One-loop effective actions and thermodynamics of near-extremal Kerr black holes

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Introduction and Motivation I

Study of the quantum effective actions in a black hole background was settled in Denef-Hartnoll-Sachdev [0908.2657], where a formula for the computation of determinants in thermal spacetimes in terms of QNMs was proposed, based on analytic properties of the effective action of Euclidean quantum gravity:

$$Z \propto \prod_{\omega_{\text{QNM}}} \frac{\sqrt{\omega_{\text{QNM}}\,\overline{\omega}_{\text{QNM}}}}{2\pi T} \prod_{n\geq 0} \left(n + \frac{i\,\omega_{\text{QNM}}}{2\pi T}\right)^{-1} \left(n - \frac{i\,\overline{\omega}_{\text{QNM}}}{2\pi T}\right)^{-1} \tag{1}$$

Introduction and Motivation II

- The low-temperature breakdown of black hole thermodynamics for extremal black holes was pointed out more than thirty years ago.
- The resolution of this puzzle did not require knowledge of the full path integral (quantum gravity) and was first achieved by a careful treatment of certain zero modes in the extremal solution.
- The fact that temperature effectively acts as a coupling constant and that the low-temperature regime is a quantum regime was understood first in the context of two-dimensional JT gravity.

Introduction and Motivation III

- The key realization resides in understanding that the near-horizon region of higher-dimensional black holes may contain a JT subsector that dominates the full path integral.
- Any higher-dimensional gravity theory admitting near-extremal solutions can admit such zero modes and, consequently, the low-temperature thermodynamics will be accordingly corrected.
- The path integral over these zero modes leads to an infrared divergence in the one-loop approximation to the Euclidean partition function. This divergence can be regulated turning on a small but finite temperature correction in the geometry, leading to the thermodynamic correction $\frac{3}{2} \log T_{\text{Hawking}}$.

Klein-Gordon equation

Let us consider the Klein-Gordon differential operator

$$\left[\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\right)-\mu^2\right]\Phi\equiv\left[\Box-\mu^2\right]\Phi=0, \qquad (2)$$

where $g_{\mu\nu}$ is the metric of the spacetime and μ is the mass of the scalar field.
We are interested in problems with enough symmetries so that we can separate the variable dependences. We use the following decomposition in Fourier modes of the wave function Φ

$$\Phi(t,r,\Omega)=\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{\ell,\vec{m}}e^{-i\omega t}S_{\omega,\ell,\vec{m}}(\Omega)R_{\omega,\ell,\vec{m}}(r). \qquad (3)$$

Decomposition of the determinant

Starting from the problem

$$\left(\Box - \mu^2\right)\Phi = \lambda\Phi, \tag{4}$$

and using (3), we obtain a system of coupled second-order differential equations for $S_{\omega,\ell,\vec{m}}(\Omega)$ and $R_{\omega,\ell,\vec{m}}(r)$, of the form

$$\begin{aligned} \mathcal{D}_{\mathrm{rad}} R_{\omega,\ell,\vec{m}}(r) &= \left(A_{\ell\vec{m}} + \lambda\right) R_{\omega,\ell,\vec{m}}(r), \\ \mathcal{D}_{\mathrm{ang}} S_{\omega,\ell,\vec{m}}(\Omega) &= -A_{\ell\vec{m}}\, S_{\omega,\ell,\vec{m}}(\Omega), \end{aligned} \tag{5}$$

for some second-order differential operators $\mathcal{D}_{\mathrm{rad}}$ and $\mathcal{D}_{\mathrm{ang}}$, and where $A_{\ell\vec{m}}$ denotes the separation constant at fixed values of the quantum numbers.

The full determinant has an expression of the form

$$\log\left(\det\left(\Box - \mu^2\right)\right) \equiv \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{\ell,\vec{m}} \log\left(\det\left(\mathcal{D}_{\mathrm{rad}} - A_{\ell\vec{m}}\right)[\omega]\right). \tag{6}$$

Radial Problem - Fuchsian Case

We first introduce a new variable z sending the points where the boundary conditions are imposed at $z = 0$ and $z = 1$, and we redefine the wave function so that the differential equation is in normal form.

Let

$$\psi_{i,\lambda}^{(\hat{z})}(z) = (z - \hat{z})^{\frac{1}{2} \pm a_{\hat{z}}} \left[1 + \mathcal{O}(z - \hat{z})\right], \quad i = 1, 2 \tag{7}$$

be the fundamental system of local solutions around $z = \hat{z}$.

Let us denote with $\psi_{1,\lambda}^{(\hat{z})}(z)$ the solution selected by the boundary condition at $z = \hat{z}$. Using the connection formulae, we can write

$$\psi_{1,\lambda}^{(0)}(z) = \mathcal{C}_{11,\lambda}\, \psi_{1,\lambda}^{(1)}(z) + \mathcal{C}_{12,\lambda}\, \psi_{2,\lambda}^{(1)}(z), \tag{8}$$

where we denote with $\mathcal{C}_{11,\lambda}, \mathcal{C}_{12,\lambda}$ the connection coefficients, which depend on λ (but are independent of z).

Radial determinant

We introduce a reference problem whose differential operator $\tilde{\mathcal{D}}_{\mathrm{rad}}$ is a hypergeometric one, obtained by keeping the indices of the singular points at $z = 0$ and $z = 1$ fixed.
We proved

$$\frac{\det\left(\mathcal{D}_{\mathrm{rad}} - A_{\ell m}\right)}{\det\left(\tilde{\mathcal{D}}_{\mathrm{rad}}\right)} = \frac{\mathcal{C}_{12,\lambda=0}}{\tilde{\mathcal{C}}_{12,\lambda=0}}. \tag{9}$$

Finally, we can compute the regularized determinant for the reference hypergeometric potential. This provides a solution for the determinant of the radial differential operator, which is of the form

$$\det\left(\mathcal{D}_{\mathrm{rad}} - A_{\ell m}\right) = 2\pi \frac{\mathcal{C}_{12,\lambda=0}}{\Gamma(1 + 2\theta_0 a_0)\Gamma(2\theta_1 a_1)}, \tag{10}$$

where a_0, a_1 denote the indices of the singularities at $z = 0$ and $z = 1$.

Heun's connection coefficients

The Heun connection coefficients were obtained by considering

- The connection formulas for semiclassical Virasoro conformal blocks, obtained by crossing symmetry from different expansions of the same correlation function with a degenerate insertion;
- The AGT correspondence, relating $2d$ Liouville CFT and $4d$ SUSY gauge theory.

Assuming $0 < t \ll 1$ and $\mathrm{Re}(a_0), \mathrm{Re}(a_1) > 0$,

$$\det(\mathcal{D}_{\mathrm{rad}} - A_{\ell m}) = \sum_{\theta'=\pm} \frac{2\pi\Gamma(-2\theta' a)\Gamma(1-2\theta' a)}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}+a_0-\theta' a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}-\theta' a+a_1+\sigma\, a_\infty\right)} \times \\ \times\, t^{a_0+\theta' a} \exp\left(\frac{1}{2}\partial_{a_0}F(t) + \frac{1}{2}\partial_{a_1}F(t) - \frac{\theta'}{2}\partial_a F(t)\right). \tag{11}$$

Wick Rotation

If we Wick rotate the spacetime metric to real-time by defining $t = i\tau$, where τ has periodicity equal to the inverse of the temperature T_H, we can introduce the thermal frequencies by setting

$$\omega_k = 2\pi i\, k\, T_H, \quad k \in \mathbb{Z}, \tag{12}$$

and we can match our results with the ones in DHS in the hypergeometric cases.

Kerr black hole

The four-dimensional asymptotically flat Kerr black hole metric in the Boyer-Lindquist coordinates is

$$\begin{aligned} \mathrm{d}s^2 = & -\mathrm{d}t^2 + \mathrm{d}r^2 + 2\, a_{\mathrm{BH}} \sin^2\theta\, \mathrm{d}r\, \mathrm{d}\phi \\ & + \left(r^2 + a_{\mathrm{BH}}^2 \cos^2\theta\right) \mathrm{d}\theta^2 + \left(r^2 + a_{\mathrm{BH}}^2\right) \sin^2\theta\, \mathrm{d}\phi^2 \\ & + \frac{2\, M\, r}{r^2 + a_{\mathrm{BH}}^2 \cos^2\theta} \left(\mathrm{d}t + \mathrm{d}r + a_{\mathrm{BH}} \sin^2\theta\, \mathrm{d}\phi\right)^2 , \end{aligned} \tag{13}$$

where M is the mass of the black hole and a_{BH} is the parameter describing its angular momentum.

The radial geometry admits two horizons: the event horizon $R_h = M + \sqrt{M^2 - a_{\mathrm{BH}}^2}$ and an inner Cauchy horizon $R_i = M - \sqrt{M^2 - a_{\mathrm{BH}}^2}$.

The temperature and the angular velocity at the event horizon read

$$T_H = \frac{R_h - R_i}{8\pi M R_h}, \qquad \Omega_H = \frac{a_{\mathrm{BH}}}{2 M R_h}. \tag{14}$$

Teukolsky formalism

Based on the Newman-Penrose formalism, Teukolsky showed that in terms of curvature invariants, the perturbation equations decouple and separate for all Petrov type-D spacetimes.
Considering the Fourier-transform of a spin-s field $\Phi(t, r, \theta.\phi)$ and expanding it in spin-weighted spheroidal harmonics, both the radial and angular equations can be rewritten as confluent Heun equations.
For the radial problem, we have

$$\mathcal{D}\,\psi \equiv \psi''(z) + \left[\frac{u - \frac{1}{2} + a_0^2 + a_1^2}{z(z-1)} + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} + \frac{\frac{1}{4} - a_0^2}{z^2} + \frac{\mu\,\epsilon}{z} - \frac{\epsilon^2}{4}\right]\psi(z) = 0, \tag{15}$$

where

$$z = \frac{r - R_i}{R_h - R_i}, \quad \epsilon = -16\,i\,\pi\,M\,\omega\,R_h\,T_H, \quad \mu = s - 2i\,M\,\omega,$$

$$a_0 = i\,\frac{\omega - m\Omega_H}{4\pi\,T_H} - 2iM\omega - \frac{s}{2}, \quad a_1 = i\,\frac{\omega - m\Omega_H}{4\pi\,T_H} + \frac{s}{2},$$

$$\mu_1 = a_0 - a_1 = -s - 2i\,M\,\omega, \quad \mu_2 = -a_0 - a_1 = 2i\,M\,\omega - i\,\frac{\omega - m\Omega_H}{2\pi\,T_H}.$$

The relevant connection formula reads

$$e^{-\epsilon z} z^{\frac{\alpha}{\epsilon}-\gamma-\delta} \mathrm{HeunC}_{\infty}(q-\gamma\epsilon, \alpha-\epsilon(\gamma+\delta), \gamma, \delta, -\epsilon; z) = \epsilon^{\frac{1}{2}-\mu} e^{-\frac{1}{2}\partial_\mu F} \times$$
$$\Big[\sum_{\sigma=\pm} \frac{\Gamma(-2\sigma a)\Gamma(1-2\sigma a)\Gamma(2a_1)\epsilon^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F+\frac{1}{2}\partial_{a_1}F}}{\Gamma\left(\frac{1}{2}-\mu_1-\sigma a\right)\Gamma\left(\frac{1}{2}-\mu_2-\sigma a\right)\Gamma\left(\frac{1}{2}-\mu-\sigma a\right)} \mathrm{HeunC}(q-\alpha, -\alpha, \delta, \gamma, -\epsilon; 1-z)$$
$$+\sum_{\sigma=\pm} \frac{\Gamma(-2\sigma a)\Gamma(1-2\sigma a)\Gamma(-2a_1)\epsilon^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F-\frac{1}{2}\partial_{a_1}F}}{\Gamma\left(\frac{1}{2}+\mu_1-\sigma a\right)\Gamma\left(\frac{1}{2}+\mu_2-\sigma a\right)\Gamma\left(\frac{1}{2}-\mu-\sigma a\right)} (1-z)^{1-\delta} \mathrm{HeunC}\left(\tilde{q}, -\alpha-(1-\delta)\epsilon, 2-\delta, \gamma, -\epsilon; 1-z\right)\Big], \tag{16}$$

where $\tilde{q} = q-\alpha-(1-\delta)(\epsilon+\gamma)$ and a is the composite monodromy parameter around the singularities at $z=0$ and $z=1$. At fixed quantum numbers ℓ, m, s, the contribution of the QNMs to the determinant is

$$\det{}^{[\mathrm{Kerr}]}_{(\ell,m,s)} = \sqrt{2\pi} \sum_{\sigma=\pm} \frac{\Gamma(-2\sigma a)\Gamma(1-2\sigma a)\epsilon^{\frac{1}{2}-\mu+\sigma a} e^{-\frac{1}{2}\left(\sigma\partial_a+\partial_{a_1}+\partial_\mu\right)F}}{\Gamma\left(\frac{1}{2}+\mu_1-\sigma a\right)\Gamma\left(\frac{1}{2}+\mu_2-\sigma a\right)\Gamma\left(\frac{1}{2}-\mu-\sigma a\right)}. \tag{17}$$

Zero-damping modes

The exact quantization condition for QNMs follows by studying the zeros of (17). In the confluent limit corresponding to the extremal Kerr case in which the horizons R_h and R_i coalesce, $\epsilon \to 0$, and $\mu_2 \to \infty$, in such a way that $\Lambda \equiv \epsilon\,\mu_2$ is finite. In this limit, an entire branch of solutions associated to the poles of the Γ function $\Gamma\left(\frac{1}{2} + \mu_2 + a\right)$ decouples from the spectrum: these are of the form

$$\omega^{\star}_{n,\ell,m} = \frac{m}{2a_{\mathrm{BH}}} - 2\pi i\, T_H \left(n + \frac{1}{2} + \frac{1}{2}\sqrt{4\,{}_sA^{(0)}_{\ell\, m} - 7m^2 + (1+2s)^2}\right) + \mathcal{O}\left(T_H^2\right), \tag{18}$$

and they have a parametrically small negative imaginary part.

Factorization of the determinant

Let us now focus on the contribution to the effective action of the Γ-function from which ZDMs can arise, and compute the corresponding leading contribution in the Hawking temperature. In the small temperature (or ϵ) regime, we can rewrite (17) as

$$\det{}^{[\mathsf{NEK}]}_{(\ell,m,s)} = \det{}^{[\mathsf{ZDM}]}_{(\ell,m,s)} \left[\det{}^{[\mathsf{EK}]}_{(\ell,m,s)} + \mathcal{O}\left(\epsilon\right)\right], \tag{19}$$

where

$$\det{}^{[\mathsf{ZDM}]}_{(\ell,m,s)} = \frac{\sqrt{2\,\pi}\,\epsilon^{\frac{1}{2}-\mu}\,\left(\epsilon/\Lambda\right)^{a}}{\Gamma\left(\frac{1}{2}+\mu_2-a\right)}, \tag{20}$$

$$\det{}^{[\mathsf{EK}]}_{(\ell,m,s)} = \sum_{\sigma=\pm} \frac{\Gamma\left(-2\sigma a\right)\Gamma\left(1-2\sigma a\right)\Lambda^{\sigma a} e^{-\frac{1}{2}\left(\sigma\partial_a+\partial_{a_1}+\partial_{\mu}\right)F}}{\Gamma\left(\frac{1}{2}+\mu_1-\sigma a\right)\Gamma\left(\frac{1}{2}-\mu-\sigma a\right)}. \tag{21}$$

Temperature corrections

For $s = 1, 2$,

$$\prod_{\ell \geq s} \prod_{m=-\ell}^{\ell} \prod_{k \geq 0} \left[\frac{1}{\det^{[\mathsf{ZDM}]}_{(\ell,m,s)}} \Big|_{\omega=\omega_k^{(M)}} \right] \propto \prod_{\ell \geq s} \prod_{m=-\ell}^{\ell} \prod_{k \geq 0} \Gamma\left(\frac{1}{2} + \mu_2 - a\right) \Big|_{\omega=\omega_k^{(M)}} \propto T_H^{\frac{s}{2}-\frac{1}{4}}, \tag{22}$$

where a ζ-regularization for infinite products has been used. The electromagnetic/gravitational perturbations are described by the Teukolsky equation with $s = \pm 1, 2$ which correspond to the two helicity states of the on-shell photon/graviton. These give the same contributions thanks to the symmetry of the separation constant

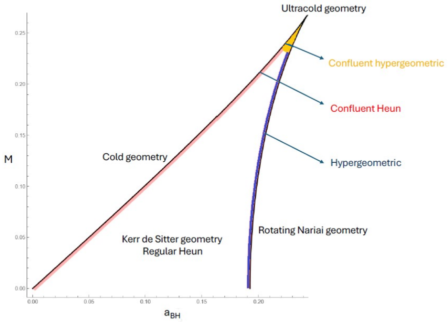
$$_{-s}A_{\ell m} = {}_{s}A_{\ell m} + 2s. \tag{23}$$

Taking into account that the one-loop partition function is proportional to the inverse square root of the determinant, we conclude

$$Z^{\mathsf{NEK}}_{\text{1-loop}} \sim \frac{1}{\sqrt{|\det^{[\mathsf{ZDM}]}|^2}} \frac{1}{\sqrt{|\det^{[\mathsf{EK}]}|^2}} \propto T_H^{s-\frac{1}{2}}. \tag{24}$$

Outlook

In 2506.08959 we extend our analysis to near-extremal (A)dS4 Kerr.



We find that the presence of log(T) corrections is a distinctive feature of the confluence limit of the radial Teukolsky equation, describing *cold near-extremal geometries*. In particular, the result is universal, depending only on the approaching inner and outer horizons, independently of the asymptotic geometry.

THANK YOU!