Renormalized Angular Momentum Across Black Hole Perturbation Frameworks

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Talk based on: 2508.12046

September 3, 2025

Main ideas

- Radial action $\mathcal{I}_r$ for null geodesics in Kerr geometry can be resummed in terms of hypergeometrics.
- Mano-Suzuki-Takasugi (MST) formalism provides a prove that $\mathcal{I}_r$ is related to the renormalized angular momentum ν in the eikonal limit.
- Correspondence between ν and the Seiberg-Witten (SW) fundamental cycle $\mathfrak{a}$.
- By definition since $\mathcal{I}_r$ keeps into account the monodromy at infinity, it can be identified with the Floquet index (Fioravanti, Rossi, 2508.19960).
- In this correspondence the resummation properties of $\mathcal{I}_r$ are transmitted to these other quantities.

Plan of the talk

- Recap on black holes (BH) perturbations theory in order to fix notation.
- Heun equations arising from the study of perturbations.
- SW theory and its applications via the Alday–Gaiotto–Tachikawa (AGT) duality.
- The definition of the fundamental SW cycle $\mathfrak{a}$.
- MST formalism applied to Kerr geometry (to be concrete).
- The necessity of the introduction of the renormalized angular momentum ν and its correspondence with $\mathfrak{a}$.
- Computation of $\mathcal{I}_r$ and its expression in terms of generalized hypergeometric functions.
- Appearance of $\mathcal{I}_r$ in the eikonal limit of ν.
- Some final considerations and possible future applications.

Black Holes Perturbation Theory: The Heun Equations

- Given the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- The linear perturbation can be studied

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$

- In some cases the eqs at first order in the perturbation $h_{\mu\nu}$ become ODEs of the second order.
- For simplicity, one could alternatively study the propagation of scalar waves in the curved background

$$\Box\Phi = \mu^2\Phi, \qquad \Box = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \qquad g = |\det(g_{\mu\nu})|$$

- which in some cases allows the separation between angular and radial dynamics.
- In $d = 4$ the separation is achieved through the ansatz

$$\Phi(t, r, \theta, \phi) = e^{-i\omega t + i m_\phi \phi} R(r) S(\theta) \,.$$

- $S(\theta)$ are
  - Legendre polynomials in the spherically symmetric case (*e.g.* Schwarzschild);
  - Oblate spheroidal harmonics in the axially symmetric case (*e.g.* Kerr).
- In the vast majority of the cases, $R(r)$ are ODE of Heun type (HE) with four regular singularities

$$\left[\frac{d^2}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{d}{dz} + \frac{\alpha\beta z - p}{z(z-1)(z-t)}\right] W(z) = 0$$

- where $\alpha + \beta + 1 = \gamma + \delta + \epsilon$. The normal (or canonical) form is achieved thanks to the redefinition

$$W(z) = z^{-\gamma/2}(1-z)^{-\delta/2}(t-z)^{-\epsilon/2}\psi(z) \qquad \Longrightarrow \qquad \left[\frac{d^2}{dz^2} + Q_{HE}\right]\psi(z) = 0.$$

Black Holes Perturbation Theory: The Heun Equations

- The $Q-$functions of the various confluences of the HE are (Bonelli, Iossa et al 2201.04491):

$$Q_{HE}=\frac{2\gamma-\gamma^2}{4z^2}+\frac{2\delta-\delta^2}{4(z-1)^2}+\frac{2\epsilon-\epsilon^2}{4(z-t)^2}+\frac{-\gamma\epsilon+2p-\gamma\delta t-2\alpha\beta z+\gamma\delta z+\gamma z\epsilon+\delta z\epsilon}{2(z-1)z(t-z)}$$

$$Q_{CHE}=-\frac{\epsilon^2}{4}+\frac{2\gamma-\gamma^2}{4z^2}+\frac{2\delta-\delta^2}{4(z-1)^2}+\frac{-\gamma\delta+\gamma\epsilon-2p+2\alpha z-\gamma z\epsilon-\delta z\epsilon}{2(z-1)z}$$

$$Q_{RCHE}=\frac{2\gamma-\gamma^2}{4z^2}+\frac{2\delta-\delta^2}{4(z-1)^2}+\frac{\gamma\delta+2p}{2z}+\frac{2\beta-\gamma\delta-2p}{2(z-1)}$$

$$Q_{DCHE}=-\frac{\delta^2}{4z^4}-\frac{(\gamma-2)\delta}{2z^3}+\frac{-\gamma^2+2\gamma-2\delta-4p}{4z^2}+\frac{2\alpha-\gamma}{2z}-\frac{1}{4}$$

$$Q_{RDCHE}=\frac{\epsilon}{z^3}-\frac{p}{z^2}+\frac{\beta}{z}-\frac{1}{4}$$

$$Q_{DRDCHE}=\frac{\epsilon}{z^3}-\frac{p}{z^2}+\frac{1}{z} \qquad C=Confluent,\ D=Doubly,\ R=Reduced$$

- Physically the $Q-$function plays the role of effective potential thanks to the analogy with the non relativistic Schrödinger eq.
- The function $-Q$ always has a maximum, furthermore
  - Fuzzball geometries: an internal minimum (Bianchi, GDR 2212.07504);
  - Asymptotically AdS geometries: external minimum (Berti, Cardoso, Pani 0903.5311).

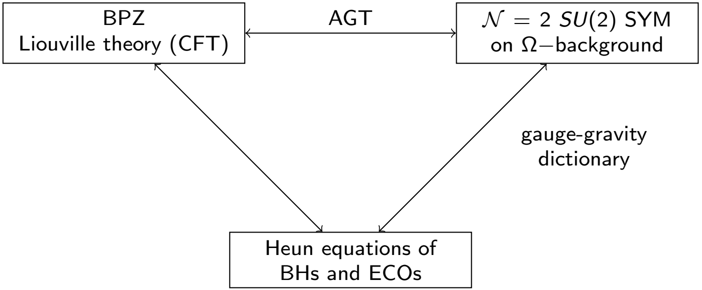
Black Holes Perturbation Theory: The Heun Equations

- Radial dynamics :
  - HE (4 regular): AdS-KN BH (Aminov, Arnaudo, Bonelli et al 2307.10141);
  - CHE (2 regular, 1 irregular): flat KN (Bonelli, Iossa et al 2105.04483, Bianchi, Consoli et al 2109.09804);
  - RCHE (2 regular, 1 irregular): CCLP, D1/D5 circular fuzzball, JMaRT (Bianchi, GDR 2212.07504, Bianchi, Di Benedetto, GDR et al 2305.00865);
  - DCHE (2 irregular) : eKN, D3-D3-D3-D3 BH;
  - RDCHE (2 irregular): eCCLP, BMPV;
  - DRDCHE a.k.a. Mathieu (2 irregular): D3, D1/D5 BHs (Bianchi, Consoli 2105.04245, Fioravanti, Gregori 2112.11434).
- Angular dynamics:
  - Spherically symmetric geometries: Spherical harmonics.
  - Axially symmetric geometries:
    - CHE: KN (Berti, Cardoso, Casalas 0511111);
    - RCHE: D1-D5 circular fuzzball, JMaRT.

(Bianchi, GDR 2203.14900)

- The classical observables of BH perturbation theory.
  - Quasi normal modes (QNMs) whose boundary conditions are:
    - BH: ingoing at horizon, outgoing at infinity;
    - Fuzzballs: regular at cap, outgoing at infinity;
    - AdS BH: ingoing at horizon, regular at infinity.
  - Tidal deformability (GDR, Fucito, Morales 2402.06621)
  - Amplification factor (Cipriani, Di Benedeto, GDR et al 2405.06566)
  - Wave form reconstruction (Bianchi, Bini, GDR 2407.10868, 2411.19612, 2502.21040, 2506.04876)

Gauge-gravity correspondence



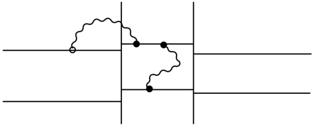
- Some correlators in Liouville CFT are related (via AGT) to results of localization for the partition function of $\mathcal{N} = 2$ SYM with $SU(2)$ gauge group.
- Some correlators in the $d = 2$ CFT satisfy a second order ODE which can be mapped to HE coming from BH perturbation theory.
- The instanton gauge theory partition functions provide a combinatorial representation of the Heun function and its connection formulae.

Hanany-Witten brane setup for $N = 2$ SYM theory

- The dynamics of $\mathcal{N} = 2$ $SU(2)$ SYM can be described in terms of the (classical) Seiberg-Witten (SW) elliptic curve

$$qy^2 P_L(x) + yP_0(x) + P_R(x) = 0$$

$$P_{L,R} = (x - m_{1,3})(x - m_{2,4}) \quad , \quad P_0 = (x - a_1)(x - a_2)$$

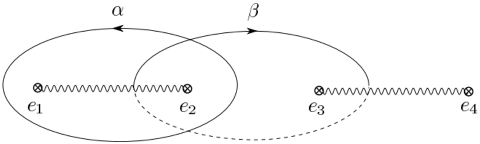


- $P_0 \to$ position of color D4 branes, $P_{L,R} \to$ position of flavour D4 branes (Witten, 1997).
- Solve for y

$$y_\pm = \frac{1}{2qP_L}\left(-P_0 \pm \sqrt{P_0^2 - 4qP_L P_R}\right) \quad , \quad P_0^2 - 4qP_L P_R = \prod_{i=1}^{4}(x - e_i)\,.$$

- The elliptic curve is a double cover of the complex plane with 2 branch cuts.
- The periods of the elliptic curve

$$\mathfrak{a} = \oint_\alpha \lambda_{SW} \quad , \quad \mathfrak{a}_D = \oint_\beta \lambda_{SW} \quad , \quad \lambda_{SW} = x^2 \frac{dx}{y(x)}\,.$$



qSW curves for BHs and fuzzballs perturbation theory

- The "classical" SW curve can be embedded in the non commutative Nekrasov-Shatashvili background

$$\left[\hat{z},\hat{y}\right] = \epsilon_1 = \hbar\,.$$

- The elliptic curve becomes an ODE

$$\left[qz^2 P_L\left(z\partial_z+\frac{\hbar}{2}\right)+zP_0\left(z\partial_z\right)+P_R\left(z\partial_z-\frac{\hbar}{2}\right)\right]f(z)=0\,.$$

- The canonical (normal) form $\psi''(z)+Q_{2,2}\psi(z)=0$ is obtained :

$$f(z)=z^{\frac{m_1+m_2}{2\hbar}}(1+z)^{-\frac{1}{2}-\frac{m_1+m_2}{2\hbar}}(1+qz)^{-\frac{1}{2}+\frac{m_3+m_4}{2\hbar}}\psi(z)$$

$$Q_{2,2}=\frac{1-\left(\frac{m_1-m_2}{\hbar}\right)^2}{4z^2}+\frac{1-\left(\frac{m_1+m_2}{\hbar}\right)^2}{4(1+z)^2}+\frac{q^2\left(1-\left(\frac{m_3+m_4}{\hbar}\right)^2\right)}{4(1+qz)^2}+$$
$$+\frac{2\left(q\left(m_3m_4-\sum_i m_i\hbar+\sum_{i<j}m_im_j\right)-m_2m_1q+m_1^2+m_2^2\right)+(q-1)\left(4u+\hbar^2\right)}{4z\hbar^2}+$$
$$-\frac{q^2\left(-2m_1m_2q+2m_3m_4(q-2)+2q\left(\sum_{i<j}m_im_j-\sum_i m_i\hbar\right)+4(q-1)u+(q+1)\hbar^2\right)}{4(q-1)\hbar^2(qz+1)}+$$
$$+\frac{-2\left(q\left(m_3m_4+\sum_i m_i\hbar-\sum_{i<j}m_im_j\right)+m_1^2(q-1)+m_2m_1q+m_2^2(q-1)\right)+4(q-1)u+(3q-1)\hbar^2}{4(q-1)(z+1)\hbar^2}$$

- $Q_{2,2}$ can be mapped to Q_{HE} through the dictionary

qSW curves for BHs and fuzzballs perturbation theory

$$\alpha = 1 + \frac{m_2 + m_4}{\hbar}, \quad \beta = 1 + \frac{m_2 + m_3}{\hbar}, \quad \gamma = 1 - \frac{m_1 - m_2}{\hbar}, \quad \delta = 1 + \frac{m_3 + m_4}{\hbar}$$

$$\epsilon = 1 + \frac{m_1 + m_2}{\hbar}, \quad p = \frac{1+3q}{4} + \frac{m_2^2 + m_3 m_4 q + m_2(m_3+m_4)q + (q-1)u}{\hbar^2} + \frac{m_2 - m_1 q}{\hbar}.$$

- The confluences of the Heun equation are obtained by decoupling

$$q_{N_f} \to 0, \quad m_{N_f} \to \infty \quad \text{with} \quad q_{N_f - 1} = -q_{N_f} m_{N_f} = \text{fixed}.$$

- The ODE can be transformed in a difference eq. whose integrability condition is (depending on $\mathfrak{a}(u,q)$):

$$P_0(\mathfrak{a}) = \frac{qM(\mathfrak{a}+\hbar)}{P_0(\mathfrak{a}+\hbar) - \frac{qM(\mathfrak{a}+2\hbar)}{P_0(\mathfrak{a}+2\hbar)-\dots}} + \frac{qM(\mathfrak{a})}{P_0(\mathfrak{a}-\hbar) - \frac{qM(\mathfrak{a}-\hbar)}{P_0(\mathfrak{a}-2\hbar)-\dots}}, \quad M(\mathfrak{a}) = P_L\left(\mathfrak{a} - \frac{\hbar}{2}\right) P_R\left(\mathfrak{a} - \frac{\hbar}{2}\right).$$

- Perturbatively solvable in q in order to find the a-cycle

$$\mathfrak{a}(u,q) = \sqrt{u} + \frac{q}{4\sqrt{u}}\left(\frac{4m_1 m_2 m_3}{4u - \hbar^2} + m_1 + m_2 + m_3 - \hbar\right) + \mathcal{O}(q^2)$$

- a-cycle can be perturbatively inverted for u(a,q) (Coulomb branch modulus).
- Integration of quantum Matone relation provides the prepotential (Matone, 9506102)

$$u = -q\partial_q \mathcal{F} \quad , \quad \mathcal{F} = \mathcal{F}_{\text{three}} + \mathcal{F}_{\text{inst}} + \mathcal{F}_{1-\text{loop}}.$$

- The integration of the quantum Matone relation doesn't keep into account the 1 loop contribution

$$\frac{\partial \mathcal{F}_{1-loop}}{\partial \mathfrak{a}} = \hbar \log\left[\frac{\Gamma^2\left(1 + \frac{2\mathfrak{a}}{\hbar}\right)}{\Gamma^2\left(1 - \frac{2\mathfrak{a}}{\hbar}\right)} \prod_{i=1}^{3} \frac{\Gamma\left(\frac{1}{2} + \frac{m_i - \mathfrak{a}}{\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{m_i + \mathfrak{a}}{\hbar}\right)}\right]$$

- The dual quantum period $\mathfrak{a}_D$

$$\mathfrak{a}_D = -\frac{1}{2\pi i}\frac{\partial \mathcal{F}}{\partial \mathfrak{a}}$$

MST solutions for homogeneous Teukolsky equation

- Solutions of the homogeneous problem: MST method → applied to Teukolsky eq. (Kerr) which is a CHE (Mano, Suzuki, Takasugi 9603020)

$$\Delta R''(r) + 2(r-M)R'(r) + \left[\frac{K^2}{\Delta} - \lambda\right]R(r) = 0, \quad K = (r^2+a^2)\omega - m_\phi a$$

$$\Delta = (r-r_+)(r-r_-) \qquad r_\pm = M \pm \sqrt{M^2-a^2}$$

- where the oblate spheroidal harmonics eigenvalue

$$\lambda = E - 2m_\phi a\omega + a^2\omega^2, \qquad E = \ell(\ell+1) - \frac{2\ell^2+2\ell-2m_\phi^2-1}{(2\ell-1)(2\ell+3)}a^2\omega^2 + \mathcal{O}(a^4\omega^4)\,.$$

- MST-type IN solution: ingoing boundary condition at $r = r_+$. In the variable $z = \frac{r_+-r}{r_+-r_-}$:

$$R(z) = e^{i\epsilon\kappa z}(-z)^{-i\frac{\epsilon+\tau}{2}}(1-z)^{i\frac{\epsilon-\tau}{2}}f(z)\,.$$

- The Teukolsky eq becomes

$$z(1-z)f''(z) + [1-i\epsilon-i\tau-(2-2i\tau)z]f'(z) + (\nu+i\tau)(\nu+1-i\tau)f(z) =$$

$$= 2i\epsilon\kappa[-z(1-z)f'(z) + (1+i\epsilon-i\tau)zf(z)] + [-\lambda+\nu(\nu+1)+\epsilon^2-i\epsilon\kappa]f(z)$$

$$\epsilon = 2M\omega, \quad q = \frac{a}{M}, \quad \kappa = \sqrt{1-q^2}, \quad \tau = \frac{\epsilon - mq}{\kappa}$$

- where ν is the renormalized angular momentum $\nu = \ell + \mathcal{O}(\omega)$.
- The solution can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} C_n f_{n+\nu}(z), \qquad f_{n+\nu}(z) = {}_2F_1\left(n+\nu+1-i\tau, -n-\nu-i\tau, 1-i\epsilon-i\tau, z\right)\,.$$

MST solutions for homogeneous Teukolsky equation

- As a consequence the ODE becomes a 3-terms recursion relation

$$\alpha_n C_{n+1} + \beta_n C_n + \gamma_n C_{n-1} = 0\,,$$

$$\alpha_n = \frac{i\epsilon\kappa\,(n+\nu+1+i\epsilon)\,(n+\nu+1-i\epsilon)\,(n+\nu+1+i\tau)}{(n+\nu+1)(2n+2\nu+3)}\,,$$

$$\beta_n = -\lambda + (n+\nu)(n+\nu+1) + \epsilon^2 + \epsilon(\epsilon - mq) + \frac{\epsilon^3(\epsilon - mq)}{(n+\nu)(n+\nu+1)}\,,$$

$$\gamma_n = -\frac{i\epsilon\kappa\,(n+\nu+i\epsilon)\,(n+\nu-i\epsilon)\,(n+\nu-i\tau)}{(n+\nu)(2n+2\nu-1)}\,.$$

- The consistency of the 3-terms recursion can be written in the form of a continuous fraction

$$\beta_0 - \frac{\alpha_{-1}\gamma_0}{\beta_{-1} - \frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2} - \frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3} - \cdots}}} - \frac{\alpha_0\gamma_1}{\beta_1 - \frac{\alpha_1\gamma_2}{\beta_2 - \frac{\alpha_2\gamma_3}{\beta_3 - \frac{\alpha_3\gamma_4}{\beta_4 - \cdots}}}} = 0\,,$$

- which can be perturbatively solved thanks to the introduction of the renormalized angular momentum (RAM)

$$\nu = \ell + \sum_{n=1}^{\infty} \nu_n (M\omega)^n\,.$$

- UP solution: outgoing b.c.s at $R(r) \underset{r\to\infty}{\sim} e^{i\omega r}$. Introduce $z = \omega(r - r_+)$ and $R(z) = z^{-1}\left(1 + \frac{\epsilon\kappa}{z}\right)^{\frac{i}{2}(\epsilon-\tau)} g(z)$.
- The Teukolsky equation becomes

$$z^2 g''(z) + [z^2 + 2\epsilon z - \nu(\nu+1)]g(z) = -\epsilon\kappa z(g''(z) + g(z)) + \epsilon\kappa(1 + i\epsilon - i\tau)g'(z)$$
$$- \frac{\epsilon\kappa(1+i\epsilon)(1-i\tau)}{z} g(z) + [\lambda - \nu(\nu+1) - 2\epsilon^2 + \epsilon mq - \epsilon^2\kappa]g(z)\,.$$

MST solutions for homogeneous Teukolsky equation

- As for the IN solution we can manage the RHS of the previous diff.eq. as the (small) source.
- Solution of the LHS can be written in terms of Coulomb functions

$$g(z) = \sum_{n=-\infty}^{\infty} \tilde{C}_n g_{n+\nu}(z) \, ,$$

$$g_{n+\nu}(z) = e^{-iz}(2z)^{n+\nu} z \frac{\Gamma(n+\nu+1+i\epsilon)}{\Gamma(2n+2\nu+2)} \Phi(n+\nu+1+i\epsilon, 2n+2\nu+2, 2iz) \, .$$

- Analogously to the IN, plugging the previous ansatz in the ODE, we have again a 3-terms recursion

$$\tilde{\alpha}_n \tilde{C}_{n+1} + \tilde{\beta}_n \tilde{C}_n + \tilde{\gamma}_n \tilde{C}_{n-1} = 0$$

- IN and UP must be matched. This happens if the coefficients of the recursion coincide

$$\alpha_n = \tilde{\alpha}_n, \quad \beta_n = \tilde{\beta}_n, \quad \gamma_n = \tilde{\gamma}_n \, .$$

- This happens automatically with IN and UP constructed in such way.
- For Teukolsky eq the RAM is

$$\nu_1 = 0, \quad \nu_2 = -\frac{2\ell^2\left(15\ell^2+15\ell-11\right)}{(2\ell-1)(2\ell+1)(2\ell+3)}, \quad \nu_3 = \frac{8\ell^3\left(5\ell^2+5\ell-3\right)q}{(\ell+1)(2\ell-1)(2\ell+1)(2\ell+3)}$$
$$\cdots$$

- It turns out that in $d=4$ (Bianchi, Bini GDR 2407.10868 and Bini, GDR, Geralico 2508.12046)

$$\boxed{\nu = \mathfrak{a} - \frac{1}{2}}$$

Kerr massless radial action

- The Kerr metric in standard Boyer-Lindquist coordinates reads

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - as^2 d\phi\right)^2 + \frac{s^2}{\rho^2}[adt - (r^2 + a^2)d\phi]^2 + \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right), \quad \rho^2 = r^2 + a^2c^2 .$$

- In the Hamiltonian formalism

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu , \quad \mathcal{H}(x^\mu, p_\nu) = p_\mu \dot{x}^\mu(p_\nu) - \mathcal{L}(x^\mu, \dot{x}^\mu(p_\nu)) = \frac{1}{2} g_{\mu\nu} p^\mu p^\nu ,$$

- The dynamics is separable thanks to the introduction of the Carter constant Q

$$p_\theta^2 = Q - c^2\left(\frac{J^2}{s^2} - a^2E^2\right) \equiv \Theta(\theta) ,$$

$$p_r^2 = \frac{(E(r^2 + a^2) - aJ)^2}{\Delta^2} - \frac{(J - aE)^2 + Q}{\Delta} \equiv \frac{R(r)}{\Delta^2}$$

- with $p_t = -E$, $p_\phi = J$.
- We are interested in the equatorial motion $\theta = \pi/2$, $p_\theta = 0$, so that $Q = 0$.
- The equatorial radial action for massless unbound geodesics is defined

$$\mathcal{I}_r = \int_{r_{\min}}^{\infty} p_r dr = \frac{J}{\hat{b}} \int_0^{u_2} du \frac{\sqrt{\mathcal{P}(u)}}{u^2\left(1 - 2u + \hat{a}^2 u^2\right)}, \quad \mathcal{P}(u) = 1 - u^2(\hat{b} - \hat{a})\left[\hat{b}(1 - 2u) + \hat{a}(1 + 2u)\right]$$

- where $u = M/r$, $\hat{b} = J/ME$, $\hat{a} = a/M$, and r_{min} the external turning points

$$\mathcal{P}(u) = 2(\hat{a} - \hat{b})^2(u - u_1)(u - u_2)(u - u_3), \qquad u_1 > u_2 > 0 > u_3 .$$

Kerr massless radial action

- The integration of the radial action can be done formally in terms of Elliptic integrals or Lauricella functions (Gonzo, Shi 2304.06066).
- We use a Post-Minkowskian (weak field/large b) expansion of the integrand for small spin parameters $\hat{a}$.
- PM expansion generates divergent integrals which require a partie finie regularization

$$\mathcal{I}_r = \frac{J}{\hat{b}} \, \mathrm{Pf}_{\epsilon_u} \int_0^{u_2} du \, \mathfrak{I}_r(u) \Big|_{\mathrm{PM}}$$

- by introducing a suitable scale ϵ_u.
- It is convenient to introduce the new integration variable $z = \hat{b}u$, so that the radial action writes as

$$\mathcal{I}_r = \hat{b}ME \, \mathrm{Pf}_{\epsilon_z} \int_0^{z_2} dz \, \mathfrak{I}_r(z) \Big|_{\mathrm{PM}} ,$$

- with $z_2 = \hat{b}u_2 = 1 + O(1/\hat{b})$, $\epsilon_z = \hat{b}\epsilon_u$, and

$$\mathcal{I}_r(z) = \frac{\sqrt{1 - z^2 + W(\hat{a}, \hat{b}; z)}}{z^2 \left(1 - \frac{2z}{\hat{b}} + \frac{\hat{a}^2 z^2}{\hat{b}^2}\right)} , \quad W(\hat{a}, \hat{b}; z) = \frac{2z^3}{\hat{b}} + \frac{z^2 \hat{a}(-4z + \hat{a})}{\hat{b}^2} + \frac{2z^3 \hat{a}^2}{\hat{b}^3} .$$

- Both the upper limit of the integral and the integrand have a large-$\hat{b}$ expansion.
- Expanding the integrand in powers of $\hat{b}$ will generate formally divergent integrals in both limits $z \to 0$ and $z \to 1$.
- Following the prescriptions of Damour, 1912.02139 one can get the correct result for the expanded integral by considering only the leading term in the PM expansion of upper limit (i.e., $z_2 = 1$).
- The radial action integral becomes

$$\mathcal{I}_r = \hat{b}ME \, \mathrm{Pf}_{\epsilon_z} \int_0^{1} dz \, \mathfrak{I}_r(z) \Big|_{\mathrm{PM}} .$$

Kerr massless radial action

- **Performing the integration order by order in $\hat{a}$ we obtain:**

$$\mathcal{I}_r = ME \sum_{n=0}^{\infty} \mathfrak{I}_r^{(n)} \hat{a}^n = ME \left[\mathfrak{I}_r^{(0)} + \mathfrak{I}_r^{(1)} \hat{a} + \mathfrak{I}_r^{(2)} \hat{a}^2 + \mathfrak{I}_r^{(3)} \hat{a}^3 + \mathfrak{I}_r^{(4)} \hat{a}^4 + O(\hat{a}^5) \right] .$$

- **The various $\mathfrak{I}_r^{(n)}$ $(n = 0, \ldots 4)$ as functions of $x = 1/\hat{b}$ are listed below, once decomposed in a π-part ($\mathfrak{I}_r^{\pi(n)}$) and a non-π ($\mathfrak{I}_r^{\not\pi(n)}$) part (or, equivalently, in x-odd and x-even parts).**

TABLE I: Contributions to the the radial action corresponding to various powers of $\hat{a}$. Here (and below) $x = \frac{1}{\hat{b}}$.

$I_r^{\pi(0)}$	$-\frac{1}{2x}\pi\,{}_3F_2\left(-\frac{1}{2},\frac{1}{6},\frac{5}{6};\frac{1}{2},1;27x^2\right)$
	$\sim -\pi\left(\frac{1}{2x}-\frac{15}{8}x-\frac{1155}{128}x^3+O(x^5)\right)$
$I_r^{\not\pi(0)}$	$-1+2\ln(2x)+\frac{32x^2}{3}\,{}_4F_3\left(1,1,\frac{5}{3},\frac{7}{3};2,\frac{5}{2},\frac{5}{2};27x^2\right)$
	$\sim -1+2\ln(2x)+\frac{32}{3}x^2+\frac{448}{5}x^4+\frac{8192}{7}x^6+O(x^8)$
$I_r^{\pi(1)}$	$-\frac{5x^2\pi}{8}\left(3\,{}_3F_2\left(\frac{7}{6},\frac{3}{2},\frac{11}{6};\frac{5}{2},3;27x^2\right)+{}_4F_3\left(\frac{7}{6},\frac{3}{2},\frac{3}{2},\frac{11}{6};\frac{1}{2},\frac{5}{2},3;27x^2\right)\right)$
	$\sim -\pi\left(\frac{5}{2}x^2+\frac{693}{16}x^4+\frac{109395}{128}x^6+O(x^8)\right)$
$I_r^{\not\pi(1)}$	$-\left[\frac{2x}{3}\,{}_4F_3\left(\frac{2}{3},1,1,\frac{4}{3};\frac{1}{2},2,\frac{5}{2};27x^2\right)-\frac{1}{3x}\left(1-{}_3F_2\left(-\frac{1}{3},\frac{1}{3},1;-\frac{1}{2},\frac{3}{2};27x^2\right)\right)\right]$
	$\sim -2x-32x^3-\frac{1792}{3}x^5+O(x^7)$
$I_r^{\pi(2)}$	$-\left[-\frac{1}{8}x\left(63x^2\,{}_3F_2\left(\frac{11}{6},\frac{13}{6},\frac{5}{2};2,\frac{7}{2};27x^2\right)+\left(8x^2-6\right)\,{}_4F_3\left(\frac{7}{6},\frac{3}{2},\frac{3}{2},\frac{11}{6};\frac{1}{2},1,\frac{7}{2};27x^2\right)+6\left(x^2\left(27\,{}_4F_3\left(\frac{11}{6},\frac{13}{6},\frac{5}{2},\frac{5}{2};\frac{3}{2},2,\frac{9}{2};27x\right.\right.\right.\right.$
	$+5\,{}_4F_3\left(\frac{11}{6},\frac{13}{6},\frac{5}{2},\frac{7}{2};\frac{1}{2},3,\frac{11}{2};27x^2\right)-55\,{}_4F_3\left(\frac{13}{6},\frac{5}{2},\frac{17}{6},\frac{7}{2};\frac{3}{2},3,\frac{11}{2};27x^2\right)\Big)+{}_4F_3\left(\frac{3}{2},\frac{11}{6},\frac{13}{6},\frac{5}{2};\frac{1}{2},2,\frac{9}{2};27x^2\right)\Big)\Big)\Big]$
	$\sim \pi\left(\frac{95}{32}x^3+\frac{15939}{128}x^5+\frac{16518645}{4096}x^7+O(x^9)\right)$
$I_r^{\not\pi(2)}$	$\frac{1}{3072x^6}\Big[-192x^2\,{}_4F_3\left(-\frac{5}{3},-\frac{4}{3},1,1;-\frac{5}{2},-\frac{1}{2},2;27x^2\right)+160x^2\,{}_4F_3\left(-\frac{4}{3},-\frac{2}{3},1,1;-\frac{5}{2},-\frac{1}{2},2;27x^2\right)$
	$-64x^2\,{}_4F_3\left(-\frac{2}{3},-\frac{1}{3},1,1;-\frac{3}{2},-\frac{1}{2},2;27x^2\right)+{}_3F_2\left(-\frac{8}{3},-\frac{7}{3},1;-\frac{7}{2},-\frac{3}{2};27x^2\right)-3072x^8\,{}_4F_3\left(\frac{2}{3},1,1,\frac{4}{3};\frac{1}{2},\frac{1}{2},3;27x^2\right)$
	$-512x^6\,{}_4F_3\left(-\frac{1}{3},\frac{1}{3},1,1;-\frac{1}{2},-\frac{1}{2},2;27x^2\right)-768x^6\,{}_4F_3\left(\frac{1}{3},\frac{2}{3},1,1;-\frac{1}{2},\frac{1}{2},2;27x^2\right)+2304x^6\,{}_4F_3\left(\frac{1}{3},\frac{2}{3},1,1;-\frac{1}{2},\frac{1}{2},3;27x^2\right)$
	$+2304x^6\,{}_4F_3\left(\frac{2}{3},1,1,\frac{4}{3};\frac{1}{2},\frac{1}{2},3;27x^2\right)-576x^4\,{}_4F_3\left(-\frac{2}{3},-\frac{1}{3},1,1;-\frac{3}{2},-\frac{1}{2},2;27x^2\right)+4608x^4\,{}_4F_3\left(-\frac{2}{3},-\frac{1}{3},1,1;-\frac{3}{2},\frac{1}{2},3;27x^2\right.$
	$-3072x^4\,{}_4F_3\left(-\frac{1}{3},\frac{1}{3},1,1;-\frac{3}{2},\frac{1}{2},3;27x^2\right)+384x^4\,{}_4F_3\left(-\frac{1}{3},\frac{1}{3},1,1;-\frac{1}{2},-\frac{1}{2},2;27x^2\right)+512x^4\,{}_4F_3\left(\frac{1}{3},\frac{2}{3},1,1;-\frac{1}{2},\frac{1}{2},3;27x^2\right)$
	$+1280x^6-64x^4+64x^2-1\Big]$
	$\sim x^2+64x^4+\frac{6784}{3}x^6+O(x^8)$
$I_r^{\pi(3)}$	$-\frac{1}{8}x^4\Big(-108\,{}_4F_3\left(\frac{11}{6},\frac{13}{6},\frac{5}{2},\frac{5}{2};1,\frac{3}{2},\frac{9}{2};27x^2\right)-5\left(630x^2\,{}_4F_3\left(\frac{17}{6},\frac{19}{6},\frac{7}{2},\frac{11}{2};\frac{3}{2},4,\frac{15}{2};27x^2\right)+20\,{}_4F_3\left(\frac{11}{6},\frac{13}{6},\frac{5}{2},\frac{7}{2};\frac{1}{2},2,\frac{11}{2};27x\right.\right.$
	$-22\left(4x^2+5\right)\,{}_4F_3\left(\frac{13}{6},\frac{5}{2},\frac{17}{6},\frac{7}{2};\frac{3}{2},2,\frac{11}{2};27x^2\right)-28\,{}_4F_3\left(\frac{13}{6},\frac{5}{2},\frac{17}{6},\frac{9}{2};\frac{1}{2},3,\frac{13}{2};27x^2\right)+91\,{}_4F_3\left(\frac{5}{2},\frac{17}{6},\frac{19}{6},\frac{9}{2};\frac{3}{2},3,\frac{13}{2};27x^2\right)\Big)\Big)$
	$\sim -\pi\left(\frac{27}{8}x^4+\frac{4455}{16}x^6+\frac{14209195}{1024}x^8+O(x^{10})\right)$
$I_r^{\not\pi(3)}$	$-\frac{1}{36864x^7}\Big[50x^2\,{}_3F_2\left(-\frac{8}{3},-\frac{7}{3},1;-\frac{7}{2},-\frac{5}{2};27x^2\right)-576x^2\,{}_4F_3\left(-\frac{8}{3},-\frac{7}{3},1,1;-\frac{9}{2},-\frac{3}{2},2;27x^2\right)$
	$+1024x^2\,{}_4F_3\left(-\frac{7}{3},-\frac{5}{3},1,1;-\frac{7}{2},-\frac{3}{2},2;27x^2\right)-224x^2\,{}_4F_3\left(-\frac{5}{3},-\frac{4}{3},1,1;-\frac{7}{2},-\frac{3}{2},2;27x^2\right)$
	$+3\,{}_3F_2\left(-\frac{11}{3},-\frac{10}{3},1;-\frac{11}{2},-\frac{5}{2};27x^2\right)-6\,{}_3F_2\left(-\frac{10}{3},-\frac{8}{3},1;-\frac{9}{2},-\frac{5}{2};27x^2\right)-12288x^8\,{}_4F_3\left(-\frac{1}{3},\frac{1}{3},1,1;-\frac{3}{2},-\frac{1}{2},3;27x^2\right)$
	$+9216x^8\,{}_4F_3\left(\frac{1}{3},\frac{2}{3},1,1;-\frac{1}{2},-\frac{1}{2},3;27x^2\right)+1920x^6\,{}_4F_3\left(-\frac{4}{3},-\frac{2}{3},1,1;-\frac{5}{2},-\frac{3}{2},2;27x^2\right)$
	$-6912x^6\,{}_4F_3\left(-\frac{2}{3},-\frac{1}{3},1,1;-\frac{3}{2},-\frac{3}{2},2;27x^2\right)+46080x^6\,{}_4F_3\left(-\frac{2}{3},-\frac{1}{3},1,1;-\frac{3}{2},-\frac{1}{2},3;27x^2\right)$
	$-15360x^6\,{}_4F_3\left(-\frac{1}{3},\frac{1}{3},1,1;-\frac{3}{2},-\frac{1}{2},3;27x^2\right)+10752x^4\,{}_4F_3\left(-\frac{5}{3},-\frac{4}{3},1,1;-\frac{7}{2},-\frac{1}{2},3;27x^2\right)$
	$-5760x^4\,{}_4F_3\left(-\frac{5}{3},-\frac{4}{3},1,1;-\frac{5}{2},-\frac{3}{2},2;27x^2\right)+2400x^4\,{}_4F_3\left(-\frac{4}{3},-\frac{2}{3},1,1;-\frac{5}{2},-\frac{3}{2},2;27x^2\right)$
	$-15360x^4\,{}_4F_3\left(-\frac{4}{3},-\frac{2}{3},1,1;-\frac{5}{2},-\frac{1}{2},3;27x^2\right)+3840x^4\,{}_4F_3\left(-\frac{2}{3},-\frac{1}{3},1,1;-\frac{5}{2},-\frac{1}{2},3;27x^2\right)$
	$+6144x^8-8832x^6+2144x^4-218x^2+3\Big]$
	$\sim -\frac{2}{3}x^3-\frac{320}{3}x^5-6400x^7+O(x^9)$

WKB solution

- WKB-type solution of scalar wave equation is obtained

$$\ell = \frac{J}{\hbar} - \frac{1}{2}\,, \qquad \omega = \frac{E}{\hbar}, \quad R_{\ell m_\phi \omega}(r) = \frac{e^{i\sum_{k=0}^{\infty}\hbar^{k-1}S_k(r)}}{\sqrt{\Delta}}\,.$$

- So that for small $\hbar$

$$\left(\frac{dS_0(r)}{dr}\right)^2 = p_r^2\,.$$

- The solution of scalar waves equation describing purely ingoing waves at horizon is

$$R^{\rm in}_{\ell m_\phi \omega} \sim e^{-ikr_*}\,, \quad r \to r_+\,,$$

$$R^{\rm in}_{\ell m_\phi \omega} \sim B^{\rm inc}_{\ell m_\phi \omega}\frac{e^{-i\omega r_*}}{r} + B^{\rm ref}_{\ell m_\phi \omega}\frac{e^{i\omega r_*}}{r}\,, \quad r \to r_\infty\,,$$

- with $k = \omega - m_\phi a/2Mr_+$, $r_* = \int \frac{r^2+a^2}{\Delta}dr$ is the tortoise coordinate.
- The scattering phase is defined

$$e^{2i\delta_{\ell m_\phi \omega}} = (-1)^{\ell+1}\frac{B^{\rm ref}_{\ell m_\phi \omega}}{B^{\rm inc}_{\ell m_\phi \omega}}\,, \quad 2\delta_\ell^{WKB} = 2\frac{\mathcal{I}_r}{\hbar} + \ell\pi$$

- IN MST formalism and for equatorial scattering $m_\phi = \ell$ we found

$$\delta_{\ell m_\phi \omega} + \delta_{\ell m_\phi -\omega} = \pi(\nu - \ell)$$

- At the leading-order in a large ℓ expansion we have the following relation

$$\frac{(\nu - \ell)}{\ell} \underset{\ell\to\infty}{\sim} \sum_i G_i \hat{a}^i\,,$$

Final results

$$
\begin{aligned}
G_0(x) &= -\frac{15}{4}x^2 - \frac{1155}{64}x^4 - \frac{51051}{256}x^6 - \frac{47805615}{16384}x^8 - \frac{3234846615}{65536}x^{10} - \frac{957220521075}{1048576}x^{12} + \mathcal{O}(x^{14}) \\
&= {}_3F_2(-\frac{1}{2}, \frac{1}{6}, \frac{5}{6}; \frac{1}{2}, 1; 27x^2) - 1\,, \\
G_1(x) &= 5x^3 + \frac{693}{8}x^5 + \frac{109395}{64}x^7 + \frac{37182145}{1024}x^9 + \frac{13233463425}{16384}x^{11} + O(x^{13}) \\
&= 5x^3{}_3F_2(\frac{7}{6}, \frac{3}{2}, \frac{11}{6}; 2, \frac{5}{2}; 27x^2)\,, \\
G_2(x) &= -\frac{95}{16}x^4 - \frac{15939}{64}x^6 - \frac{16518645}{2048}x^8 - \frac{1970653685}{8192}x^{10} - \frac{1812984489225}{262144}x^{12} + \mathcal{O}(x^{14}) \\
&= \frac{1}{16}x^4\Big(- 11088x^2\,{}_3F_2\left(\frac{13}{6}, \frac{5}{2}, \frac{17}{6}; 2, \frac{7}{2}; 27x^2\right) - 360\,{}_3F_2\left(\frac{7}{6}, \frac{3}{2}, \frac{11}{6}; 1, \frac{5}{2}; 27x^2\right) - 5\,{}_3F_2\left(\frac{7}{6}, \frac{3}{2}, \frac{11}{6}; \frac{5}{2}, 3; 27x^2\right) \\
&\quad + \frac{5\left(\left(9x^2 - 47\right)\,{}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 3; 27x^2\right) - 7\left(189x^2 + 1\right)\,{}_2F_1\left(\frac{11}{6}, \frac{13}{6}; 3; 27x^2\right)\right)}{27x^2 - 1}\Big)\,, \\
G_3(x) &= \frac{27}{4}x^5 + \frac{4455}{8}x^7 + \frac{14209195}{512}x^9 + \frac{578013345}{512}x^{11} + O(x^{13}) \\
&= \frac{1}{128}x^5\Big(- 31680x^2\,{}_3F_2\left(\frac{13}{6}, \frac{17}{6}, \frac{7}{2}; 2, \frac{9}{2}; 27x^2\right) - 1408\,{}_3F_2\left(\frac{7}{6}, \frac{11}{6}, \frac{5}{2}; 1, \frac{7}{2}; 27x^2\right) \\
&\quad + 2560\,{}_5F_4\left(\frac{7}{6}, \frac{11}{6}, 2, 2, 2; 1, 1, 1, 3; 27x^2\right) - 2048\,{}_5F_4\left(\frac{7}{6}, \frac{11}{6}, 2, 2, \frac{5}{2}; 1, 1, 1, \frac{7}{2}; 27x^2\right) \\
&\quad + 70840x^2\,{}_2F_1\left(\frac{13}{6}, \frac{17}{6}; 4; 27x^2\right) + 1760\,{}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 3; 27x^2\right) \\
&\quad + 765765x^4\,{}_2F_1\left(\frac{19}{6}, \frac{23}{6}; 5; 27x^2\right)\Big)\,, \qquad (4.32)
\end{aligned}
$$

- So that the connection between the null geodesics radial action (expanded in powers of $\hat{a}$) to the RAM ratio $\frac{\nu - \ell}{\ell}$ expanded in powers of $\hat{a}$ is

$$
\frac{G_i(x) + \delta_{i\mathbf{0}}}{2x} = -I_r^{\pi(i)}(x)\,, \qquad i = 0, \ldots 4\,.
$$

Conclusions

- These resummed expressions in black hole perturbation theory, could improve so far the accuracy in (semi-)analytical simulations (for example, Effective-One-Body-based) of the two-bodies dynamics.

Outlooks

- The Schwarzschild massive radial action can be resummed at all orders in PM expansion ($\hat{b}$) and all orders in the probe mass (work in progress)
- Generalize the scalar wave (spin-0) results to the the generic spin s case.
- Generalize to the non planar case.

Thank you for attention!