Non-local Charges from Perturbed Defects

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Motivation: to investigate symmetries of "new" flows

Prototype new flow: $M_{3,10} + \varphi_{1,7} \longrightarrow M_{3,8}$

Conjectured TBA equation for ground state energy

Consistent with preservation of topological defects

Can we find other indications of integrability?

Consider some simple integrable flows :

Massless flow

TCIM $\to$ IM $\qquad \mathcal{M}_{4,5} + \varphi_{1,3} \longrightarrow \mathcal{M}_{4,3}$

Massive flow

LY $\to$ SLY $\qquad \mathcal{M}_{2,5} + \varphi_{1,2} \longrightarrow$ SLYM

Measures of Integrability :

	TBA system	local conserved quantities	Classical integrable model
$\mathcal{M}_{4,5} + \varphi_{1,3} \longrightarrow \mathcal{M}_{4,3}$	✓	$1, 3, 5, \dots$	$a_1^{(1)}$ Sine-Gordon
$\mathcal{M}_{2,5} + \varphi_{1,2} \longrightarrow$ SLYM	✓	$1, 5, 7, 11, \dots$	$a_2^{(1)}$ ZMS-BD-Tz.
$\mathcal{M}_{3,10} + \varphi_{1,7} \longrightarrow \mathcal{M}_{3,8}$	(✓)		

Local conserved quantities

[Zam. '89]

In CFT, holomorphic (chiral) current $J(z)$ $\bar\partial J = 0$

In perturbed CFT, $\delta S = \mu \int \varphi(z)\, d^2z$ $\quad \bar\partial J = \mu\, (J\varphi)_1$

If $(J\varphi)_1 = \partial X$ then $\quad \bar\partial J = \partial(\mu X)$

$$Q = \int J\, dz + \int \mu X\, d\bar z$$

We have checked up to $\Delta(J) = 20$: No non-zero J

$$(J\varphi_{17})_1 = \partial(\dots)$$

$$\boxed{\text{In } M_{3,10}}$$

Results of search for local charges for ϕ_{17} perturbations.

$M(p,q)$	c	h	4	5	6	7	8	9	10	11	12	13	14	15	16
$(2,9)$	$-\frac{46}{3}$	$-\frac{1}{5}$	—	—	1	—	1	—	—	—	1	—	1	—	—
$(2,11)$	$-\frac{232}{11}$	$-\frac{9}{11}$	—	—	—	—	1	—	—	—	1	—	1	—	—
$(3,8)$	$-\frac{21}{4}$	$\frac{3}{2}$	—	—	1	—	2	—	2	1	3	1	5	3	6
$(5,8)$	$-\frac{7}{20}$	$\frac{9}{2}$	—	—	—	—	1	—	—	—	—	—	—	—	—
$(7,8)$	$\frac{25}{28}$	$\frac{15}{2}$	—	—	—	—	—	—	—	—	—	—	1	—	—
$(13,18)$	$\frac{14}{39}$	$\frac{17}{3}$	—	—	—	—	—	—	—	—	—	—	1	—	1

Table 6: Models $M(p,q)$ with $p,q \leq 20$ for which there are currents with spins ≤ 16 for $\psi_{1,7}$ perturbations which solve the first-order conservation condition (C.8), together with the number of independent currents for each spin.

Rigid Defects

Further conserved quantities : rigid defects

Well-known in Lee-Yang model

- Free field construction [BLZ '94]
- Intrinsic CFT construction [Runkel '08, '10]

Starting point : topological defects

- Topological defect symmetry preserved along flow
  Defect commutes with perturbing field
- Topological defect symmetry broken by perturbation
  Defect does <u>not</u> commute with perturbing field

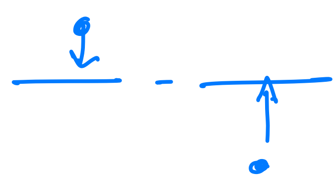
Defects in Scaling Lee-Yang model

$\mathcal{M}_{2,5}$: Single non-trivial primary field φ

$$h = \bar{h} = -1/5$$

Single non-trivial defect $\mathcal{D} \equiv \mathcal{D}_{(1,2)}$

- $\mathcal{D}$ does not commute with the perturbation

$[\varphi(z), \mathcal{D}] =$  $\neq 0$

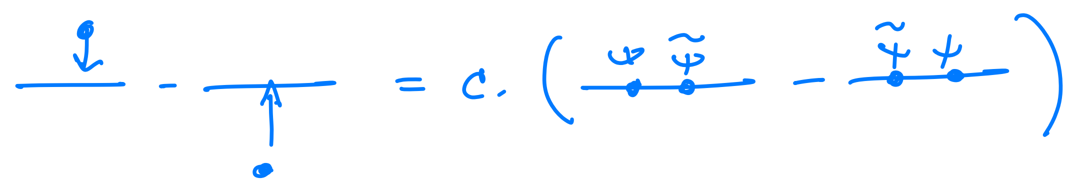
Defects in Scaling Lee-Yang model

$\mathcal{M}_{2,5}$: Single non-trivial primary field φ

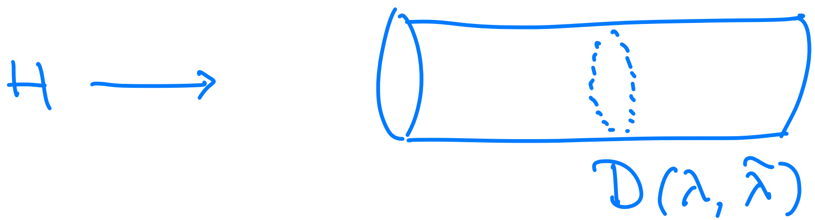
$h = \bar{h} = -1/5$

Single non-trivial defect $D \equiv D_{(1,2)}$

- D does not commute with the perturbation
- Fields $\psi \; (h = -1/5, \bar{h} = 0)$ $\tilde{\psi} \; (h = 0, \bar{h} = -1/5)$ s.t.

$[\varphi(z), D] =$ 

Can show the perturbed defect commutes with the Hamiltonian



$$H = \left(L_0 + \bar{L}_0 - \frac{c}{12}\right) + \mu \int \varphi \, d\theta$$

$$D(\lambda, \tilde{\lambda}) = e^{\int \lambda \psi + \tilde{\lambda} \tilde{\psi} \, d\theta} \cdot D$$

$$[H, D(\lambda, \tilde{\lambda})] = 0 \iff \mu = \lambda \tilde{\lambda}$$

Defects Commute:

$$\left[D\left(\lambda, \frac{\mu}{\lambda}\right) , D\left(\tilde{\lambda}, \frac{\mu}{\tilde{\lambda}}\right) \right] = 0$$

Infinite set (family) of commuting non-local charges

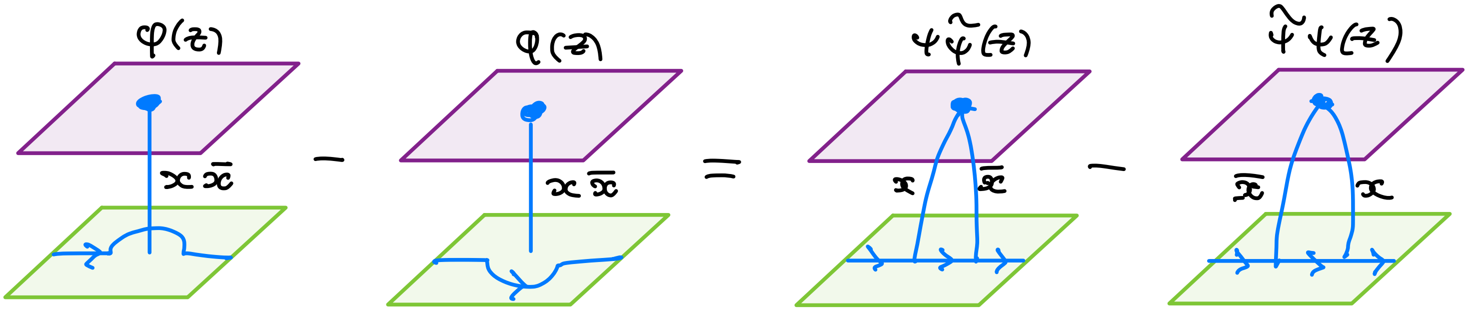
- $D(\lambda, \tilde{\lambda})$ commutes with $H \iff \lambda\tilde{\lambda} = \mu$
- $D\left(\lambda, \frac{\mu}{\lambda}\right) \longrightarrow \mathbb{1}$ if $\lambda \to \infty$
- Have asymptotic expansion over local charges as $\lambda \to \infty$
- Satisfies functional relation $D^{(+)} \cdot D^{(-)} = 1 + D$

Bulk Commutation condition

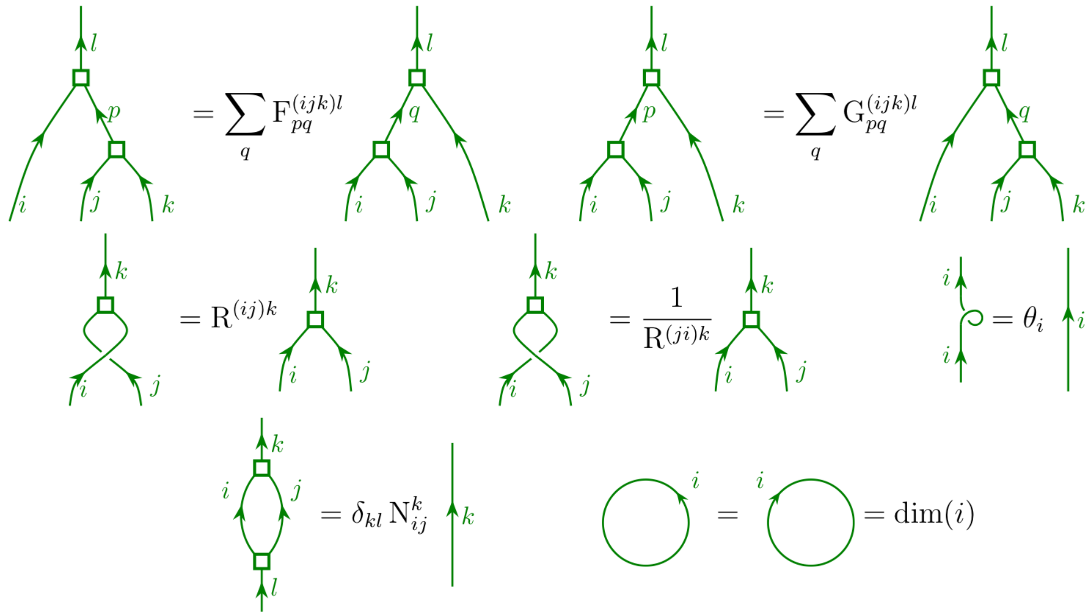
- Bulk field φ (h, h)
- Defect D
- Defect fields ψ $(h, 0)$ and $\tilde{\psi}$ $(0, h)$

$$\mu \left[D, \varphi \right] = \lambda \tilde{\lambda} \left[\psi, \tilde{\psi} \right]$$

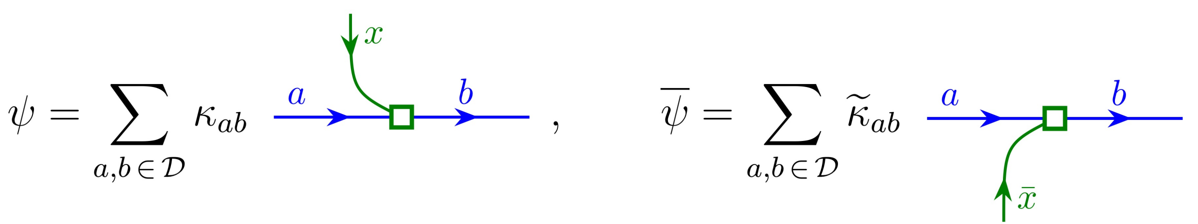
Nice interpretation in SymTFT



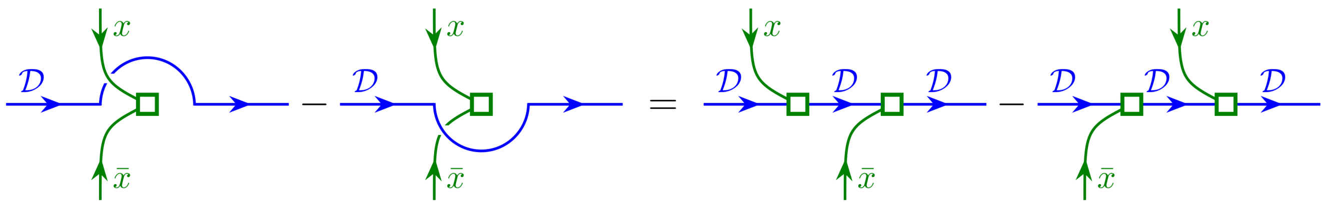
Standard TFT rules :



Basis of defect fields



Bulk commutation relation



Equations for $\kappa, \widetilde{\kappa}$

$$\boxed{\delta_{ac}\left(\mathrm{R}^{(xa)b} - \frac{1}{\mathrm{R}^{(ax)b}}\right)\mathrm{F}^{(ax\bar{x})a}_{1b} = \delta_{b\in\mathcal{D}}\,\kappa_{ab}\widetilde{\kappa}_{bc} - \sum_{d\in\mathcal{D}}\mathrm{F}^{(xa\bar{x})c}_{db}\,\widetilde{\kappa}_{ad}\kappa_{dc}}$$

Results of Bulk Commutation Test

Depends on : The field $\varphi_{r,s}$

The defect $D = \{ D_{r',s'} \ ; \ \mathbb{1} \oplus D_{r,s} \ ; \ \dots \}$

The model $M_{p,q}$

Typically $\psi, \tilde{\psi}$ determined by the condition (up to normalisation)

Results of Bulk Commutation Test

Depends on : The field $\phi_{r,s}$

The defect $D = \{ D_{r',s'} \, ; \, \mathbb{1} \oplus D_{r,s} \, ; \, \dots \}$

The model $\mathcal{M}_{p,q}$

	ϕ	D	$\mathcal{M}_{p,q}$
Known cases :	$\phi_{1,3}$	D_{12}	: all models $\mathcal{M}_{p,q}$, $q \geq 4$
	ϕ_{12}	$\mathbb{1} + D_{12}$	: all models $\mathcal{M}_{p,q}$, $q \geq 4$
	ϕ_{15}	D_{13}	: all models $\mathcal{M}_{p,q}$, $q \geq 6$

In all cases the defects mutually commute.

$\mathcal{M}_{3,10} + \varphi_{1,7}$

9 fundamental defects $D_{1,s}$ $1 \le s \le 9$

$\psi, \tilde{\psi}$ exist on $D_{1,4}$ $D_{1,5}$ $D_{1,6}$

Bulk commutation relation $\begin{cases} \checkmark & \text{satisfied for } D_{1,5} \\ \times & \text{not satisfied for } D_{1,4}, D_{1,6} \end{cases}$

Relation to local conserved charges : behaviour for $\lambda \to \infty$
(later, if time)

$\varphi_{1,7}$ in general models $\mathcal{M}_{p,q}$

$D_{1,5}$: $\psi, \bar{\psi}$ on $D_{1,5}$ requires $q \geq 9$

Bulk commutation only satisfied for $q = 9, 10, 18$

In each case, the perturbed defects mutually commute

$\Phi_{1,5}$ in $\mathcal{M}_{3,10}$

Defect	Perturbation (13)	(15)	(17)	(19)
(12)	✓			
(13)	✓	✓		
(14)	✓	✗	✗ *	
(15)	✓	✓	✓ *	✓ *
(16)	✓	✗	✗ *	
(17)	✓	✓		
(18)	✓			

* : requires regularisation, results only formal

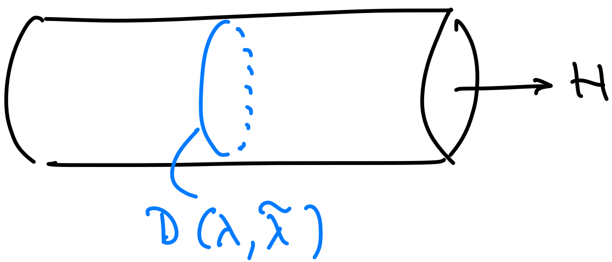
$\varphi_{1,s}$ in general models $\mathcal{M}_{p,q}$

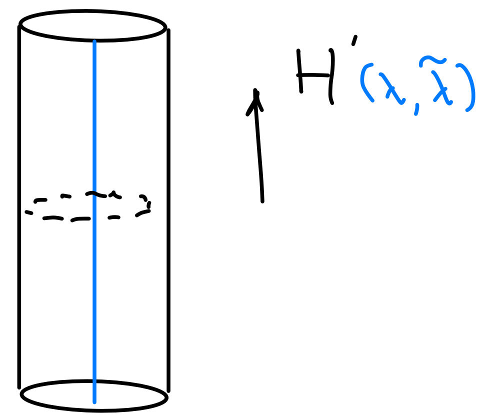
Bulk commutation ok for

	D in	$\mathcal{M}_{p,q}$ for q in
$\varphi_{1,9}$	$D(15)$	$\{10, 11, 12\}$
	$D(16)$	$\{11, 12\}$
	$D(17)$	$\{12, 14\}$
$\varphi_{1,11}$	$D(16)$	$\{12, 13\}$
	$D(17)$	$\{13, 14\}$
$\varphi_{1,13}$	$D(17)$	$\{14, 15, 16\}$
	$D(18)$	$\{15, 16\}$
	$D(19)$	$\{16, 18\}$

Behaviour as $\lambda \to \infty$

The nature of the endpoint $D \to ?$ can be identified by looking at the spectrum of the Hamiltonian in the crossed channel



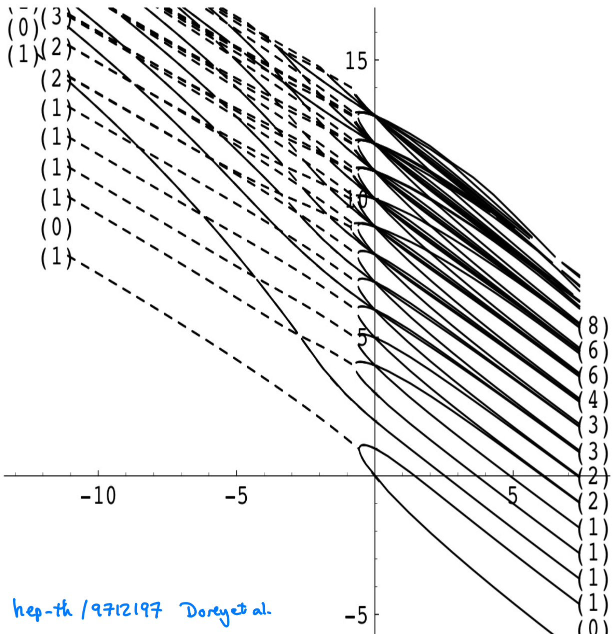


(Actually easier to look at models on a strip..)

Simpler still to look at

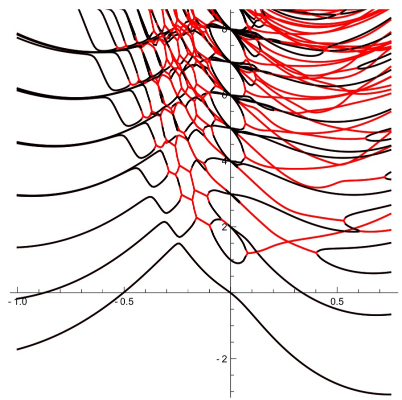
- $\lambda \to \infty$ limit in case $\mu = 0$.
- Model on a strip.

$\mathcal{M}_{2,5}: D_{12} + \psi_{12}$



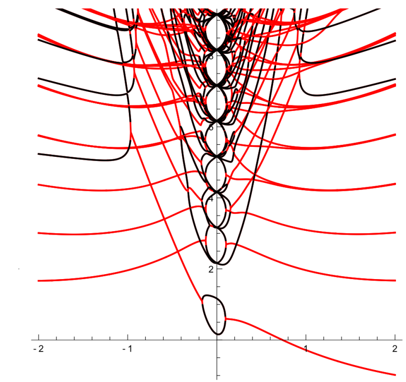
hep-th/9712197 Dorey et al.

$\mathcal{M}_{3,10}: D_{14} + \psi_{17}$



$D_{(1,4)} + \lambda\psi_{(1,7)}$

$\mathcal{M}_{3,10}: D_{15} + \psi_{17}$



$D_{(1,5)} + \lambda\psi_{(1,7)}$

Further cases

So far, mostly considered perturbations of a single elementary defect

Also looked at $D = \mathbb{1} + D_{(15)}$ ψ_{15} is a "defect changing field"

Found many more cases which satisfy bulk commutation relation.

Analysis of commutativity of perturbed defects still to be done.

Outlook

- Check if formal calculations survive after regularisation and renormalisation

- Check if can establish T-function type relations in "new" cases

- Do defects commute in other cases ?

- Extend to boundary / strip systems [WIP with A. Konechny]

- Do these charges, if they exist, actual have anything to do with integrability ?

Thank You.