W-algebras and Bethe ansatz in 2d CFT

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$\mathfrak{su}(2)$ XXX spin chain Bethe equations

$$1 = q \prod_{l=1}^{N} \frac{u_j + a_l - \epsilon}{u_j + a_l} \prod_{k \neq j} \frac{u_j - u_k + \epsilon}{u_j - u_k - \epsilon}$$

[Bethe 1931]

$$\Downarrow$$

CFT / $\mathcal{W}_N$ / $\mathcal{W}_{1+\infty}$ Bethe equations

$$1 = q \prod_{l=1}^{N} \frac{u_j + a_l - \epsilon_3}{u_j + a_l} \prod_{k \neq j} \frac{(u_j - u_k + \epsilon_1)(u_j - u_k + \epsilon_2)(u_j - u_k + \epsilon_3)}{(u_j - u_k - \epsilon_1)(u_j - u_k - \epsilon_2)(u_j - u_k - \epsilon_3)}$$

[Nekrasov, Shatashvili 2009; Litvinov 2013; Bonelli, Sciarappa, Tanzini, Vaško 2014; Feigin, Jimbo, Miwa, Mukhin 2016; Kozlowski, Sklyanin, Torrielli 2016]

Bethe equations for gaussian matrix model (Hermite)

$$0 = x_j + \sum_{k \neq j} \frac{1}{x_j - x_k}$$

$$\Downarrow$$

Bethe equations for 2d free boson (Wronskian Hermite)

$$0 = x_j + \frac{\ell(\ell+1)}{x_j^3} + \sum_{k \neq j} \frac{2}{(x_j - x_k)^3}$$

[Oblomkov 1999]

$$\Downarrow$$

BLZ equations for Virasoro algebra ($\mathbb{Z}_{N+2}$ orbifold of ...)

$$0 = \frac{N}{2} x_j^{N-1} + \frac{\ell(\ell+1)}{x_j^3} + \sum_{k \neq j} \frac{2}{(x_j - x_k)^3}$$

[Bazhanov,Lukyanov,Zamolodchikov 2003, Fioravanti 2004]

W algebras - motivation

$\mathcal{W}$-algebas: extensions of the Virasoro algebra (2d CFT) by higher spin currents - appear in many different contexts:

- integrable hierarchies of PDE (KdV/KP) $\rightsquigarrow$ $\mathcal{W}$ is quant. KP
- (old) matrix models
- instanton partition functions and AGT
- holographic dual of 3d higher spin gravity
- quantum Hall effect
- topological strings (topological vertex..)
- $4d$ $\mathcal{N} = 4$ SYM at codimension 2 junction of three codimension 1 defects (Gaiotto, Rapčák)
- superconformal indices of 4d $\mathcal{N} = 2$ SCFTs
- geometric representation theory (equivariant cohomology of moduli spaces of instantons)

Zamolodchikov $\mathcal{W}_3$ algebra

$\mathcal{W}_3$ algebra constructed by Zamolodchikov (1984) has a stress-energy tensor (Virasoro algebra) with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$

together with spin 3 primary field $W(w)$

$$T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + reg.$$

To close the algebra we need to find the OPE of W with itself consistent with associativity (Jacobi, crossing symmetry...).

The result:

$$W(z)W(w) \sim \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{1}{(z-w)^2}\left(\frac{32}{5c+22}\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right) + \frac{1}{z-w}\left(\frac{16}{5c+22}\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right) + reg.$$

Λ is a quasiprimary 'composite' (spin 4) field,

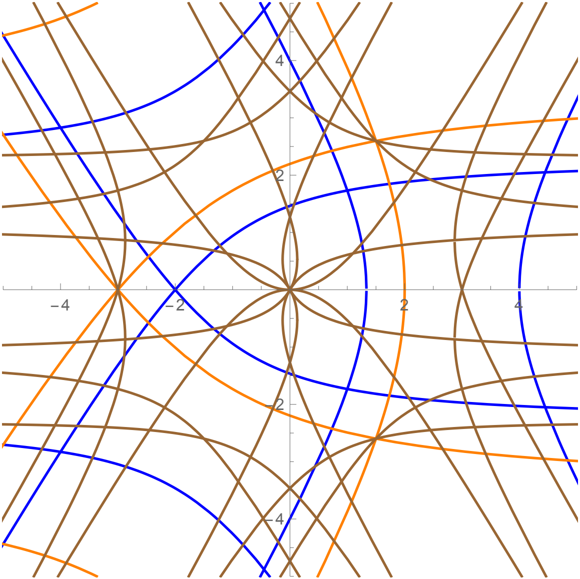
$$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z).$$

The algebra is non-linear, not a Lie algebra in the usual sense

W_N series and $\mathcal{W}_\infty$ algebra

- W_N: an interesting family of W-algebras associated to $\mathfrak{sl}(N)$ Lie algebras (spins $2, 3, \ldots, N$, Virasoro $\leftrightarrow \mathfrak{sl}(2)$)
- $\mathcal{W}_\infty$: interpolating algebra for W_N series; spins $2, 3, \ldots$
- Gaberdiel-Gopakumar: solving associativity conditions for this field content $\rightsquigarrow$ two-parameter family: central charge c and rank parameter λ (proof cf. Andy Linshaw)
- choosing $\lambda = N \to$ truncation of $\mathcal{W}_\infty$ to $\mathcal{W}_N = \mathcal{W}[\mathfrak{sl}(N)]$, i.e. $\mathcal{W}_\infty$ is interpolating algebra for the whole $\mathcal{W}_N$ series
- adding spin 1 field, we have $\mathcal{W}_{1+\infty} \rightsquigarrow$ *many simplifications*
- **triality** symmetry of the algebra (Gaberdiel & Gopakumar) $\mathcal{W}_\infty[c, \lambda_1] \simeq \mathcal{W}_\infty[c, \lambda_2] \simeq \mathcal{W}_\infty[c, \lambda_3]$

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \qquad c = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1)$$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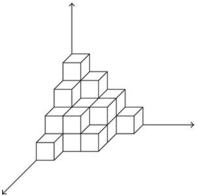
- MacMahon function as vacuum character of the algebra (enumerating all the local fields in the algebra)

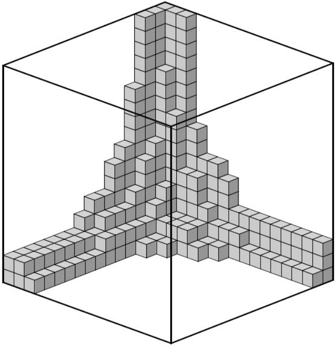
$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \cdots$$

- The same generating function is well-known to count the plane partitions (3d Young diagrams)

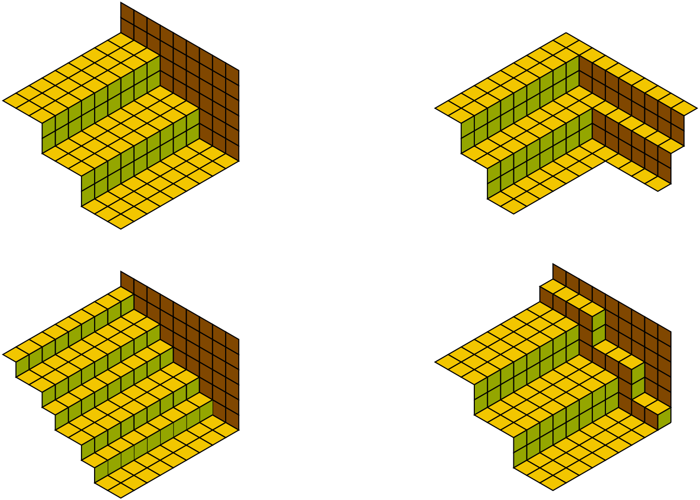


- triality acts by permuting the coordinate axes
- restriction to $\mathcal{W}_N$ corresponds to max N boxes in one of the directions

- this can be generalized to degenerate primaries (not only vacuum rep) by allowing 2d Young diagram asymptotics
- counting exactly as in topological vertex $\rightsquigarrow$ topological vertex can be interpreted as being a character of degenerate $\mathcal{W}_{1+\infty}$ representations
- the three asymptotic Young diagrams specify the representation (conformal primary)



- box counting generalizes also to $\mathcal{W}_N$ minimal models (Ising...)
  $\rightsquigarrow$ periodic configurations - lozenge tilings on cylinder



The OPEs in $\mathcal{W}_N$ or $\mathcal{W}_{1+\infty}$ are very complicated, but can be written for all spins in terms of bilocal quantities (TP 2014)

$$
\begin{aligned}
U_3(z)U_5(w) \quad \sim \quad & \frac{1}{(z-w)^7}\Big(\frac{1}{2}\alpha(n-3)(n-2)(n-1)n\left(4\alpha^2\left(\alpha^2(n(5n-9)+1)-3n+4\right)+1\right)\mathbb{1}\Big) \\
& +\frac{1}{(z-w)^6}\Big(\frac{1}{6}(n-3)(n-2)(n-1)\left(6\alpha^4 n(2n-3)+\alpha^2(10-9n)+1\right)U_1(w)\Big) \\
& +\frac{1}{(z-w)^5}\Big(-\alpha(n-3)(n-2)(n-1)\left(-4\alpha^2+3\alpha^2 n-1\right)(U_1U_1)(w) \\
& +\alpha(n-3)(n-2)\left(4\alpha^2n^2-4\alpha^2 n-n-2\right)U_2(w) \\
& -\frac{1}{2}\alpha^2(n-3)(n-2)(n-1)\left(4\alpha^2 n(2n-3)-3n+2\right)U_1'(w)\Big) \\
& +\frac{1}{(z-w)^4}\Big(-\alpha(n-3)(n-2)(n-1)\left(\alpha^2(3n-4)-1\right)(U_1'U_1)(w) \\
& -\frac{1}{2}(n-3)(n-2)\left(2\alpha^2(n-1)-1\right)(U_1U_2)(w) \\
& +(n-3)\left(\alpha^2\left(n^2+2\right)-3\right)U_3(w) \\
& -\frac{1}{4}\alpha^2(n-3)(n-2)(n-1)\left(4\alpha^2n(2n-3)-3n+2\right)U_1''(w) \\
& +\alpha(n-3)(n-2)\left(\alpha^2(n-1)n-1\right)U_2'(w)\Big) \\
& +\cdots
\end{aligned}
$$

¿ Is there a better way to organize the algebra ?

Yangian of $\widehat{\mathfrak{gl}(1)}$

The Yangian of $\widehat{\mathfrak{gl}(1)}$ (Arbesfeld-Schiffmann-Tsymbaliuk) is an associative algebra with generators $\psi_j, e_j, f_j, j \geq 0$ and relations

$$
\begin{aligned}
0 &= [e_{j+3}, e_k] - 3\,[e_{j+2}, e_{k+1}] + 3\,[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\
&\quad + \sigma_2\,[e_{j+1}, e_k] - \sigma_2\,[e_j, e_{k+1}] - \sigma_3\,\{e_j, e_k\} \\
0 &= [f_{j+3}, f_k] - 3\,[f_{j+2}, f_{k+1}] + 3\,[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}] \\
&\quad + \sigma_2\,[f_{j+1}, f_k] - \sigma_2\,[f_j, f_{k+1}] + \sigma_3\,\{f_j, f_k\} \\
0 &= [\psi_{j+3}, e_k] - 3\,[\psi_{j+2}, e_{k+1}] + 3\,[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] \\
&\quad + \sigma_2\,[\psi_{j+1}, e_k] - \sigma_2\,[\psi_j, e_{k+1}] - \sigma_3\,\{\psi_j, e_k\} \\
0 &= [\psi_{j+3}, f_k] - 3\,[\psi_{j+2}, f_{k+1}] + 3\,[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}] \\
&\quad + \sigma_2\,[\psi_{j+1}, f_k] - \sigma_2\,[\psi_j, f_{k+1}] + \sigma_3\,\{\psi_j, f_k\} \\
0 &= [\psi_j, \psi_k] \\
\psi_{j+k} &= [e_j, f_k]
\end{aligned}
$$

'initial/boundary conditions'

$$
\begin{aligned}
&[\psi_0, e_j] = 0, \qquad &[\psi_1, e_j] = 0, \qquad &[\psi_2, e_j] = 2e_j, \\
&[\psi_0, f_j] = 0, \qquad &[\psi_1, f_j] = 0, \qquad &[\psi_2, f_j] = -2f_j
\end{aligned}
$$

and finally the Serre-like relations

$$
0 = \mathrm{Sym}_{(j_1,j_2,j_3)} \left[e_{j_1}, [e_{j_2}, e_{j_3+1}]\right], \quad 0 = \mathrm{Sym}_{(j_1,j_2,j_3)} \left[f_{j_1}, [f_{j_2}, f_{j_3+1}]\right].
$$

Parameters $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{C}$ constrained by $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and

$$
\begin{aligned}
\sigma_2 &= \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3 \\
\sigma_3 &= \epsilon_1\epsilon_2\epsilon_3.
\end{aligned}
$$

We have both commutators and anticommutators in defining quadratic relations (but no $\mathbb{Z}_2$ grading) - for $\sigma_3 \neq 0$ not a Lie (super)-algebra.

Introducing generating functions (Drinfel'd currents)

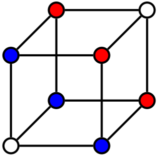
$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \quad f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \quad \psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}$$

the first set of formulas above (almost!) simplify to

$$e(u)e(v) \sim \varphi(u-v)e(v)e(u), \quad f(u)f(v) \sim \varphi(v-u)f(v)f(u),$$
$$\psi(u)e(v) \sim \varphi(u-v)e(v)\psi(u), \quad \psi(u)f(v) \sim \varphi(v-u)f(v)\psi(u)$$

with rational *structure function* (scattering phase in BAE)

$$\boxed{\varphi(u) = \frac{(u+\epsilon_1)(u+\epsilon_2)(u+\epsilon_3)}{(u-\epsilon_1)(u-\epsilon_2)(u-\epsilon_3)}}$$



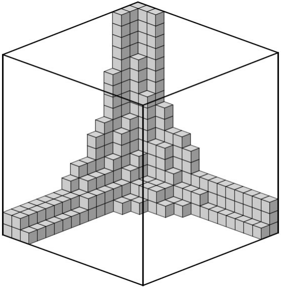
The representation theory of the algebra is much simpler in this Yangian formulation and φ controls basically everything

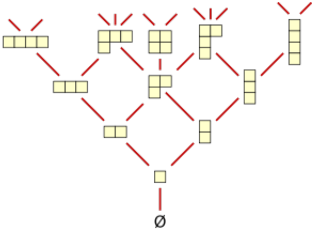
$\psi(u)$, $e(u)$ and $f(u)$ in representations act like

$$\psi(u)\,|\Lambda\rangle = \psi_0(u) \prod_{\square\in\Lambda} \varphi(u-\epsilon_\square)\,|\Lambda\rangle$$

$$e(u)\,|\Lambda\rangle = \sum_{\square\in\Lambda^+} \frac{E(\Lambda\to\Lambda+\square)}{u-\epsilon_\square}\,|\Lambda+\square\rangle$$

where the states $|\Lambda\rangle$ are associated to geometric configurations of boxes (plane partitions,...) and where $\epsilon_\square = \sum_j \epsilon_j x_j(\square)$ is the weighted geometric position of the box.





Classical KdV/KP

- in the classical limit, the theory reduces to the theory of integrable hierarchies of PDEs (KdV, KP)
- the classical object associated to Virasoro algebra is the one-dimensional Schrödinger operator

$$L^2 = \partial_x^2 + u(x)$$

- there exists an infinite dimensional family of continuous deformations of $u(x)$ which preserve the spectrum of L^2 and are organized into commuting flows
- the first such deformation is the trivial rigid translation of the potential

$$\partial_{t_1} u = \partial_x u$$

- the next one is already rather non-trivial and is captured by the Korteweg-de-Vries equation (Boussinesq 1877)

$$4\partial_{t_3} u = 6u\partial_x u + \partial_x^3 u.$$

- the space of Schödinger potentials is a Hamiltonian system if we equip it with Poisson bracket

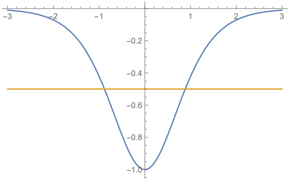
$$\{u(x), u(y)\} = -\delta'''(x-y) - 4u(x)\delta'(x-y) - 2u'(x)\delta(x-y)$$

(whose Fourier transform is just the classical Virasoro algebra)

- the deformations are generated by local Hamiltonians which are at the same time conserved quantities capturing the spectral data of the family of Schrödinger operators

$$I_1 = \int u(x)dx, \qquad I_3 = \int u^2(x)dx, \quad \ldots$$

- e.g. KdV soliton (Pöschl-Teller potential) with a single bound state



Quantum KdV

- the KdV conserved charges survive quantization in the form

$$I_1 = \int T(x)dx = L_0 - \frac{c}{24}$$

$$I_3 = \int (TT)(x)dx = L_0^2 + 2\sum_{m=1}^{\infty} L_{-m}L_m - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880}$$

$$I_5 = \int \left[(T(TT))(x) - \frac{c-2}{12}(\partial T \partial T)(x)\right] dx = \ldots$$

so it makes sense to ask what their spectrum is

- since L_0 is part of the family, the problem is to diagonalize finite dimensional matrices level by level
- a surprising description of their spectrum was found by Bazhanov-Lukyanov-Zamolodchikov (in the context ODE/IM correspondence initiated by Dorey and Tateo)

- consider a Schrödinger operator

$$\partial_x^2 + x^N + u + \frac{\ell(\ell+1)}{z^2}$$

associated to a CFT primary state (central charge c and conformal dimension Δ are encoded in N and ℓ) and dress it by allowing for additional collection of regular singular points

$$\sum_{j=1}^{M(N+2)} \left[\frac{2}{(x-x_j)^2} + \frac{\gamma_j}{x(x-x_j)} \right]$$

where M is the Virasoro level

- the requirement of trivial monodromy around these singularities leads to a system of BLZ Bethe equations

$$0 = N x_j^{N-1} + \frac{2\ell(\ell+1)}{x_j^3} + \sum_{k \neq j} \frac{4}{(x_j - x_k)^3}$$

in coordinates convenient for AD spectral curve *[Fioravanti 2004]*

- given any solution of BLZ Bethe equations, the eigenvalues of I_j are determined, for instance

$$I_3 = \frac{(N-3)(N-1)(2N+3)}{1920(N+2)} - \frac{(-\ell(\ell+1)-2(N+2)M)^2}{8(N+2)} - \frac{(N+1)\left(-\ell(\ell+1)-2(N+2)M\right)}{16(N+2)} + \frac{N}{2}\sum_j x_j^{N+2}$$

- generalization to higher ranks? $\partial^N + x^K - u = 0$ *[Dorey,Dunning,Masoero,Suzuki,Tateo 2006, Feigin,Frenkel 2007 Masoero,Raimondo 2018]*
  $\rightsquigarrow$ the trivial monodromy condition selecting differential operators associated to simultaneous eigenstates of local IM applies just as in the Virasoro case
- how to find the expressions for I_j in terms of Bethe roots? ODE/IM: using WKB! [Kudrna,Prochazka 2025]

Argyres-Douglas VOAs

- for concreteness, in the following we focus on minimal models that have simultaneously $\mathcal{W}_N$ and $\mathcal{W}_K$ symmetry with $\gcd(K, N) = 1$
- these are non-unitary $\mathcal{W}_N$ or $\mathcal{W}_K$ minimal models with central charge

$$c = -\frac{(K-1)(N-1)(NK+K+N)}{K+N} < 0$$

  the simplest being the Lee-Yang model with $K = 2$, $N = 3$ and $c = -22/5$
- these VOAs also control twisted OPEs of Schur operators in 4d $\mathcal{N} = 2$ Argyres-Douglas SCFTs of (A_{K-1}, A_{N-1}) type so we call them Argyres-Douglas VOAs
- by specializing to these we reduce from two complex parameters to two integer parameters, but most quantities of our interest are analytic functions of these parameters so we do not lose any information

WKB calculation

- to calculate the eigenvalues of $\mathcal{I}_j$ associated to a given differential operator, we use the formal WKB expansion [Dorey,Dunning,Negro,Tateo 2019]
- we look for a formal wave function annihilated by

$$\hbar^N \partial_x^N + x^K - 1 + \sum_{j=2}^{N-1} \frac{b_j}{x^j} \hbar^{N-j} \partial_x^{N-j} + (descendants)$$

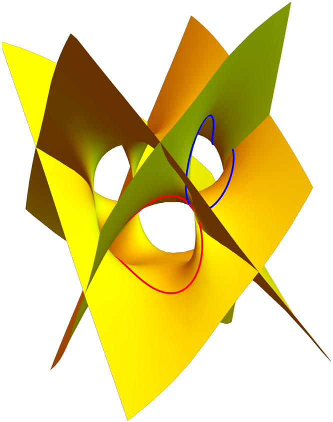
in the WKB form

$$\psi(x) = \exp\left[\frac{1}{\hbar} \sum_{n=0}^{\infty} \hbar^n \int^x Y_n(x') dx'\right]$$

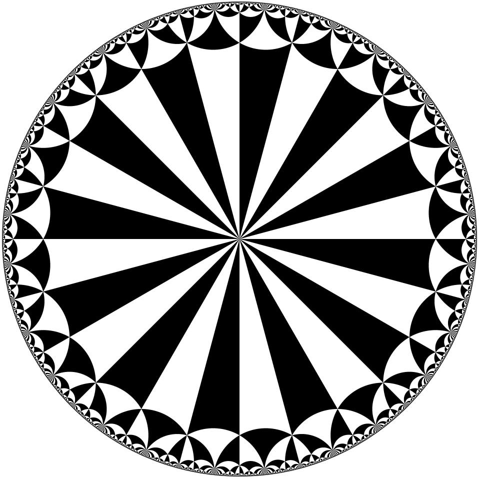
- the zeroth order equation determines WKB curve $\mathcal{W} \subset \mathbb{C}^2$,

$$Y_0^N + x^K = 1$$

which is sometimes called Catalan curve and for $\gcd(K, N) = 1$ has genus $(K-1)(N-1)/2$

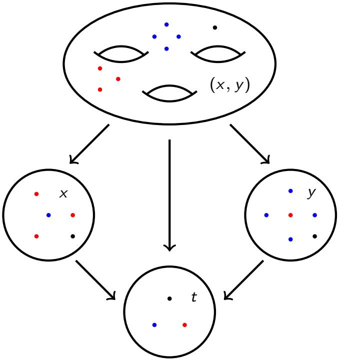


Projection of $\mathcal{W} \subset \mathbb{C}^2 \simeq \mathbb{R}^4$ to 3d subspace for $(\mathcal{W}_3, \mathcal{W}_4)$ minimal model ($c = -114/7$, WKB curve of genus 3)



The fundamental 24-gon for $(\mathcal{W}_3, \mathcal{W}_4)$ model ($c = -114/7$, WKB curve of genus 3)

- we want to identify the integrals of motion $\mathcal{I}_n$ with period integrals of Y_n along cycles of $\mathcal{W}$
- fortunately our WKB curves have large discrete group of automorphisms $\mathbb{Z}^K \times \mathbb{Z}^N$
- we can therefore quotient $\mathcal{W}$ by this group and obtain a rational curve $\mathcal{M}$ which we call the *mirror curve*
  (it is the mirror curve to $\mathbb{C}^3$ which is very closely related to $\mathcal{W}_{1+\infty}$)



- when projected to $\mathcal{M}$, the differentials $Y_n(x)\,dx$ are not single-valued, but branch at $0, 1, \infty$ with monodromies

$$\exp\left(\frac{2\pi i}{\lambda_j}\right)$$

- the CY condition

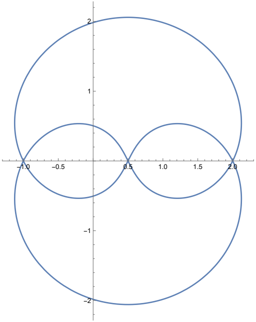
$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0$$

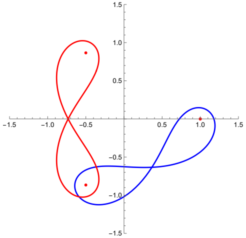
or the equivalent Nekrasov-like condition

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$$

precisely guarantee that there are only three singular points at $0, 1, \infty$

- for generic values of parameters, there is essentially a unique contour on $\mathcal{M}$ over which we can integrate $Y_n(x)\,dx$, the Pochhammer contour
- its lifts to WKB curve $\mathcal{W}$ are pairs of figure eight contours around the branch points (one in x-plane and one in y-plane)





- all the possible lifts of the Pochhammer contour give an overcomplete generating set of $H_1(\mathcal{W}, \mathbb{Z})$
- this set is not a symplectic basis of A- and B-cycles, but has the advantage of respecting the symmetries of the problem
- the intersection pairing reflects the basic structure constant of the algebra or the fundamental box of Nekrasov box counting

$$\mathcal{C}_{j,k} \cap \mathcal{C}_{j+1,k} = +1, \ldots$$

- all of the lifts of the Pochhammer contour give up to a phase the same period integrals and we find an easy identification

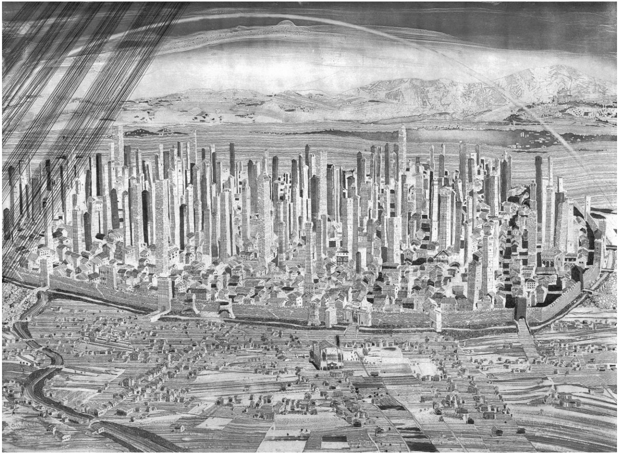
$$\mathcal{I}_n \sim \#^{n-1} \frac{\Gamma\left(\frac{n-1}{\lambda_1}\right)\Gamma\left(\frac{n-1}{\lambda_2}\right)\Gamma\left(\frac{n-1}{\lambda_3}\right)}{(\lambda_1\lambda_2\lambda_3)^{1/2}} \int_{\mathcal{C}} Y_n(x)dx$$

that applies not only to the highest weight states but also to descendants, i.e. we can identify the quantum periods with the eigenvalues of the local integrals of motion $\mathcal{I}_n$

- in this way we can find easily expressions for eigenvalues of $\mathcal{I}_n$ up to $n \sim 15 - 20$ in terms of Bethe roots for Virasoro algebra, $\mathcal{W}_3$ and $\mathcal{W}_4$
- for Argyres-Douglas VOAs which have simultaneously $\mathcal{W}_K$ and $\mathcal{W}_N$ symmetry we find exact agreement between calculations using both versions of Bethe equations and also explicit calculations using commutation relations of $\mathcal{W}_\infty$

Questions

- how are the (solutions of?) ILW and BLZ Bethe ansatz equations related?
- another set of Bethe ansatz equations based on affine Gaudin model (nested BA structure?)
- how can the ILW generating function be regularized to extract interesting information in $q \to 1$ limit? qq-characters?
- refined characters & modularity (Dijkgraaf, Maloney-Ng-Ross-Tsiares) & thermodynamic properties and geometrization
- quantum periods (non-perturbative completion of the local expansion?), TBA, mirror symmetry in topological string



Thank you