Conformal blocks for black hole perturbation theory

BRUNO CARNEIRO DA CUNHA – 11TH CFT AND INTEGRABLE MODELS – UNIBO – 05/09/25

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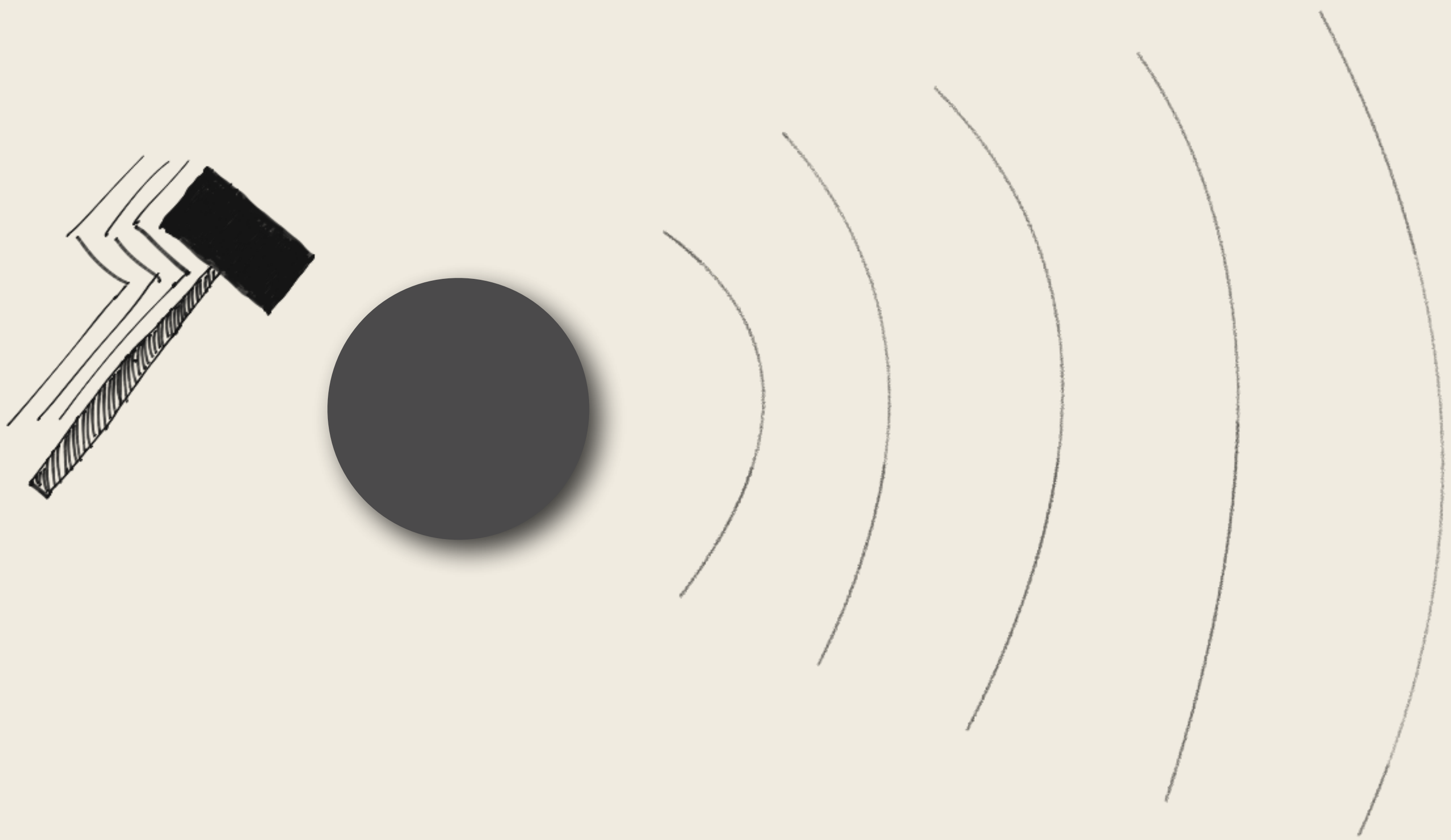
Quasi-normal modes: stability, analysis of merger events, relaxation times in AdS/CFT, etc.



$$\omega = \mathrm{Re}(\omega) + i\,\mathrm{Im}(\omega)$$

Oscillation

Decay time

Linear regime leads to separable equations in (vacuum) space-times

$$\nabla^2\Phi = \mu^2\Phi, \qquad \Phi(t,r,\theta,\phi) = e^{-i\omega t}R(r)S(\theta)e^{im\phi}$$

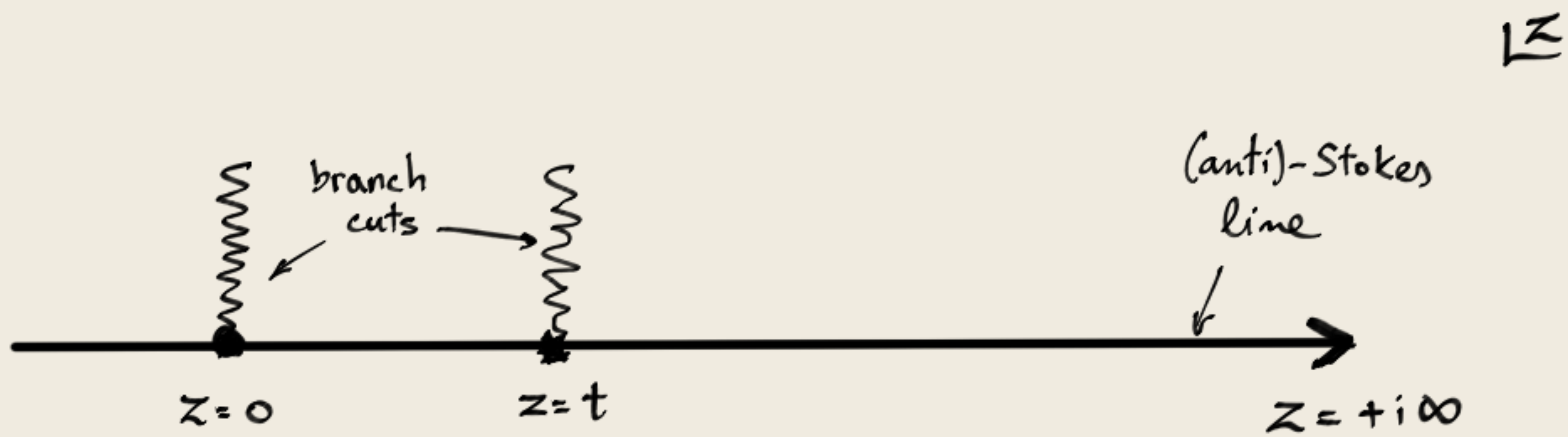
$$\partial_u\left[(1-u^2)\partial_u S\right] + \left(\frac{m^2}{1-u^2} - a^4(\omega^2-\mu^2)u^2 - 2am\omega - \lambda\right)S = 0, \qquad u = \cos\theta$$

$$\partial_r(\Delta\partial_r R) + \left[\frac{(\omega(r^2+a^2)-am)^2}{\Delta} - \lambda - \mu^2 r^2\right]R = 0,$$

$$\Delta = r^2 - 2Mr + a^2 = (r-r_+)(r-r_-)$$

Mathematically, these are known as the Confluent Heun Equation (CHE)

$$\frac{d^2y}{dz^2} + \left[\frac{1-\theta_1}{z} + \frac{1-\theta_2}{z-t}\right]\frac{dy}{dz} - \left[\frac{1}{4} + \frac{\theta_\star}{2z} + \frac{t\,c}{z(z-t)}\right]y(z) = 0,$$



$$y(z) = \begin{cases} z^{0}(1+\dots) \\ z^{\theta_1}(1+\dots) \end{cases} \qquad y(z) = \begin{cases} (z-t)^{0}(1+\dots) \\ (z-t)^{\theta_2}(1+\dots) \end{cases} \qquad y(z) = \begin{cases} e^{-z/2} z^{\alpha}(1+\dots) \\ e^{+z/2} z^{\beta}(1+\dots) \end{cases}$$

waves in tortoise coordinates

spherical waves

$\alpha - \beta = \theta_\star$

Angular equation: singular points are South/North poles:
regular solutions, Sturm-Liouville problem.

Radial equation: singular points are Killing horizons!
QNMs are also solutions to Sturm-Liouville problem!

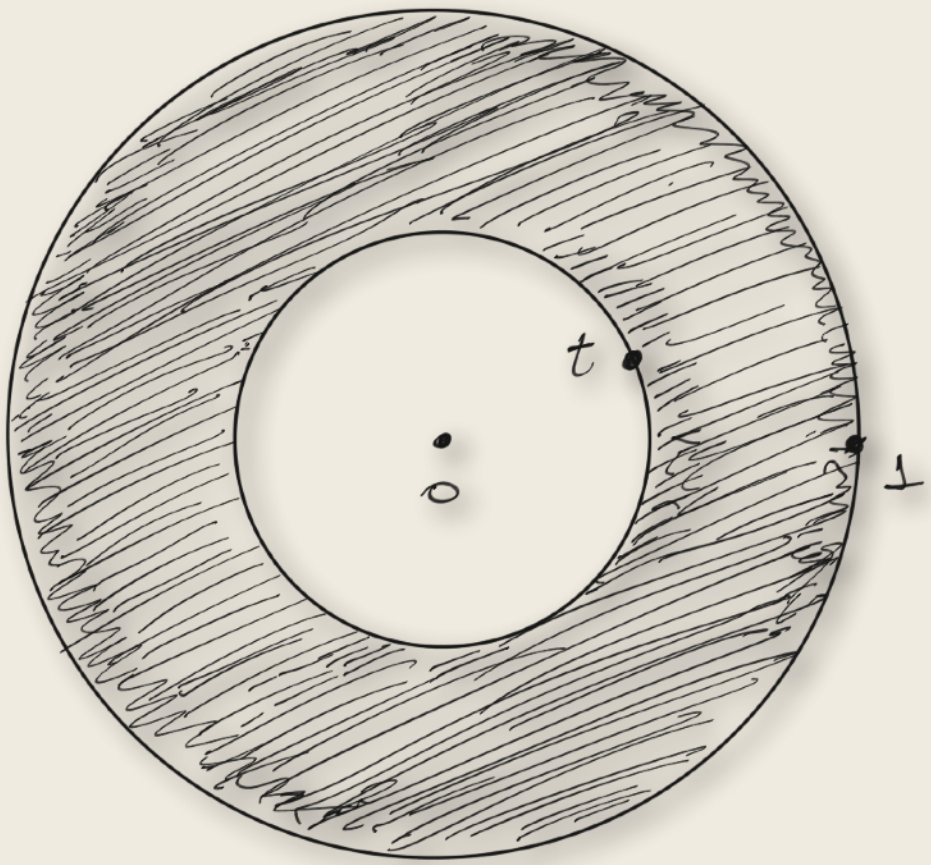
Complex conjugation changes m: no guarantee of stability

$$V(r) = \frac{1-\theta_-^2}{4(r-r_-)^2} + \frac{1-\theta_+^2}{4(r-r_+)^2} + \frac{\beta_1}{r-r_+} + \frac{\beta_2}{r-r_-} + \omega^2 - \mu^2$$

$$\beta_1 + \beta_2 = (2\omega^2 - \mu^2)(r_+ - r_-)$$

$$\beta_1 = \frac{1}{r_+ - r_-}\left(\lambda + \frac{1}{2} + \mu^2 r_-^2 + 2r_-^2\frac{(3r_+ - r_-)(r_+ + r_-)}{(r_+ - r_-)^2}\omega^2 - 2a^2 m\frac{4a\omega - m}{(r_+ - r_-)^2}\right)$$

Method of choice: Leaver's method



Floquet solutions:

$$y(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}(\sigma+\theta_0+\theta_t)-1} \sum_{n=-\infty}^{\infty} a_n z^n$$

3-term recurrence equation

$$A_n a_{n-1} - (B_n + t\, C_n) a_n + t\, D_n a_{n+1} = 0,$$

$$\frac{t\, A_0 D_{-1}}{B_{-1} + t\, C_{-1} - \frac{t\, A_{-1} D_{-2}}{B_{-2} + t\, C_{-2} - \frac{t\, A_{-2} D_{-3}}{B_{-3} + \ldots}}} + \frac{t\, D_0 A_1}{B_1 + t\, C_1 - \frac{t\, D_1 A_2}{B_2 + t\, C_2 - \frac{t\, D_2 A_3}{B_3 + \ldots}}} = B_0 + t\, C_0\,.$$

$$\mathscr{F}_0(\vec{\theta}; \sigma; c, t) = 0$$

what is σ?

$$\mathscr{F}_0(\vec{\theta}; \sigma; c, t) = 0$$

Works for small t, can write asymptotic expansions for angular eigenvalue and find QNM frequencies for generic (non-extremal parameters); Great numerics!

$$c(\sigma, t) = \frac{c_{-1}(\sigma)}{t} + c_0(\sigma) + c_1(\sigma)t + c_2(\sigma)t^2 + \dots$$

$$\theta_{\mathrm{Rad},1} = \theta_- = -\,i\frac{\omega - m\Omega_-}{2\pi T_-}, \qquad \theta_{\mathrm{Rad},2} = \theta_+ = +\,i\frac{\omega - m\Omega_+}{2\pi T_+}, \qquad \theta_{\mathrm{Rad},\star} = \theta_* = \frac{2iM(2\omega^2 - \mu^2)}{\sqrt{\omega^2 - \mu^2}},$$

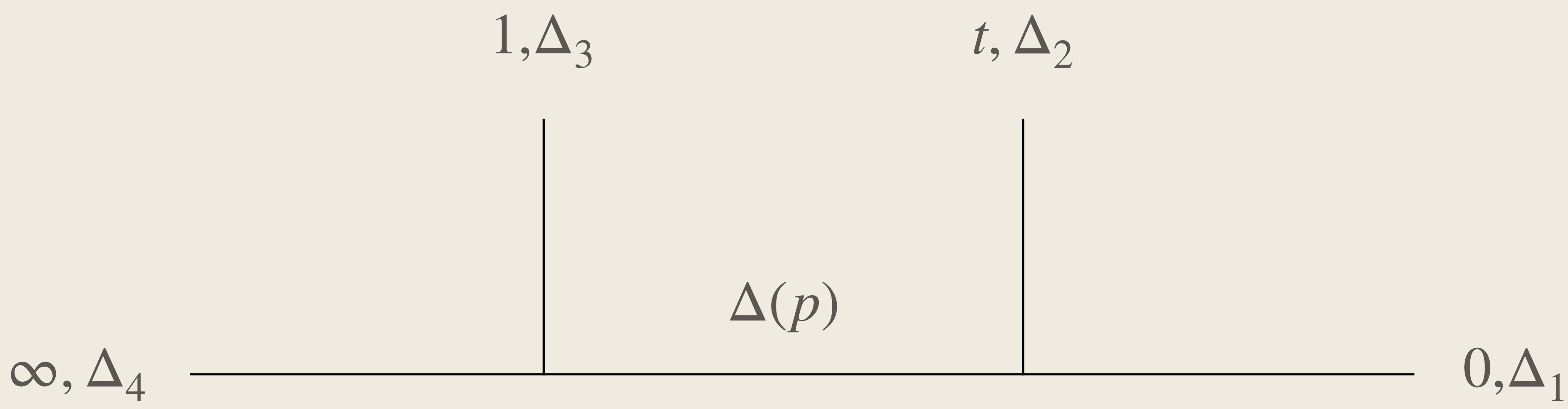
Diverge at the extremal limit!

Entropy intake of the black hole in the perturbation process

Liouville field theory

$$\langle V_{\Delta_2}(x_2)V_{\Delta_1}(x_1)\rangle = \frac{\delta_{1,2}}{|x_1 - x_2|^{2\Delta}}, \qquad \langle V_{\Delta_3}(x_3)V_{\Delta_2}(x_2)V_{\Delta_1}(x_1)\rangle = \frac{C_{123}}{|x_1 - x_2|^{(123)}|x_1 - x_3|^{(132)}|x_2 - x_3|^{(321)}},$$

$$\langle \Delta_4 | V_3(1)V_2(t) | \Delta_1 \rangle = \sum_p \langle \Delta_4 | V_3(1)\Pi_p V_2(t) | \Delta_1 \rangle = \sum_p C_{12p}C_{p34}t^{\Delta_p - \Delta_2 - \Delta_1}\mathscr{F}_p(\Delta, t)$$



Zamolodchikov's recursions, elliptic recursion

Degenerate representations of the Virasoro algebra

$$\Delta_{(m,n)} = \frac{c-1}{24} - \frac{(mb+nb^{-1})^2}{4}$$

$$\left[\frac{1}{b^2}\frac{\partial^2}{\partial z^2} + \sum_{i=k}^{N}\left(\frac{\Delta_k}{(z-z_k)^2} + \frac{\partial_{z_k}}{z-z_k}\right)\right]\langle V_{(2,1)}(z)\prod_{k=1}^{N} V_{\alpha_k}(z_k)\rangle = 0$$

$$\Delta_k = \alpha_k(Q-\alpha_k)$$

$$V_{-\frac{b}{2}}(z) \simeq V_{(2,1)}(z)$$

$$V_{-\frac{b}{2}}(z)V_\alpha(w) = A_+(z-w)^{b\alpha}(V_{\alpha+\frac{b}{2}}(w) + \mathcal{O}(z-w)) + A_-(z-w)^{b(Q-\alpha)}(V_{\alpha-\frac{b}{2}}(w) + \mathcal{O}(z-w)),$$

Analytic!

Pick up phases as z goes round w!

"monodromy"

In the semiclassical limit ($b \to 0$) the equation becomes a (Fuchsian) ODE

$$\Delta_k = \alpha_k(Q - \alpha_k) = \frac{\delta_k}{b^2}, \qquad \Delta_p = \frac{\delta_p}{b^2}, \qquad \mathscr{F}_p(\Delta_k, t) \simeq \exp\left(\frac{1}{b^2}\mathscr{W}_p(\delta_k, t)\right)$$

$$\langle V_{-\frac{b}{2}}(z)\prod_{i=1}^{N} V_{\Delta_k}(z_k)\rangle \to \psi(z; z_k)\ \exp\left(\tfrac{1}{b^2}\mathscr{W}_p(\delta_k, z_k)\right)$$

$$V(z; z_k) = \sum_k \left[\frac{\delta_k}{(z - z_k)^2} + \frac{c_k}{z - z_k}\right], \qquad\qquad c_k = \frac{\partial}{\partial z_k}\mathscr{W}_p(\delta_k, z_k),$$

δ_p related to σ defined above: CFT version of Riemann-Hilbert map

Many methods to compute
- Isomonodromy ($c=1$ CBs!)
- AGT in the NS limit
- Continued fractions

Each ODE is associated with a correlation function in chiral Liouville Field Theory

Inner, outer horizons → primary operators $\alpha_k = \frac{\theta_k}{b} = \frac{i}{2\pi b}\delta S_k$ unitary!

Infinity: Whittaker operator

$$V_{\alpha,\gamma}(z) = \lim_{\Lambda\to\infty}\left(\frac{\gamma}{\Lambda}\right)^{2\alpha_+\alpha_-} V_{\alpha_+}(z+\gamma/\Lambda)V_{\alpha_-}(z), \qquad \alpha_\pm = \alpha \pm \Lambda/2$$

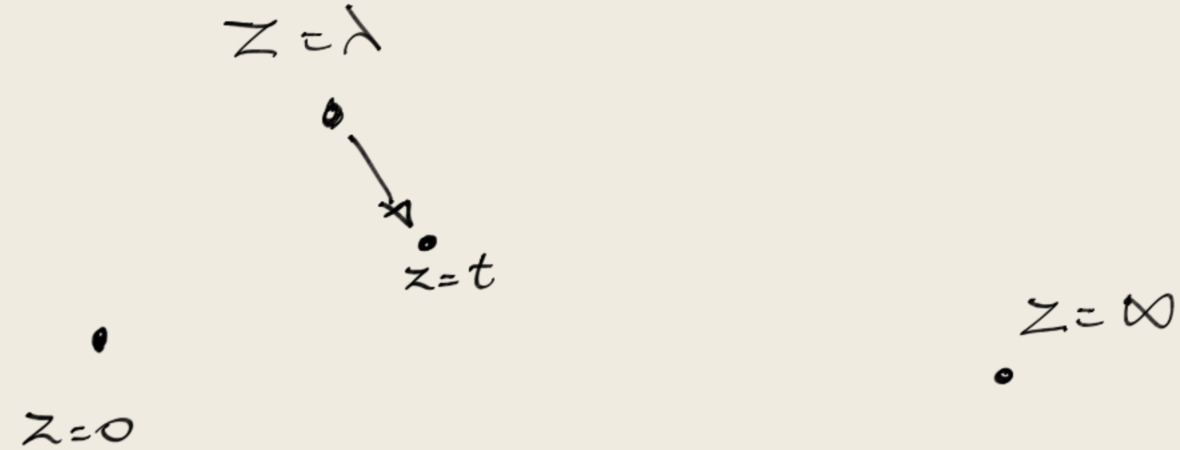
$$L_0|\alpha,\gamma\rangle = \left(\Delta(\alpha)+\gamma\frac{\partial}{\partial\gamma}\right)|\alpha,\gamma\rangle, \qquad L_1|\alpha,\gamma\rangle = \gamma(Q-\alpha)|\alpha,\gamma\rangle, \qquad L_2|\alpha,\gamma\rangle = -\frac{\gamma^2}{4}|\alpha,\gamma\rangle$$

$$\alpha = \frac{\theta_\star}{b} = \frac{2iM(2\omega^2-\mu^2)}{b\sqrt{\omega^2-\mu^2}}, \qquad \gamma = \frac{t}{b} = \frac{4i\sqrt{\omega^2-\mu^2}(r_+-r_-)}{b}$$

Semiclassical limit can be computed for small and large t!

Isomonodromy vs. Continued Fractions

Painlevé V Garnier system



$$\frac{d\lambda}{dt} = \frac{\partial K}{\partial \mu}, \qquad \frac{d\mu}{dt} = -\frac{\partial K}{\partial \lambda}$$

$$\tau_V(\vec{\theta}; \sigma, \eta; t_0) = 0, \qquad t_0 \frac{d}{dt} \log \tau_V(\vec{\theta}_-; \sigma - 1, \eta; t_0) - \frac{\theta_0(\theta_t - 1)}{2} = t_0 c(t_0),$$

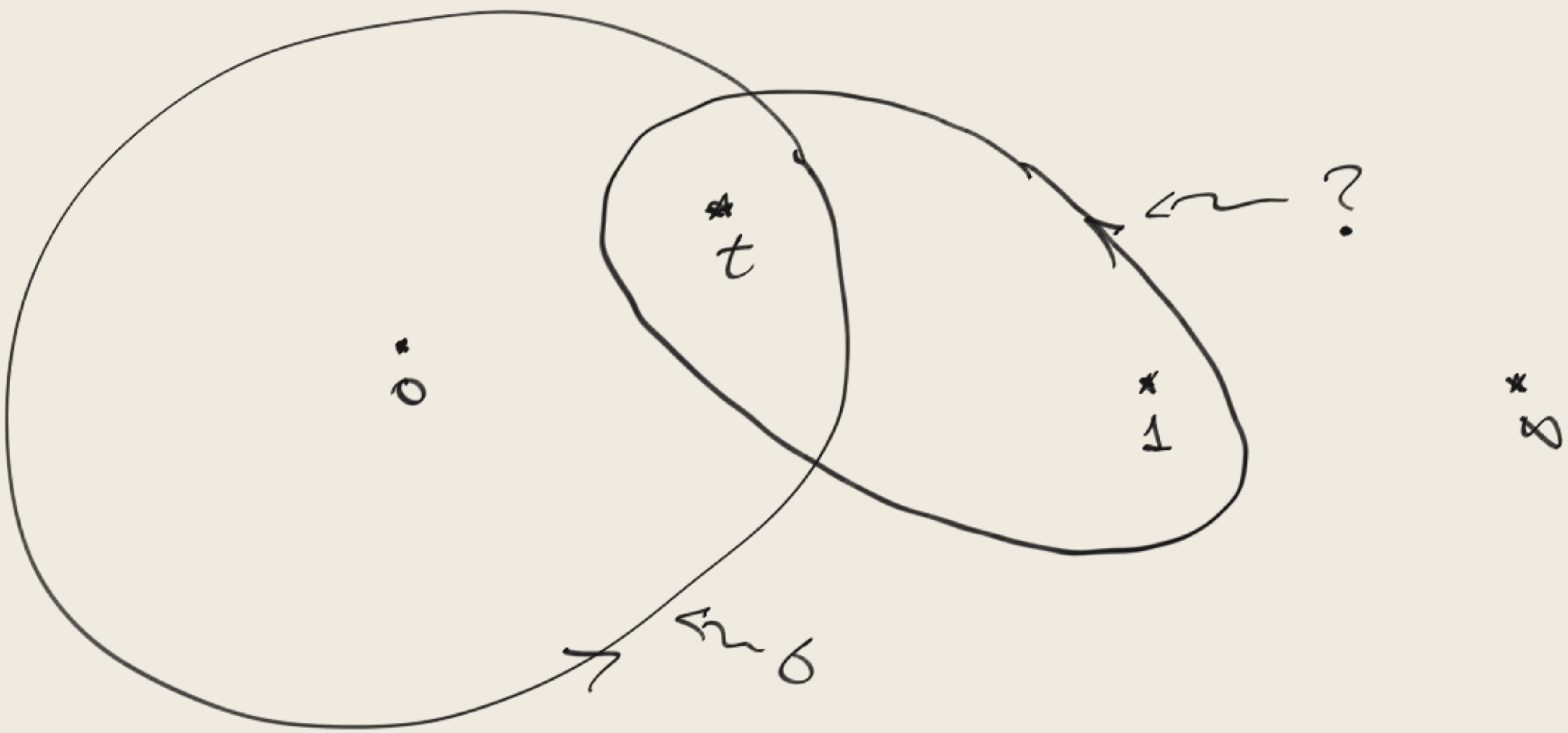
defines Lagrangian submanifold

surprisingly, series expansion!

$$c(\sigma, t_0) = c_{-1}(\sigma) t_0^{-1} + c_0(\sigma) + c_1(\sigma) t_0 + \ldots + c_n(\sigma) t_0^n + \ldots = \frac{\partial \mathscr{W}_s}{\partial t_0}$$

$$c^c_{-1}(\sigma) = \delta_\sigma - \delta_\star, \qquad c^c_0(\sigma) = \frac{(1 - \theta_\star)(\delta_\sigma - \delta_1 + \delta_2)}{4\delta_\sigma},$$

Fusion rules: going round in complex plane



Fixed by other parameter, canonically conjugate to σ:

$$\eta = \frac{1}{\pi i}\frac{\partial}{\partial\sigma}\mathscr{W}_s(\sigma, t) \qquad \text{sets} \qquad \sigma_{t1}(\sigma,\eta), \sigma_{01}(\sigma,\eta)$$

Small and large parameter expansions

$$c(\sigma,t_0) = c_{-1}(\sigma)t_0^{-1} + c_0(\sigma) + c_1(\sigma)t_0 + \ldots + c_n(\sigma)t_0^n + \ldots = \frac{\partial \mathscr{W}_s}{\partial t_0}$$

$$c^c_{-1}(\sigma) = \delta_\sigma - \delta_\star, \qquad c^c_0(\sigma) = \frac{(1-\theta_\star)(\delta_\sigma - \delta_1 + \delta_2)}{4\delta_\sigma},$$

Substitute basis with "same Stokes multipliers" into CHE

$$y(z) = \sum_{n\in\mathbb{Z}} a_n e^{-\frac{1}{2}z} z^{\theta_0} F_n(z), \qquad F_n(z) = {}_1F_1(\tfrac{1}{2}\theta_1 + \tfrac{1}{4}(1+\theta_\star) - \nu - n; 1+\theta_1; z)\,.$$

Continuous fraction gives conformal block at infinity

$$c(\nu,t_0) = \bar{c}_0(\nu) + \bar{c}_1(\nu)t_0^{-1} + \bar{c}_2(\nu)t_0^{-2} + \bar{c}_3(\nu)t_0^{-3} + \ldots,$$

$$\bar{c}_0(\nu) = -\nu + \tfrac{1}{4}(1-\theta_\star), \qquad \bar{c}_1(\nu) = 2\nu^2 + \delta_0 + \delta_1 + \tfrac{1}{8}(1+\theta_\star)^2 - \tfrac{1}{2},$$

Continuous fraction gives conformal block at infinity

$$c(\nu, t_0) = \bar{c}_0(\nu) + \bar{c}_1(\nu)t_0^{-1} + \bar{c}_2(\nu)t_0^{-2} + \bar{c}_3(\nu)t_0^{-3} + \ldots,$$

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– Asymptotic series for irregular conformal block;

– continued fractions good even beyond that limit;

– Resurgence: $\sigma \to -2\nu - \theta_1 - \tfrac{1}{2}(1-\theta_\star)$ in $\mathscr{F}_0(\vec{\theta}; \sigma; c, t) = 0$

Last ingredient: eigenvalue problems can be written in terms of monodromy parameters!

Angular equation (eigenvalue problem):

$$\sigma_{\mathrm{Ang}} = \theta_{1,\mathrm{Ang}} + \theta_{2,\mathrm{Ang}} + 2(\ell + 1), \qquad \ell = 0,1,2,\ldots$$

Radial equation (QNM problem):

$$e^{\pi i \eta} = e^{\pi i \sigma} \frac{\sin\frac{\pi}{2}(\theta_\star + \sigma)}{\sin\frac{\pi}{2}(\theta_\star - \sigma)} \frac{\sin\frac{\pi}{2}(\theta_t + \theta_0 + \sigma)\sin\frac{\pi}{2}(\theta_t - \theta_0 + \sigma)}{\sin\frac{\pi}{2}(\theta_t + \theta_0 - \sigma)\sin\frac{\pi}{2}(\theta_t - \theta_0 - \sigma)}$$

Different from CFT 4-point function…

To sum up:

- Monodromy parameters are related to global analytic aspects of solutions to ODEs that arise in QM problems;
- They can be computed efficiently from conformal blocks of CFTs at any point in parameter space;
- We will use the conformal blocks as a tool to compute QNMs of black holes; Other uses include scattering coefficients, greybody factors, Love numbers, etc.;
- Method allows for following QNMs up to and including the extremal limit. Price to pay: computing the monodromy parameters as function of frequency;
- Monodromy parameters as A & B periods of Seiberg-Witten curve. Anything to be learned from SYM analogy? Blow-up formulas?
- Heun (and higher Fuchsian) problem solved by Trieste formula. So what?

Results

Angular eigenvalue expansion $\tilde{\alpha} = a\sqrt{\omega^2 - \mu^2}$

$$\lambda(\tilde{\alpha}) = \ell(\ell+1) + \left(\frac{2((\ell+1)^2 - m^2)(\ell+1)^4}{(2\ell+1)(\ell+1)^3(2\ell+3)} - \frac{2(\ell^2 - m^2)\ell^4}{(2\ell-1)\ell^3(2\ell+1)} - 1\right)\tilde{\alpha}^2 + \mathcal{O}(\tilde{\alpha}^3),$$

$$\lambda(\tilde{\alpha}) = -\tilde{\alpha}^2 + 4q\tilde{\alpha} + \frac{1}{2}(m^2-1) - \frac{q^2}{2} - \frac{1}{\tilde{\alpha}}\left(\frac{q^3}{8} - \frac{1}{8}q\left(m^2-1\right)\right) + \frac{1}{\tilde{\alpha}^2}\left(\frac{1}{32}q^2\left(3m^2-5\right) - \frac{1}{64}\left(m^4 - 2m^2 + 1\right) - \frac{5q^4}{64}\right) + \mathcal{O}(\tilde{\alpha}^{-3})$$

$$q = 2\ell - m + 1$$

Small frequency known, large frequency simple extension of known results

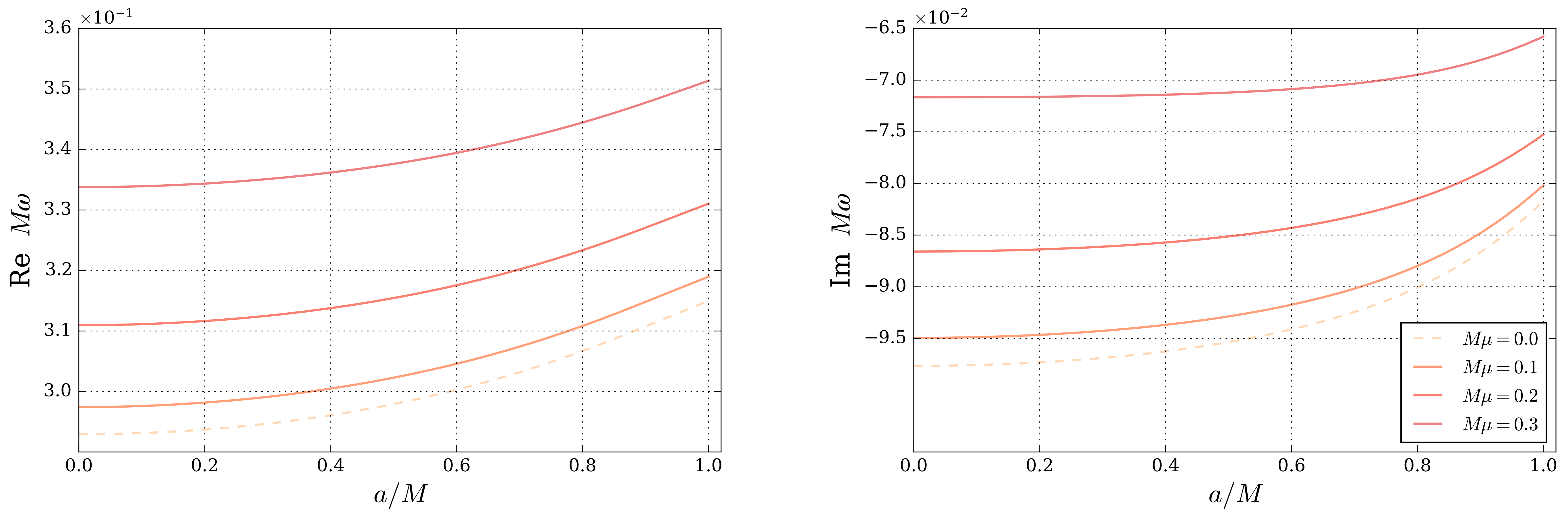
Radial equation: QNMs are functions of a/M and $M\mu$: no natural small parameters.

$M\omega_{n,\ell,m}$ found through an equilibrium condition: $\lambda_{\ell,m} = \frac{\partial}{\partial u_0}\mathscr{W}_{\text{ang}} = \frac{\partial}{\partial t}\mathscr{W}_{\text{rad}} + \dots$

Studied numerically. “Non-linear secular equation” $\det(\text{Id} - \text{K}(\omega)) = 0$

“Non-co-rotating" modes $m \neq \ell$

Damped modes

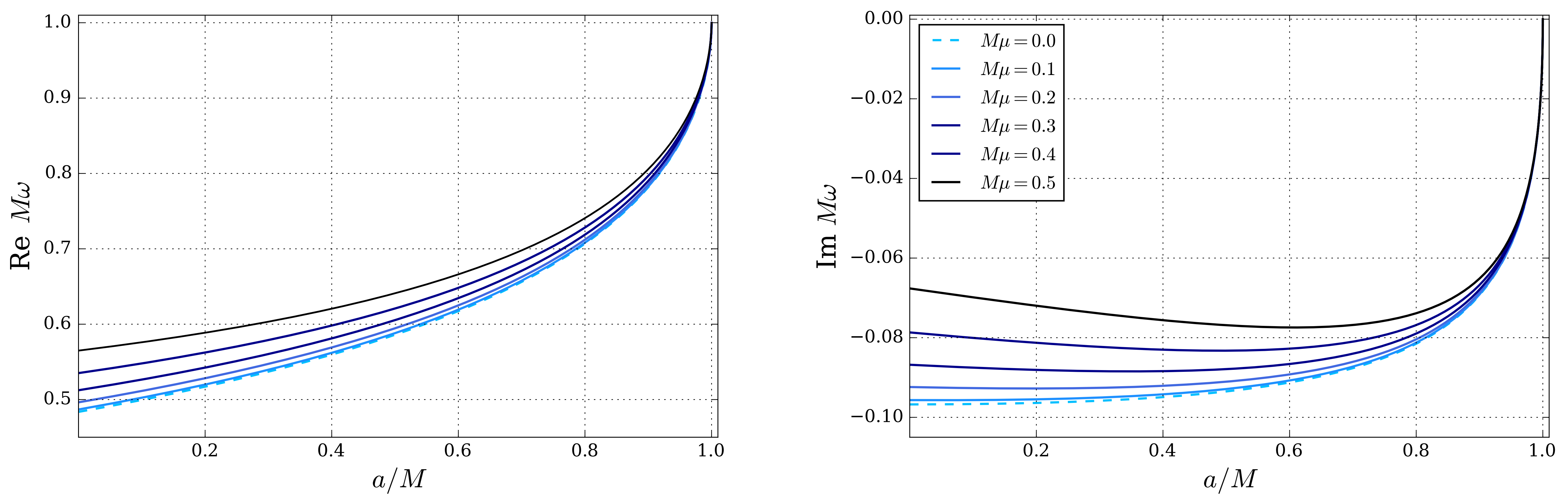


$\ell = 1,\ m = 0$

Co-rotating modes $\ell = m \; (\neq 0)$

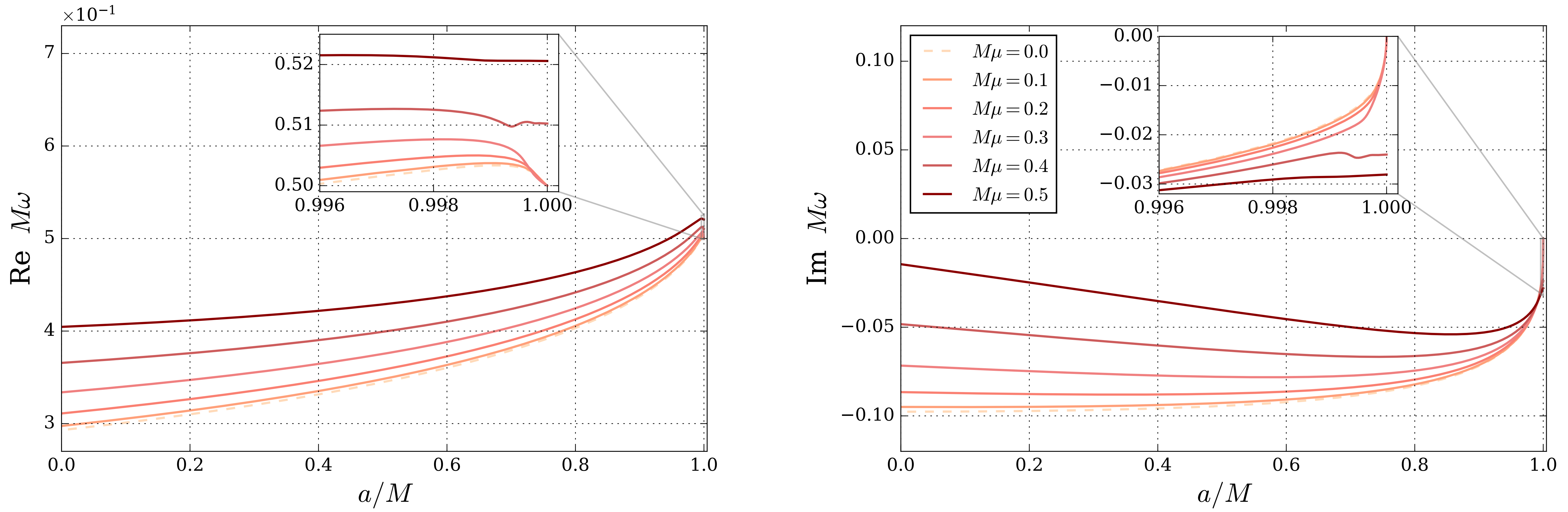
$$M\omega_{n,\ell,\ell} \to \frac{m}{2}$$

Zero-damped modes

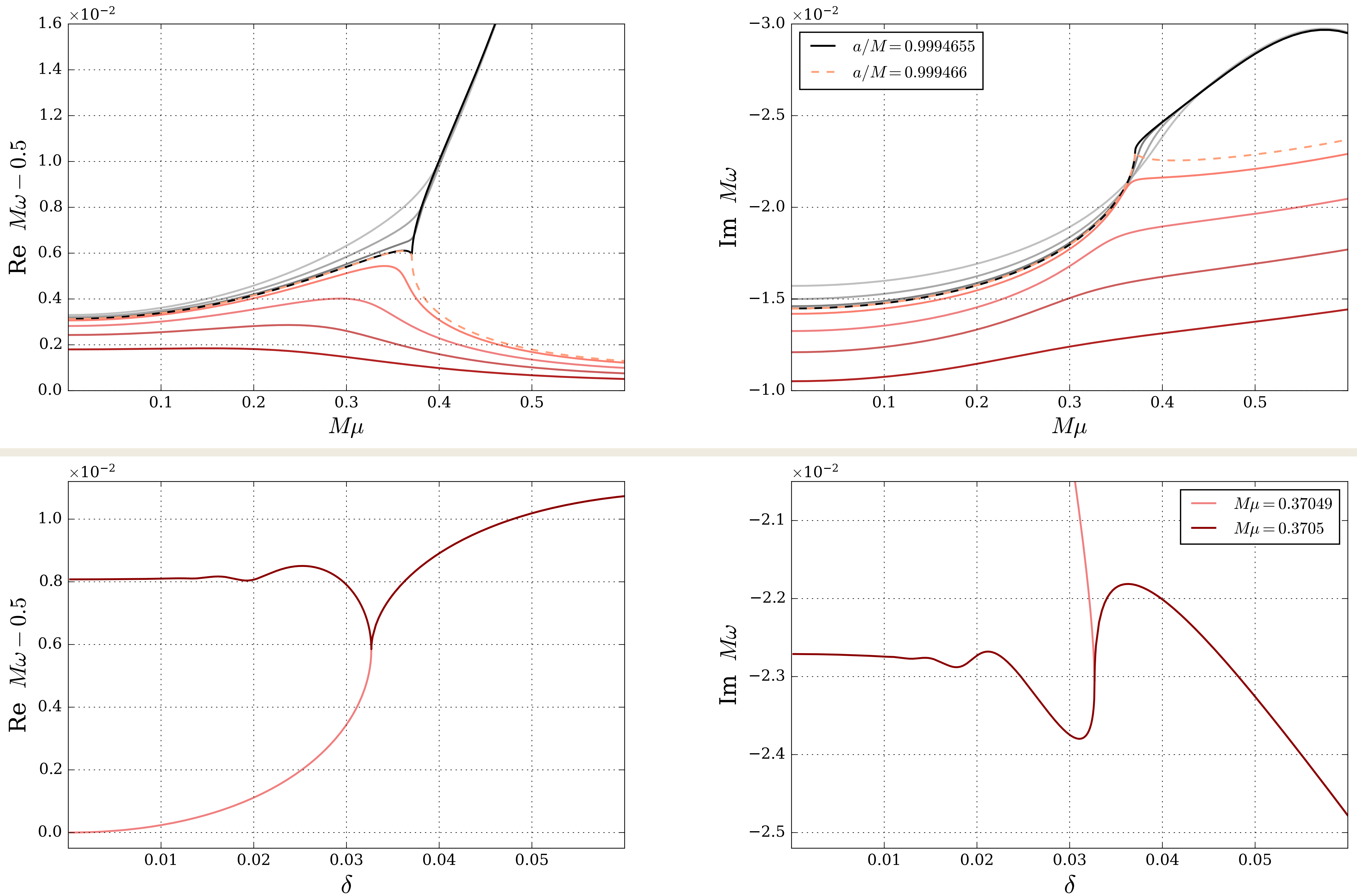


$$\omega_{n\ell m} \simeq \frac{m}{2M} - i2\pi T_{BH}\left(n + \frac{1}{2}\right) \mp i\pi T_{BH}\sqrt{4\lambda_0 + 4M^2\mu^2 - 7m^2 + 1} + \mathcal{O}(T_{BH}, \log T_{BH})$$

$\ell = m = 1$ modes



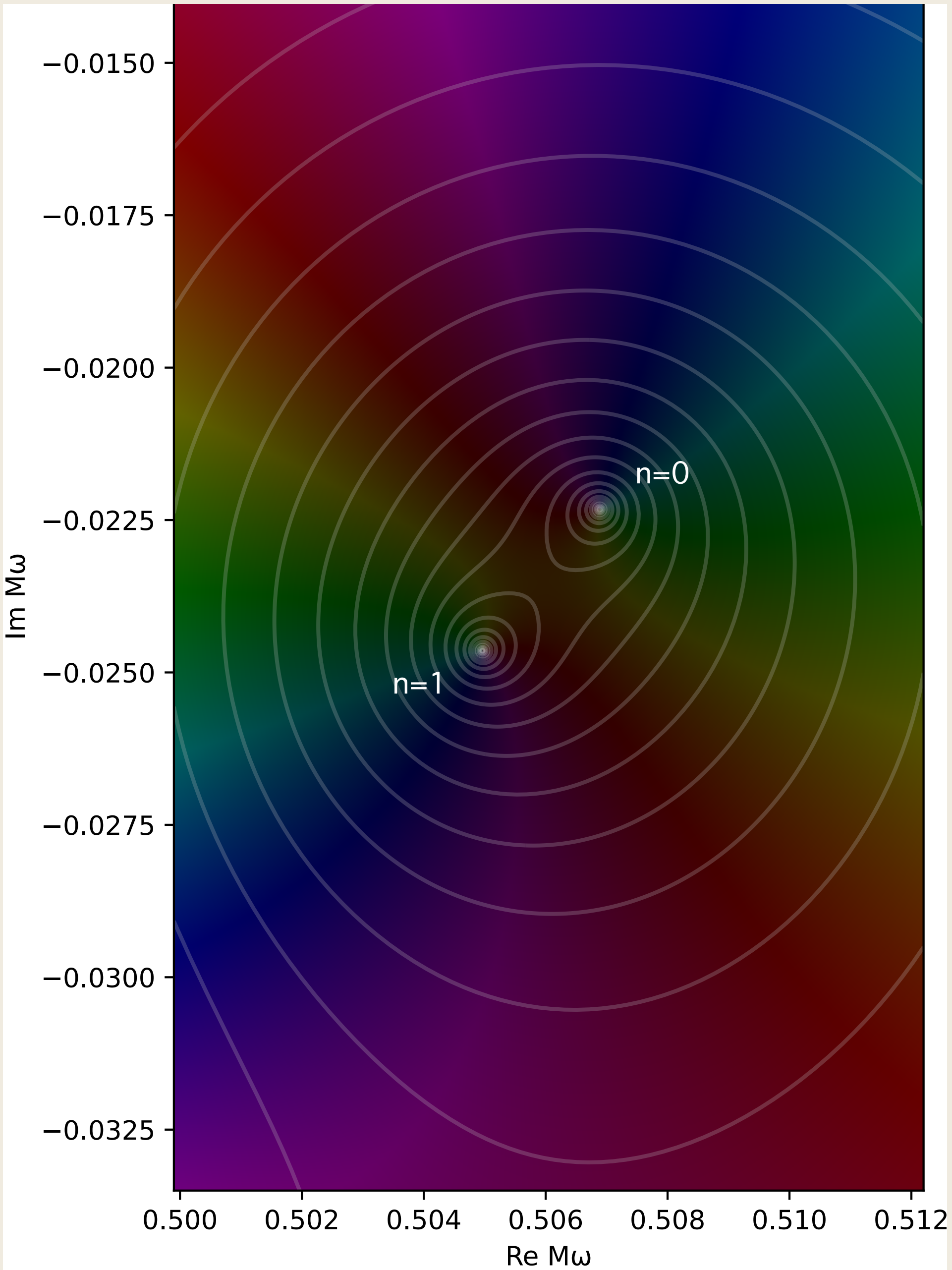
Non-trivial structure in the near-extremal regime



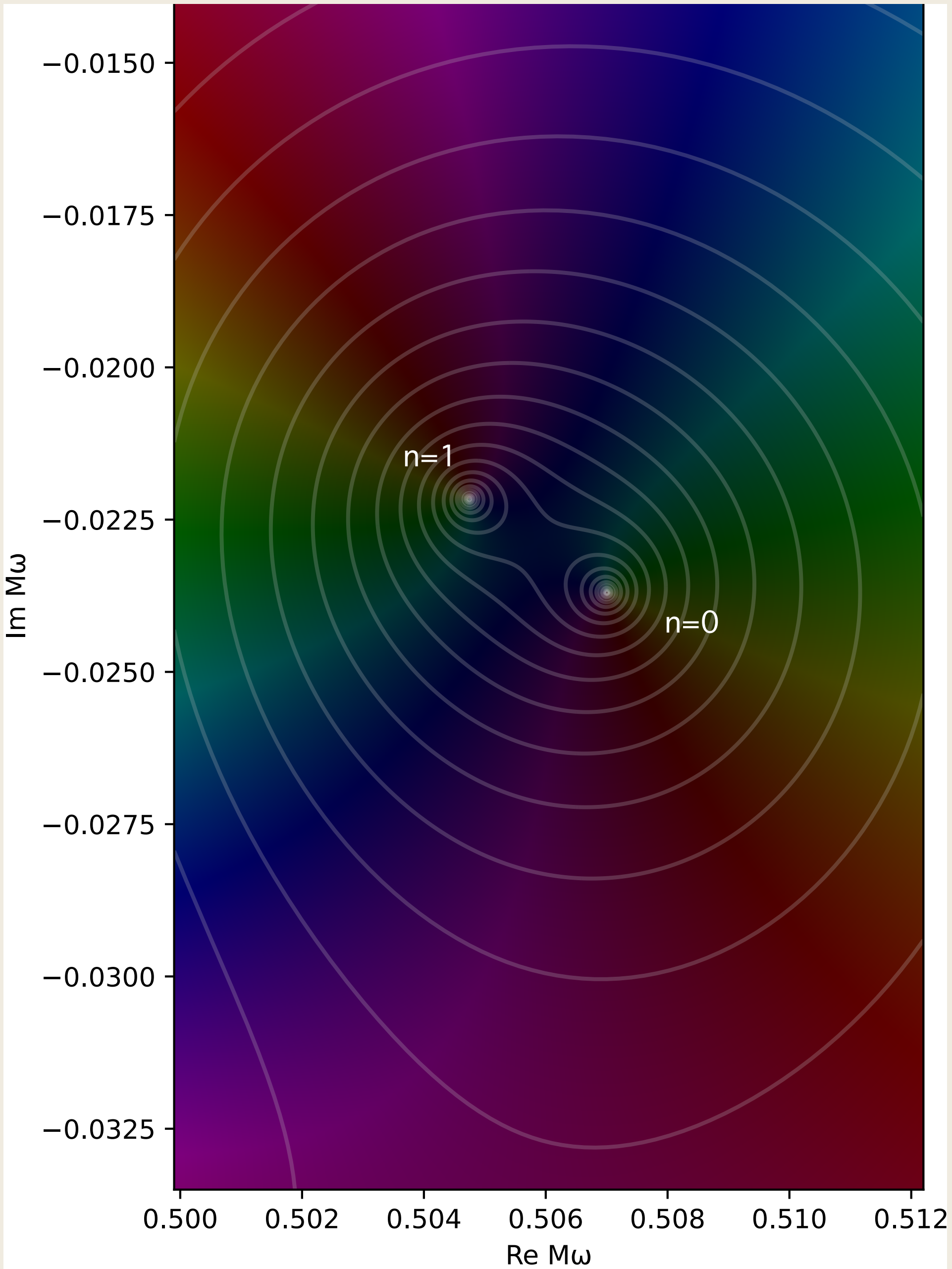
$$\frac{a}{M} = \cos\delta$$

Transition from ZDM to DM!

$n = 0$ (as ordered in the massless Schwarzschild case) no longer the fundamental mode!



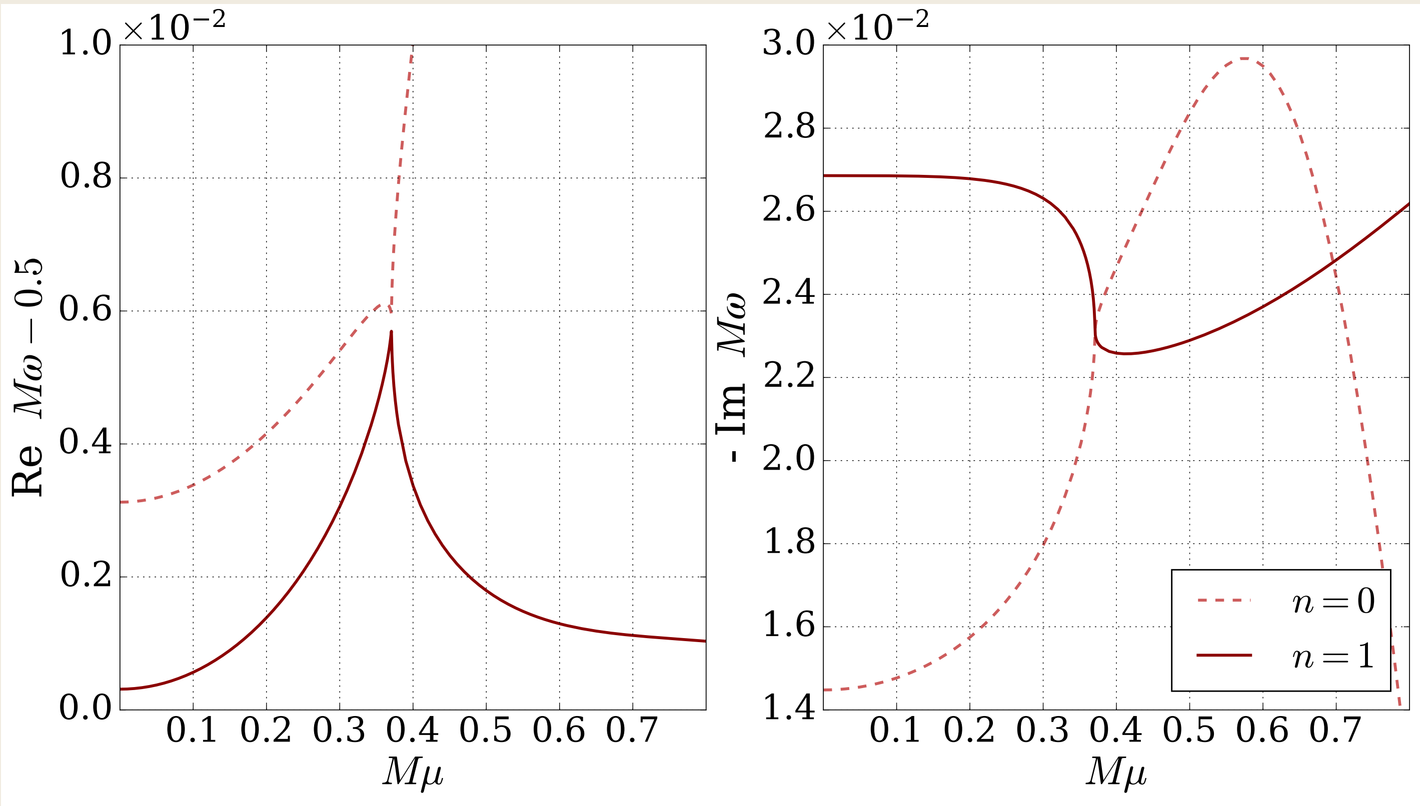
$$\frac{a}{M} = 0.99944$$



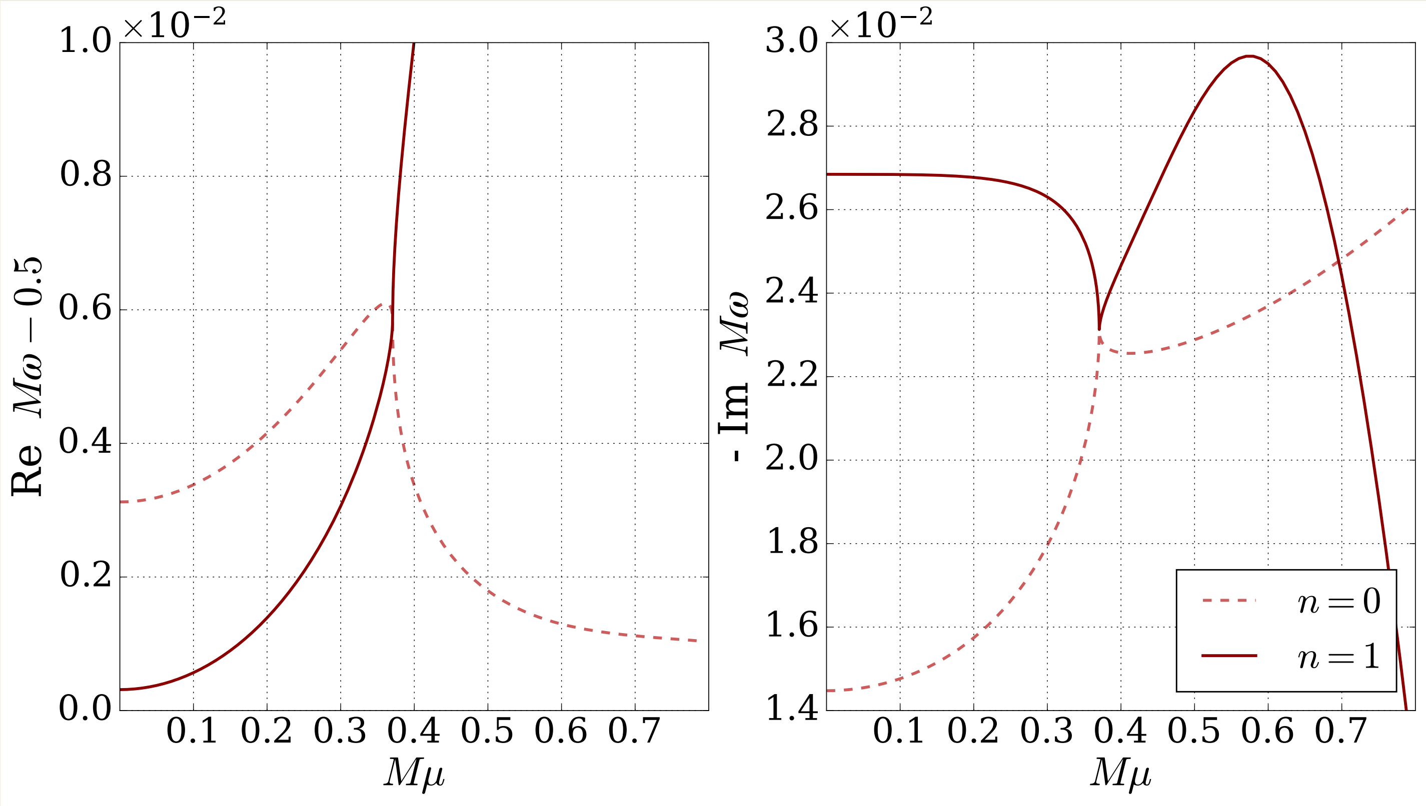
$$\frac{a}{M} = 0.99948$$

Change of behavior from ZDM to DM at $(M\mu)_c \simeq 0.3704981$ and $(a/M)_c \simeq 0.9994660$

Suspiciously close to the "overtaking"… Let's analyze the 1st excited mode:

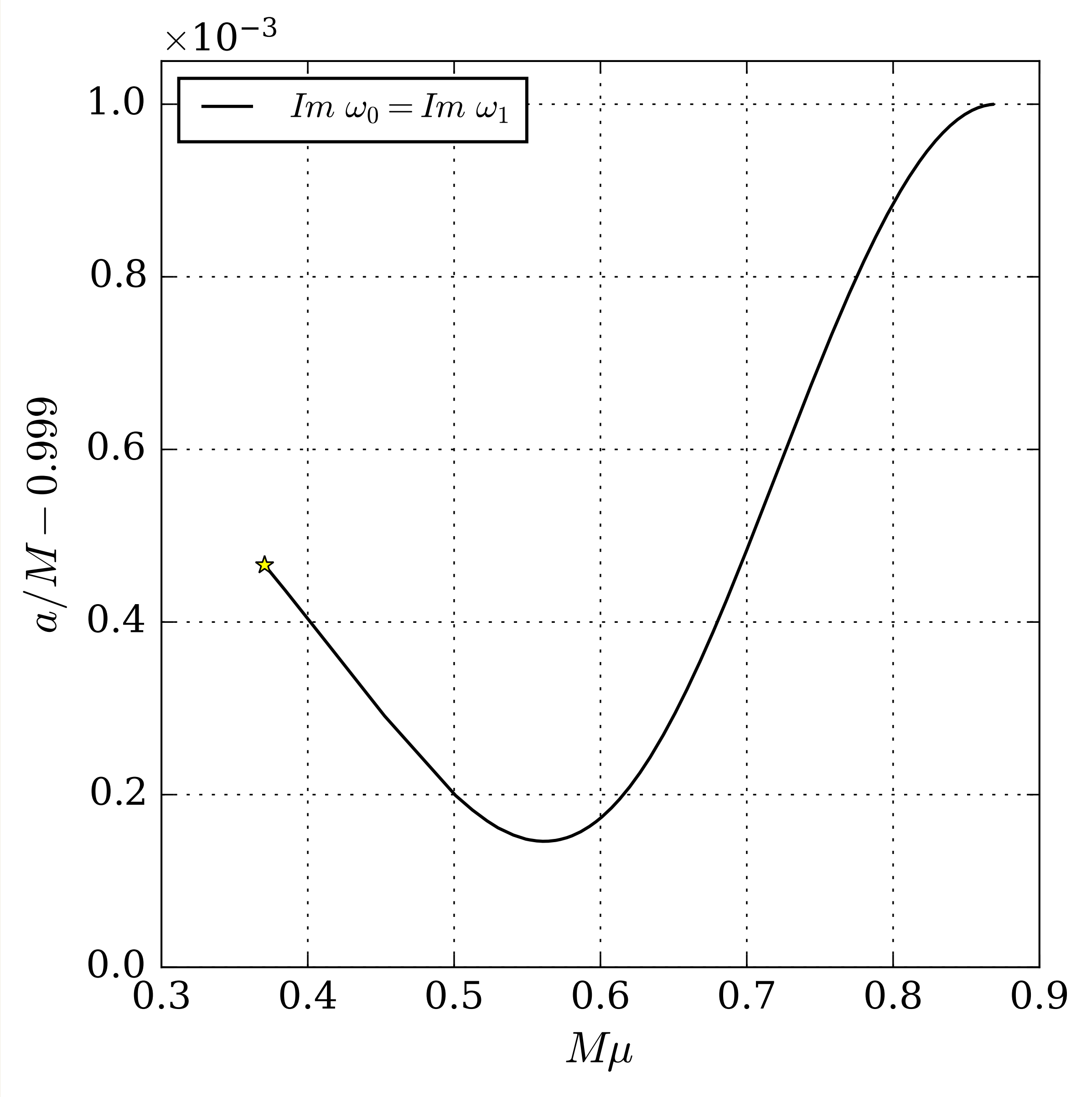


$(a/M) \lesssim (a/M)_c$

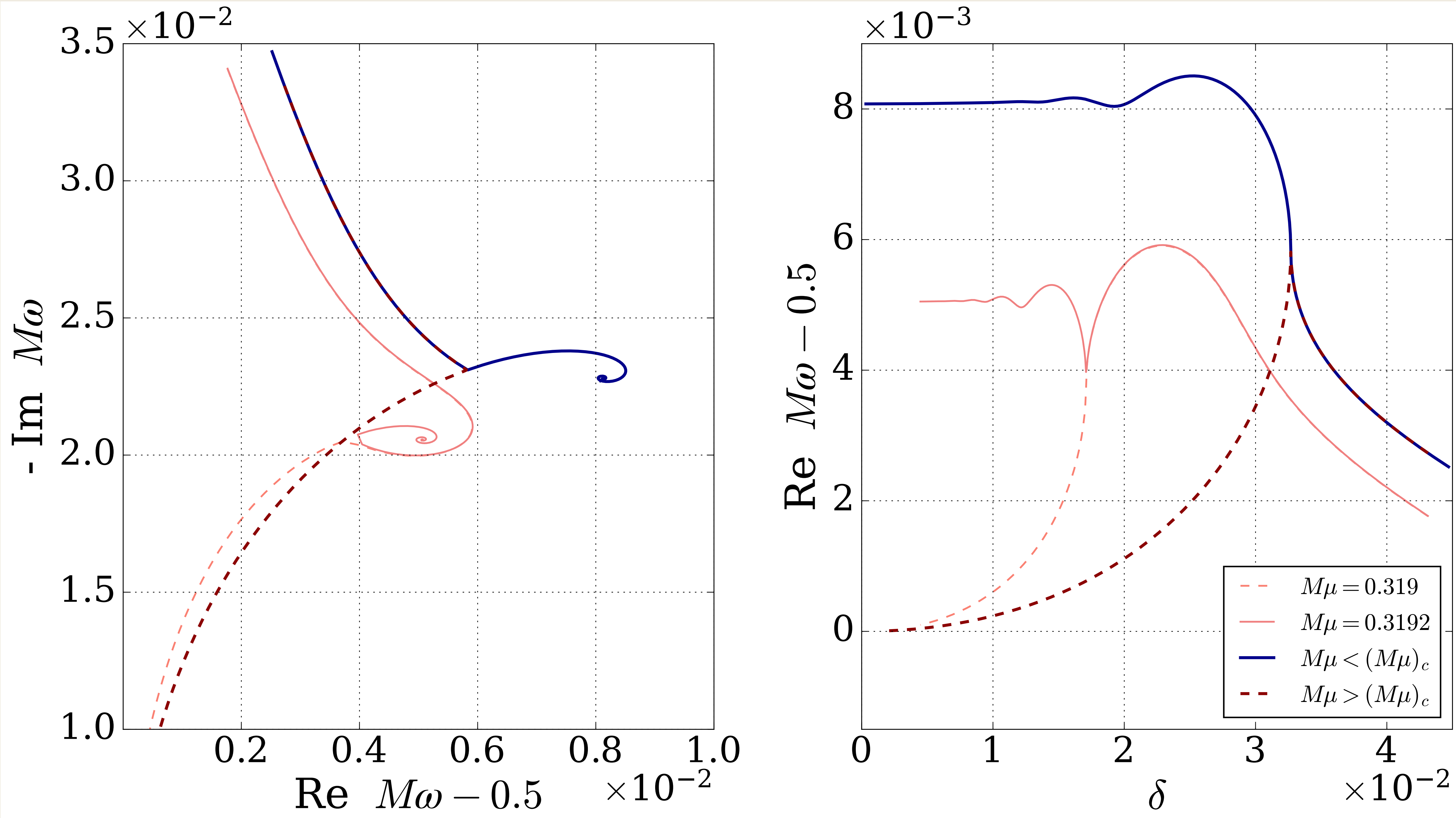


$(a/M) \gtrsim (a/M)_c$

"Curve of coexistence"

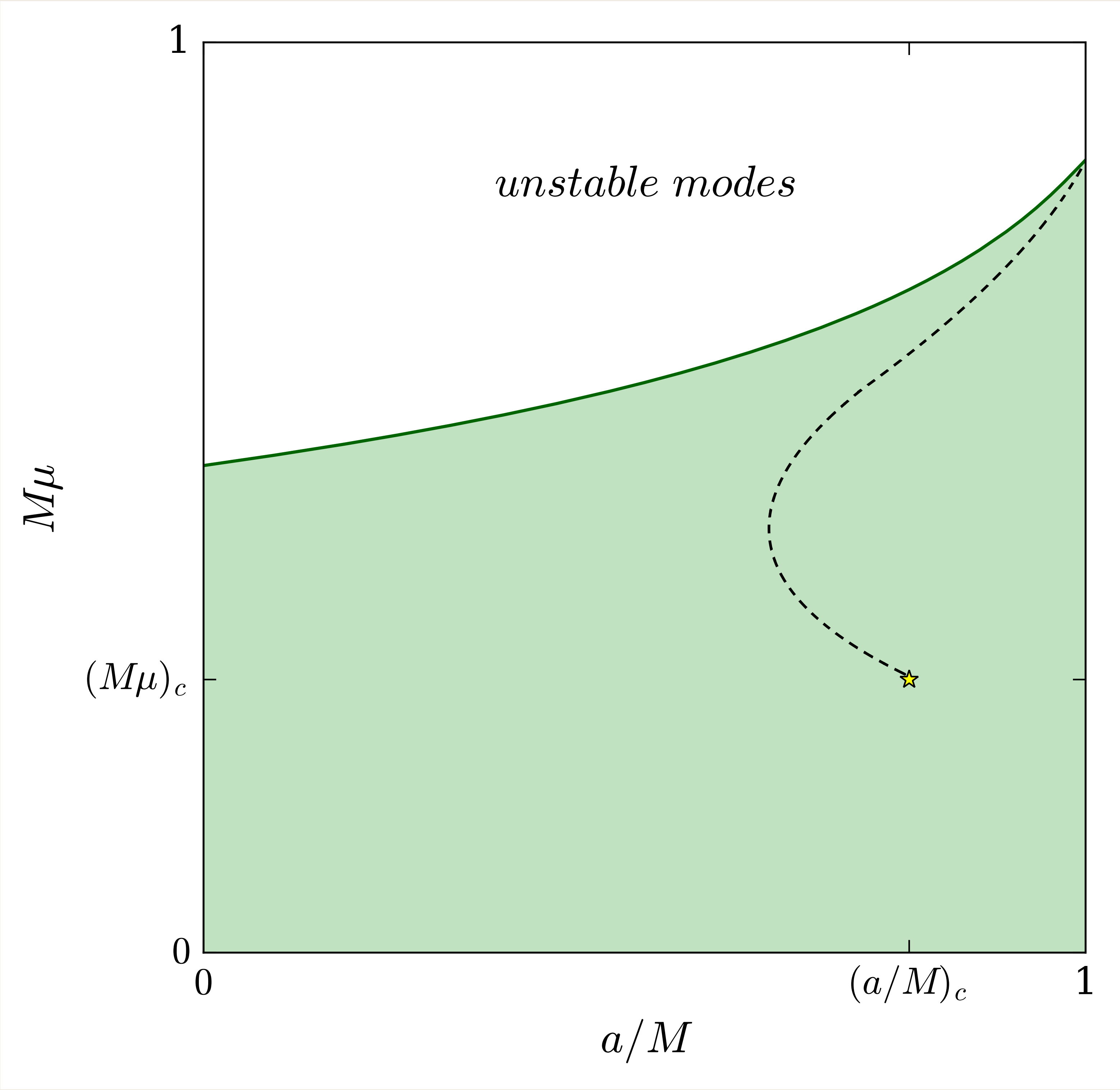


Crossing the line, and there's a discontinuity in the longest-lived QNM frequency



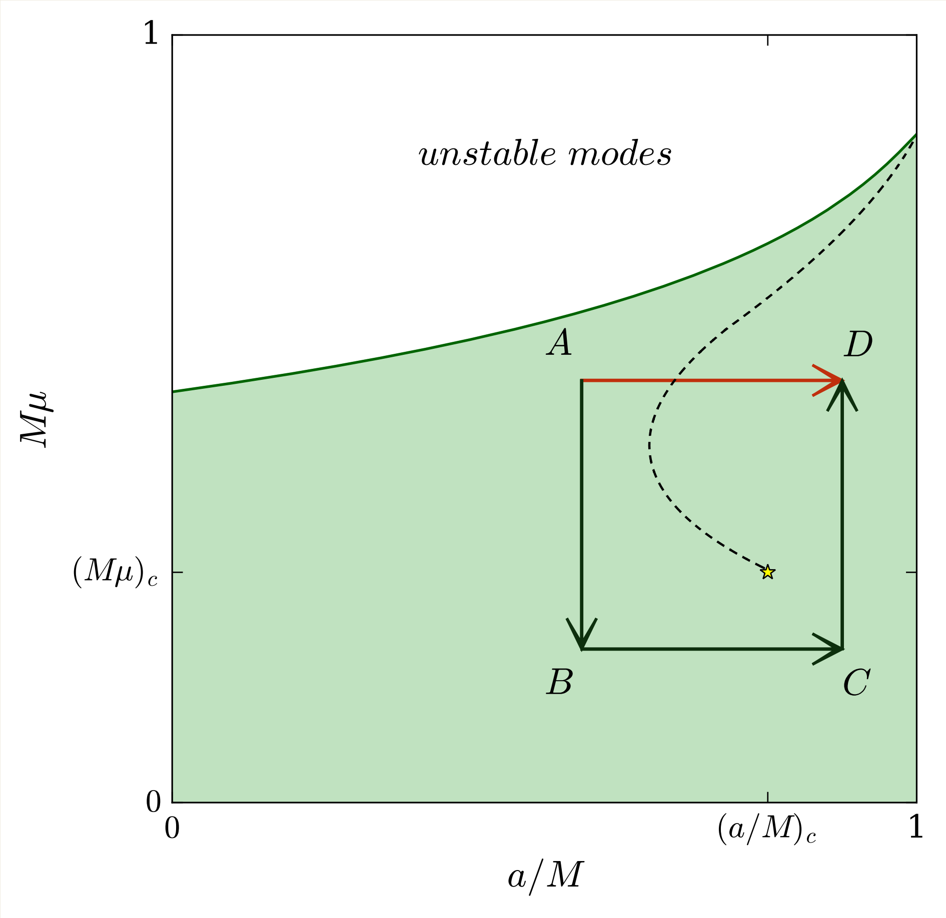
$\times 10^{-2}$
3.5
3.0
2.5
2.0
1.5
1.0
- Im $M\omega$
0.2
0.4
0.6
0.8
1.0
Re $M\omega - 0.5$
$\times 10^{-2}$
$\times 10^{-3}$
8
6
4
2
0
Re $M\omega - 0.5$
0
1
2
3
4
δ
$\times 10^{-2}$
$M\mu = 0.319$
$M\mu = 0.3192$
$M\mu < (M\mu)_c$
$M\mu > (M\mu)_c$

Suspiciously similar to phase transitions in thermodynamics



Cross-over?

Adiabatic changes (quasi-static processes)



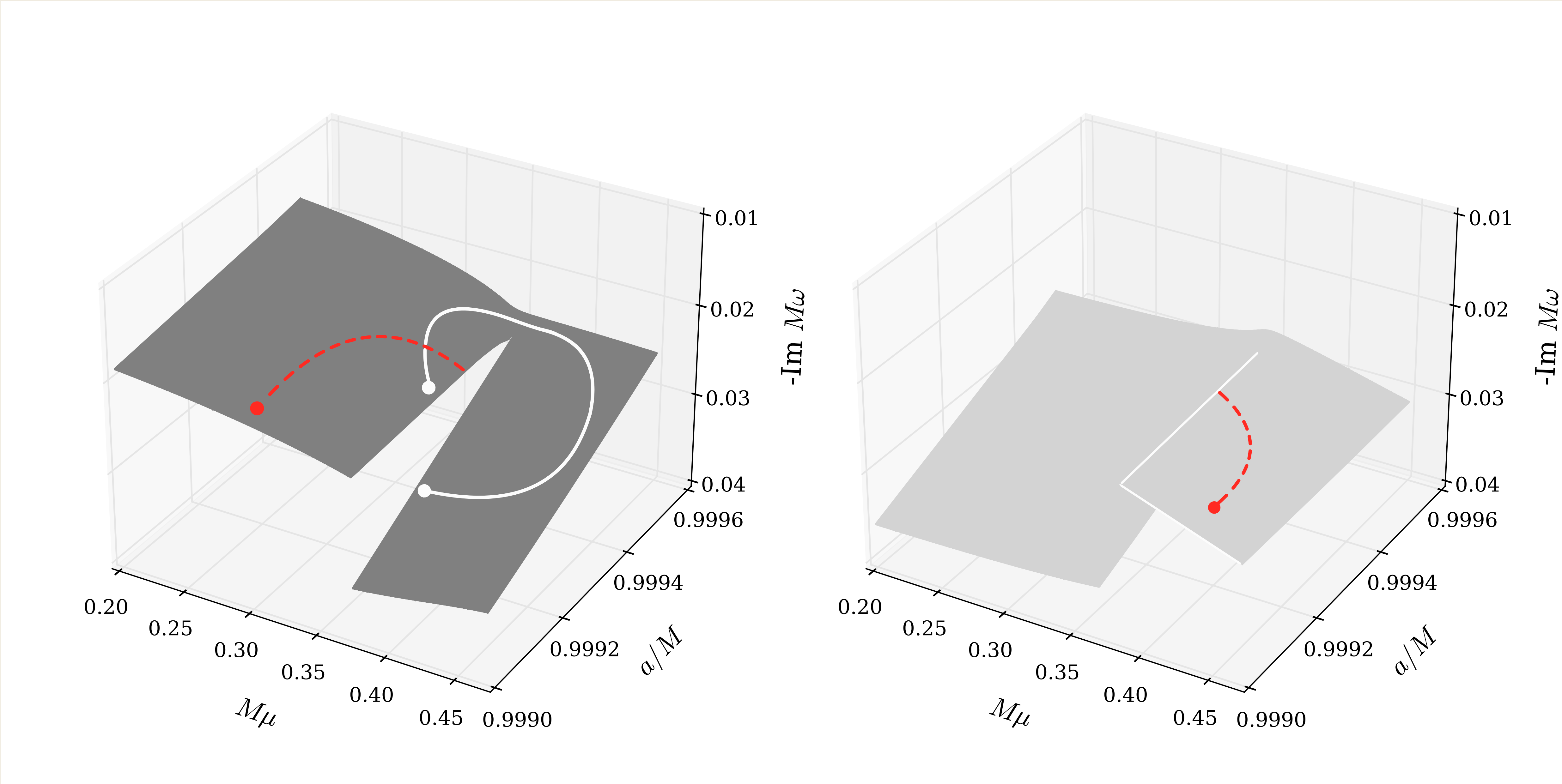
$$A: \quad (a/M)(A) = 0.999, \quad (M\mu)(A) = 0.45,$$
$$B: \quad (a/M)(B) = 0.999, \quad (M\mu)(B) = 0.30,$$
$$C: \quad (a/M)(C) = 0.9999, \quad (M\mu)(C) = 0.30,$$
$$D: \quad (a/M)(D) = 0.9999, \quad (M\mu)(D) = 0.45,$$

$$\omega_\alpha(A) \simeq 0.514974 - 0.026418i,$$
$$\omega_\beta(A) \simeq 0.503306 - 0.032998i$$

$$\omega_\alpha(AD) \simeq 0.514827 - 0.026255i,$$
$$\omega_\beta(AD) \simeq 0.500386 - 0.009456i$$

$$\omega_\alpha(ABCD) \simeq 0.500386 - 0.009456i,$$
$$\omega_\beta(ABCD) \simeq 0.514827 - 0.026255i$$

“Critical point” is an exceptional point. Hysteresis is geometrical phase.



$$(\omega - \omega_\star)\,\delta\omega = ((a/M) - (a/M)_c)\,\delta(a/M) + ((M\mu) - (M\mu)_c)\,\delta(M\mu)$$

Manifold of frequencies is two-sheeted.

Comments

- Thermodynamical analogue: van der Waals fluid! Line of coexistence of phases ending at a critical point;
- Are there consequences for stability analysis? Degeneracy = symmetry?
- Exceptional points and geometric phases usually associated with QM of open systems; ("the universe's least practical quantum NOT gate!"); PT broken by separation of variables;
- CFT formulation to be taken seriously? Operators unitary for real frequency, Cardy's formula works… Angular eigenvalue can be understood as equilibrium between two systems;
- Generic features of interacting "non-Hermitian" systems. Do they happen for other black holes (*e.g.* AdS)?
- Conformal blocks = new class of special functions?