Integrability of 1/2 BPS Nahm pole defects in $\mathcal{N} = 4$ SYM

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Based on:

A. Chalabi, C.K., C. Su, Phys.Lett. B866 (2025), (ArXiv:2503.22598)
C.K., & K. Zarembo, JHEP08 (2023) 184 (ArXiv:2305.03649),
JHEP 02 (2025) 179 (ArXiv:2412.01972)
Earlier work w/ de Leeuw, Gombor, Linardopoulos,...

11'th Bologna Workshop on Conformal Field Theory and Integrable Models
University of Bologna, Italy
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AdS/CFT

Conformal operator, $\mathcal{O}$ $\longleftrightarrow$ String state (AdS/CFT)

$\updownarrow$ $\updownarrow$

Eigenstates of integrable super spin chain: $|\mathbf{u}\rangle$ Minahan. Zarembo '02

Main examples: $\mathcal{N}=4$ SYM (4D), ABJM theory (3D), ...
Planar limit

AdS/dCFT

Co-dimension d defect $\longleftrightarrow$ Probe brane

$\updownarrow$ $\updownarrow$

(Integrable) boundary state $|\Psi_0\rangle$ of spin chain De Leeuw, C.K. Zarembo '15

$\langle\Psi_0|\mathbf{u}\rangle$ is the one-point function $\langle\mathcal{O}\rangle$

Plan of the talk

I. ½ BPS Nahm pole defects in N=4 SYM

II. Integrability properties (in particular of GW-surface defects)

III. Prediction of wrapping (= finite size) corrections from localization

IV. Summary & Outlook

1/2 BPS Nahm pole defects in $\mathcal{N}=4$ SYM

Nahm pole defects :
of co-dimension d
(All 1/2 BPS)

$$\langle \Phi \rangle \sim \frac{\Phi^{\text{cl}}}{r}$$ ⟵ distance to defect

- $d=1$: Domain wall defect:

$$\langle \phi_i \rangle \neq 0, \qquad i = 1,2,3.$$

Karch & Randall '02
de Leeuw, CK. & Zarembo '15

- $d=2$: Gukov-Witten surface defect:

Gukov & Witten '08
Drukker, Gomis & Matsuura '08
Choi, Gomis & Garcia '24
De Leeuw & Holguin '24
Holguin '25
Chalabi, C.K & Su '25

$$\langle \phi_i \rangle \neq 0, \qquad i = 1,2, \qquad \langle A_3 \rangle \neq 0.$$

- $d=3$ 't Hooft line (monopole):

Kapustin '05
C.K & Zarembo '23, '24

$$\langle \phi_i \rangle \neq 0, \qquad i = 1, \qquad \langle A_i \rangle \neq 0, \qquad i = 1,2,3.$$

One-point functions

$$\langle \mathcal{O}^{\text{bulk}}_{\Delta}(x) \rangle = \frac{C}{|x_\perp|^{\Delta}}$$

Due to vevs scalar operators can have non-zero 1-pt fcts at tree-level

$$\langle \mathcal{O}_{\Delta}(x) \rangle = (\underbrace{\text{Tr}\ \left(\Phi_{i_1}\ldots\Phi_{i_L}\right)}_{\sim|s_{i_1}\ldots s_{i_L}\rangle} + \ldots)|_{\Phi_i\to\Phi_i^{\text{cl}}}$$

$$\langle \mathcal{O}_{\Delta}(x) \rangle = \frac{1}{|x_\perp|^{\Delta}}\frac{\langle \text{MPS}|\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle^{1/2}}, \qquad \langle \text{MPS}| = \sum_{\{s_i\}} \text{Tr}(\Phi^{\text{cl}}_{i_1}\ldots\Phi^{\text{cl}}_{i_L})\langle s_{i_1}\ldots s_{i_L}|$$

De Leeuw, C.K. Zarembo '15

1/2 BPS Nahm pole defects in $\mathcal{N} = 4$ SYM

	Domain wall	GW surface Non-rigid	GW surface rigid	't Hooft line
Parameters	Discrete Magnetic flux	Continuous	Discrete	Discrete Monopole charge
Probe brane Geometry	D5 $AdS_4 \times S^2$	D3 $AdS_3 \times S^1$	D3 $AdS_3 \times S^1$	D1 AdS_2
Symmetries	$OSp(4^*\|4)$	$PSU(1,1\|2)^2 \times SO(2)$	$SU(1,1\|2)^2$	$OSp(4^*\|4)$
Background, Φ^{cl}	Non-Commutative	Commutative	Non-commutative	Commutative

Gukov Witten surface defects in $\mathcal{N} = 4$ SYM

Ordinary case

Gukov & Witten '08   Drukker, Gomis & Matsuura '08

$$\Phi^{\rm cl} = \frac{1}{\sqrt{2}z}\mathrm{diag}((\beta_1 + \mathrm{i}\gamma_1)\mathbb{1}_{N_1}, \ldots, (\beta_M + \mathrm{i}\gamma_M)\mathbb{1}_{N_M}) \,, \quad z = re^{i\psi}$$

$$A^{\rm cl} = \mathrm{diag}(\alpha_1\mathbb{1}_{N_1}, \alpha_2\mathbb{1}_{N_2}, \ldots, \alpha_M\mathbb{1}_{N_M})d\psi, \qquad \alpha_i \text{ real and periodic}$$

Transverse coordinates: r, ψ

Rigid case

Gukov & Witten '08

$$A^{\rm cl} = \frac{t_3}{\log\frac{r}{r_0}}d\psi \,, \qquad \Phi^{\rm cl} = \frac{t_1 + \mathrm{i}t_2}{\sqrt{2}z\log\frac{r}{r_0}} \,,$$

t_1, t_2, t_3 k-dimensional irreducible representation of $SU(2)$

Corresponds to $\beta, \gamma \to 0$

Probe brane description

$AdS_5 \times S^5$ background

$$ds^2 = \frac{1}{y^2}\left(dy^2 + dx_0^2 + dx_1^2 + dr^2 + r^2 d\psi^2\right) + d\Omega_5$$

$$d\Omega_5 = \cos^2\theta\, d\Omega_3 + d\theta^2 + \sin^2\theta\, d\phi^2$$

World volume coordinates of D3-brane: x_0, x_1, r, ψ, $(AdS_3 \times S^1)$

Embedding functions:

$$\theta = \frac{\pi}{2}, \qquad \phi = \phi(\psi) = \phi_0 - \psi, \qquad y = y(r) = \frac{r}{\kappa}$$

Integration constants related to β and γ as

$$\beta + i\gamma = \frac{\sqrt{\lambda}}{2\pi}\kappa\, e^{i\phi_0}$$

Rigid case: $\beta, \gamma \to 0$, i.e. $\kappa \to 0$, cone in AdS_5 shrinks

Tree-level integrability properties of GW defects

Chalabi, CK & Su '25

	Ordinary	Rigid
$SU(2)$-sector	Trivial	Integrable $\forall\, k \in \boldsymbol{N}, k \geq 2$ *
$SO(6)$ sector	Integrable $\forall\, \beta_i, \gamma_i$	Integrable for $k = 2$
SL(2)-sector	Non-integrable *	Integrable for $k = 2$

* Also noted in Holguin '25

Integrability test: $Q_{2n+1}|B\rangle = 0, \quad \forall\, n \in \mathbb{N}$ Pozsgay '17

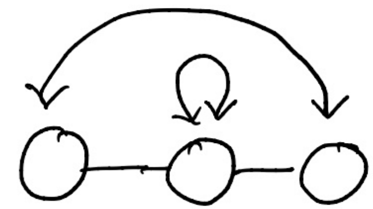
Overlap formula: KT-relation & recursive strategy Gombor '24

Tree-level one-point functions for rigid GW-defects

For $SO(6)$-sector, $k=2$

Chalabi, C.K & Su '25

$$\langle \mathcal{O}_L(x)\rangle \propto \frac{\langle \mathrm{MPS}\,|\mathbf{u}\rangle}{\langle \mathbf{u}\,|\mathbf{u}\rangle^{\frac{1}{2}}} = \sqrt{\frac{Q_2(i/2)}{Q_2(0)}\frac{\det G_+}{\det G_-}}$$



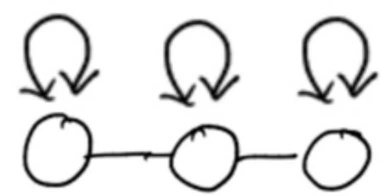
NB: A-chiral overlap

$SO(2)\times SO(4)$ symmetry

Derived following Gombor '24,'25

Domain wall case

$$\langle \mathcal{O}_L(x)\rangle \propto \sqrt{\frac{Q_1(i/2)Q_2(i/2)Q_3(i/2)}{Q_1(0)Q_2(0)Q_3(0)}\frac{\det G_+}{\det G_-}}$$



Chiral overlap

$SO(3)\times SO(3)$ symmetry

Higher loop integrability

- Domain wall defect:
  - String boundary conditions integrable incl. vevs. Linardopoulos & Zarembo '19
  - Asymptotic all loop formula derived Komatsu & Wang '20 Gombor & Bajnok '20
- Gukov-Witten surface defect:
  - String boundary conditions integrable without vevs Dekel & Oz '11
  - Only hope for rigid case $k = 2$ from field theory Chalabi, C.K. & Su '25
- 't Hooft line:
  - String boundary conditions integrable without vevs Dekel & Oz '11
  - Asymptotic all loop formula up to reflection phase exists Gombor & Bajnok '23

Missing: Finite size (= wrapping) corrections

(TBA, QSC, NLIE,...)

Wrapping corrections from localization

Example: 't Hooft loop

Consider: $\langle \mathcal{O}_L \rangle_T \equiv \frac{\langle T(C)\mathcal{O}_L(x)\rangle}{\langle T(C)\rangle} = \frac{C_L}{(2|x_\perp|)^L}$

with $\mathcal{O}_L = \frac{1}{\sqrt{L}} \left(\frac{4\pi^2}{\lambda}\right)^{\frac{L}{2}} \operatorname{tr} Z^L$

Exact results for WL give prediction for TL correlators for protected (1/2 BPS) operators

Kristjansen & Zarembo '23

S-duality of $\mathcal{N} = 4$ SYM, gauge group $U(N)$

Montonen & Olive '77

$$\lambda \longleftrightarrow \frac{16\pi^2 N^2}{\lambda}$$

Wilson loop $W(C) \longleftrightarrow$ 't Hooft loop $T(C)$

Chiral primary, $\mathcal{O}_L \longleftrightarrow$ Chiral primary $\mathcal{O}_L$

't Hooft loop correlators

$\langle \mathcal{O}_L \rangle_W \equiv \frac{\langle W(C) \mathcal{O}_L(x) \rangle}{\langle W(C) \rangle}$ known exactly from localization.

$$\langle \mathcal{O}_L(x) \rangle_W = \frac{1}{\sqrt{L}} \left(\frac{\sqrt{\lambda}}{2NR} \right)^L \mathrm{e}^{\frac{\lambda}{8N}} \sum_{k=1}^{L} L_{N-k}^{L} \left(-\frac{\lambda}{4N} \right)$$

Pestun '07

Okuyama & Semenoff '07

- $\lambda \to \frac{16\pi^2 N^2}{\lambda}$
- Integral representation of Laguerre polynomials
- Large-N saddle point approximation

Kristjansen & Zarembo '25

Large-N perturbative prediction for $\langle \mathcal{O}_L \rangle_T$

$$\langle \mathcal{O}_L \rangle_T = \frac{1}{(2r)^L} \frac{1}{\sqrt{L}} \left[\left(\sqrt{1 + \frac{\pi^2}{\lambda}} + \frac{\pi}{\sqrt{\lambda}} \right)^L - \left(\sqrt{1 + \frac{\pi^2}{\lambda}} - \frac{\pi}{\sqrt{\lambda}} \right)^L \right] \equiv \frac{1}{(2r)^L} \mathcal{C}_L$$

In Zhukovski variables: $x + \frac{1}{x} = 2u, \;\; u = \frac{\pi i}{\sqrt{\lambda}}$

$$\mathbb{C}_L = \frac{1}{i^L \sqrt{L}} \left[x^L + \frac{(-1)^{L+1}}{x^L} \right] = \text{regular} + \text{wrapping}$$

Predicted structure of higher loop corrections

Kristjansen & Zarembo '25

$$\mathcal{C}_2 = \frac{1}{\sqrt{2}}\left(\frac{2\pi}{\sqrt{\lambda}}\right)^2\left(1+\frac{\lambda}{2\pi^2}-\frac{\lambda^2}{8\pi^4}+\frac{\lambda^3}{16\pi^6}+\dots\right),$$

$$\mathcal{C}_3 = \frac{1}{\sqrt{3}}\left(\frac{2\pi}{\sqrt{\lambda}}\right)^3\left(1+\frac{3\lambda}{4\pi^2}\right),$$

$$\mathcal{C}_4 = \frac{1}{2}\left(\frac{2\pi}{\sqrt{\lambda}}\right)^4\left(1+\frac{\lambda}{\pi^2}+\frac{\lambda^2}{8\pi^4}-\frac{\lambda^4}{128\pi^8}+\dots\right),$$

$$\mathcal{C}_5 = \frac{1}{\sqrt{5}}\left(\frac{2\pi}{\sqrt{\lambda}}\right)^5\left(1+\frac{5\lambda}{4\pi^2}+\frac{5\lambda^2}{16\pi^4}\right),$$

$$\mathcal{C}_6 = \frac{1}{\sqrt{6}}\left(\frac{2\pi}{\sqrt{\lambda}}\right)^6\left(1+\frac{3\lambda}{2\pi^2}+\frac{9\lambda^2}{16\pi^4}+\frac{\lambda^3}{32\pi^6}-\frac{\lambda^6}{2048\pi^{12}}+\dots\right).$$

Notice:

- The perturbative series truncates for odd L
- For even L wrapping corrections set in from $\mathcal{O}(\lambda^{2L})$
- The non-wrapping part truncates for all L
- For domain walls: Same structure except $\mathcal{C}_{2n+1}=0$.
- For Gukov-Witten defects truncation without wrapping.
- For the Coulomb branch only wrapping

Choi, Gomis, Garcia '24

Coronado, Komatsu, Zarembo '25

Understanding the perturbative series

Closed form of non-wrapping (nw) contributions

$$\mathcal{C}_L^{nw} = \frac{1}{\sqrt{L}} \left(\frac{2\pi}{\sqrt{\lambda}} \right)^L \left\{ 1 + \sum_{n=1}^{[L/2]} \binom{L-n}{n} \frac{L}{L-n} \left(\frac{\lambda}{4\pi^2} \right)^n \right\}$$

Wish to recover this from
quantizing around the monopole background

NB: Same structure for the domain wall defect

$$\mathcal{C}_L^{nw} = \frac{1}{\sqrt{L}} \left(\frac{2\pi}{\sqrt{\lambda}} \right)^L \left\{ 1 + \sum_{n=1}^{[L/2]} \binom{L-n}{n} \frac{L}{L-n} \frac{2B_{L-2n+1}\left(\frac{1+k}{2}\right)}{L-2n+1} \left(\frac{\lambda}{4\pi^2} \right)^n \right\}$$

de Leeuw, Ipsen, C.K. &Wilhelm '17 Komatsu & Wang '20

Future goal:
Understand the wrapping corrections in the same detail
and compute one-point functions of non-protected operators
exactly using integrability

Quantizing around the monopole background

1. Expand around classical fields $(\Phi_1^{\rm cl}, A_\mu^{\rm cl})$

$$
\begin{array}{c} \\ A_\mu, \Phi_i, \Psi = \end{array}
\begin{array}{c} \quad 1 \qquad N-1 \\ \left[\begin{array}{c|ccc} \alpha & \beta & \beta & \beta \\ \hline \beta & \gamma & \gamma & \gamma \\ \beta & \gamma & \gamma & \gamma \\ \beta & \gamma & \gamma & \gamma \end{array}\right] \end{array}
\begin{array}{c} 1 \\ \\ N-1 \end{array}
$$

Due to vevs, non-trivial mixing problem for β components

2. Gauge fix

3. Invert the quadratic part of the action (determine propagators)

Spectral decomposition

$$S = \Phi\, G^{-1}\, \Phi$$

$$G(x,y) = \sum_k \Psi_k(x)\, \frac{1}{\lambda_k}\, \Psi_k^\dagger(y), \qquad G^{-1}\Psi_k(x) = \lambda_k \Psi_k(x)$$

Quantum mechanical problem (field components β)

Quantum mechanical problems involved

I. For $\Phi_2, \Phi_3, \ldots \Phi_6, A_0$ and ghost c

Scalar particle in monopole potential — Dirac '31

II. For Φ_1, $\vec{A}$:

Scalar coupled to spin-1 particle in monopole potential

Spin-1 particle & monopole
Olsen, Osland & Wu '90

III. For $\Psi^I_\alpha \;\; (\alpha, I \in \{1,2,3,4\})$

Fermions in (non-standard) monopole potential

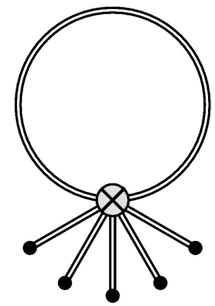
Standard case
Kazama, Yang &
Goldhaber '77

All are beautiful, exactly solvable quantum mechanical systems

Computation of one-point functions — ex: 't Hooft loops

Need propagator for $Z = \Phi_1 + i\Phi_2$ at coinciding points

$$\langle Z^{1a}(x) Z^{b1}(x') \rangle = \frac{\lambda \delta^{ab}}{2N} G^{-}(x,x').$$



$$G^{-}(x,x') = G(x,x') - \widetilde{G}(x,x')$$

Monopole charge

Jacobi pol.

$$G(x,x') = \frac{1}{4\pi (rr')^{\frac{d}{2}-1}} \left(\frac{1+\eta}{2}\right)^q \sum_j (2j+1) P_{j-q}^{(0,2q)}(\eta) D_{j+\frac{d}{2}-1}(\xi),$$

AdS$_2$ propagator

$$\begin{aligned}\widetilde{G}(x,x') &= \frac{1}{4\pi (rr')^{\frac{d}{2}-1}} \left(\frac{1+\eta}{2}\right)^q \sum_j P_{j-q}^{(0,2q)}(\eta) \\ &\times \left[\frac{q^2}{j} D_{j+\frac{d}{2}-2} + \frac{(2j+1)(j^2+j-q^2)}{j(j+1)} D_{j+\frac{d}{2}-1} + \frac{q^2}{j+1} D_{j+\frac{d}{2}} \right].\end{aligned}$$

The tadpole diagram

Telescoping sum

$$G^{-}(x,x')\Big|_{\eta=\eta'} = \frac{q}{4\pi(rr')^{\frac{d}{2}-1}}\left(D_{q+\frac{d}{2}-2} - D_{q+\frac{d}{2}-1}\right)$$

$$\lim_{x\to x',d\to 4} G^{-}(x,x')\Big|_{\eta=\eta'} = \frac{1}{8\pi^2 r^2}$$

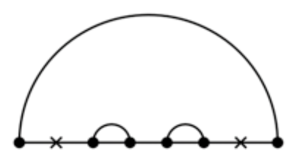
NB: Non-commutativity of limits!

$$\sim \left(\frac{1}{2r}\right)^2 \cdot \frac{\lambda}{4\pi^2}$$

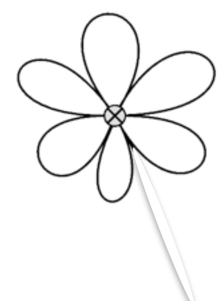
Origin of non-wrapping part of corrections

$$\mathcal{C}_L^{nw} = \frac{1}{\sqrt{L}}\left(\frac{2\pi}{\sqrt{\lambda}}\right)^L \left\{1 + \sum_{n=1}^{[L/2]} \binom{L-n}{n} \frac{L}{L-n}\left(\frac{\lambda}{4\pi^2}\right)^n\right\}$$

$$\binom{L-n}{n}\frac{L}{L-n} = \binom{L-n}{n} + \binom{L-2-(n-1)}{n-1}$$



Only contributions from flower diagrams with varying numbers of leaves



- Same phenomenon observed for the domain wall defect.
- Same phenomenon for ordinary Gukov Witten defects but no wrapping.

Choi, Gomis, Garcia '24

Summary & Outlook

- (At least) three examples of dCFT's (co-dimension 1,2,3) amenable to localization, integrability and potentially bootstrap
- Quantization completed for all three cases
- Localization performed, implies intriguing structure of perturbative exp.
- One-pt fcts of non-protected operators obtained from integrability up to wrapping interactions for best understood case ($d = 1$)
- Scene is set for full solution by integrability incl. wrapping
- Boundary bootstrap still to be implemented for non-vanishing vevs.

Thank you