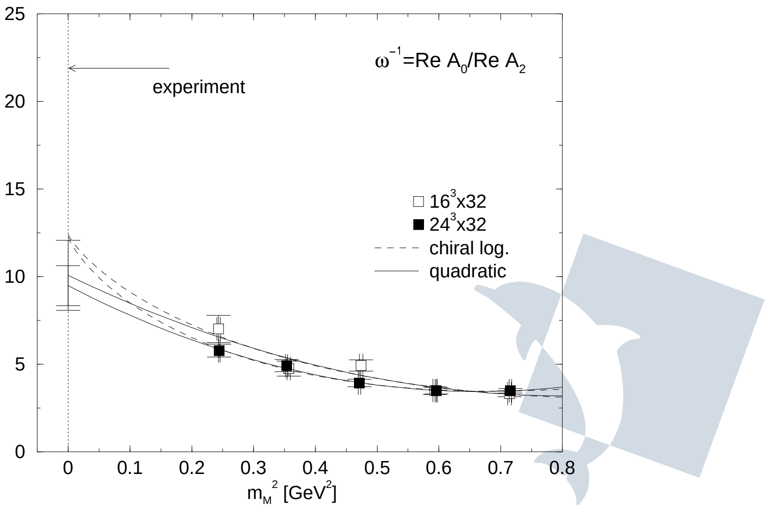
- Chiral perturbation theory is widely used to extrapolate lattice results computed at unphysically heavy values of $m_{u,d}$ to their physical values.
- At Lowest Order in the Chiral Expansion one can obtain the $K \to \pi\pi$ decay amplitude from $K \to \pi$ and $K \to$ vacuum matrix elements.

In 2001, two collaborations published some very interesting (quenched) results on non-leptonic kaon decays in general and on the $\Delta I = 1/2$ rule and ε'/ε in particular:

Collaboration(s)	$\mathrm{Re}\, A_0/\mathrm{Re}\, A_2$	ε'/ε
RBC	25.3 ± 1.8	$-(4.0 \pm 2.3) \times 10^{-4}$
CP-PACS	9÷12	(-7 ÷ -2)$\times 10^{-4}$
Experiments	22.2	$(17.2 \pm 1.8) \times 10^{-4}$

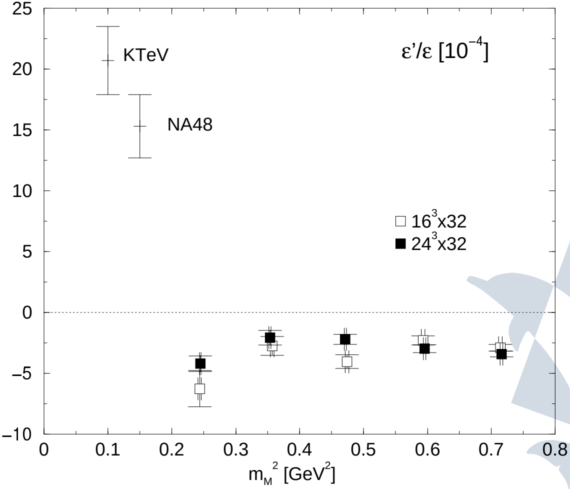
Re A_0/Re A_2 as a function of the meson mass.

CP-PACS



ε'/ε as a function of the meson mass.

CP-PACS



Comments

- Results from RBC and CP-PACS are very interesting and provide valuable benchmarks for future calculations.
- These collaborations were able to control the *Ultraviolet Problem*, i.e. the subtraction of power divergences due to the mixing of operators in the weak Hamiltonian and lower dimensional operators.
- The simulations were quenched, and relied on the validity of lowest order χPT in the region of approximately 400-800 MeV.
- One natural suggestion is to improve the precision to NLO in the chiral expansion.
  This requires the evaluation of $K \to \pi\pi$ decay amplitudes directly.
- For ε'/ε there is a significant partial cancellation from the $\Delta I = 1/2$ and $\Delta I = 3/2$ contributions. Does this amplify the relative errors?

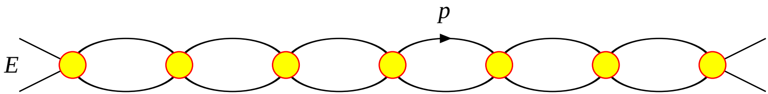
Finite-Volume Corrections for Two-Pion States

- M.Lüscher (1986-91) derived the two-hadron spectrum in a finite-volume in the rest frame.
- K.Rummukainen and S.Gottlieb (1996) generalized the derivation of the spectrum to a frame with non-zero momentum.
- L.Lellouch & M.Lüscher (2000) derived the finite-volume corrections to $K \to \pi\pi$ matrix elements in the rest frame.
- D.Lin, G.Martinelli, CTS & M.Testa (2001) rederived the spectrum (validating the results beyond the first 7 states) and the LL formula, interpreting the effects as being due to the density of two-pion states in a finite volume.
- C.Kim, CTS & S.Sharpe (2005) and N.Christ, C.Kim & T.Yamazaki (2005) generalized all the results to a moving frame.
- See also S.Beane, P.Bedaque, A.Parreno and M.Savage (2004).

I now sketch the derivation of the results from the perspective of Kim, CTS, Sharpe (2005).

Finite-Volume Corrections for Two-Pion States

For two-particle states the finite-volume corrections decrease as powers of the volume and not exponentially. They are numerically significant and hence need to be controlled.



where $E^2 = 4(k^2 + m^2)$.

Performing the p_0 integration by contours we obtain summations over loop-momenta of the form:

$$\frac{1}{L^3}\sum_{\vec{p}}\frac{f(p^2)}{p^2-k^2}$$

where $f(p^2)$ is non-singular.

For simplicity I am assuming here that only the s-wave $\pi\pi$ phase-shift is significant and that we are in the centre-of-mass frame. The generalization to higher partial waves is technical but straightforward.

$$\frac{1}{L^3}\sum_{\vec{p}}\frac{f(p^2)}{p^2-k^2}$$

- The *large* finite-volume effects, i.e. those which decrease only as powers of L, come from the difference between the momentum sums in finite-volume and the corresponding integrals in infinite volume.
- The required relation between the FV sums and infinite-volume integrals is the Poisson Summation Formula, which in 1-dimension is:

$$\frac{1}{L}\sum_{p} g(p) = \sum_{l=-\infty}^{\infty}\int\frac{dp}{2\pi}e^{ilLp}g(p)$$

  If $g(p)$ is non-singular then only the term with $l=0$ on the rhs contributes, up to exponentially small terms in L.
- From the above it follows that this is not the case for two-hadron final states $\Rightarrow$ finite-volume corrections $\sim 1/L^n$.
- For two-hadron final states we start with

$$\frac{1}{L^3}\sum_{\vec{p}}\frac{f(p^2)-f(k^2)e^{\alpha(k^2-p^2)}}{p^2-k^2} = \int\frac{d^3p}{(2\pi)^3}\,\frac{f(p^2)-f(k^2)e^{\alpha(k^2-p^2)}}{p^2-k^2}\,.$$

  This is the key formula to understanding FV effects in two-pion states.

$$\frac{1}{L^3}\sum_{\vec{p}}\frac{f(p^2)-f(k^2)e^{\alpha(k^2-p^2)}}{p^2-k^2}=\int\frac{d^3p}{(2\pi)^3}\frac{f(p^2)-f(k^2)e^{\alpha(k^2-p^2)}}{p^2-k^2}\,.$$

- We rewrite the above formula as:

$$\frac{1}{L^3}\sum_{\vec{p}}\frac{f(p^2)}{p^2-k^2}=\mathscr{P}\int\frac{d^3p}{(2\pi)^3}\frac{f(p^2)}{p^2-k^2}+f(k^2)Z(k^2)$$

where $\mathscr{P}$ represents *principal value* and

$$Z(k^2)\equiv\frac{1}{L^3}\sum_{\vec{p}}\frac{e^{\alpha(k^2-p^2)}}{p^2-k^2}-\mathscr{P}\int\frac{d^3p}{(2\pi)^3}\frac{e^{\alpha(k^2-p^2)}}{p^2-k^2}\,.$$

- This does not have the physical $i\varepsilon$ prescription and so we write

$$\mathscr{P}\int\frac{d^3p}{(2\pi)^3}\frac{f(p^2)}{p^2-k^2}=\int\frac{d^3p}{(2\pi)^3}\frac{f(p^2)}{p^2-k^2-i\varepsilon}-\frac{ik}{4\pi}f(k^2)\,.$$

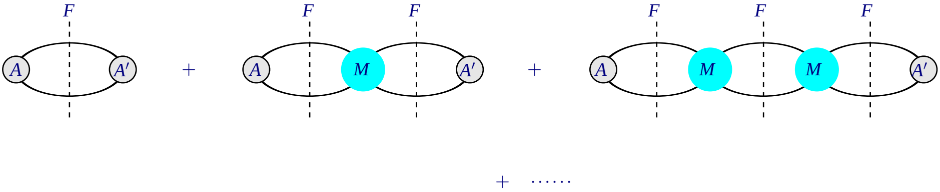
$$\frac{1}{L^3}\sum_{\vec{p}}\frac{f(p^2)}{p^2-k^2} = \int\frac{d^3p}{(2\pi)^3}\frac{f(p^2)}{p^2-k^2-i\varepsilon}$$

$$-\frac{ik}{4\pi}f(k^2)+f(k^2)\left\{\frac{1}{L^3}\sum_{\vec{p}}\frac{e^{\alpha(k^2-p^2)}}{p^2-k^2}-\mathscr{P}\int\frac{d^3p}{(2\pi)^3}\frac{e^{\alpha(k^2-p^2)}}{p^2-k^2}\right\}$$

- The finite-volume correction only depends on the function f evaluated at the external energy corresponding to k^2 .
- The expression $F\equiv\mathscr{Z}-ik/(4\pi)$ is purely kinematical and can readily be evaluated.
- The FV correction exhibited above appears in every loop and we need to resum these corrections (geometric series).
- In infinite volume there is a cut with a branch point at the two-pion threshold.

  In finite volume the cut $\Rightarrow$ series of poles. The positions of these poles correspond to the allowed energy levels (Lüscher Quantization Condition).

Thus the lattice correlation function can be represented by:



where M is the **physical** $\pi\pi$ scattering amplitude and A and A' are the matrix elements of the operators used to prepare the two-pion states. F represents the factor $Z - ik/(4\pi)$.

Thus the correlation function is equal to the one in infinite volume $+$

$$-A'FA + A'F\frac{iM}{2}FA - A'F\frac{iM}{2}F\frac{iM}{2}FA + \cdots = A'F\frac{1}{1+iMF/2}A\,,$$

and the quantization condition corresponds to those values of E or k such that

$$1 + \frac{iMF}{2} = 0\,.$$

- The quantization condition can be rewritten in terms of the s-wave phase-shift δ as:

$$\tan(\delta(k^2)) = -\tan[\phi(k)] \equiv -\frac{k}{4\pi}\left\{\frac{1}{L^3}\sum_{\vec{p}}\frac{e^{\alpha(k^2-p^2)}}{k^2-p^2} - \mathscr{P}\int\frac{d^3p}{(2\pi)^3}\,\frac{e^{\alpha(k^2-p^2)}}{k^2-p^2}\right\}^{-1}.$$

- We have now generalized this to the case in which the two-pions have non-zero momentum ($\vec{P}$ say).

Rummukainen & Gottlieb (1996); Kim, CTS & Sharpe (2005); Christ, Kim & Yamazaki (2005)

Kim, CTS, Sharpe

$$\tan(\delta(k^*)) = -\tan[\phi^P(k^*)] = \frac{k^{*2}}{4\pi}\left[c(k^{*2})\right]^{-1}$$

$$\text{with}\quad c(k^{*2}) \equiv \frac{1}{L^3}\sum_{\vec{p}}\frac{\omega_p^*}{\omega_p}\,\frac{e^{\alpha(k^{*2}-p^{*2})}}{k^{*2}-p^{*2}} - \mathscr{P}\int\frac{d^3p^*}{(2\pi)^3}\,\frac{e^{\alpha(k^{*2}-p^{*2})}}{k^{*2}-p^{*2}}.$$

Rummakainen, Gottlieb

$$c(k^{*2}) \to \frac{1}{\gamma L^3}\sum_{\vec{p}}\frac{1}{k^{*2}-r^2}\quad\text{with}\quad r^2 = \frac{1}{\gamma^2}\left(p_{\parallel}-\frac{P}{2}\right)^2 + \tilde{p}_{\perp}^2.$$

The two are equivalent (up to exponentially small terms)!

Relation between Matrix Elements in Finite and Infinite Volume

$$|A|^2 = V^2\,\frac{m_K E^2}{k^{*\,2}}\left\{\delta'(k^*)+\phi^{P'}(k^*)\right\}\;|M|^2$$

where the $'$ represents the derivative with respect to k^*.

$$A = {}_\infty\langle\pi\pi;E,\vec{P}\,|\,\mathscr{H}_W(0)\,|\,K;\vec{P}\rangle_\infty \quad \text{and} \quad M = {}_V\langle\pi\pi;E,\vec{P}\,|\,\mathscr{H}_W(0)\,|\,K;\vec{P}\rangle_V\,.$$

are the $K\to\pi\pi$ matrix elements in infinite and finite volumes respectively and the external states have energy and momentum $(E,\vec{P})$.

Kim, CTS, Sharpe (2005); Christ,Kim,Yamazaki (2005)

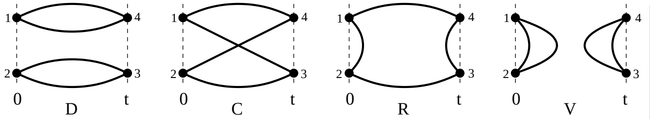
Preliminary results for $\Delta I = 3/2\; K\to\pi\pi$ decays using this technique have been presented using a quenched simulation on a course lattice ($a^{-1}=1.3\,$GeV). T.Yamazaki (for RBC), hep-lat/0509135, hep-lat/0610051.

We therefore have all the necessary techniques to control the finite-volume effects in both the spectrum and in the matrix elements.

- Similar issues and results also hold for other two-hadron states (e.g. $\pi-N$ and $N-N$).

Summary

- Finite volume effects for the two-pion spectrum and $K \to \pi\pi$ amplitudes understood in rest and moving frames.
  - For $I = 2$ final states, there is now no barrier to calculating the matrix elements precisely.
  - For $I = 0$ $\pi\pi$ states we need to learn how to calculate the disconnected diagrams with sufficient precision.



$\Delta I = 2$ Transitions at NLO in the Chiral Expansion

At NLO in χPT for $K \to \pi\pi$ matrix elements the generic structure is of the form:

$$\langle \pi\pi | \mathcal{O}_W | K \rangle = \mathrm{LO} * (1 + \mathrm{Logs}) + \mathrm{NLO\ counterterms}.$$

The Logs are calculable in one-loop χPT. The idea is to use lattice computations of $K \to \pi\pi$ matrix elements, for a range of masses and momenta, in order to

- determine the LO and NLO low-energy constants;
- use these to determine the physical decay amplitudes.

Exploratory SPQR Simulation at NLO in the Chiral Expansion

We performed an exploratory quenched study with the SPQR kinematics, obtaining the matrix elements of the EWP successfully:

Ph. Boucaud et al., hep-lat0412029

$$
\begin{aligned}
{}_{I=2}\langle \pi\pi\,|O_7(2\,\mathrm{GeV})\,|K^0\rangle &= (0.12\pm0.02)\,\mathrm{GeV}^3 \quad \text{and} \\
{}_{I=2}\langle \pi\pi\,|O_8(2\,\mathrm{GeV})\,|K^0\rangle &= (0.68\pm0.09)\,\mathrm{GeV}^3
\end{aligned}
$$

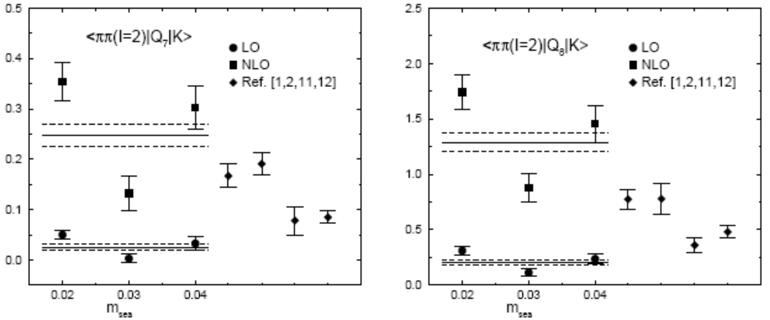
We were unable to determine the LEC's for O_4 sufficiently well to perform the chiral extrapolation.

- Finite-volume energy shift measurable. $\surd$
- Matrix Elements at simulated massed well determined. $\surd$
- NPR implemented successfully. $\surd$
- Quark masses too high for demonstrably reliable chiral extrapolation. $\times$
  LL factor not implemented $\times$
  Quenched Simulation. $\times$

EWP at NLO in the Chiral Expansion

- It is possible to evaluate the $K \to \pi\pi$ matrix elements of the EWP operators at NLO in χPT, by evaluating $K \to \pi$ and $K \to$ vacuum matrix elements.

Laiho & Soni

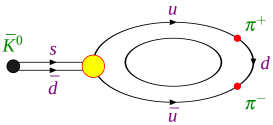


$\Delta I = 3/2\ K \to \pi\pi$ Matrix Elements (GeV^3) from DWF with $N_F = 2$.

VERY PRELIMINARY, J.Noaki, Lattice 2005

Lack of Unitarity in Quenched QCD

In full QCD we have the following contribution to the $\Delta I = 1/2$ decay $\bar{K}^0 \to \pi^+\pi^-$:



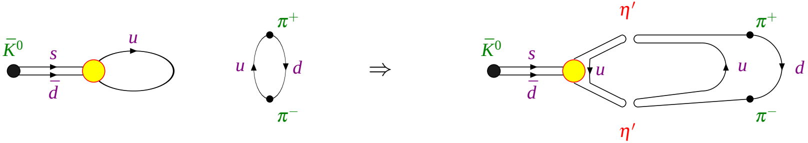
In the quenched theory this contribution is absent. This is achieved, e.g. by introducing ghost-quarks (with the opposite statistics) to cancel the effect. Internal particles are not the same as the external ones $\Rightarrow$ FSI depend on the operator.
Is there some meaningful way of overcoming this?

This effect is not present for ΔI=3/2 decays.

This effect is also present for partially quenched QCD, when $m_K > 2m_\pi$.

Hairpin Diagrams and Double Poles

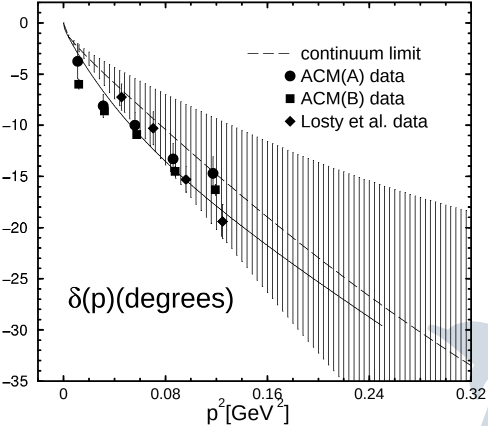
As an example consider the following $\Delta I = 1/2$ contribution to decay $\bar{K}^0 \to \pi^+\pi^-$ in quenched QCD:



Qualitatively the η' propagator is rewritten as the first two terms of the pion propagator.
Double Pole $\Rightarrow$ more singular long-distance behaviour.

At one-loop order in the chiral expansion there are no such contributions to $\Delta I = 3/2$ transitions.

Phase-Shifts from Finite-Volume Effects



Comparison of the $N_f = 2$ lattice results for the I=2 Scattering Phase Shift $\delta(p)$ with experiments.

CP-PACS Collaboration, T.Yamazaki et al. *hep-lat/0402025*

(Partially) Twisted Boundary Conditions and the Lellouch-Lüscher Factor.

- The Lellouch-Lüscher factor relating the $K \to \pi\pi$ matrix elements in finite-volume to the physical decay amplitudes contains the derivative of the phase-shift.
- The phase-shift can be determined from the two-particle spectrum in finite-volume, but only at discrete momenta.
- Typical Example:

$$L = 24a \quad \text{with} \quad a^{-1} = 2\,\text{GeV} \quad \Rightarrow \quad \frac{2\pi}{L} = .52\,\text{GeV}$$

Momentum resolution is very poor!

- Using twisted boundary conditions

$$q(x_i + L) = e^{i\theta_i} q(x_i)$$

the momentum spectrum is modified (relative to periodic bcs)

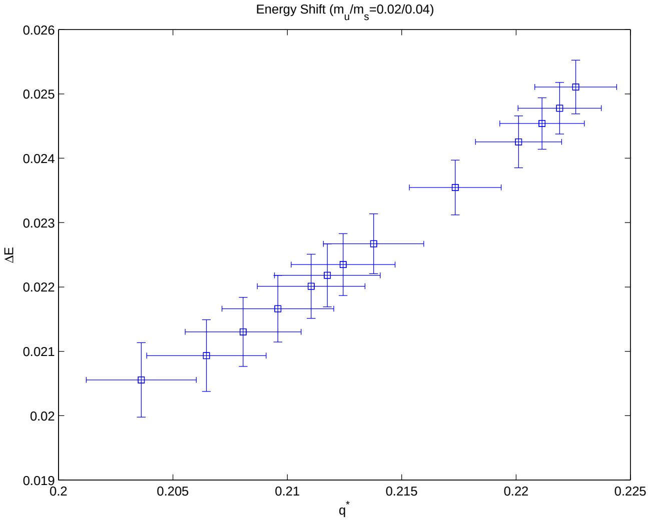
$$p_i = n_i \frac{2\pi}{L} + \frac{\theta_i}{L}\,.$$

- For quantities which do not involve Final State Interactions (e.g. masses, decay constants, form-factors) the Finite-Volume corrections are exponentially small also with Twisted BC's. CTS & G. Villadoro (2004)
- Moreover they are also exponentially small for *partially twisted boundary conditions* in which the sea quarks satisfy periodic BC's but the valence quarks satisfy twisted BC's. CTS & G. Villadoro (2004); Bedaque & Chen (2004)

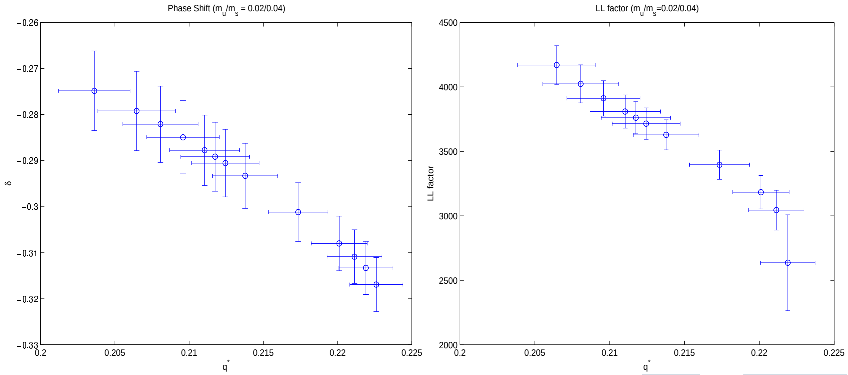
We do not need to perform new simulations for every choice of $\{\theta_i\}$.

- The technique can also be applied to evaluate $\delta'(q^{*2})$ for $I=2$ decays. C.Kim & CTS (preliminary)
  - Consider the propagation of two π^+ mesons with momenta θ/L and $(\theta-2\pi)/L$, where we can vary θ.
  - From the spectrum we can determine $\delta(q^{*2})$, where q^* is the corresponding centre-of-mass relative momentum.
  - Since θ can be varied with small intervals, we can evaluate $\delta'(q^2)$.
- Exploratory results from a DWF simulation at $N_f=2+1$ (UKQCD/RBC Configurations) .

Energy Shift as a Function of q^*



Phase-Shift and Lellouch-Lüscher Factor



Rôle of the Charm Quark in the $\Delta I = 1/2$ Rule

There is a suggested procedure to study the rôle of the charm quark in the $\Delta I = 1/2$ rule. L.Giusti, P.Hernández, M.Laine, P.Weisz and H.Wittig, hep-lat/0407007

P.Hernández, hep-lat/0610129, L.Giusti et al., hep-ph/0607220.

Step 1: $SU(4)$ or GIM limit. $m_c = m_s = m_u = m_d \ll \Lambda_{\chi\mathrm{PT}}$.
Two LECs, g^+ and g^-, which have been evaluated by matching a quenched QCD simulation with χPT

$$g^+ = 0.51 \pm 0.09 \quad \text{and} \quad g^- = 2.6 \pm 0.5 \quad \Rightarrow \quad \frac{A_0}{A_2} = \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{3g^-}{2g^+}\right) \simeq 6\,.$$

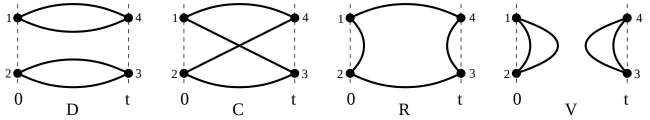
These authors conclude that: *Even though the enhancement is not large enough to match the experiment, it already indicates that penguin operator/contractions cannot be the whole story.*

Step 2: $\Lambda_{\chi\mathrm{PT}} \gg m_c \gg m_s = m_u = m_d$.
The matching to this effective theory can be done analytically. This has been done at LO but *unfortunately NLO couplings are needed to have predictability.*

Step 3: $m_c \geq \Lambda_{\chi\mathrm{PT}} \gg m_s = m_u = m_d$.
For the future.

Summary and Conclusions

- There has been a considerable amount of theoretical progress in formulating $K \to \pi\pi$ decays in a form suitable for lattice simulations.
- There is the opportunity of achieving significant numerical results for $K \to \pi\pi$ decay amplitudes.
  - For $I = 2$ final states, there is now no barrier to calculating the matrix elements precisely.
  - For $I = 0$ $\pi\pi$ states we need to learn how to calculate the disconnected diagrams with sufficient precision.



- It would be interesting to repeat the RBC/CP-PACS LO χPT study in unquenched simulations and at lower masses. This is underway – see Bob Mawhinney's talk.
- Is it possible to approach non-leptonic B-decays in lattice simulations?