

Integrals of Motion in CFT and ODE/IM correspondence

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Introduction

Linear Problem and WKB analysis

Integrals of motion in W CFT

$$E_6^{(1)}$$

Outlook

Introduction

Integrals of motion in CFT

- 2-dim Integrable QFT has an infinite number of integrals of motion (IM)
[Zamolodchikov]
- In 2d CFT +integrable perturbation, the IMs I_s have been constructed as a quantum version of soliton hierarchy
[Zamolodchikov,Sasaki-Yamanaka, Eguchi-Yang, Kupershmidt-Mathieu,...]

$$[I_s, I_{s'}] = 0, \quad I_s = \oint P_s(z) dz$$

- Integrable structure in 2d CFT [Bazhanov-Lukyanov-Zamolochikov]
T-function $\implies$ IMs on a cylinder
- Application to thermal properties of CFT [Essler et al. ,Dymarsky-Pavlenko, Ashok et.al. , Downing et al.]

ODE/IM correspondence

[Dorey-Tateo 9812211, Bazhanov-Lukyanov-Zamolodchikov 9812247, ...]

- Spectral problem of Ordinary Differential Equations (ODE) $\iff$ Functional relations in quantum integrable models (IM)
- T-function: (generating function of IMs) $\iff$ Wronskian of the solutions at infinity
- WKB solutions of the ODE are applied to calculate the Wronskians
- Schrödinger Eq. $\iff$ Virasoro minimal model [BLZ]
3rd order ODE $\iff$ W_3 minimal model [Bazhanov-Hibberd-Koroshkin]
- CFT data $(c, \Delta_2, \dots)$ $\iff$ ODE data (potential, monodromy)

ODE/IM

linear differential system $(\hat{g}^\vee) \iff W_{\hat{g}}$ minimal model

Linear Problem and WKB analysis

Linear Problem

- $\mathfrak{g}$: a (simply-laced) Lie algebra of rank r
- $\{H^i, E_\alpha\}$; the Cartan-Weyl basis of $\mathfrak{g}$
- $\alpha_1, \dots, \alpha_r$: simple roots
- extended root $\alpha_0 = -\theta$: θ is the highest root

first order Linear differential system

$$[\epsilon \partial_x + A(x)] \Psi(x) = 0$$

$$A(x) = \epsilon \frac{1}{x} (l \cdot H) \Psi + \sum_{i=1}^r E_{\alpha_i} + p(x) E_{\alpha_0}$$

- $l \cdot H = \sum_{i=1}^r l_i \alpha_i^\vee \cdot H$, l_i "angular momentum" parameter
- $p(x) = x^{h^\vee M} - E$ "potential" term ($E = 1$)
- $\epsilon = \hbar$ Planck constant
- Ψ take values in the representation of $\mathfrak{g}$

- The linear system is obtained from the Lax system $[L_z, L_{\bar{z}}] = 0$ from the affine Toda field equation by taking the light-cone ($\bar{z} \rightarrow 0$) and the conformal limit. [Lukyanov-Zamolodchikov, Ito-Locke, ...]
- For $A_1^{(1)}$, the linear system is equivalent to the Schrödinger equation with potential $p(x)$ with centrifugal term

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^{2M} - E \right] y(x, E, \ell) = 0$$

WKB analysis of $A_r^{(1)}$ type Linear problem

$(r+1)$ -dim representation of A_r , $E_{\alpha_i} = E_{i,i+1}$, $E_{\alpha_0} = E_{r+1,1}$

$$\left[\epsilon^{r+1} \left(\partial_x - \frac{l_1}{x} \right) \left(\partial_x - \frac{l_2}{x} \right) \cdots \left(\partial_x - \frac{l_r}{x} \right) \left(\partial_x - \frac{l_{r+1}}{x} \right) + (-1)^r p(x) \right] \psi(x, \epsilon) = 0$$

[Dorey-Dunning-Tateo]

$$\left[(\epsilon \partial_x)^h + \sum_{i=2}^h \epsilon^i \frac{L_i}{x^i} (\epsilon \partial_x)^{h-i} + (-1)^r p(x) \right] \psi(x, \epsilon) = 0$$

WKB expansion

$$\psi(x, \epsilon) = \exp\left(\frac{1}{\epsilon} \int^x dx S(x, \epsilon)\right),$$

$$S(x, \epsilon) = \sum_{i=0}^{\infty} \epsilon^i S_i(x).$$

Riccati eq. for S_i

$$(\epsilon \partial_x + S)^r S + \sum_{i=2}^{r+1} \frac{L_i}{x^i} (\epsilon \partial_x + S)^{r-i} S + (-1)^r p(x) = 0,$$

WKB solutions and period integral

$$S_0(x) = -p(x)^{\frac{1}{h}}, \quad S_1(x) = \frac{h-1}{2h} \partial_x \log(p),$$

$$S_2(x) = \frac{L_2}{h} p^{-\frac{1}{h}} x^{-2} - \frac{(h-1)(h+1)M(hM-1)}{12} p^{-\frac{1}{h}-1} x^{hM-2}$$

$$+ \frac{(h-1)(h+1)(2h+1)M^2}{24} p^{-\frac{1}{h}-2} x^{2hM-2}, \dots$$

period integral $Q_i = \oint_{\mathcal{C}} dx S_i(x)$



$$J(a, b) \equiv \int_{\mathcal{C}} \left(x^{hM} - 1\right)^a x^b dx = -\frac{e^{\pi i a} 2\pi i}{hM} \frac{\Gamma\left(-a - \frac{b+1}{hM}\right)}{\Gamma(-a)\Gamma\left(1 - \frac{b+1}{hM}\right)},$$

WKB periods

$$Q_2 = \frac{1}{h} J_{1,2} \left(L_2 + \frac{1}{24} (h-1)h(hM-1) \right),$$

$$Q_3 = \frac{1}{h} J_{2,3} \left(-L_3 - (h-2)L_2 \right), \dots$$

WKB analysis of $D_r^{(1)}$ type Linear problem

$2r$ -dim vector representation [Dorey-Dunning-Masoero-Suzuki-Tateo]

$$\epsilon^{2r-2} \left(\partial_x - \frac{l_1}{x} \right) \cdots \left(\partial_x - \frac{l_r}{x} \right) \partial_x^{-1} \left(\partial_x + \frac{l_r}{x} \right) \cdots \left(\partial_x + \frac{l_1}{x} \right) \psi = 4\sqrt{p(x)} \partial_x \sqrt{p(x)} \psi,$$

- WKB analysis of the ODE including a pseudo-differential operator ∂_x^{-1}
- Diagonalization (abelianization)

WKB analysis of pseudo-ODE

- Set $l_r \rightarrow 0$, $D_r^{(1)} \rightarrow B_{r-1}^{(1)}$

$$\epsilon^{2r-2} \left(\partial_x - \frac{l_1}{x} \right) \cdots \left(\partial_x - \frac{l_{r-1}}{x} \right) \partial_x \left(\partial_x + \frac{l_{r-1}}{x} \right) \cdots \left(\partial_x + \frac{l_1}{x} \right) \psi(x) - 4\sqrt{p(x)} \partial_x \sqrt{p(x)} \psi(x) = 0.$$

- WKB expansion of $B_{r-1}^{(1)}$ -type ODE

$$(\epsilon \partial_x + S)^{2r-2} S + \sum_{i=2}^{2r-1} \frac{L'_i}{x^i} (\epsilon \partial_x + S)^{2r-2-i} S - 4\sqrt{p(z)} \partial_z \sqrt{p(z)} = 0.$$

L'_i is expressed by symmetric polynomials in $l_1^2, \dots, l_{r-1}^2$

Solve the Riccati eq.

- Replace L'_i by symmetric polynomials in $l_1^2, \dots, l_{r-1}^2, \textcolor{red}{l}_r^2$ ($B_{r-1}^{(1)} \rightarrow D_r^{(1)}$)
- WKB expansion of $D_r^{(1)}$

Diagonalization of the Lax operator

$$A_{\text{diag}}(z) = T^{-1}(z)A(z)T(z) + \epsilon T^{-1}(z)\partial_z T(z).$$

$$\Psi(z) = \text{diag} \left[\exp \left(-\frac{1}{\epsilon} \int dz A_{\text{diag}}(z) \right) \right].$$

step-by-step diagonalization

$$T(z) = T_d = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ g_1 & g_2 & \dots & g_{d-1} & 1 \end{pmatrix}$$

$$\mathbf{Gau}_{T_d}[A(z)] = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ \mathbf{Gau}_{T_d}[A(z)]_{d,1} & \dots & \mathbf{Gau}_{T_d}[A(z)]_{d,d-1} & \mathbf{Gau}_{T_d}[A(z)]_{d,d} \end{pmatrix}$$

Eliminate $g_1, \dots, g_{d-2}$ and $p(x)$ by solving

$$\mathbf{Gau}_{T_d}[A(z)]_{d,1} = \dots = \mathbf{Gau}_{T_d}[A(z)]_{d,d-1} = 0 \text{ (Riccati Eqs.)}$$

$$\mathbf{Gau}_{T_d}[A(z)]_{d,d} = \epsilon \left(\frac{l \cdot H}{x} \right)_{d,d} + g_{d-1} = S(z, \epsilon)$$

This procedure is the same as the construction of classical conserved charges in the Drinfeld-Sokolov reduction of the soliton hierarchy.

WKB periods for $D_r^{(1)}$ -type linear problem

WKB expansion

$$S_0(x) = 2^{\frac{2}{h}} p(x)^{\frac{1}{h}},$$

$$S_1(x) = -\frac{1}{2} \partial_x \log(p),$$

$$S_2(x) = -4^{-1/h} \frac{L'_2}{h} p^{-\frac{1}{h}} x^{-2} - 2^{-\frac{2}{h}-3} \frac{(h+1)(h+2)(2h+1)M^2}{3} p^{-\frac{1}{h}-2} x^{-2+2hM} \\ + \frac{4^{-\frac{h+1}{h}} (h+1)(h+2)(hM-1)M}{3h} p^{-\frac{1}{h}-1} x^{-2+hM},$$

$$S'_r = 2p^{-\frac{1}{2}} x^{-r} K_r, \quad K_r = K_r := x^r \left(\partial_x - \frac{l_1}{x} \right) \cdots \left(\partial_x - \frac{l_r}{x} \right) \cdot 1$$

WKB period integrals

$$Q_2 = -\frac{2^{-\frac{2}{h}}}{h} J_{1,2} \left(L'_2 + \frac{1}{24} h(h+2)(hM-1) \right),$$

$$Q_4 = -\frac{2^{-\frac{6}{h}}}{h} J_{3,4} \left[L'_4 - \frac{h-3}{2h} L_2'^2 + \frac{3(h-2)}{2} L'_3 - \frac{1}{8} [(M-4)h^2 - (6M-9)h - 6] L'_2 \right. \\ \left. + \frac{1}{1920} h(h-6)(h+2)(Mh-3)(Mh-1)(2(M+1)h-1) \right],$$

$$Q'_r = 2J_{-\frac{1}{2}, -r}(\epsilon_1, l + \rho) \cdots (\epsilon_1, l + \rho),$$

Integrals of motion in W CFT

W minimal model

$W\hat{\mathfrak{g}}$ minimal model: spin $i + 1$ current (i : exponent of $\mathfrak{g}$)

$WA_r^{(1)}: W_2, W_3, \dots, W_{r+1}$

$WD_r^{(1)}: W_2, W_4, \dots, W_{2r-2}$ and W'_r

$WE_6^{(1)}: W_2, W_5, W_6, W_8, W_9, W_{12}$

- OPE closure

$$A(z)B(w) = \sum_{1 \leq k \leq \Delta_A + \Delta_B} \frac{\{AB\}_k(w)}{(z-w)^k} + \circ AB \circ (w) + O(z-w).$$

- quantum Miura transformation [Fateev-Lukyanov]
- Quantum Drinfeld-Sokolov reduction of affine Lie algebra $\hat{\mathfrak{g}}$ [Feigin-Frenkel]

Free Field realization of W algebras

$\phi_i(z)$: r free bosons with OPE $\phi_i(z)\phi_j(w) \sim -\delta_{ij} \log(z-w)$

energy-momentum tensor

$$T(z) = -\frac{1}{2} \circ (\partial_z \phi)^2(z) \circ -i\alpha_0 \rho \cdot \partial_z^2 \phi(z).$$

The central charge

$$c = r - 12\alpha_0^2 \rho^2,$$

Quantum Miura transformation WA_r [Fateev-Lukyanov]

$$(\alpha_0 \partial_z)^{r+1} - \sum_{k=2}^{r+1} \tilde{W}_k(z) (\alpha_0 \partial_z)^{r+1-k} =: (\alpha_0 \partial_z - \epsilon_1 \cdot i \partial_z \phi) \dots (\alpha_0 \partial_z - \epsilon_{r+1} \cdot i \partial_z \phi) : (z) .$$

primary field $V_\Lambda(z) =: e^{i\Lambda \cdot \phi(z)} :$

$$\tilde{\Delta}_2 = \Delta_\Lambda = \frac{1}{2} \Lambda(\Lambda + 2\rho) = -\sigma_2 - \frac{1}{4} \binom{r+2}{3} \alpha_0^2$$

$$\tilde{\Delta}_3 = \sigma_3 + (r-1)\alpha_0\sigma_2 + \binom{r+2}{4} \alpha_0^3$$

symmetric polynomials in $p_i = (\epsilon_i, \Lambda + \alpha_0 \rho)$

$$\sigma_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq r+1} p_{i_1} p_{i_2} \dots p_{i_k}.$$

Free field realization of the W-algebras (2)

$$WD_r \text{ algebra : } W_2^{(r)}, \dots, W_{2r-2}^{(r)}, R_r^{(r)}, c = r - r(2r-1)(2r-2)\alpha_0^2$$

$$R_r(z) = \circ (\alpha_0 \partial_z - i\epsilon_1 \cdot \partial_z \phi(z)) \dots (\alpha_0 \partial_z - i\epsilon_n \cdot \partial_z \phi(z)) \circ \cdot 1.$$

$$R_r(z)R_r(w) = \frac{A_r}{(z-w)^{2r}} + \sum_{k=1}^{r-1} \frac{A_{r-k}}{(z-w)^{2(r-k)}} \left(\tilde{W}_{2k}(z) + \tilde{W}_{2k}(w) \right),$$

$$WD_r = WD_{r-1} \oplus [\tilde{\phi}_r]$$

$$\tilde{\Delta}_2 = \frac{1}{2}\sigma_1 - \frac{r(2r-2)(2r-1)}{24}\alpha_0^2,$$

$$\begin{aligned} \tilde{\Delta}_4 = & \frac{1}{2}\sigma_2 - \left(\frac{1}{12}(r-7)(r-2)(2r-3)\alpha_0^2 + \frac{1}{4} \right) \sigma_1 + \frac{r(2r-1)(r-1)}{24}\alpha_0^2 \\ & + \frac{1}{720}(r-2)(r-1)r(2r-3)(2r-1)(5r-71)\alpha_0^4, \end{aligned}$$

$$\sigma_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq r} p_{i_1}^2 p_{i_2}^2 \dots p_{i_k}^2, \quad p_i = (\epsilon_i, \Lambda + \alpha_0 \rho)$$

CFT on a cylinder

- On the complex plane, the integrals of motion is constructed as $I_s = \oint dz J_s(T, W, \dots) dz$ with $[I_s, I_{s'}] = 0$.
- the IMs $\hat{J}_s(u)$ on the cylinder by the conformal map $z = e^u$ ($u = \tau + i\sigma$)
- A (non-primary) field $A(z) = \sum_n A_n z^{-n-\Delta_A}$ on the complex plane transforms as ($\frac{dz}{du} = z$)

$$\hat{A}(u) = z^{\Delta_A} A(z) (+\delta A(z)) =: z^{\Delta_A} A_R(z)$$

- normal-ordered product on the cylinder

$$:\hat{A}\hat{B}: (v) = \frac{1}{2\pi i} \oint_v du \frac{T(\hat{A}(u)\hat{B}(v))}{u-v} = \frac{1}{2\pi i} \oint_w \frac{dz}{z} \frac{z^{\Delta_A} w^{\Delta_B} R(A_R(z)B_R(w))}{\log \frac{z}{w}}.$$

Normal ordered product on the cylinder

Bernoulli polynomial $\psi_n(x)$: $\frac{(z+1)^x}{\log(1+z)} = \sum_{n=0}^{\infty} \psi_n(x) z^{n-1}$

$$A_R(z)B_R(w) = \sum_{k=1}^{\Delta_A + \Delta_B} \frac{\{A_R B_R\}_k(w)}{(z-w)^k} + \{A_R B_R\}_0(w) + \cdots,$$

[Dymarksy-Pavlenko-Solov'yev, Novaes]

$$: \hat{A} \hat{B} : (v) = \hat{A}_-(v) \hat{B}(v) + \hat{B}(v) \hat{A}_+(v) + \sum_{k=1}^{\Delta_A + \Delta_B} f_k(\Delta_A - 1) \{A_R B_R\}_k(w) w^{\Delta_A + \Delta_B - k},$$

$$f_k(x) = \psi_k(x) - \frac{(x)_k}{k!}$$

$$\begin{aligned} \hat{A}_-(v) \hat{B}(v) + \hat{B}(v) \hat{A}_+(v) &= w^{\Delta_A + \Delta_B} \{A_R B_R\}_0(w) \\ &+ \sum_{n=1}^{\Delta_A - 1} \sum_{k=1}^{\Delta_A + \Delta_B} w^{\Delta_A + \Delta_B - k} \frac{(\Delta_A - n - 1)_{k-1}}{(k-1)!} \{A_R B_R\}_k(w). \end{aligned}$$

Example [BLZ]: $\hat{T}(u) = z^2 T(z) - \frac{c}{24}$

$$: \hat{T} \hat{T} : (v) = w^4 \{T T\}_0(w) - \frac{c-10}{12} w^2 T(w) + \frac{3}{2} w^3 \partial T(w) + \frac{22c + 5c^2}{2880}.$$

$$(: \hat{T} \hat{T} :)_0 = 2 \sum_{n=1}^{\infty} L_{-n} L_n + L_0^2 - \frac{c+2}{12} L_0 + \frac{5c^2 + 22c}{2880}.$$

Quantum IM for $WA_r^{(1)}$

charge densities

$$\hat{J}_2(v) = \hat{T}(v),$$

$$\hat{J}_3(v) = \hat{W}_3(v),$$

$$\hat{J}_4(v) = \hat{W}_4 + a_1 : \hat{T} \hat{T} : (v), \quad a_1 = \frac{3(r-4) [(r+2)c + 8r^2 - 18r + 4]}{2(5c+22)(r-2)(r-1)r}$$

$$\hat{J}_5(v) = \hat{W}_5 + b_1 : \hat{T} \hat{W}_3 : (v), \quad b_1 = \frac{8(r-5) [(r+3)c + 15r^2 - 33r + 6]}{(7c+114)(r-2)(r-1)r}$$

integrals of motion

$$\mathbf{I}_s = \int_0^{2\pi} \frac{d\sigma}{2\pi} \hat{J}_s(i\sigma).$$

$$(\hat{W}_s)_0 |\{\Delta_s\}\rangle = \Delta_s |\{\Delta_s\}\rangle.$$

$$\mathbf{I}_s |\{\Delta_s\}\rangle = I_s |\{\Delta_s\}\rangle.$$

$$I_2 = \Delta_2 - \frac{c}{24},$$

$$I_3 = \Delta_3,$$

$$I_4 = \Delta_4 + a_1 \left(\Delta_2^2 - \frac{c+2}{12} \Delta_2 + \frac{5c^2 + 22c}{2880} \right),$$

$$\hat{J}_2(v) = \hat{T}(v),$$

$$\hat{J}_4(v) = \hat{W}_4 + a_1 : \hat{T}\hat{T} : (v), \quad a_1 = -\frac{3(r-4)(2cr+c+16r^2-10r)}{2(5c+22)(r-1)r(2r-1)}$$

$$\hat{J}_6(v) = \hat{W}_6 + x_1 : T(: TT :) + x_2 : \partial T \partial T :,$$

$$I_2 = \Delta_1 - \frac{c}{24},$$

$$I_4 = \Delta_4 + a_1 \left(\Delta_2^2 - \frac{c+2}{12} \Delta_2 + \frac{5c^2+22c}{2880} \right).$$

WD_4

$$\begin{aligned} I_6 = \Delta_6 + & \frac{(656+11c)(23+52c)}{2646(2c-1)(68+7c)} \Delta_2^3 - \frac{(c+4)(656+11c)(23+52c)}{21168(2c-1)(68+7c)} \Delta_2^2 \\ & + \frac{(656+11c)(-96+364c+231c^2+26c^3)}{254016(2c-1)(68+7c)} \Delta_2 - \frac{c(656+11c)(60+13c)}{128024064}. \end{aligned}$$

ODE/IM correspondence

- ODE : $Q_2 = -h(M+1)J_{1,2} \left[-\frac{1}{h^2(M+1)}s_2 - \frac{r}{24} \right]$
CFT: $I_2 = -\sigma_2 - \frac{r}{24}$
- $Q_3 = \frac{1}{h}J_{2,3}s_3$
 $I_3 = \sigma_3$
- $Q_4 = J_{3,4} \left[\frac{1}{h}s_4 - (h-3) \left(\frac{1}{2h^2}s_2^2 + \frac{1+M}{8h}s_2 - \frac{(1+M)^2}{1920}h^2(h-1)\left(\frac{2hM^2}{1+M} - 9\right) \right) \right]$
 $I_4 = -\sigma_4 + \frac{r-2}{2(r+1)}\sigma_2^2 + \frac{r-2}{8(r+1)}\sigma_2 + \frac{(r-2)r(9-2(r+1)\alpha_0^2)}{1920(1+r)}.$

ODE/IM

$$s_k = (h\sqrt{1+M})^k \sigma_k, \quad \Lambda + \alpha_0 \rho = \frac{1}{h\sqrt{1+M}}(l + \rho).$$

$$\alpha_0^2 = \frac{1}{h^2(M+1)}, \quad c_{\text{eff}} = r - \frac{12}{h^2(M+1)}(l + \rho^\vee)^2$$

$$E_6^{(1)}$$

Diagonalization of the $E_6^{(1)}$ linear problem

[KI-Kono-Zhu, work in progress]

27-dimensional representation

$$E_{\alpha_1} = E_{1,2} + E_{12,15} + E_{14,17} + E_{16,19} + E_{18,21} + E_{20,22}$$

$$E_{\alpha_2} = E_{2,3} + E_{10,12} + E_{11,14} + E_{13,16} + E_{21,23} + E_{22,24}$$

$$E_{\alpha_3} = E_{3,4} + E_{8,10} + E_{9,11} + E_{16,18} + E_{19,21} + E_{24,25}$$

$$E_{\alpha_4} = E_{4,5} + E_{6,8} + E_{11,13} + E_{14,16} + E_{17,19} + E_{25,26}$$

$$E_{\alpha_5} = E_{5,7} + E_{8,9} + E_{10,11} + E_{12,14} + E_{15,17} + E_{26,27}$$

$$E_{\alpha_6} = E_{4,6} + E_{5,8} + E_{7,9} + E_{18,20} + E_{21,22} + E_{23,24}$$

$$E_{-\theta} = E_{20,1} + E_{22,2} + E_{24,3} + E_{25,4} + E_{26,5} + E_{27,7}$$

$$\left(\epsilon \partial_x + \epsilon \frac{l \cdot H}{x} + \sum_{i=1}^r E_{\alpha_i} + p(x) E_{\alpha_0} \right) \Psi(x) = 0$$

gauge transformation $A^T = T_{27}^{-1} A T_{27} + \epsilon T_{27}^{-1} \partial T_{27}$

Riccati Eqs. $(A^T)_{27,1} = \cdots (A^T)_{27,26} = 0$ for gauge parameters $g_1, \dots, g_{26}$.

WKB analysis

$$g_{26}(x) = \sum_{n=0}^{\infty} \epsilon^n g_{26}^{(n)}(x)$$

$$g_{26} = s_{26}, \quad p(x) = \frac{-45 - 26\sqrt{3}}{9} (s_{26})^{12}$$

$$g_{26}^{(1)} = -v_5 + 8 \frac{\partial s_{26}}{s_{26}}$$

$$g_{26}^{(2)} = \frac{3 + \sqrt{3}}{24} \left(\frac{2W_2}{s_{26}} + 39 \frac{\partial^2 s_{26}}{s_{26}^2} \right) + \partial(*)$$

$$g_{26}^{(3)} = \partial(*)$$

$$g_{26}^{(4)} = \partial(*)$$

$$g_{26}^{(5)} = -\frac{3 + 2\sqrt{3}}{6} \frac{W_5}{s_{26}^4} + \partial(*)$$

W_2, W_5 spin 2, 5 Casimirs of l

Outlook

- WKB analysis for exceptional , non-simply-laced and twisted affine Lie algebras
- Eigenvalues of IMs in $WE_6^{(1)}$ minimal model
- quantum period in E -type Argyres-Douglas theories
 E -type TBA equation
 $(A_2, A_3) \sim (E_6, A_1)$
- Langlands duality $W\hat{\mathfrak{g}}^\vee$ -minimal model $\iff$ linear problem based on $\hat{\mathfrak{g}}$