Painlevé equations, gauge theories and gravity

An integrable models perspective

Marco Rossi

Università della Calabria and INFN Cosenza

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Painlevé equations

- In the 19th century special functions were introduced: Airy, Bessel, Hypergeometric functions. They are all solutions of linear second order ODEs with at most poles in the Riemann sphere.

- At the end of the 19th century elliptic functions were defined. They satisfy nonlinear ODEs

- Painlevé aimed to define new special functions as solutions of nonlinear ODEs. In general, solutions of nonlinear ODES have movable singular points, i.e. singular points whose position and type depend on initial conditions. For instance elliptic functions have movable poles.

- Painlevé property is to allow, as in the case of elliptic functions, only movable poles (no movable branch points) and fixed singular points of any type.

- Degree one second order ODEs $\ddot{q} = F(\dot{q}, q, s)$, with F rational function of $\dot{q}, q$, whose solutions $q(s)$ satisfy Painlevé property have been classified by Painlevé and Gambier in 1900-1910: six types of Painlevé equations

Plan of the talk: explore connections with gravity and SYM

• By using a Lax pair presentation, associate to a Painlevé equation a second order differential equation

• The ODE obtained in this way appears to coincide with the ODE appearing when the metric of some massive object, like a black hole (BH), is subjected to the small perturbation.

• The same ODEs realise quantisations of Seiberg-Witten differentials for $\mathcal{N} = 2$ SYM in the Nekrasov-Shatashvili background.

• We will study and construct solutions of these ODEs and their connection coefficients.

• We will discuss Painlevé III_3 and VI.

Painlevé III_3 equation

One way to introduce Painlevé equations is through a Lax pair. For Painlevé III_3 the Lax pair is

$$(\partial_z - \mathbf{A}_z)\psi(s,z) = 0\,, \quad (\partial_s - \mathbf{A}_s)\psi(s,z) = 0$$

$$\mathbf{A}_z = \begin{pmatrix} \frac{pq}{z} & 1 - \frac{s}{zq} \\ \frac{1}{z} - \frac{q}{z^2} & -\frac{pq}{z} \end{pmatrix}\,, \quad \mathbf{A}_s = \begin{pmatrix} 0 & \frac{1}{q} \\ \frac{q}{sz} & 0 \end{pmatrix}\,.$$

q is a function of s, $q(s)$

$$\dot{q} = \frac{dq}{ds}\,, \quad pq = \frac{1}{2} - \frac{s\dot{q}}{2q}$$

From the equality $\partial_z\partial_s\psi(s,z) = \partial_s\partial_z\psi(s,z)$, which means

$$[\partial_z - \mathbf{A}_z, \partial_s - \mathbf{A}_s] = 0\,,$$

one gets for $q(s)$ the constraint

$$\ddot{q} = \frac{\dot{q}^2}{q} - \frac{\dot{q}}{s} + \frac{2q^2}{s^2} - \frac{2}{s}\,,$$

which is Painlevé III_3 equation. Symmetry: $q \to s/q$.

Going on a singularity

- Define t, time, from $s = (t/8)^4$. Around a movable double zero $t = \bar{t}$

$$q(t) = \frac{\bar{t}^2}{256}(t-\bar{t})^2 + \frac{3\bar{t}}{256}(t-\bar{t})^3 + \kappa(t-\bar{t})^4 + O(t-\bar{t})^5$$

$\bar{t}, \kappa$ determined by initial conditions

- Consider the equation $(\partial_z - \mathbf{A}_z)\psi(s,z) = 0$ in the limit $t \to \bar{t}$. For the two components $\psi(s,z) = \begin{pmatrix} \psi_1(s,z) \\ \psi_2(s,z) \end{pmatrix}$ we find in the variable $y = \ln z$ the limiting equation

$$\frac{d^2\psi_i}{dy^2} = \left(2e^{2\theta}\cosh y + P^2\right)\psi_i(y)\,, \quad \bar{t} = 8e^{\theta}\,, \quad P^2 = -\frac{35}{64} + 48\kappa\,,$$

which is the Modified Mathieu Equation.

M. Bershtein, P. Gavrylenko, A. Grassi, (2022)

Remark: two irregular singular points at $y = \pm\infty$.

- It is the quantisation of Seiberg-Witten differential of $\mathcal{N} = 2$ SYM with no matter in Nekrasov-Shatashvili background.

Basis of solutions

A basis of solutions is (U_0, V_0), where

$$V_0(y) \simeq \frac{1}{\sqrt{2}} \exp\left(-\frac{\theta}{2} + \frac{y}{4}\right) \exp\left(-2e^{\theta - \frac{y}{2}}\right) , \quad y \to -\infty$$

and $U_0(y) = V_0(-y)$.

Another basis is given by Floquet solutions $\psi_\pm(y + 2\pi i) = e^{\pm 2\pi ik}\psi_\pm(y)$.
k is the Floquet index and $\psi_\pm(y) = \psi_\mp(-y)$.

They diverge at infinity:

- when $y \to -\infty$

$$\psi_\pm(y) \simeq e^{\mp\frac{\varphi}{2}} e^{\frac{y}{4}} \exp\left(2e^{\theta} e^{-\frac{y}{2}}\right)$$

- when $y \to +\infty$

$$\psi_\pm(y) \simeq e^{\pm\frac{\varphi}{2}} e^{-\frac{y}{4}} \exp\left(2e^{\theta} e^{\frac{y}{2}}\right)$$

Connecting the solutions

The connection between U_0, V_0 and $\psi_\pm$ is D. Fioravanti, M.R. '24

$$U_0(y) = \frac{\sqrt{2}e^{\frac{\theta}{2}}}{W[\psi_+,\psi_-]}\left[e^{-\frac{\varphi}{2}}\psi_+(y;k) - e^{\frac{\varphi}{2}}\psi_-(y;k)\right] ,$$

$$V_0(y) = \frac{\sqrt{2}e^{\frac{\theta}{2}}}{W[\psi_+,\psi_-]}\left[e^{-\frac{\varphi}{2}}\psi_-(y;k) - e^{\frac{\varphi}{2}}\psi_+(y;k)\right]$$

Connection depends on the Wronskian $W[\psi_+,\psi_-] = -4e^{\theta}\sin 2\pi k$ and on

$$\varphi = \int_{-\infty}^{0} dy\left(\Pi_+(y) + e^{\theta}e^{-\frac{y}{2}} - \frac{1}{4}\right) + \int_{0}^{+\infty} dy\left(\Pi_+(y) - e^{\theta}e^{\frac{y}{2}} + \frac{1}{4}\right) , \quad \Pi_+ = \frac{d}{dy}\ln\psi_+(y)$$

the phase acquired by ψ_+ in going from $y=-\infty$ to $y=+\infty$.

$\Pi_+(y)$ satisfies the Riccati equation

$$\Pi_+(y)^2 + \frac{d}{dy}\Pi_+(y) = 2e^{2\theta}\cosh y + P^2$$

We want to compute k,φ when $\theta\to-\infty$. Easy for $k = \int_0^{2\pi i}\frac{dy}{2\pi i}\Pi_+(y) = P - \frac{e^{4\theta}}{P(4P^2-1)} + \ldots$, but φ has problem of conflicting $y\to+\infty$ and $\theta\to-\infty$ limits.

Computing the solutions and the connection coefficients

- Write $\varphi = \varphi_< + \varphi_>$, with

$$\varphi_< = \int_{-\infty}^{0} dy \left(\Pi_+(y) + e^{\theta} e^{-\frac{y}{2}} - \frac{1}{4} \right) , \quad \varphi_> = \int_{0}^{+\infty} dy \left(\Pi_+(y) - e^{\theta} e^{\frac{y}{2}} + \frac{1}{4} \right)$$

- Concentrating on $\varphi_>$, rewrite it in terms of $\Pi_>(y) = \Pi_+(y - 2\theta)$ as

$$\varphi_> = \int_{2\theta}^{+\infty} dy \left(\Pi_>(y) - e^{\frac{y}{2}} + \frac{1}{4} \right) .$$

- Solve the Riccati equation satisfied by $\Pi_>(y)$ *in terms of* θ, k

$$\Pi_>(y)^2 + \frac{d}{dy}\Pi_>(y) = e^y + P^2(k) + e^{4\theta} e^{-y} ,$$

by expanding $P^2(k) = \sum_{n=0}^{+\infty} p_2^{(n)}(k) e^{4n\theta}$ and $\Pi_>(y) = \sum_{n=0}^{+\infty} \Pi_>^{(n)}(y) e^{4n\theta}$ when $y > 2\theta$.

- One finds a system of first order ODEs

$$\Pi_>^{(0)}(y)^2 + \frac{d}{dy}\Pi_>^{(0)}(y) = e^y + k^2 , \quad 2\Pi_>^{(0)}(y)\Pi_>^{(1)}(y) + \frac{d}{dy}\Pi_>^{(1)}(y) = e^{-y} + \frac{2}{4k^2 - 1}$$

$$\sum_{m=0}^{n} \Pi_>^{(m)}(y)\Pi_>^{(n-m)}(y) + \frac{d}{dy}\Pi_>^{(n)}(y; k) = p_2^{(n)}(k)$$

Solutions

The first ODE has solution $\Pi^{(0)}_{>}(y) = \frac{d}{dy} \ln J_{2k}(2ie^{\frac{y}{2}})$. For $n \geq 1$ explicit solution for $n = 1, 2$ brings to conjecture

$$\Pi^{(n)}_{>}(y) = \frac{d}{dy}\left[P^{(n)}_0(e^{-y}) + \sum_{m=1}^{n} P^{(n)}_m(e^{-y})\frac{d^{m-1}}{dy^{m-1}}\Pi^{(0)}_{>}(y;k)\right],$$

with $P^{(n)}_m(e^{-y})$ degree n polynomials. E.g.
$P^{(1)}_0(x) = x/(1-4k^2)$, $P^{(1)}_1(x) = 2x/(1-4k^2)$.

The wave function $\psi_{>}(y) = \exp\int^y \Pi_{>}(y')dy'$ is then reconstructed for $y > 2\theta$ and this gives $\psi_{+}(y) = \psi_{>}(y+2\theta)$ for $y > 0$.

Passing to $\varphi_<$:

$$\varphi_< = \int_{-\infty}^{0} dy \left(\Pi_+(y) + e^{\theta} e^{-\frac{y}{2}} - \frac{1}{4} \right)$$

We define $\Pi_<(y) = \Pi_+(y + 2\theta)$

$$\varphi_< = \int_{-\infty}^{-2\theta} dy \left(\Pi_<(y) + e^{\theta} e^{-\frac{y}{2}} - \frac{1}{4} \right)$$

which satisfies

$$\Pi_<(y;\theta,P)^2 + \frac{d}{dy}\Pi_<(y;\theta,P) = e^{-y} + P^2(k) + e^{4\theta} e^{y} .$$

Symmetry $\Pi_<(y;\theta,k) = -\Pi_>(-y;\theta,-k)$ holds. Then, one finds $\psi_<(y)$ for $y < -2\theta$ and this gives $\psi_+(y) = \psi_<(y - 2\theta)$ for $y < 0$. The relation $\psi_-(y) = \psi_+(-y)$ gives the other Floquet.

(Partial) conclusions

- We reconstruct the wave function as a series of powers of $e^{4\theta}$.
- We express the connection coefficient $\varphi = \varphi_> + \varphi_<$ as a series of powers of $e^{4\theta}$:

$$\varphi(\theta,k) = -4k\theta + \ln \frac{\Gamma(1+2k)}{\Gamma(1-2k)} + \frac{8k}{(1-4k^2)^2} e^{4\theta} + O(e^{8\theta})$$

- Claim: $\varphi = A_D/\hbar$, $k = a/\hbar$ (periods of $\mathcal{N} = 2$ SYM without matter in the Nekrasov-Shatashvili background).

Painlevé VI

If one goes to a movable pole of Painlevé VI equation one finds that the second Lax operator in $z = e^y$ becomes the Heun Equation $\frac{d^2}{dy^2}\psi(y) = V(y)\psi(y)$,

$$V(y) = -\frac{1}{4\left(e^{\theta} - 4e^{\frac{\theta}{2}}\cosh y + 4\right)^2}\Bigg[-16\left(e^{\theta}+4\right)P^2 - 16e^{\theta} + 24e^{\theta}\left(q_1q_2+q_3q_4\right) -$$

$$-e^{2\theta}\left(q_1^2+q_2^2+q_3^2+q_4^2\right) +$$

$$+4e^{\frac{\theta}{2}+y}\left(\frac{e^{\theta}}{2} + e^{\theta}q_1^2 - \left(e^{\theta}+8\right)q_2q_1 + e^{\theta}q_2^2 - e^{\theta}q_3q_4 + 8P^2 + 2\right) +$$

$$+4e^{\frac{\theta}{2}-y}\left(\frac{e^{\theta}}{2} + e^{\theta}q_3^2 - \left(e^{\theta}+8\right)q_4q_3 +\right.$$

$$\left. + e^{\theta}q_4^2 - e^{\theta}q_1q_2 + 8P^2 + 2\right) - 4\left(q_1-q_2\right)^2 e^{\theta+2y} - 4\left(q_3-q_4\right)^2 e^{\theta-2y}\Bigg]$$

The Heun equation has four regular singular points at $y = \pm\infty, \pm(\frac{\theta}{2} - \ln 2)$.

- It is related to radial part of perturbations of the metric of Kerr-(Anti)-de Sitter BH H. Suzuki, E. Takasugi and H. Umetsu (1998).
- It is the quantisation of Seiberg-Witten differential of $\mathcal{N} = 2$ SYM with $N_f = 4$ in Nekrasov-Shatashvili background.

Expansion for $\Pi_+ = d/dy \ln \psi_+(y)$ (Floquet $+$) when $\theta \to -\infty$ and $|y| < \ln 2 - \theta/2$.

- Right region $0 < y < \ln 2 - \theta/2$

Define $\Pi_>$, $\Pi_> \left(y + \frac{\theta}{2}\right) = \Pi_+(y)$ and expand $\Pi_>(y) = \sum_{n=0}^{\infty} \Pi_>^{(n)}(y) e^{n\theta}$. By solving a system of first order (Riccati) ODEs, the proposed expressions are

$$\Pi_>^{(0)}(y) = \frac{d}{dy} \ln \left[(e^y - 2)^{\frac{1-q_1-q_2}{2}} e^{ky} {}_2F_1 \left(\frac{1}{2} + k - q_1, \frac{1}{2} + k - q_2; 1 + 2k; \frac{e^y}{2} \right) \right]$$

and

$$\Pi_>^{(n)}(y) = \frac{d}{dy} \left[P_0^{(n)}(e^{-y}) + \sum_{m=1}^{n} P_m^{(n)}(e^{-y}) \frac{d^{m-1}}{dy^{m-1}} \Pi_>^{(0)}(y) \right], \quad n \geq 1,$$

with $P_m^{(n)}(x)$ polynomials of degree n:

$$P_0^{(1)}(e^{-y}) = \left(-\frac{1}{4} + \frac{q_3 q_4}{1 - 4k^2} \right) \frac{1}{2} e^{-y}, \quad P_1^{(1)}(e^{-y}) = \left(-\frac{1}{4} + \frac{q_3 q_4}{1 - 4k^2} \right) \left(e^{-y} - \frac{1}{2} \right).$$

- Left region $\theta/2 - \ln 2 < y < 0$

Define $\Pi_<$, $\Pi_< \left(y - \frac{\theta}{2}\right) = \Pi_+(y)$. As in other cases

$$\Pi_<(y; k, q_1, q_2, q_3, q_4) = -\Pi_>(-y; -k, q_3, q_4, q_1, q_2)$$

Finally the symmetry $\psi_-(y, q_1, q_2, q_3, q_4) = \psi_+(-y, q_3, q_4, q_1, q_2)$ gives also the other Floquet solution.

Results to be compared with Cipriani, Di Russo, Fucito, Morales, Poghosyan, Poghossian (2025).

The Confluent Heun Equation (CHE) and Schwarzschild BH

The CHE is the confluence limit of the Heun equation

$$
\begin{aligned}
- & \frac{d^2}{dy^2}\psi(y) + \Big\{ \frac{1}{4}e^{2y}(q_1-q_2)^2 + e^y(-\frac{1}{2}+2q_1q_2+e^\theta q_3 - 2P^2) + (e^{2\theta}-6e^\theta q_3+4P^2) + \\
+ & e^{-y}(8e^\theta q_3 - 4e^{2\theta}) + 4e^{-2y}e^{2\theta}\Big\}\frac{1}{(e^y-2)^2}\psi(y) = 0\,.
\end{aligned}
$$

Related to $\mathcal{N}=2$ SYM with $N_f=3$. By the change of notations

$$
R = \frac{r\sqrt{r-2M}}{\sqrt{2M}}\psi\,, \qquad r = 4Me^{-y}\,, \qquad e^\theta = -4iM\omega\,, \qquad P^2 = l(l+1) - 8M^2\omega^2 + \frac{1}{4}\,,
$$

$$
q_1 = 2 - 2iM\omega\,, \qquad q_2 = -2iM\omega\,, \qquad q_3 = -2-2iM\omega\,,
$$

it maps into the Teukolsky equation for Schwarzschild background (in its homogeneous version, with spin $s=-2$) Teukolsky, Phys. Rev. Lett. 29 (1972) 1114; Teukolsky, Astrophysical Journal 185 (1973) 635

$$
r^4 f^2(r)\frac{d}{dr}\left[\frac{\frac{d}{dr}R(r)}{r^2 f(r)}\right] + \left(\frac{\omega^2 r^2 + 4i\omega(r-M)}{f(r)} - 8i\omega r - (l+2)(l-1)\right)R(r) = 0\,, \qquad f(r) = 1 - \frac{2M}{r}\,.
$$

Expressions for the wave function

The CHE has regular singular points at $y = \ln 2$ which means $r = 2M$, i.e. the BH horizon and at $y = +\infty$ which means $r = 0$. Irregular at $y = -\infty$ which means $r = +\infty$.

We are interested in the solution in between $y = -\infty$ and $y = \ln 2$.
By confluence limit of solutions of Heun, one finds solutions of the CHE
E.g. the Floquet solutions $\psi_\pm(y + 2\pi i) = e^{\pm 2\pi i k}\psi_\pm(y)$.
Define $\Pi_\pm(y) = d/dy \ln \psi_\pm(y)$ and $\Pi_\pm(y) = \sum_{n=0}^{+\infty} \Pi_\pm^{(n)}(y) e^{n\theta}$

- $a < y < \ln 2$ (near horizon, from confluence limit in right region of Heun)

$$\Pi_\pm^{(0)}(y)| = \frac{d}{dy} \ln \left[(e^y - 2)^{\frac{1-q_1-q_2}{2}} e^{\pm k y} {}_2F_1\left(\frac{1}{2} \pm k - q_1, \frac{1}{2} \pm k - q_2; 1 \pm 2k; \frac{e^y}{2} \right) \right] .$$

and

$$\Pi_\pm^{(n)}(y) = \frac{d}{dy} \left[P_0^{(n)}(e^{-y}) + \sum_{m=1}^{n} P_m^{(n)}(e^{-y}) \frac{d^{m-1}}{dy^{m-1}} \Pi_\pm^{(0)}(y) \right] , \quad n \geq 1$$

with $P_m^{(n)}(x)$ polynomials of degree n:

$$P_0^{(1)}(e^{-y}) = \frac{2q_3}{1-4k^2} e^{-y} , \quad P_1^{(1)}(e^{-y}) = \frac{4q_3}{1-4k^2} \left(e^{-y} - \frac{1}{2} \right)$$

• $-\infty < y < a$ (far from horizon, from confluence limit in left region of Heun)

$$\Pi_{\pm}^{(0)}(y)| = \frac{d}{dy}\ln\left[\exp\left(-e^{\theta-y} \pm k(y-\theta)\right)\,{}_1F_1\left(\frac{1}{2} \mp k + q_3, 1 \mp 2k; 2e^{\theta-y}\right)\right]$$

and

$$\Pi_{\pm}^{(n)}(y) = \frac{d}{dy}\left[P_0^{(n)}(e^{y-\theta}) + \sum_{m=1}^{n} P_m^{(n)}(e^{y-\theta})\frac{d^{m-1}}{dy^{m-1}}\Pi_{\pm}^{(0)}(y)\right] , \quad n \geq 1$$

with $P_m^{(n)}(x)$ polynomials of degree n:

$$P_0^{(1)}(e^{y-\theta}) = \left(-\frac{1}{4} + \frac{q_1 q_2}{1-4k^2}\right)\frac{1}{2}e^{y-\theta} , \quad P_1^{(1)}(e^{y-\theta}) = \left(-\frac{1}{4} + \frac{q_1 q_2}{1-4k^2}\right)e^{y-\theta}$$

Similar expressions are in Fucito, Morales, Russo (2024) and Cipriani, Di Russo, Fucito, Morales, Poghosyan, Poghossian (2025).

• In terms of Floquet basis one can construct 'physical' solutions (i.e. incoming at the horizon, $\psi_{in}(y) = \tilde{C}\left[\psi_-(y) - e^{-\varphi}\psi_+(y)\right]$, outgoing at infinity $\psi_{up}(y) = \tilde{D}\left[\psi_+(y) - e^{-\varphi}\psi_-(y)\right]$, with $\varphi = A_D/\hbar$).

Exact relations for the incoming solution $R_{in} = \frac{r\sqrt{r-2M}}{\sqrt{2M}}\psi_{in}$:

$$R_{in}(r) \simeq Br^3 e^{i\omega r^*} + \frac{e^{-i\omega r^*}}{2i\omega r}, \quad r^* = r + 2M\ln\frac{r}{2M}, \quad r \to +\infty$$

$$R_{in}(r) \simeq -i2^{5/2}(iM\omega)^{\frac{5}{2}+2iM\omega}\frac{1}{MQ(\theta, q_1, q_2, q_3)}\left(1 - \frac{2M}{r}\right)^{2-2iM\omega} \quad r \to 2M$$

$$B = -\frac{\omega^3}{2}\frac{Q(\theta + i\pi, q_1, q_2, -q_3)}{Q(\theta, q_1, q_2, q_3)}(2M\omega)^{4iM\omega}.$$

with Q the Wronskian of two well specified solutions of the CHE, $\psi_{0,0} \sim \psi_{in}$, $\psi_{-,0} \sim \psi_{up}$: $\quad Q = W[\psi_{0,0}, \psi_{-,0}]$

$\psi_{0,0}(y) \simeq \frac{1}{\sqrt{2}}(e^y - 2)^{\frac{1+q_1+q_2}{2}} \quad y \to \ln 2$

$\psi_{-,0}(y) \simeq e^{-(q_3+1/2)\theta+(q_3+1/2)y}\exp\left(-e^{\theta-y}\right) \quad y \to -\infty$

Summary and Perspectives

- In the framework of connections Painlevé / gravity /gauge theories we studied ODEs of the Heun type. In particular Floquet solutions, which we constructed perturbatively by using a system of (Riccati) first order ODEs.
- The relative connection coefficients φ coincide with $A_D/\hbar$, where A_D is dual gauge period of $\mathcal{N} = 2$ SYM in the Nekrasov-Shatashvili background.
- Still to be done: understand (also numerically) the polynomials appearing in the solutions: possible recursive relations between them.
- Study and construct subdominant solutions (in/up basis in gravity).
- Other Painlevé, other confluences of Heun equations

MME in gravity and gauge theories

• By the map $e^{-\frac{y}{2}} = \frac{r}{L}, 2e^{\theta} = -i\omega L, P = \frac{l+2}{2}$, the above Modified Mathieu Equation becomes the ODE which describes scalar field perturbation $\phi(r) = \sqrt{r}\psi_{\pm}(y)$ of the D3 brane with supergravity background ($AdS_5 XS^5$ both with radius of curvature L) Gubser, Hashimoto '98

$$ds^2 = H(r)^{-1/2}(dt^2 + dx^2) + H(r)^{1/2}(dr^2 + r^2 d\Omega_5^2)\,, \quad H(r) = 1 + L^4/r^4$$

In specific, one gets the radial wave equation for the l-th partial wave of energy ω

$$\frac{d^2\phi}{dr^2} + \left[\omega^2\left(1 + \frac{L^4}{r^4}\right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2}\right]\phi(r) = 0$$

$V_0 \sim e^{i\omega r}$ is the upgoing solution at $r = \infty$, $U_0 \sim e^{i\omega L^2/r}$ is the incoming at $r = 0$

• The Modified Mathieu Equation is the quantisation of the SW differential for $\mathcal{N} = 2$ SYM without matter in the NS background.

Painlevé gauge correspondence

- Painlevé I —> H_0 (Argyres-Douglas)
- Painlevé II—>H_1
- Painlevé III_3 —>$N_f = 0$ (MME)
- Painlevé III_2—>$N_f = 1$
- Painlevé III_1—>$N_f = 2$ (DCHE)
- Painlevé IV–>H_2
- Painlevé V—>$N_f = 3$ (CHE)
- Painlevé VI—>$N_f = 4$ (Heun)

Confluence limit Heun->CHE $y = y' - \theta/2$, $q_4 e^{\theta} = 4e^{\theta_3}, \theta \to -\infty$)

$$\varphi = -\theta_3 k + 2k\ln 2 + \ln\frac{\Gamma(1+2k)}{\Gamma(1-2k)} + \frac{1}{2}\sum_{i=1}^{3}\ln\frac{\Gamma\left(\frac{1}{2}-k+q_i\right)}{\Gamma\left(\frac{1}{2}+k+q_i\right)} + \frac{8kq_1q_2q_3}{(1-4k^2)^2}e^{\theta_3} + \ldots.$$