An integrable deformed Landau-Lifshitz model with particle production?

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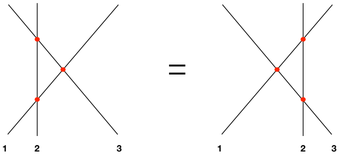
Based on arXiv:2506.13598 with M. de Leeuw and J.M. Nieto García

Introduction

Quantum integrable systems are solutions of the Yang-Baxter Equation

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v)$$



4×4 R-matrix: all solutions have been found!

Regular: $R(u,u) = P$

- Difference form [de Leeuw, Pribytok, Ryan (2019)]
- Non-difference form [Corcoran, de Leeuw (2023)]

Non-regular: $R(u,u) = \text{const.}$

- Difference + Non-difference form [de Leeuw, Posch (2024)]

Past work on solutions of YBE: [Kulish, Sklyanin, (1982)], [Sogo, Uchinami, Akutsu, Wadati (1982)], [Drinfeld (1985)], [Jimbo, (1985], [Fonseca, Frappat, Ragoucy (2015)], [R.S. Vieira (2018)]

$d^2 \times d^2$ R-matrix, $d = 3, 4$, using Machine Learning [Lal,Majumder, Sobko (2025)]

The classification of “de Leeuw, Pribytok, Ryan (2019)”

All regular 4×4 R-matrix of difference form

The Heisenberg XXX spin chain appears in the list!

New non-Hermitian deformations of XXX spin chain: **Class 5 and 6**

Main question:

> *If the continuum limit of the XXX spin chain is the Landau-Lifshitz model, what is the continuum limit of the Class 5 and 6 spin chains?*

Answer:

A non-unitary, integrable deformation of the LL theory.

Class 5 is a Drinfeld twist of the XXX R-matrix, and its continuum limit has soft particle production.

Motivations

Integrable deformations of the AdS/CFT

XXX spin chain and LL theory played important role in AdS/CFT.

[Kruczenski (2004)], [Kruczenski, Ryzhov, Tseytlin (2004)]

Conjecture [Araujo, Bakhmatov, Colgain, Sakamoto, Sheikh-Jabbari, Yoshida (2018)], [van Tongeren (2016)], [Meier, van Tongeren (2023)]

Jordanian deformation of $AdS_5 \times S^5$	$\Longleftrightarrow$	Drinfeld twist of XXX spin chain

Class 5 solution precisely enters in the r.h.s. of the duality
$\rightarrow$ Expect the deformed LL model to enter on the l.h.s.

Non-diagonalisable transfer matrix

There is a growing interest on this topic, e.g.

- Non-hermitian integrable systems [Maity, Padmanabhan, Korepin (2025)]
- Eclectic spin chain [Ahn, Staudacher (2022)], [Nieto García, (Wyss) (2021/22/23)]
- (progress on) Bethe ansatz [Nieto García (2023)], [Borsato, García Fernández (2025)]
- Non-relativistic string theory [Fontanella, Nieto García, Ohlsson Sax (2022)]

Plan of the talk

1. Class 5 model as a Drinfeld twist
2. Continuum limit
3. Integrability via “boost operator”
4. Soft particle production

Class 5 model as a Drinfeld twist

$$R_5(u) = \begin{pmatrix} 2u+1 & a_2 u & -a_2 u & a_2 a_3 u^2 \\ 0 & 2u & 1 & -a_3 u \\ 0 & 1 & 2u & a_3 u \\ 0 & 0 & 0 & 2u+1 \end{pmatrix} ,$$

a_2, a_3 free parameters

$a_2 = a_3 = 0 \;\rightarrow\;$ XXX

$\exists$ a basis transformation, where effectively $a_3 = 0$. Here, we find:

$$R_5^{a_3=0}(u) = F_{21}\, R^{\mathrm{XXX}}(u)\, F_{12}^{-1} , \qquad F = \mathbb{I}\otimes\mathbb{I} + \frac{a_2}{8}\sigma^+ \otimes (\mathbb{I} + \sigma^z)$$

[2-cocyle condition: $F_{12}(\Delta\otimes\mathbb{I})F = F_{23}(\mathbb{I}\otimes\Delta)F$, Δ trivial co-product]

Two-site Hamiltonian density $h_{12} = R^{-1}\frac{\mathrm{d}}{\mathrm{d}u}R\big|_{u=0}$; total Hamiltonian:

$$\mathcal{H}_5 = \sum_{j=1}^{L} h_{j,j+1} , \qquad h_{L,L+1} = h_{L,1}$$

$$h_{j,j+1} = \underbrace{2P_{j,j+1}}_{\text{XXX}} + \frac{a_2 - a_3}{2} \underbrace{\left(\mathbb{I}_j \otimes \sigma^+_{j+1} - \sigma^+_j \otimes \mathbb{I}_{j+1}\right)}_{\text{boundary term}} + \frac{a_2 + a_3}{2} \underbrace{\left(\sigma^z_j \otimes \sigma^+_{j+1} - \sigma^+_j \otimes \sigma^z_{j+1}\right)}_{\text{physical deformation}}$$

Continuum limit

Follow Fradkin [Fradkin, Cambridge University Press (2013)]

Introduce $SU(2)$ coherent states [Perelomov (1986)]

$$|\theta,\phi\rangle = e^{\frac{i}{2}\phi\,\sigma^z} e^{\frac{i}{2}\theta\,\sigma^y}\,|0\rangle\ .$$

$\theta,\phi \ \to\ $ fields of 2d sigma model, $\mathcal{L}_{\text{kin}} = -\cos\theta\,\phi_t$ universal,

$$S_5 = \int \mathrm{d}t\mathrm{d}x\Big(\mathcal{L}_{\text{kin}} + \langle\theta,\phi|\otimes\langle\theta+\delta\theta,\phi+\delta\phi|\,h_{12}\,|\theta,\phi\rangle\otimes|\theta+\delta\theta,\phi+\delta\phi\rangle\Big)\,,$$

$\delta\phi,\delta\theta$ given by Taylor expansion

$$\delta\theta = \varepsilon\theta_x + \frac{1}{2}\varepsilon^2\theta_{xx} + \mathcal{O}(\varepsilon^3)\,, \qquad \delta\phi = \varepsilon\phi_x + \frac{1}{2}\varepsilon^2\phi_{xx} + \mathcal{O}(\varepsilon^3)\,.$$

$\delta x \equiv \varepsilon$ is the *lattice spacing*. Take the continuum limit $\varepsilon \to 0$, get

$$S_5 = -\int \mathrm{d}t\mathrm{d}x\Big\{\underbrace{\cos\theta\,\phi_t + \frac{1}{2}\Big[(\theta_x)^2 + \sin^2\theta(\phi_x)^2\Big]}_{\text{LL}} + \alpha\,\underbrace{\frac{e^{-i\phi}}{2}\Big(\theta_x - \frac{i}{2}\sin 2\theta\,\phi_x\Big)}_{\text{deformation}}\Big\}\,.$$

Spacetime symmetries: Aristotelian (time + space translations)

Broken on-shell anisotropic dilatation ($t \to \lambda^2 t, x \to \lambda x$) of LL

Integrability via "boost operator"

Quantum systems. Introduce the *boost operator* [Tetel'man (1982)]

$$\mathcal{B}[\mathcal{H}] \equiv \sum_{n=-\infty}^{+\infty} n\, h_{n,n+1}\,,$$

and generate the higher conserved charges as ($\mathcal{Q}_2 \equiv \mathcal{H}$; $\mathcal{Q}_p$ range p)

$$\mathcal{Q}_{r+1} \equiv [\mathcal{B}[\mathcal{H}], \mathcal{Q}_r]\,, \qquad [\mathcal{Q}_p, \mathcal{Q}_q] = 0\,, \qquad \forall\, p, q$$

Evidence that: if $\exists\ \mathcal{Q}_3 \to$ integrable (solid proof?)

Classical field theories. similar construction works for LL [Fuchssteiner (1984)] and for our deformed LL [de Leeuw, AF, Nieto Garcia (2025)]

Boost functional

$$B[H] \equiv \int_{-\infty}^{+\infty} \mathrm{d}x\, x\, \mathscr{H}\,, \qquad \to \qquad Q_{r+1} = \{B[H], Q_r\}\,.$$

$B[H]$ is *strong* and *hereditary* symmetry

Apply [Fuchssteiner (1979)] theorem on *strong* + *hereditary* symmetries

Guarantees: if $\exists\ Q_3 \to$ integrable

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discretisation + quantisation

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Soft particle production

Study perturbation theory + quantisation

Similarly to LL, introduce complex field

[Minahan, Tirziu, Tseytlin (2006)]
[Roiban, Tirziu, Tseytlin, (2006)]
[Klose, Zarembo (2006)]

$$\eta = \frac{\sin\theta}{\sqrt{2+2\cos\theta}} e^{i\phi}$$

Expand in number of fields, get (up to $\mathcal{O}(\eta^5)$)

$$\mathcal{L}_5^\eta = -\alpha\eta_x^* + \underbrace{\left(i(\eta_t^*\eta - \eta^*\eta_t) - 2\eta_x^*\eta_x\right)}_{\text{LL}} + \frac{1}{2}\alpha\eta^*\left(2\eta_x^*\eta - 3\eta^*\eta_x\right) - \underbrace{\left((\eta_x^*)^2\eta^2 + (\eta^*)^2(\eta_x)^2\right)}_{\text{LL}}$$

Structure: “LL + cubic potential”

Quantisation goes like for LL (quadratic part is the same).

Cubic term gives *soft particle production*:

$$\mathcal{S}(k;p,p') = \frac{i\alpha k}{4|k|}\left[\delta(p)\delta(k-p') + \delta(p')\delta(k-p)\right].$$

$\eta \to \eta \times \eta$ with one of outgoing particles with zero momentum

$\eta \times \eta \to \eta$ not allowed, theory is non-unitary

• Is this incompatible with integrability? No!

Very often we heard: "no particle production for integrable theories"

Based on assumptions: i) 1+1 d ✓; ii) Lorentz invariance ✗; iii) massive particles ✗

• Particle production is justifiable from the quantum spin chain:
$SU(2)$ symmetry is broken, $\sum \mathfrak{s}_n^+$ only unbroken generator.
$\rightarrow$ Number of excitations, counted by $\sum \mathfrak{s}_n^z$, not conserved!

• Similar to QED: $\exists$ potential V that does not die off at $|t| \rightarrow \infty$

This creates a cloud of soft particles. Faddeev and Kulish (FK): "move" V inside the states. States are dressed by a cloud of soft particles

$$|\Psi\rangle_{\mathrm{FK}} \equiv T e^{-i \int \mathrm{d}t\, V} |\Psi\rangle_{\mathrm{I}}$$

For us, V is the cubic term. Equivalently, the FK transformation is the continuum limit of the Drinfeld twist.

Conclusions

Constructed a: i) non-unitary, ii) integrable deformation of LL theory, iii) with soft particle production

- Deformed Lax pair?
- Classify integrable LL deformations via boost functional $\rightarrow$ Only continuum limits of Classes 5 & 6? Is there more?
- its role in Jordanian deformations of the AdS/CFT?
- deformed solitons?
- physical relevance of these soft particles? e.g. for gravitons in flat spacetime, emergence of asymptotic BMS supertranslations

Thank you!