RG flows between Rational CFTs

Non-invertible symmetries and integrability

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11th Bologna Workshop on Conformal Field Theory and Integrable Models
September 4th 2025

Based on [2501.07511] with Stefano Negro
+ WIP with Tomáš Procházka

A symmetry revival

Generalized Global Symmetries

Davide Gaiotto (Perimeter Inst. Theor. Phys.), Anton Kapustin (Stony Brook U.), Nathan Seiberg (Princeton, Inst. Advanced Study), Brian Willett (Princeton, Inst. Advanced Study)
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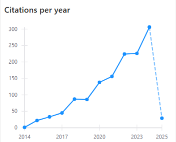

Lot of activity: plethora of generalization

p-form, higher-groups, non-invertible, emanent, mixed anomalies, gauging, SymTFT, higher-category ...

[Frolich, Fuchs, Runkel, Schwiegert, Tachikawa, Bhardwaj, Chang, Lin, Shao, Wang, Yin, Copetti, Cordova, Komatsu, Schäfer-Nameki, Bottini, Tiwari, Cordova, Dumitrescu, Intriligator, Benini, Antinucci] $+ \ldots\ldots \infty$

Crucial observation:

Symmetry = Topological Operator

Symmetries strike back

Symmetry: most important guiding principle in physics

> More symmetries = More non-perturbative data

A lot new information: e.g.

- Constraints on RG flows and S-Matrix:
  $\longrightarrow$ Phases of 2d Adjoint QCD, Modified crossing...
  [Copetti, Cordova, Komatsu][Komargodski, Ohmori Seifnashri, Roumpedakis][Tanaka, Nakayama]
- Organizing principles
  $\longrightarrow$ Particle-soliton degeneracy[Cordova, García-Sepúlveda, Holfester]
- Explain experimental data
  $\longrightarrow$ Pion decay $\pi^0 \to \gamma\gamma$, axion coupling [Cordova, Shao. . .]

Today's talk

What can we learn by having the full symmetry data?

Usually hard in QFT: need all top. ops. , fusion, anomalies...

Rational 2d CFT: we know all topological defects

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What can we learn by having the full symmetry data?

Usually hard in QFT: need all top. ops. , fusion, anomalies...

Rational 2d CFT: we know all topological defects

This Talk:

- Predict new RG flows
  - $\rightarrow$ Virasoro[FA, Negro][Tanaka, Nakayama] & $\mathcal{W}$ algebra [FA, Procházka]
  - $\rightarrow$ Putative NLIE description [FA, S. Negro]

Next talk:

- construction new commuting non-local conserved charges $\leftrightarrow$ **Perturbed** Verlinde lines [Runkel] [Ambrosino, Runkel, Watts]
  $\rightarrow$ Novel criterion for integrability?

Virasoro Minimal Models RG flows

A primer on Virasoro Minimal models

$\mathcal{M}(p,q)$: Rational 2D CFT, A-series

- Central charge: $c = 1 - \frac{6(p-q)^2}{pq}$ Unitary iff $q = p+1$
- $\frac{(p-1)(q-1)}{2}$ primaries $\phi_{(r,s)} = \phi_{(p-r,q-s)}$
- weigths $h_{(r,s)}$ and C^t_{rs} all known
- Fusion Category: $\phi_\rho \otimes \phi_\sigma = \sum_\delta \mathcal{N}^\delta_{\rho\sigma}\, \phi_\delta$

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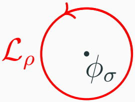
Topological lines = Verlinde Lines $\{\mathcal{L}_\rho\} \overset{\text{1 to 1}}{\Longleftrightarrow} \{\phi_\rho\}$

Non-invertible symmetry: $\mathcal{L}_\rho \times \mathcal{L}_\sigma = \sum_\delta \mathcal{N}^\delta_{\rho\sigma}\, \mathcal{L}_\delta$

Ward identity:



$= \left[\frac{S_{\rho\sigma}}{S_{0\sigma}}\right] \quad \phi_\sigma$

2D Ising CFT $\mathcal{M}(3,4)$

$c = \frac{1}{2}$, Primaries : $1_{0,0}, \epsilon_{\frac{1}{2},\frac{1}{2}}, \sigma_{\frac{1}{16},\frac{1}{16}}$, TLD : $\mathbf{1}, \eta, \mathcal{N}$

Fusion Algebra TY_2: $\eta^2 = \mathbf{1}, \quad \mathcal{N} \times \eta = \mathcal{N}, \quad \mathcal{N}^2 = \mathbf{1} + \eta$

$\eta \longleftrightarrow \mathbb{Z}_2$ $\qquad \mathcal{N} \longleftrightarrow$ KW duality defect

Duality in QFT $\qquad \mathrm{KW} : \mathrm{Ising}_T \longleftrightarrow \mathrm{Ising}_{T^{-1}}$

Symmetry in CFT $\qquad \mathrm{KW} : \mathrm{Ising}_{T_c} \circlearrowleft$

TDL **honest** symmetries of CFT

Derfomation by primary field

Relevant deformation of minimal models

$$\mathcal{T}_{\mathrm{UV}} = \mathcal{M}(p,q) + g_\rho \int \phi_\sigma \ , \qquad (h_\rho < 1)$$

Gapped Gapless

$\mathcal{T}_{\mathrm{IR}} =$ TQFT CFT

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TODAY

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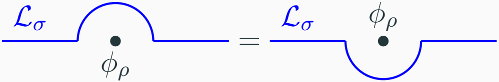
TODAY

c-theorem with: $c_{\mathrm{eff}} = 1 - \frac{6}{pq}$ $(\mathcal{PT}$-sym) [Ravanini]

Invariants under RG flows

When: $[\mathcal{L}_\sigma, \phi_\rho] \,|\Phi\rangle = 0 \Leftrightarrow$



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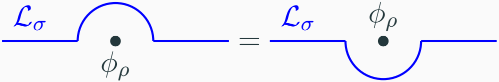
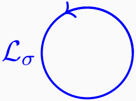
When: $[\mathcal{L}_\sigma, \phi_\rho] |\Phi\rangle = 0 \Leftrightarrow$ 

Then: preserved symmetry along flow &

[Chang, Lin, Shao, Wang, Yin] [Nakayama, Tanaka]

<u>Quantum dimension</u> $= \langle 0| \mathcal{L}_\sigma |0\rangle = d_\rho =$ 

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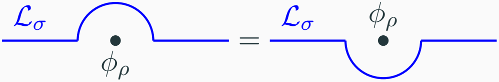

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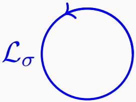
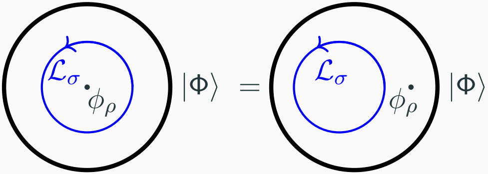


RG invariant!



Impose anomaly matching

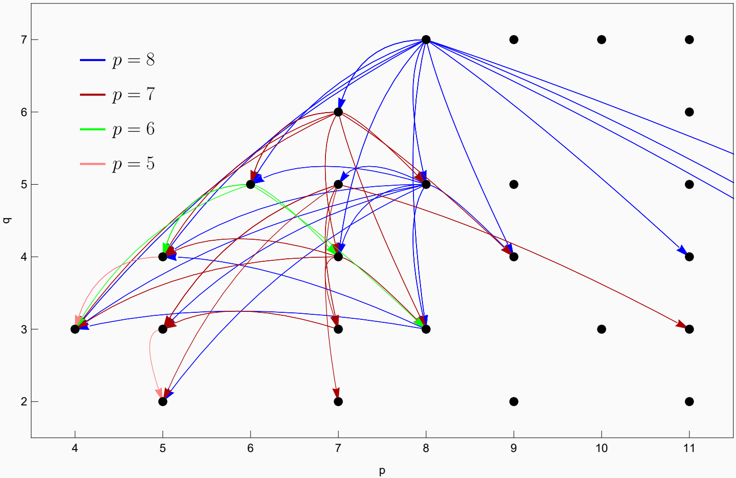
Strategy: Fix $\mathcal{T}_{UV} = \mathcal{M}(p,q)$

1. For any relevant def. compute all RG invariants
2. Generate all $\mathcal{M}(p',q')$ with $c_{\mathrm{eff}}^{\mathrm{IR}} < c_{\mathrm{eff}}^{\mathrm{UV}}$
3. Select putative $\mathcal{T}_{\mathrm{IR}}$ fullfliing all anomalies

Produces flows:

$$\mathcal{M}(p,q) \xrightarrow{\phi_{(r,s)}} \mathcal{M}(p',q')$$

Result



Main series: $\mathcal{M}(kp + l, q) \xrightarrow{\phi_{(1,2k+1)}} \mathcal{M}(kp - l, q)$ [Nakayama, Tanaka]

New flows

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Integrable flows:

- $k,l=1$: $\phi_{(1,3)}$ [Fendley, Saleur, Al. Zamolodchikov][Al. Zamolodchikov]
- $k=1/2, k=2$: $\phi_{(1,2)}, \phi_{(1,5)}$ [Dorey, Dunning, Tateo]
- $k=3$: $\phi_{(1,7)}$ $\qquad \mathcal{M}(3,10) \to \mathcal{M}(3,8)$ [Narovlansky, Sun, Tarnopolsky]
- $\dots$

Preserve $SU(2)_{q-2}$ fusion category:

$$\{\mathcal{L}_{(1,1)} = \mathbf{1}, \mathcal{L}_{(2,1)}, \cdots, \mathcal{L}_{(q-1,1)}\}\,, \qquad \mathcal{L}_{(q-1,1)} \times \mathcal{L}_{(q-1,1)} = \mathbf{1}$$

$$[\mathcal{L}_{(n,1)}, \phi_{(1,2m+1)}] = 0\,, \qquad m = 1, \cdots, k$$

To $(\mathcal{W})$ infinity and beyond!

Virasoro algebra: generated by modes of spin 2 field $T(z)$

$\mathcal{W}_N$ algebra: additional currents of spin $s = 2, \cdots, N$

$$\{T(z), W^{(3)}, \cdots, W^{(N)}\}$$

[Zamolodchikov², Fateev, Lukyanov. . .]

$$T(z)W^{(s)}(u) \sim \frac{sW^{(s)}(u)}{(z-u)^2} + \frac{\partial W^{(s)}(u)}{z-u} + \cdots .$$

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$$T(z)W^{(s)}(u) \sim \frac{sW^{(s)}(u)}{(z-u)^2} + \frac{\partial W^{(s)}(u)}{z-u} + \cdots .$$

E.g. $\mathcal{W}^{(3)}$:

$$W(z)W(u) \sim \frac{c/3}{(z-u)^6} + \frac{2T(u)}{(z-u)^4} + \frac{\partial T(u)}{(z-u)^3} + \frac{1}{(z-u)^2}\left(\frac{3}{10}\partial^2 T(u) + \frac{32}{22+5c}\Lambda(u)\right) + \cdots$$

$\mathcal{W}_N$ Minimal models

Also $\mathcal{W}_N$ admit minimal truncations:

$\mathcal{W}_N$ minimal models: $\mathcal{W}_N(p,q)$

- Rational CFT: $c^{(N)}_{p,q} = (N-1)\left[1 - \frac{N(N+1)(p-q)^2}{pq}\right]$
- Unitary iff $p - q = \pm 1$
- Primaries: pairs $(\mathrm{A}_N, \mathrm{A}_N)$ reps / Identifications:

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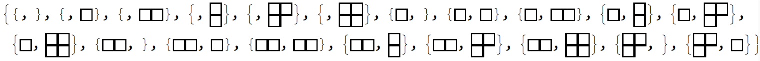
  E.g. for $\mathcal{W}_3(4,5)$ (simplest)



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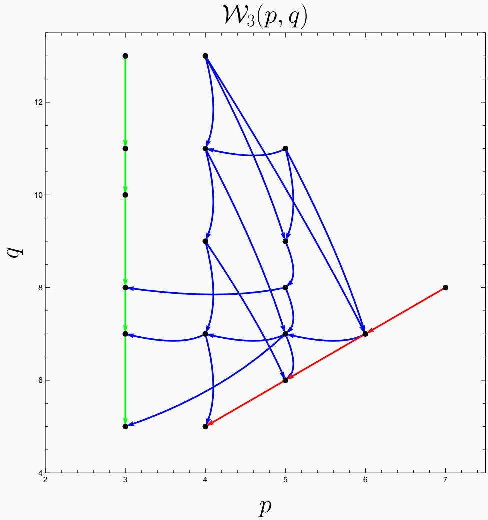
Can we conjecture flows?

YES! Using non-invertible sym [WIP with Prochazka & Negro]

A quick preview: example for $\mathcal{W}_3$



- Unitary series: [Poghossian, Poghosyan]
  - $\mathcal{W}_3(q+1,q) \to \mathcal{W}_3(q,q-1)$
  - Triggered by $(\cdot, \boxplus) \sim \phi_{(1,3)}$
- Extend to new ∞ families:
  - $\mathcal{W}_3(kq+l,q) \to \mathcal{W}_3(kq-l,q)$
  - Def. $(\cdot, \text{[Young diagram]}) \sim \phi_{(1,5)} \cdots$
- Commutes with $SU(N)_{q-N}$
  - $\# = \binom{q-1}{q-N}$: $(\star\star\star,\cdot)$

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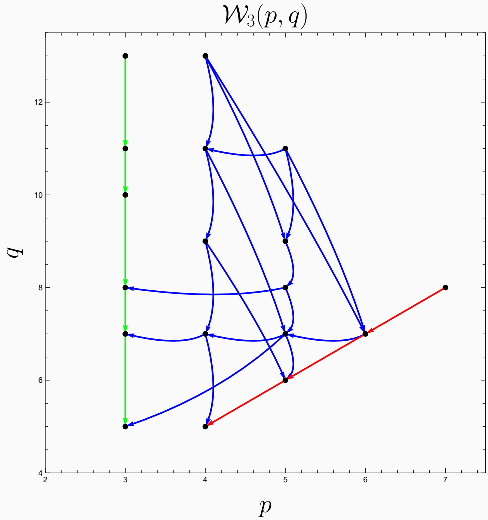

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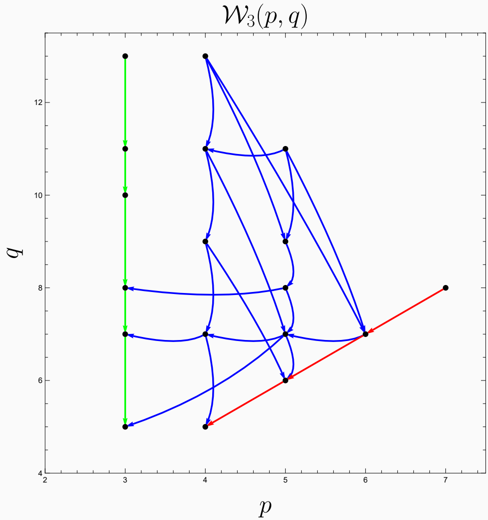
  - E.g. $SU(3)_3$:

{{, }, {□, }, {□□, }, {[Young diagram], }, {[Young diagram], }, {[Young diagram], }}

A quick preview: example for $\mathcal{W}_3$



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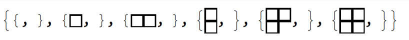

Generalizes to $\mathcal{W}_N$!

New questions

We found:

- New flows
- Putative non-perturbative description

What is special about flows with $\phi_{(1,r)}$?

Remember: $r = 2, 3, 5$ are integrable: red. of BD and SG

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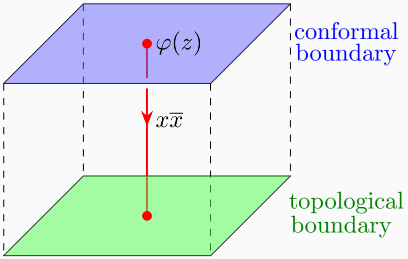
We found:[Ambrosino, Runkel, Watts]

Non-top. defects = Non-local conserved charges

Sym TFT for 2d RCFT

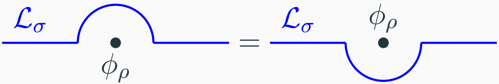
TQFT_{d+1} encoding all symmetries of QFT_d

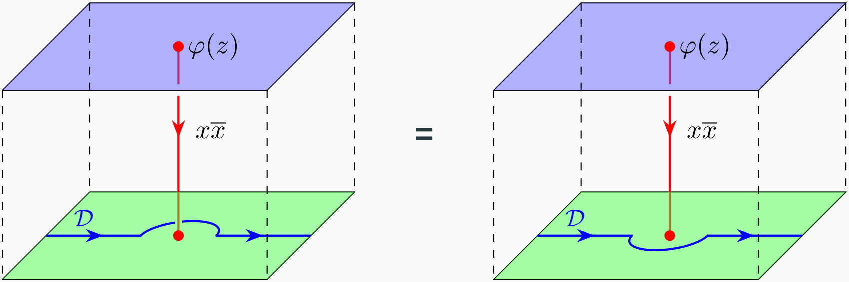


- Topological ops in QFT: Neumann @ top boundary
- Non topological: Dirichlet boundary conditions

All lines = topological anyons of 3D TFT

Example: commutation condition

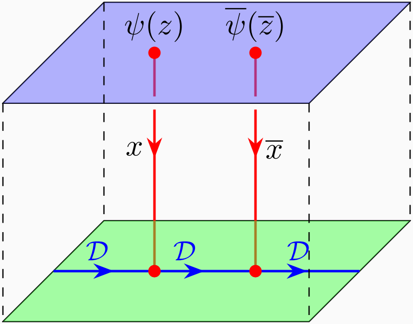




Perturbed defect operators

Perturb a topological defect $\mathcal{D}$ by (chiral) defect operators:

$$\underbrace{\psi(z)}_{\text{hol}}, \qquad \underbrace{\overline{\psi}(\bar{z})}_{\text{antihol.}}, \qquad \psi, \overline{\psi} \in \mathcal{D}$$



$$\mathcal{D}(\lambda\psi + \widetilde{\lambda}\overline{\psi}) = \exp\left(\int_0^L (\lambda\psi(s+it) + \tilde{\lambda}\bar{\psi}(s+it))\, ds\right),$$

s coordinate along defect, t position of defect

Commuting defects and where to find them

Hamiltonian of perturbed CFT: $H(\mu) = H_0 + H_{\text{pert}}(\mu)$

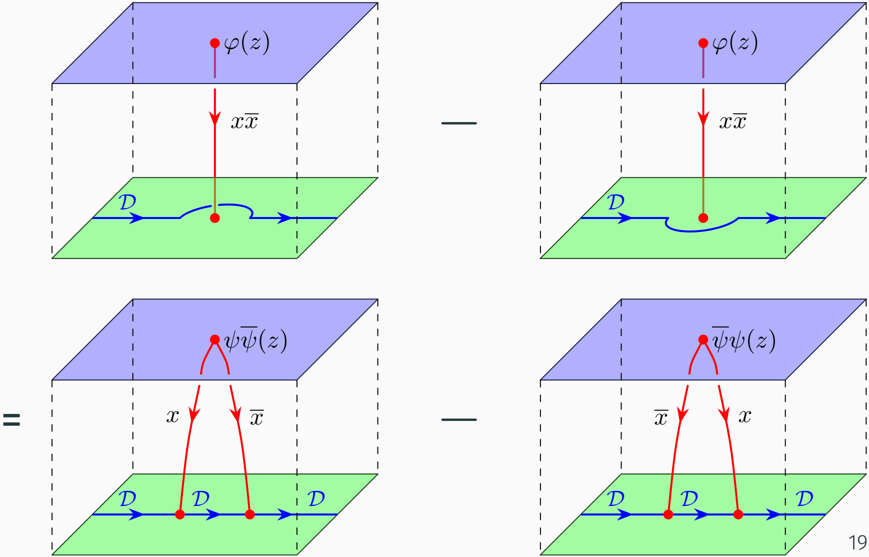
$$H_0 = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) \;, \quad H_{\text{pert}}(\mu) = 2i\mu \int_0^L \varphi(s) ds \;.$$

For $\mathcal{D}$ topological in H_0, we want:

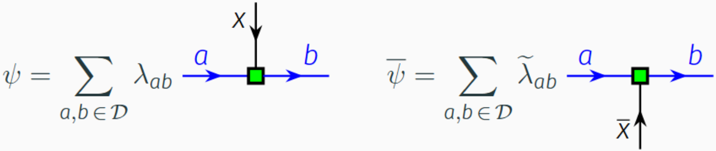
$$\left[H(\mu), \mathcal{D}(\lambda\psi + \overline{\lambda\psi})\right] = 0, \qquad \varphi \sim \psi\overline{\psi}$$

Non topological but rigid translation-invariant

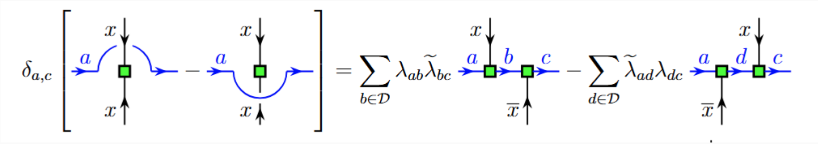
Condition for translation-invariant



Conditions



$\psi = \sum_{a,b \in \mathcal{D}} \lambda_{ab}$
$\overline{\psi} = \sum_{a,b \in \mathcal{D}} \widetilde{\lambda}_{ab}$



$\delta_{a,c}$
$= \sum_{b \in \mathcal{D}} \lambda_{ab} \widetilde{\lambda}_{bc}$
$- \sum_{d \in \mathcal{D}} \widetilde{\lambda}_{ad} \lambda_{dc}$

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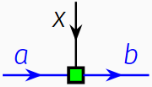
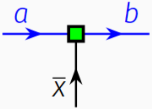
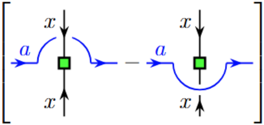
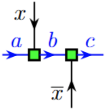
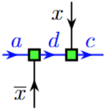
$\psi = \sum_{a,b\in\mathcal{D}} \lambda_{ab}$ 

$\overline{\psi} = \sum_{a,b\in\mathcal{D}} \widetilde{\lambda}_{ab}$ 

$\delta_{a,c}$ 

$= \sum_{b\in\mathcal{D}} \lambda_{ab}\widetilde{\lambda}_{bc}$ 

$- \sum_{d\in\mathcal{D}} \widetilde{\lambda}_{ad}\lambda_{dc}$ 

$$\delta_{ac}\left(\mathrm{R}^{(ax)b} - \frac{1}{\mathrm{R}^{(ax)b}}\right)\mathrm{F}^{(axx)a}_{1b} = \delta_{b\in\mathcal{D}}\,\lambda_{ab}\widetilde{\lambda}_{bc} - \sum_{d\in\mathcal{D}} \mathrm{F}^{(xax)c}_{db}\,\widetilde{\lambda}_{ad}\lambda_{dc}$$

Solutions of conditions for minimal models

$$\forall a, c \in \mathcal{D}, \qquad b \in a \otimes x$$

$$\delta_{ac} \left(\mathrm{R}^{(ax)b} - \frac{1}{\mathrm{R}^{(ax)b}} \right) \mathrm{F}^{(axx)a}_{1b} = \delta_{b\in\mathcal{D}}\, \lambda_{ab} \widetilde{\lambda}_{bc} - \sum_{d\in\mathcal{D}} \mathrm{F}^{(xax)c}_{db}\, \widetilde{\lambda}_{ad} \lambda_{dc}$$

Give condition on $\lambda, \widetilde{\lambda}$ in $\mathcal{T}(\mu) = \mathcal{M}(p,q) + \mu \int \phi$

Sols of the form $\lambda \cdot \widetilde{\lambda} = \mu$ for given p, q

Solutions in Minimal models

Minimal model $M(p,q)$	perturbing bulk field φ	weights $h=\bar{h}$ of φ	topological defect solving (4)
$q \geq 3$	$(1,2)$	$h_{1,2} = \frac{3}{4}t - \frac{1}{2}$	$(1,1) \oplus (1,2)$
$q \geq 4$	$(1,3)$	$h_{1,3} = 2t - 1$	$(1,2)$
$q \geq 6$	$(1,5)$	$h_{1,5} = 6t - 2$	$(1,3)$
$q = 9, 10, 18$	$(1,7)$	$h_{1,7} = 12t - 3$	$(1,5)$

$$\left[\mathcal{D}(\lambda, \mu/\lambda), \mathcal{D}(\lambda', \mu/\lambda')\right] = 0, \qquad \lambda\lambda'$$

Infinitely many non-local conserved charges!

$(1,7)$ deformation has no local conserved charges [BLZ]

Solutions in Minimal models

Minimal model $M(p,q)$	perturbing bulk field φ	weights $h=\bar{h}$ of φ	topological defect solving (4)
$q \geq 3$	$(1,2)$	$h_{1,2} = \frac{3}{4}t - \frac{1}{2}$	$(1,1) \oplus (1,2)$
$q \geq 4$	$(1,3)$	$h_{1,3} = 2t - 1$	$(1,2)$
$q \geq 6$	$(1,5)$	$h_{1,5} = 6t - 2$	$(1,3)$
$q = 9, 10, 18$	$(1,7)$	$h_{1,7} = 12t - 3$	$(1,5)$

$$\left[\mathcal{D}(\lambda, \mu/\lambda), \mathcal{D}(\lambda', \mu/\lambda')\right] = 0, \qquad \lambda\lambda'$$

Infinitely many non-local conserved charges!

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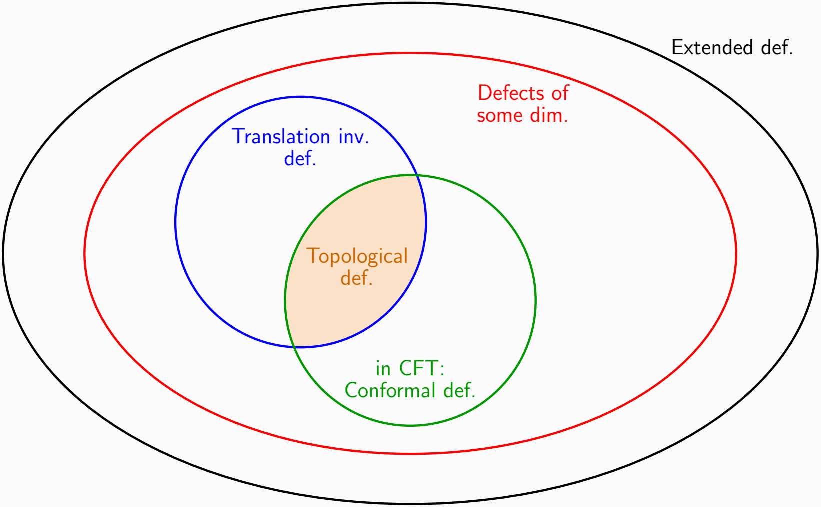
Generalizes to any (1,r)

General picture



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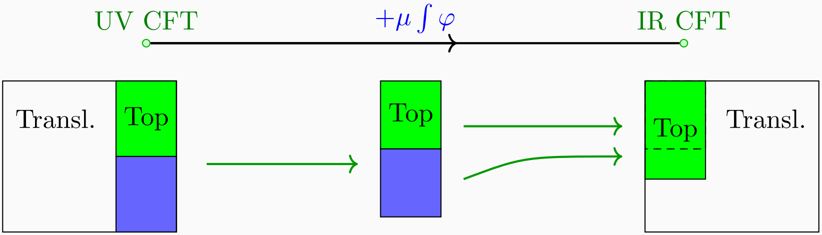

Non perturbative description via TBA-like equations

NLIE description

Integrable flows admit NLIE/TBA encoding energy levels non-perturbatively[Zamoldchikov, Dorey, Dunning, Tateo]. Ground state

$$f_s(r) = \sum_{\sigma=\pm} \frac{3ir\sigma}{2\pi^2} \int_{\mathcal{C}_s^\sigma} d\theta \left[e^{-\theta} L_L^\sigma(\theta) - e^{\theta} L_R^{-\sigma}(\theta)\right], \qquad f_s(r) = 6RE_s(R)/\pi$$

$$L^{\pm}_{\substack{R\\L}}(\theta) = \log\left[1 + \exp\left(\pm f_{\substack{R\\L}}(\theta)\right)\right]$$

$$f_{R/L}(\theta) = i\alpha' - i\frac{r}{2}e^{\pm\theta} \mp \sum_{\sigma=\pm} \sigma \int_{\mathcal{C}_s^\sigma} d\theta' \left[\phi(\theta-\theta') L_R^{\mp\sigma}(\theta') + \chi(\theta-\theta') L_L^{\pm\sigma}(\theta')\right]$$

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Assume that structure persist: Can we find $\chi(\omega), \phi(\omega)$?

Bootstrapping kernels

Assume kernels $\chi(\omega), \phi(\omega)$:

- Trigonometric form (analogy with integrable case)
- Decay at $\pm\infty \oplus$ Analytic properties
  (poles at $\pm i$ & non-van. @ 0)
- Scaling limits $f_s(r \to 0) \propto c_{\mathrm{eff}}^{\mathrm{UV}}$ & $f_s(r \to \infty) \propto c_{\mathrm{eff}}^{\mathrm{IR}}$

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$\Rightarrow$2-parameter family κ, ξ

$$\phi(\theta) = -\int_{\mathbb{R}} \frac{d\omega}{2\pi} e^{i\theta\omega} \frac{\sinh\left(\frac{1}{\kappa}\pi\omega\right)\cosh\left(\frac{2\xi-\kappa}{2\kappa}\pi\omega\right)}{\sinh\left(\frac{\xi-1}{\kappa}\pi\omega\right)\cosh\left(\frac{1}{2}\pi\omega\right)},$$

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ξ, κ encode central charges:

$c_{\rm eff}^{\rm UV}(p,q) = 1 - 3\left(\frac{\alpha'}{\pi}\right)^2 \frac{(\xi-1)^2}{\xi(\xi+1)}$

$c_{\rm eff}^{\rm IR}(p',q') = 1 - 3\left(\frac{\alpha'}{\pi}\right)^2 \frac{\xi-1}{\xi}$

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Obey Diofantine quantization conditions

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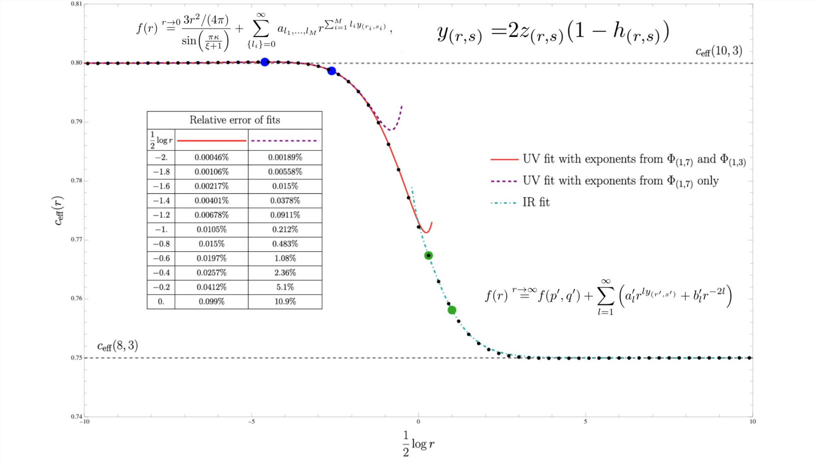
Varying quantized ξ, κ recovers RG flows!

Example: $\mathcal{M}(3,10) \to \mathcal{M}(3,8)$

Recent interest in this flow [Klebanov et al] [Miro' et al]

Solutions of our NLIE match UV and IR expansion



Relative error of fits

$\frac{1}{2}\log r$	UV fit with exponents from $\Phi_{(1,7)}$ and $\Phi_{(1,3)}$ (solid red line)	UV fit with exponents from $\Phi_{(1,7)}$ only (dashed purple line)
−2.	0.00046%	0.00189%
−1.8	0.00106%	0.00558%
−1.6	0.00217%	0.015%
−1.4	0.00401%	0.0378%
−1.2	0.00678%	0.0911%
−1.	0.0105%	0.212%
−0.8	0.015%	0.483%
−0.6	0.0197%	1.08%
−0.4	0.0257%	2.36%
−0.2	0.0412%	5.1%
0.	0.099%	10.9%

Concluding remarks

News from Minimal Models!![BPZ, 1984]

- New flows predicted by non-invertible symmetries
  - → From Virasoro to $\mathcal{W}_N$ and beyond! 
  - → Truncations of $\mathcal{W}_{1+\infty}$: $Y_{N,0,0} \cap Y_{0,p,q}$, compact irrational CFT [Antunes, Behan, Rong], etc dots...
  - → Mapping the space of RCFTs
- Putative new NLIE
- Virasoro ADE classification[Cappelli,Itzykson,Zuber][Nakayama,Tanaka]
  - → Modular invariant classification for $\mathcal{W}_N$?
- We need to go beyond the topological defect framework!
- From 2d to 4d and back
- Novel integrable flows? New non-local charges

Thanks for your attention!