ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

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Istituto Nazionale di Fisica Nucleare

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Entanglement dynamics and Page curves in random permutation circuits*

11th Bologna Workshop on CFT and Integrable Models - Bologna, 2–5 Sept 2025





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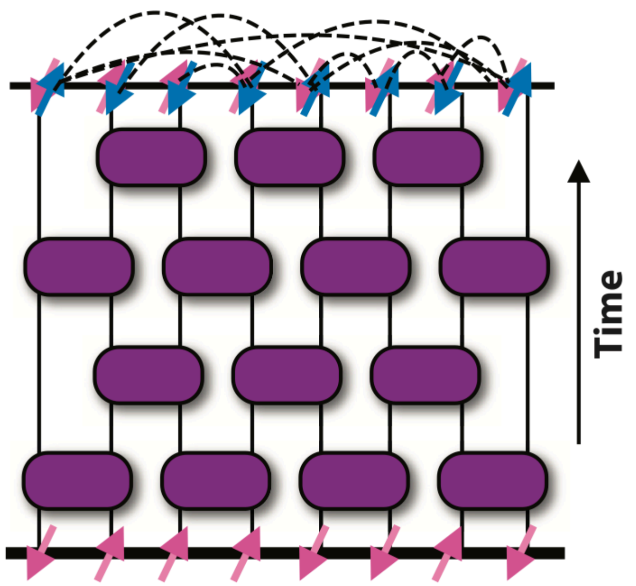
*https://arxiv.org/abs/2505.06158

Random ensembles

- Important tool of quantum information theory and many-body physics

Haar random unitary

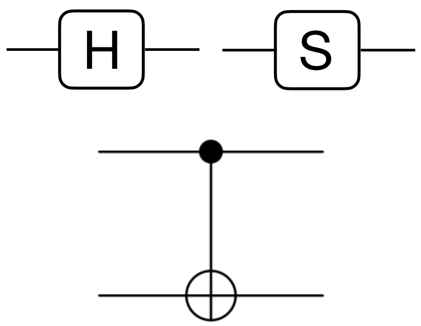


Entanglement dynamics,

thermalization…

Random Clifford

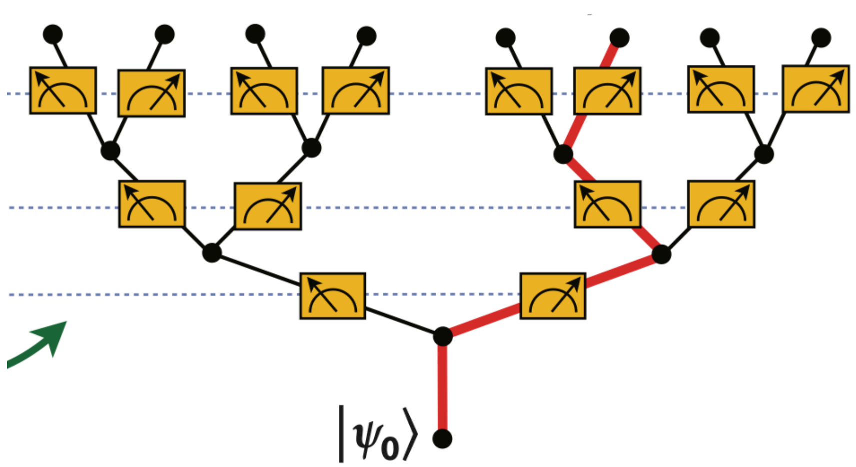
$$\{H, S, CNOT\}$$



Stabilizerness,

error correction…

Measurements



MIPT, …

- **Haar random circuits:** our benchmark. Maximally chaotic

  ➜ Exact solutions via Weingarten calculus!

M. P. Fisher, V. Khemani, A. Nahum, and S. Vijay, Ann. Rev. Cond. Matt. Phys. 14, 335 (2023).

M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information: 10th Anniversary Edition (Cambridge University Press, 2011).

Permutation gates

- Permutation $\pi \in S_{2^k}$ permutes all possible binary strings of length k.

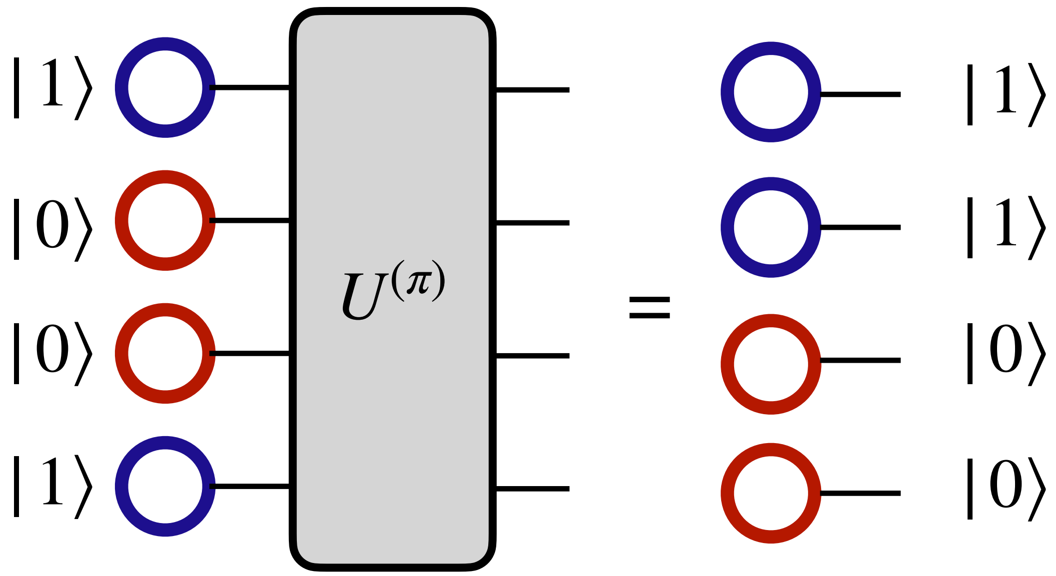
  e.g. $(1,0,0,1) \rightarrow (1,1,0,0)$, $(1,1,1,1) \rightarrow (0,1,0,0)$, etc.

- A **permutation gate** $U^{(\pi)}$ **acting on** k **qubits** is defined as

  $U^{(\pi)} | s \rangle = | \pi(s) \rangle$ $\qquad$ $| s \rangle = | s_1 \rangle \otimes | s_2 \rangle \otimes \ldots \otimes | s_k \rangle$

  e.g. $U^{(\pi)} | 1111 \rangle = | 0100 \rangle$

  $U^{(\pi)} | 1001 \rangle = | 1100 \rangle$ etc.

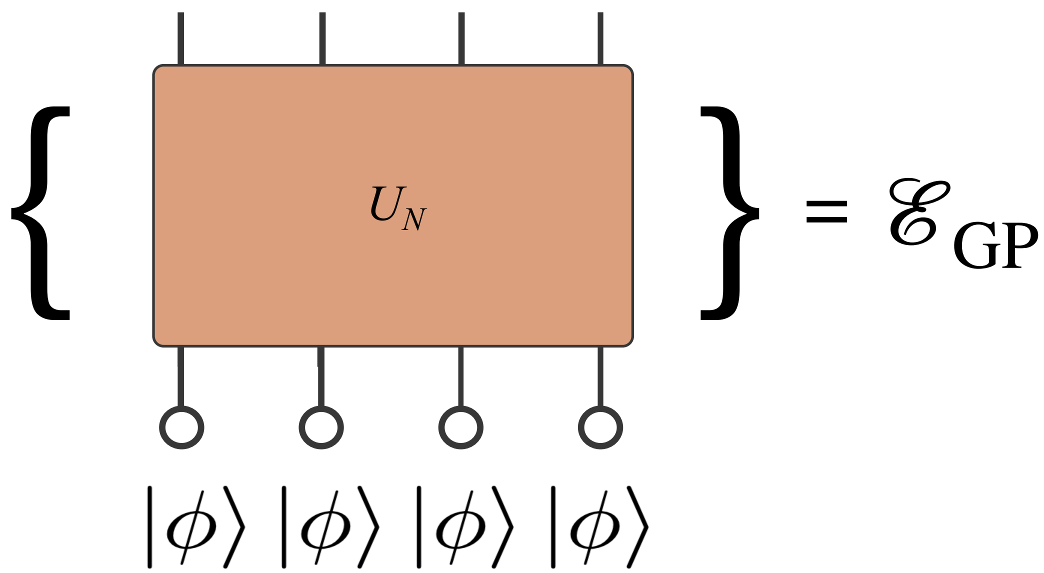


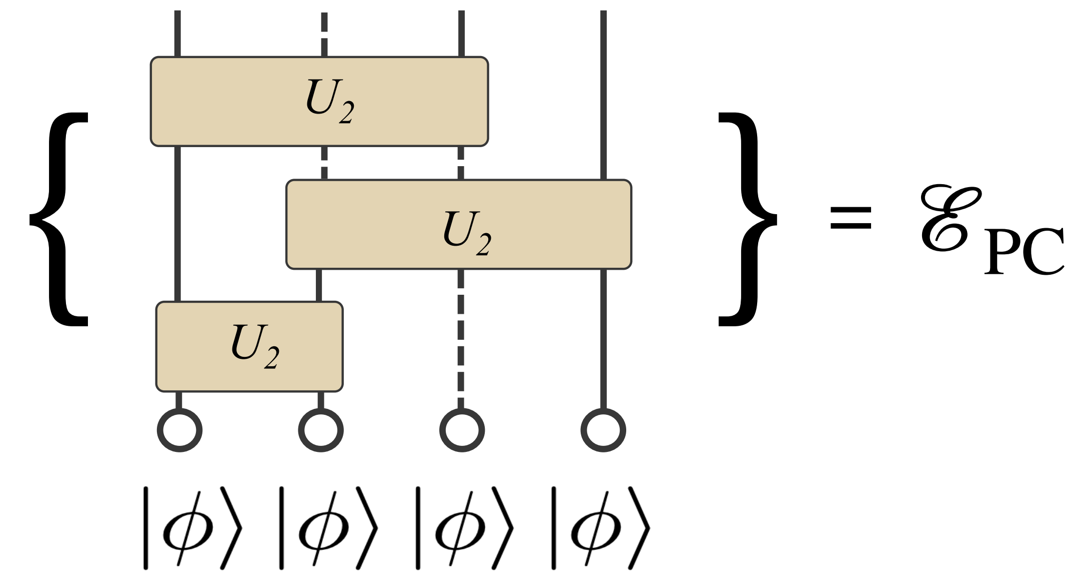
- Permutation gates **act classically on the computational basis**,

  But they produce entanglement when acting on generic states

Random permutation circuits (RPC)

- We consider two ensembles of random permutations: $\mathscr{E}_{\mathrm{GP}}$ and $\mathscr{E}_{\mathrm{PC}}$





Global gates (acting on N qubits)

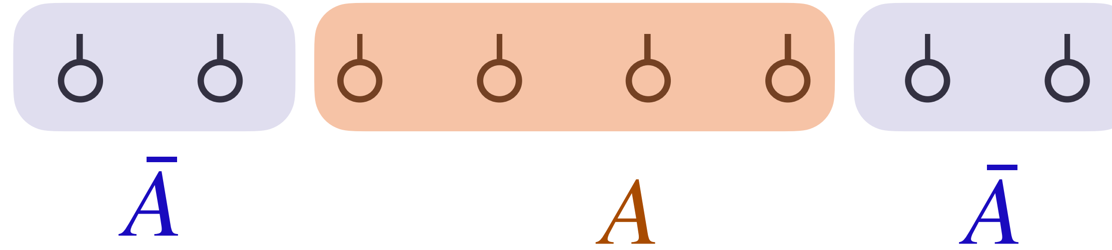
2-local circuits of finite depth

1. With probability p apply a 2-body gate $\mathscr{U}_{i,j}$ to the qubits at sites i, j
2. Average over the set $\{\mathscr{U}_{i,j}\}$ and over the sites i, j

- The dynamics is chaotic, but there are (few) constants of motion!

B. Bertini, K. Klobas, P. Kos, and D. Malz, arXiv:2508.10890 (2025).

Main results



We focus on **entanglement dynamics**

$$\mathbb{E}[S_\alpha(\rho_A(t))] = \mathbb{E}\left[(1-\alpha)^{-1}\log_2 \mathrm{Tr}(\rho_A^\alpha(t))\right] \qquad \rho_A(t) = \mathrm{Tr}_{\bar{A}}[\,|\psi_t\rangle\langle\psi_t|\,]$$

Averaged over random ensembles $\mathscr{E}_{\mathrm{GP}}$ or $\mathscr{E}_{\mathrm{PC}}$

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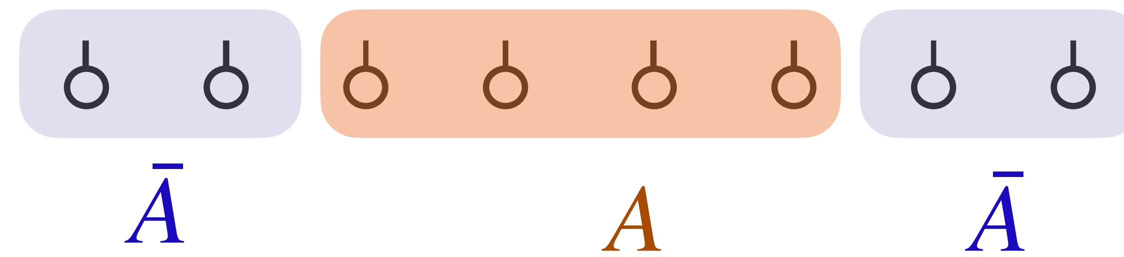

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1. **Bounds on the entanglement** that can be generated by permutation circuits acting on *arbitrary initial states*. We show that they are **typically saturated**.

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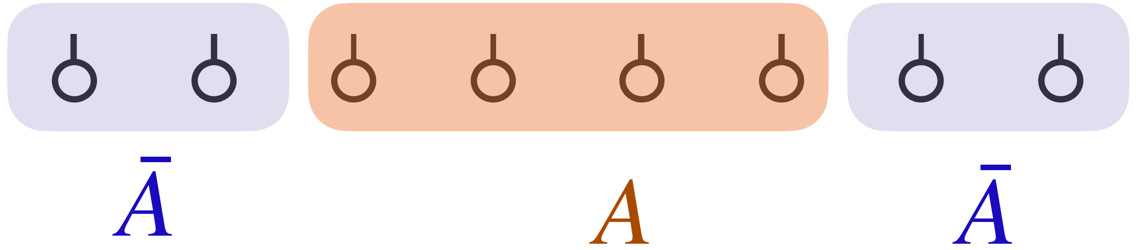

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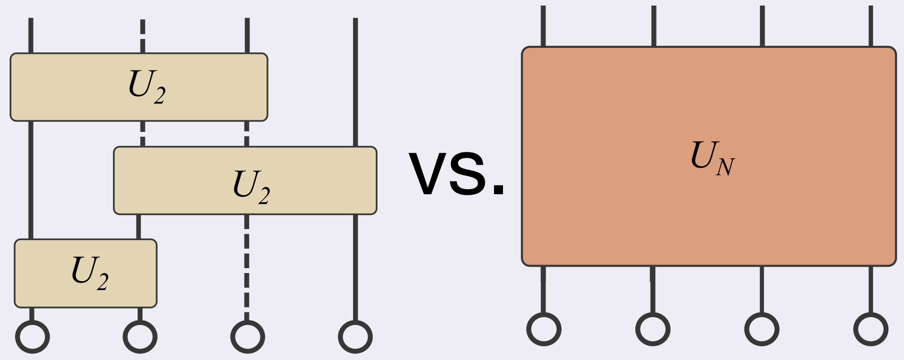

: Infinite-time **Page curves** for averaged Rényi-2 entropies are different for finite N but **typically coincide** in the thermodynamic limit.

Bounds

- IPR and participation entropy: $$\begin{cases} I_\alpha(|\psi\rangle) = \sum_s |\langle s|\psi\rangle|^{2\alpha} \\ S_\alpha^{\mathrm{PE}}(|\psi\rangle) = (1-\alpha)^{-1}\log_2 I_\alpha(|\psi\rangle) \end{cases}$$

Measures of **anti-localization** (spread over the computational basis elements) $$0 \leq S_\alpha^{\mathrm{PE}}(|\psi\rangle) \leq \log D$$

- Overlap with the "maximally anti-localized" state: $z(|\psi\rangle) = |\langle a_N|\psi\rangle|$

$$|a_N\rangle = |+\rangle^{\otimes N} = (|0\rangle + |1\rangle)^{\otimes N}/2^{N/2}$$

$S_\alpha^{\mathrm{PE}}(|\psi\rangle)$, $z(|\psi\rangle)$: **constants of motion!** + can be efficiently computed for low-entangled, shortly-correlated states

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Rényi-α entropies are bounded*

$$S_\alpha(\rho_A(t)) \le \min[|A|, S_\alpha^{\mathrm{PE}}(|\psi_0\rangle), (1-\alpha)^{-1}\log_2 z^{2\alpha}] \qquad |A| \le N/2$$

High localization prevents maximal entanglement saturation!

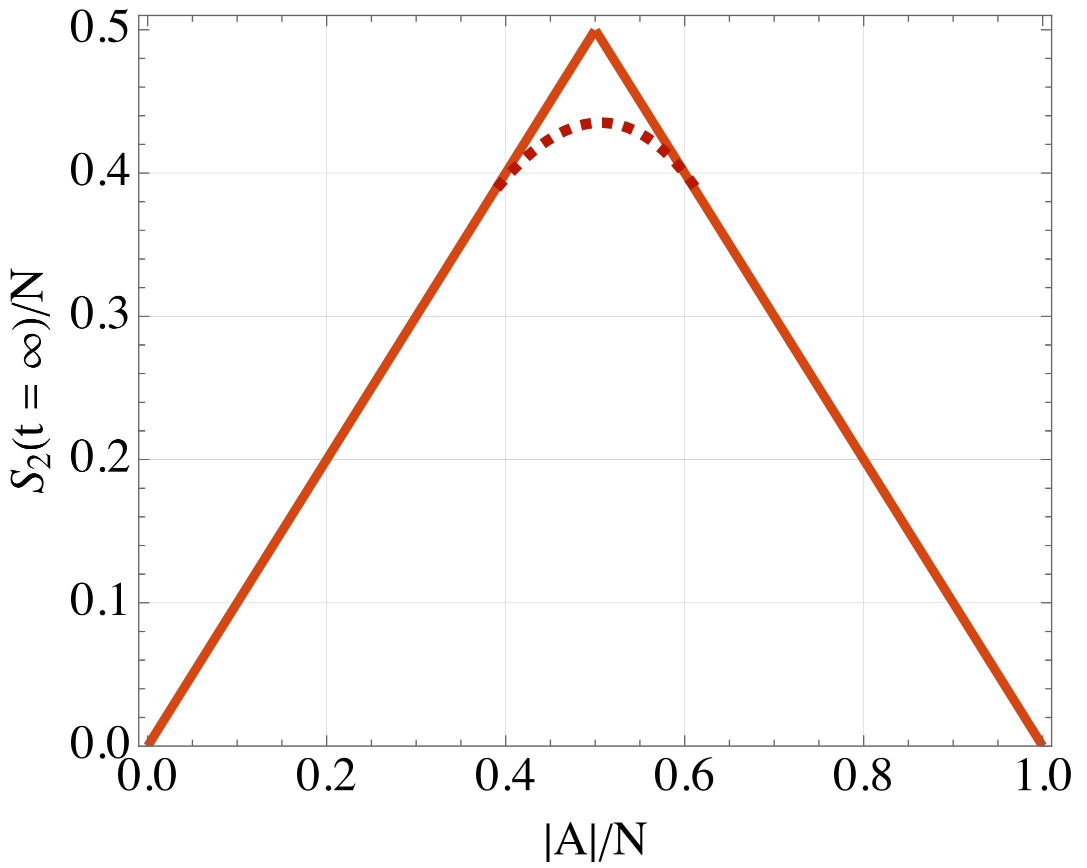
*** and bound *typically* saturated**

S. Gopalakrishnan and B. Zakirov, Quantum Science Tech. 3, 044004 (2018))

Page curve of 2-local permutation circuit

- Entropy at infinite time (circuit depth) vs system size: **Page curve**
- Haar random circuit:

$$\mathbb{E}_{\text{Haar}}[S_\alpha(t=\infty)] = \min(\log 2^{|A|}, \log 2^{N-|A|}) + O(1)$$



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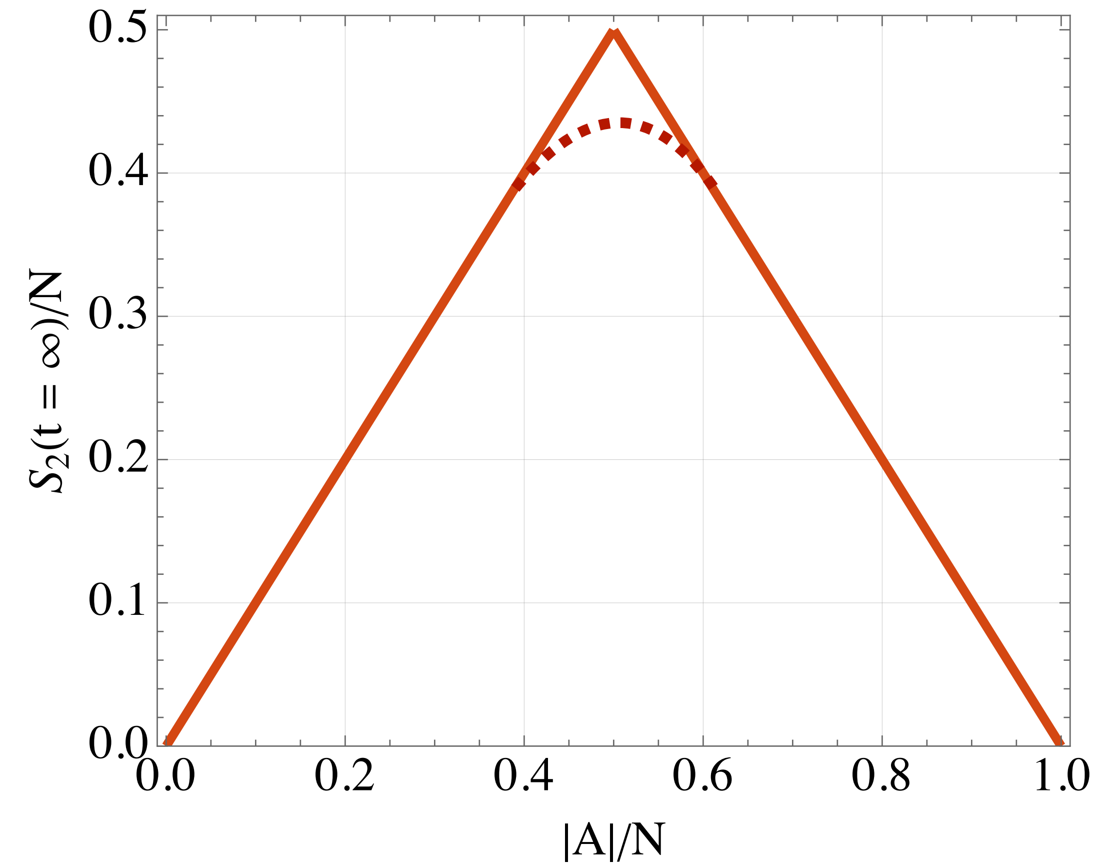
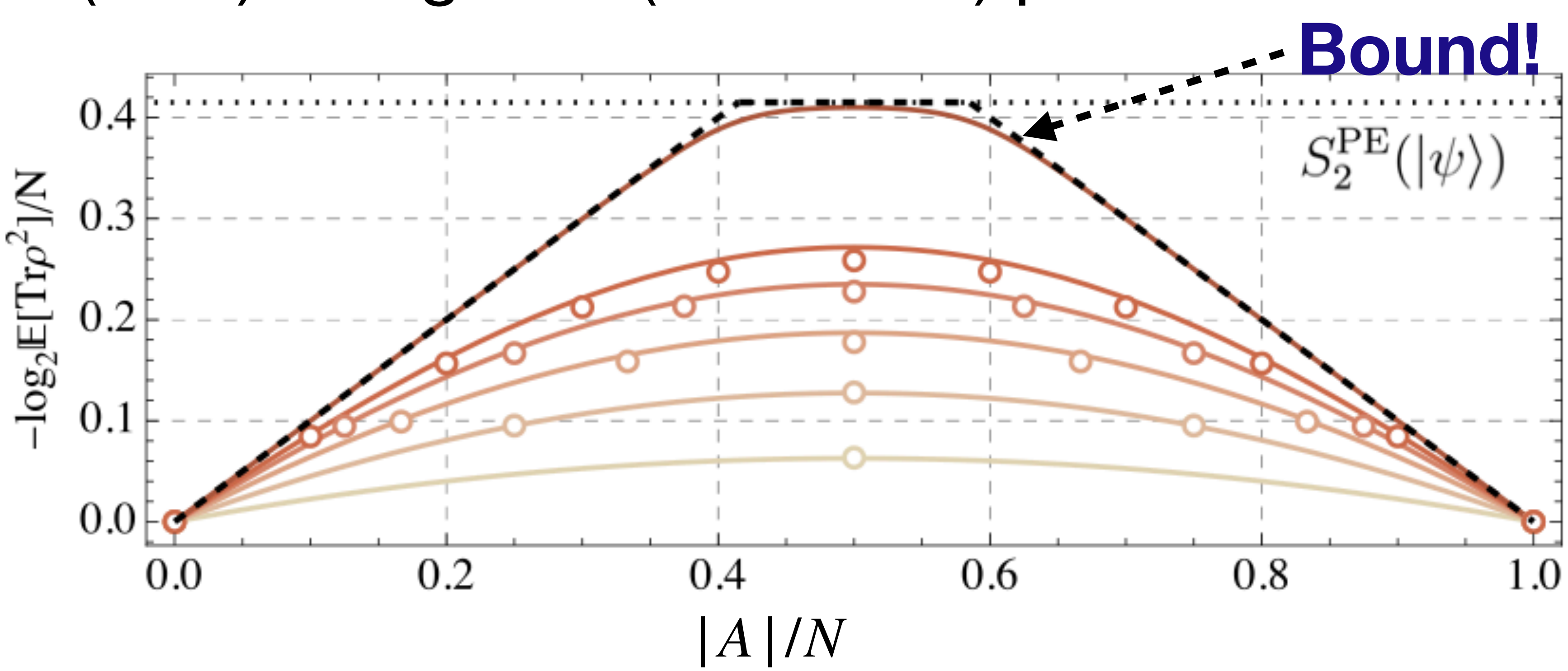


- Permutation circuit: averaged late time Rényi-2 entropies for local (dots) and global (solid lines) permutations



Initial product state

$$|\phi(\theta)\rangle = (\cos\theta|0\rangle + \sin\theta|1\rangle)^{\otimes N}$$

$$\theta = \pi/8$$

$$N = 2,4,6,8,10,60$$

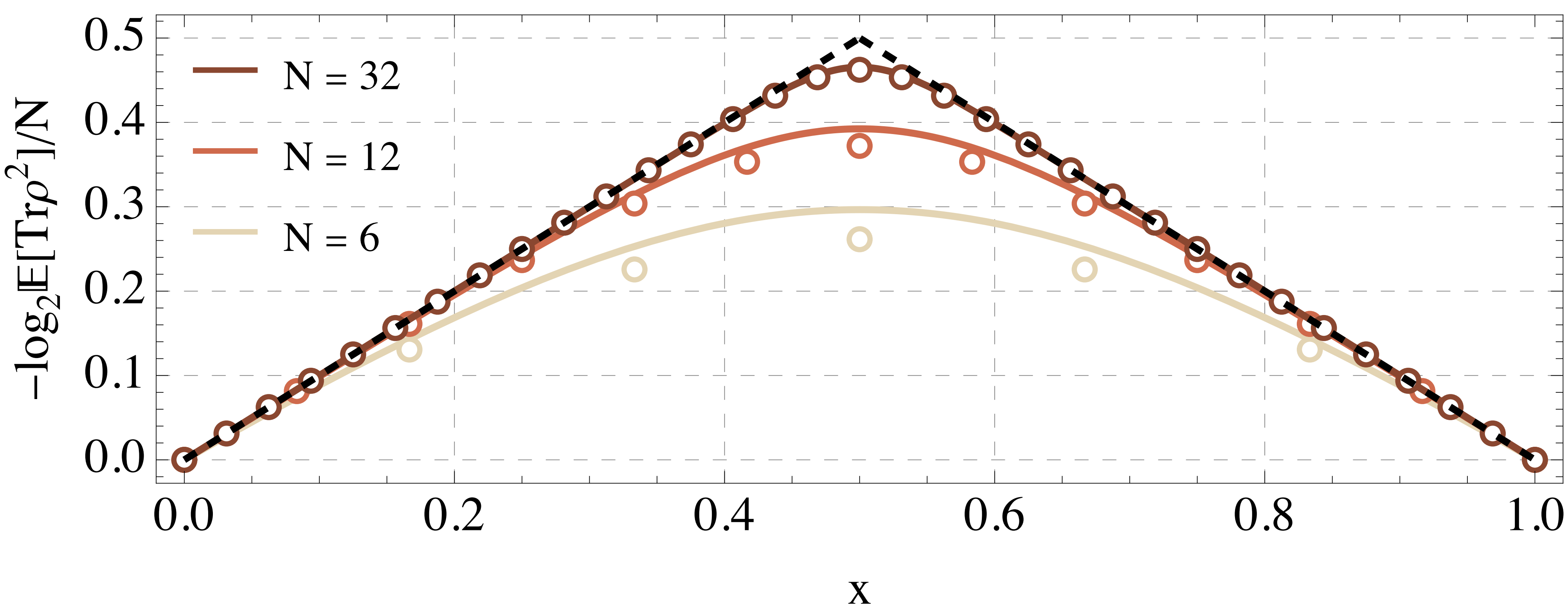
2-Local vs Global permutations

- Haar random: in sufficiently connected local circuits, averaged entropies coincide with those computed in the global ensemble at any finite size

- Different situation for 2-local permutation circuits:

$$-\mathbb{E}_{\mathrm{GP}}[\log \mathrm{Tr}\rho_A^2] \neq -\mathbb{E}_{\mathrm{PC}}[\log \mathrm{Tr}\rho_A^2(t=\infty)]$$ at finite size

But they **typically** coincide as $N \to \infty$



Initial **random product state**

$$\mathbb{E}_{\mathrm{GP}}[\mathrm{Tr}\rho_A^2] = q^{-xN} + q^{-(1-x)N} + \left(1 - q^{-xN} - q^{-(1-x)N}\right)\mathbb{E}[I_2] + \mathcal{O}(q^{-N})$$

Program

We compute the **averaged bipartite entanglement** in the global permutation ($\mathscr{E}_{\mathrm{GP}}$) and the 2-local permutation circuit ($\mathscr{E}_{\mathrm{PC}}$) ensembles via a replica space approach

we are interested in three aspects:

1. How much time $\mathscr{E}_{\mathrm{PC}}$ needs to reach **stationary values**?

2. Do these coincide with $\mathscr{E}_{\mathrm{GP}}$?

3. What about $k-$**local** permutations ($k > 2$) and adding random phases (**automaton circuits**)?

Haar random circuits: the stationary values of the circuit dynamics coincide with the global ensemble. + scrambling time $t_s \sim O(\log N)$

2-local circuit dynamics in replica space

1

Replica space: $|\rho_A(t) \otimes \rho_A(t)\rangle = \mathbf{1} \otimes \rho_A(t) \otimes \mathbf{1} \otimes \rho_A(t)|\mathscr{I}\rangle$

where $|\mathscr{I}\rangle = |I^+\rangle^{\otimes N} \in \mathscr{H}^{\otimes 4}$ and $|I^+\rangle = \sum_{a,b=0}^{1} |aabb\rangle$

2

Purity: $\mathrm{Tr}_A \mathbb{E}[\rho_A^2(t)] = \langle G_{|A|,N-|A|}|\rho_A(t) \otimes \rho_A(t)\rangle$

where $|G_{n_-,n_+}\rangle = |I^-\rangle^{\otimes n_-} \otimes |I^+\rangle^{\otimes n_+}$ with $|I^-\rangle = \sum_{a,b=0}^{1} |abba\rangle$

3

Lindbladian dynamics: $\frac{d}{dt}\mathbb{E}\big[|\rho(t)\rangle \otimes \rho(t)\big] = -\mathscr{L}\mathbb{E}\big[|\rho(t)\rangle \otimes \rho(t)\big]$

where $\mathscr{L} = \frac{2\lambda}{N-1} \sum_{\langle j,k\rangle} (1 - \mathscr{U}_{j,k})$

and $\mathscr{U}_{j,k} = \mathbb{E}[U^*_{j,k} \otimes U_{j,k} \otimes U^*_{j,k} \otimes U_{j,k}]$

Local dimensional reduction

Solving $\frac{d}{dt}\mathbb{E}\left[\,|\rho(t)\rangle \otimes \rho(t)\right] = -\mathscr{L}\mathbb{E}\left[\,|\rho(t)\rangle \otimes \rho(t)\right]$ is hard (exponentially large Hilbert space, local dimensions $d_{\text{eff}} = 15$)

Simplifications:

1 **Site-permutation invariance**: reduces to polynomial, still large

$$D_{\text{symm}} = \binom{N + d_{\text{eff}} - 1}{d_{\text{eff}} - 1} \sim N^{14}$$

2 **Permutations = Cliffords**: reduces dimensionality of $\mathscr{H}_{\text{eff}}$ for typical (random) initial states

$$|\psi_0^{\text{rand}}\rangle = \otimes_j |\phi_j\rangle = \otimes_j v_j|0\rangle \qquad \text{with } v_j \text{ Haar random}$$

$d_{\text{eff}} = 4 \;\rightarrow\; D_{\text{symm}} \sim N^3 \;\rightarrow$ Effective numerical implementation

2-local circuit dynamics for typical states

Lindbladian dynamics for random initial states:

$$|\rho(t) \otimes \rho(t)\rangle = e^{-\tilde{\mathscr{L}}t}\mathbb{E}_{\phi_j}[\otimes_j \Pi_j |(|\phi_j\rangle^* |\phi_j\rangle)^{\otimes 2}\rangle]$$

with $\tilde{\mathscr{L}} \equiv (\otimes_j \Pi_j)\mathscr{L}(\otimes_j \Pi_j)$ and $\Pi_j = \frac{1}{4}\sum_{\alpha=0}^{3} \sigma_j^{\alpha *} \otimes \sigma_j^{\alpha} \otimes \sigma_j^{\alpha *} \otimes \sigma_j^{\alpha}$

Differential equations encoding the *purity dynamics*

$$\frac{dG_{p,q,r,s}(t)}{dt} = \sum_{\{\ell_k\}_{k=1}^4} c[\{\ell_k\}] G_{p+\ell_1,q+\ell_2,r+\ell_3,s+\ell_4}(t) \,.$$

$$G_{p,q,r,s} = \langle G_{p,q,r,s} | \rho_A(t) \otimes \rho_A(t)\rangle$$

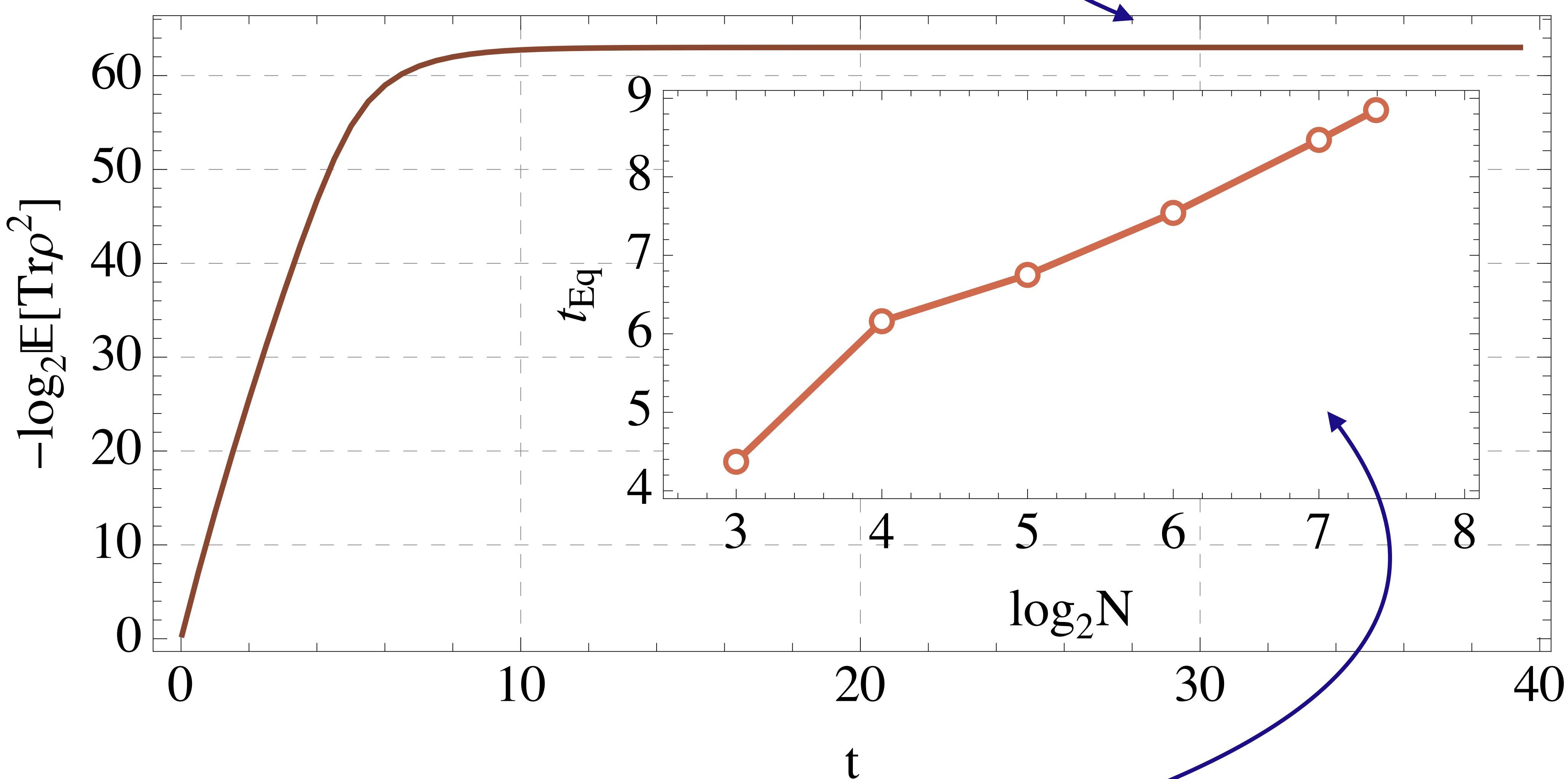
$$|G_{p,q,r,s}\rangle = |I^-\rangle^{\otimes p} \otimes |I^+\rangle^{\otimes q} \otimes |I^0\rangle^{\otimes r} \otimes |I^x\rangle^{\otimes s}$$

$$|I^0\rangle = \sum_{a=0,1} |aaaa\rangle$$

$$|I^x\rangle = \sum_{a,b=0,1} |abab\rangle$$

2-local circuit dynamics for typical states

Stationary value corresponding to $\mathscr{E}_{\rm PC}$



Fast scrambler! $t_{\rm Eq} \sim O(\log N)$

Remark - breaking the Clifford property

- Although permutations = Cliffords, our initial states are not stabilizers → non-trivial behaviour

- To break Clifford property:

  1. go from 2-local to ***k*-local** ($k > 2$) gates

  2. Add **random phases**

$$U_{ij}^{(\pi)} | s_1, s_2 \rangle = e^{i\varphi_{s_1,s_2}} | \pi(s)_1, \pi(s)_2 \rangle$$

- k-local: known to coincide for all N

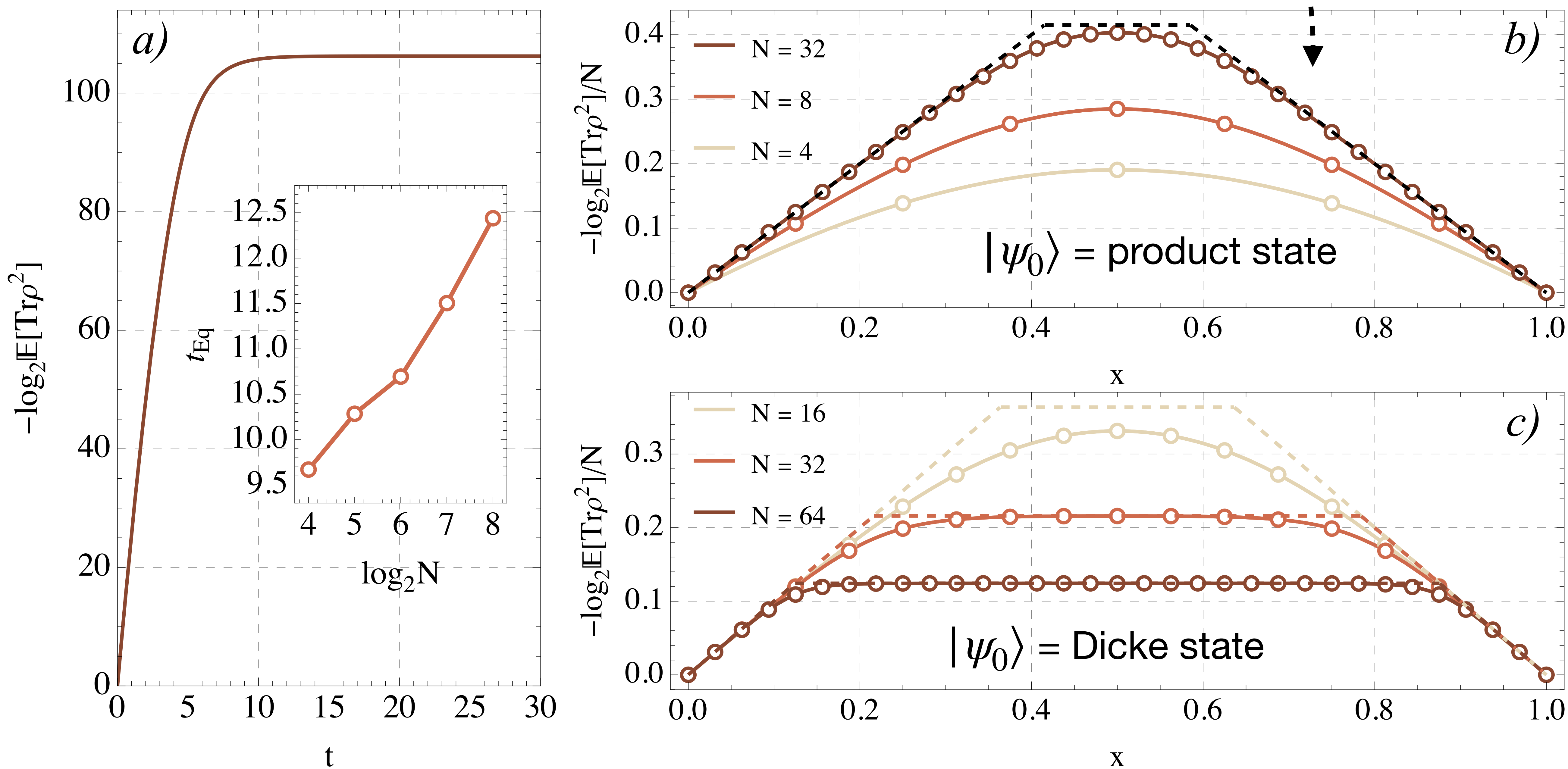
- phases: z is not conserved → bounds are given in terms of part. entropy

$$S_\alpha(\rho_A(t)) \leq \min[\, |A|\, , S_\alpha^{\mathrm{PE}}(|\psi_0\rangle)], \quad |A| \leq N/2$$

Random permutations + phases

Random phases → simplifications → we have results for very general states, including entangled states (plots)

$$S_\alpha(\rho_A(t)) \leq \min[\,|A|, S_\alpha^{\mathrm{PE}}(|\psi_0\rangle)]$$



Outlook

- Bound for the von Neumann entropy:

$$S_1(t) \leq \min\left[|A|, S_1^{\mathrm{PE}}(|\psi\rangle), \frac{|A|}{2}\sqrt{1-z^2} + O(1)\right], \quad |A| \leq N/2$$

is it *typically* tight?

- Which initial states saturate the bound apart from random product states?

- Modifying circuits? eg: by adding measurements

We expect a non-trivial initial state dependence.

Preprint from arxiv