Applications of the modified algebraic Bethe ansatz :

Full Counting Statistics for the XXX Twisted Spin Chain
And
Rectangular 6-Vertex Model with General Boundary Conditions

arXiv:2310.05850
arXiv:1908.00032
arXiv:1906.06897 in collaboration with **R. Pimenta, N. Slavnov, B. Vallet.**
arXiv:1805.11323
arXiv:1804.00597
arXiv:1506.06550

To appear (2025) **M. Cornillault and A. Hutsalyuk**

Samuel Belliard. IDP Tours France.

Bologna Workshop on:
CFT AND INTEGRABLE MODELS
and their applications from gauge/gravity dualities to statistical mechanics and quantum information

INFN Istituto Nazionale di Fisica Nucleare

INSTITUT DENIS POISSON ORLÉANS • TOURS Mathématiques & Physique Théorique

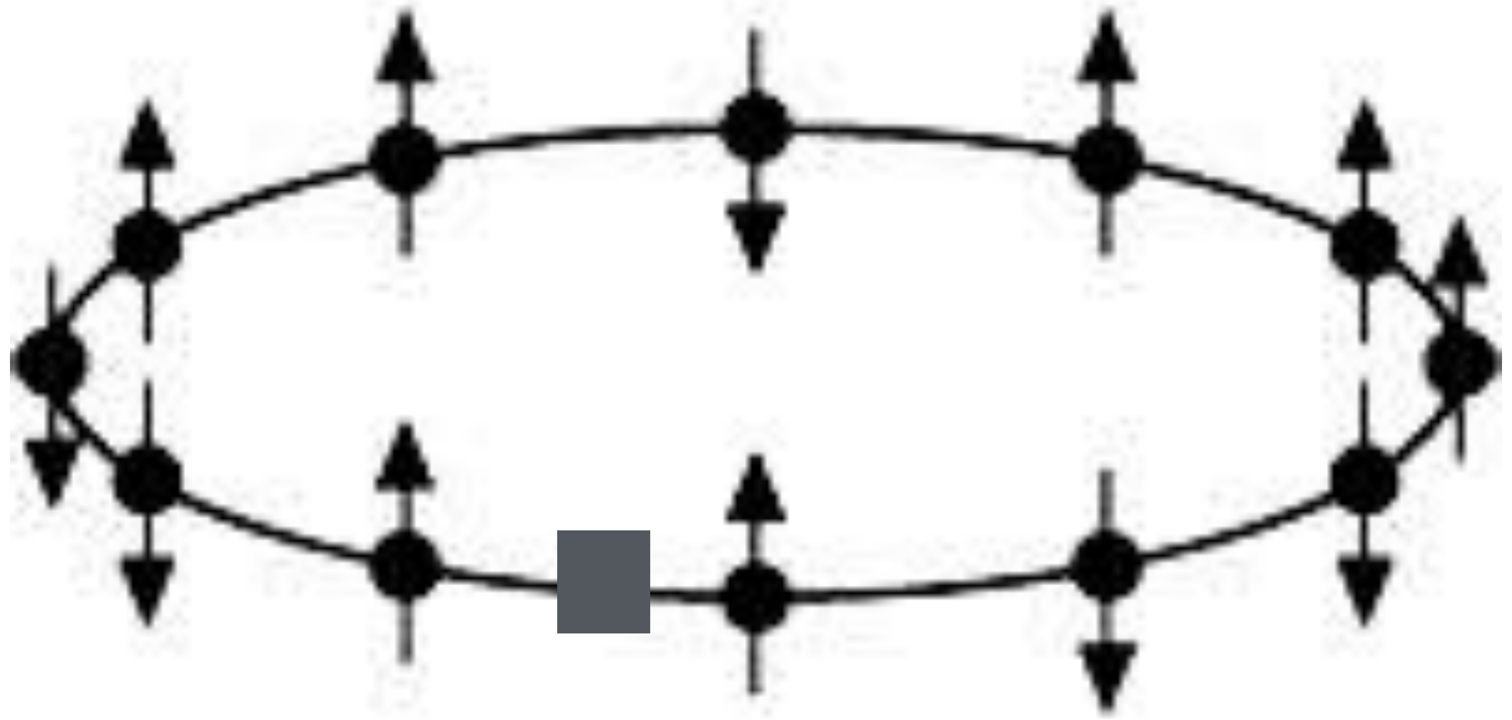
Plan of the Talk

- MABA for the twisted XXX spins chain
- Modified Slavnov's formula
- Full counting statistics for the twisted XXX spins chain (with A. Hutsalyuk)
- Rectangular 6-vertex model with general boundaries conditions (with M. Cornillault)
- Open problems

MABA for twisted XXX spins chain

Conjecture (B, Pimenta 2015)

Proof (B, Slavnov 2018)



Twisted one dimensional spin chains

[Heisenberg 28],[Bethe 31],[Sklyanin, Takhtadzhyan, Faddeev 79], [Slavnov 89],…

$$R(u) = u + P \qquad R_{ab}(u-v)R_{ac}(u-w)R_{bc}(v-w) = R_{bc}(v-w)R_{ac}(u-w)R_{ab}(u-v).$$

R matrix

Star-triangle, Yang-Baxter equation.

[Yang 67][Baxter 72]

Quantum group : The Yangian

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v)$$

Monodromy matrix

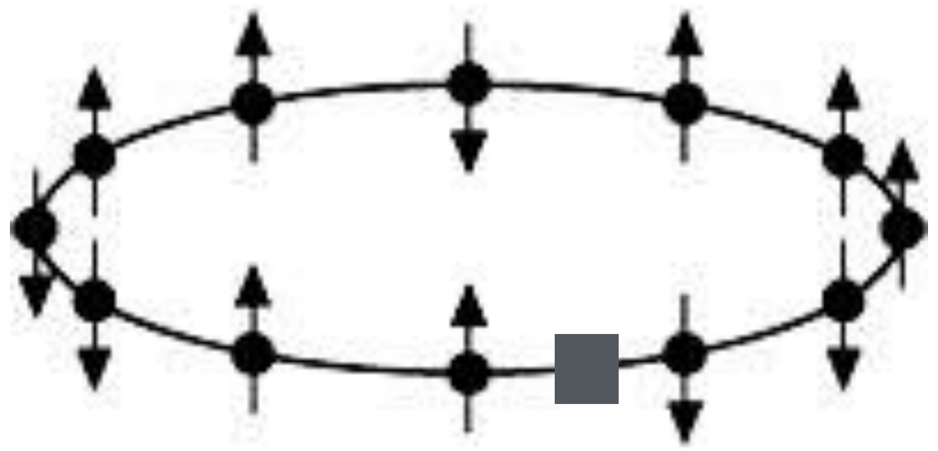
$$T_a(u) = R_{a1}(u)\dots R_{aN}(u) \qquad T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Transfer matrix

$$t(u) = tr(KT(u))$$

Twisted XXX spins chain

$$H = 2\,\frac{d}{du}\Big(\ln\left(t(u)\right)\Big)\Big|_{u\to 0} - N$$



$$H = \sum_{k=1}^{N}\left(\sigma_k^x\otimes\sigma_{k+1}^x + \sigma_k^y\otimes\sigma_{k+1}^y + \sigma_k^z\otimes\sigma_{k+1}^z\right)$$

$$\sigma_{N+1}^{\alpha} = w_{\alpha\beta}\sigma_1^{\beta}$$

Modified operators

« Mysterious » Twist decomposition

$$K = BDA = \begin{pmatrix} \tilde{\kappa} & \kappa^{+} \\ \kappa^{-} & \kappa \end{pmatrix}$$

$$D = \begin{pmatrix} \tilde{\kappa} - \rho_1 & 0 \\ 0 & \kappa - \rho_2 \end{pmatrix}$$

$$A = \sqrt{\mu} \begin{pmatrix} 1 & \frac{\rho_2}{\kappa^-} \\ \frac{\rho_1}{\kappa^+} & 1 \end{pmatrix}, \qquad B = \sqrt{\mu} \begin{pmatrix} 1 & \frac{\rho_1}{\kappa^-} \\ \frac{\rho_2}{\kappa^+} & 1 \end{pmatrix}, \qquad \mu = \frac{1}{1 - \frac{\rho_1 \rho_2}{\kappa^+ \kappa^-}},$$

One constraint, one free parameter

$$\kappa^+ \kappa^- - (\rho_1 \kappa + \rho_2 \tilde{\kappa}) + \rho_1 \rho_2 = 0.$$

Modified monodromy matrix

$$\bar{T}(z) = AT(z)B = \begin{pmatrix} \nu_{11}(z) & \nu_{12}(z) \\ \nu_{21}(z) & \nu_{22}(z) \end{pmatrix}$$

Modified transfert matrix

$$t(z) = \mathrm{Tr}(D\bar{T}(z)) = (\tilde{\kappa} - \rho_1)\nu_{11}(z) + (\kappa - \rho_2)\nu_{22}(z).$$

Modified representation

$$\nu_{11}(z)|0\rangle = \lambda_1(z)|0\rangle + \frac{\rho_2}{\kappa^+}\nu_{12}(z)|0\rangle,$$
$$\nu_{22}(z)|0\rangle = \lambda_2(z)|0\rangle + \frac{\rho_1}{\kappa^+}\nu_{12}(z)|0\rangle,$$
$$\nu_{21}(z)|0\rangle = \left(\frac{\rho_1}{\kappa^+}\lambda_1(z) + \frac{\rho_2}{\kappa^+}\lambda_2(z)\right)|0\rangle + \frac{\rho_1\rho_2}{(\kappa^+)^2}\nu_{12}(z)|0\rangle.$$

Off shell Bethe vectors

$$\nu_{12}(\bar{u})|0\rangle$$

$$\#\bar{u} = M$$

Off shell action of the transfer matrix

$$t(z)\nu_{12}(\bar{u})|0\rangle = \frac{\kappa^-}{\mu}\nu_{12}(z)\nu_{12}(\bar{u})|0\rangle + \Lambda_1^M(z,\bar{u})\nu_{12}(\bar{u})|0\rangle$$
$$+\sum_{i=1}^{M} g(u_i,z)E_1^M(u_i,\bar{u}_i)\nu_{12}(z)\nu_{12}(\bar{u}_i)|0\rangle,$$

$$\Lambda_1^M(z,\bar{u}) = (\tilde{\kappa}-\rho_1)\lambda_1(z)f(\bar{u},z) + (\kappa-\rho_2)\lambda_2(z)f(z,\bar{u}),$$
$$E_1^M(u_i,\bar{u}_i) = (\kappa-\rho_2)\lambda_2(u_i)f(u_i,\bar{u}_i) - (\tilde{\kappa}-\rho_1)\lambda_1(u_i)f(\bar{u}_i,u_i).$$

For finite dimensional representation

$$\sum_{i=1}^{N} 2s_i = S$$

$$\frac{\kappa^-}{\mu}\nu_{12}(z)\nu_{12}(\bar{u}) = (\rho_1+\rho_2)\left(F(z)g(z,\bar{u})\nu_{12}(\bar{u}) + \sum_{i=1}^{S} g(u_i,z)F(u_i)g(u_i,\bar{u}_i)\nu_{12}(z)\nu_{12}(\bar{u}_i)\right)$$

Fixed number of Bethe roots

$$\#\bar{u} = \sum_{i=1}^{N} 2s_i = S$$

$$t(z)\nu_{12}(\bar{u})|0\rangle = \Lambda(z,\bar{u})\nu_{12}(\bar{u})|0\rangle + \sum_{i=1}^{S} g(u_i,z)E(u_i,\bar{u}_i)\nu_{12}(z)\nu_{12}(\bar{u}_i)|0\rangle,$$

$$\Lambda(z,\bar{u}) = (\tilde{\kappa}-\rho_1)\lambda_1(z)f(\bar{u},z) + (\kappa-\rho_2)\lambda_2(z)f(z,\bar{u}) + (\rho_1+\rho_2)F(z)g(z,\bar{u}),$$
$$E(u_i,\bar{u}_i) = (\kappa-\rho_2)\lambda_2(u_i)f(u_i,\bar{u}_i) - (\tilde{\kappa}-\rho_1)\lambda_1(u_i)f(\bar{u}_i,u_i) + (\rho_1+\rho_2)F(u_i)g(u_i,\bar{u}_i).$$

Produce the inhomogeneous TQ equation (Off diagonal Bethe ansatz Cao & al, 13)

Slavnov formula

$$S_K(\bar{u};\bar{v}) = \langle 0|\bar{B}(\bar{u})|0\rangle \frac{\Delta(\bar{v})\Delta'(\bar{u})d(\bar{u})}{g(\bar{v},\bar{u})} \det\left(c\frac{\partial\Lambda_K(v_k|\bar{u})}{\partial u_j}\right),$$

Conjecture (B, Pimenta 2015)

Proof from linear system method (Cramer's rule). (B, Slavnov 2019)

$$\sum_{i=0}^{N} M_{ji}X_i = 0.$$

$$X_i = S_K(\bar{u}_i,\bar{v})$$

Full counting statistic

(B, Hutsalyuk to appear)

$$\exp\left(Q^{(\ell)}(\bar{\beta})\right) = \exp\left(\sum_{j=1}^{\ell} Q_j(\bar{\beta})\right),$$

$$Q(\bar{\beta}) = (\beta_x\sigma^x + \beta_y\sigma^y + \beta_z\sigma^z)$$

$$\left\langle \exp\left(Q^{(\ell)}(\bar{\beta})\right) \right\rangle_K = \frac{\langle K| \exp\left(Q^{(\ell)}(\bar{\beta})\right) |K\rangle}{\langle K|K\rangle},$$

$$\tilde{K} = K\exp(-Q(\bar{\beta})),$$

$$(t_{\tilde{K}}(0))^{-l}(t_K(0))^l = \prod_{i=1}^{l} \tilde{K}_i^{-1}K_i = \prod_{i=1}^{l} \exp(Q_i(\bar{\beta})).$$

Inverse problem (Maillet Terras 00)

$$\left\langle \exp\left(Q^{(\ell)}(\bar{\beta})\right) \right\rangle_K = \sum_{\tilde{K}} \left(\frac{\Lambda_K(0)}{\Lambda_{\tilde{K}}(0)}\right)^l \frac{\langle K|\tilde{K}\rangle\langle\tilde{K}|K\rangle}{\langle K|K\rangle\langle\tilde{K}|\tilde{K}\rangle}.$$

$$\left\langle \exp\left(Q^{(\ell)}(\bar{\beta})\right)\right\rangle_K = \sum_{\tilde{K}} \left(\frac{\Lambda_K(0)}{\Lambda_{\tilde{K}}(0)}\right)^l \frac{\langle K|\tilde{K}\rangle\langle\tilde{K}|K\rangle}{\langle K|K\rangle\langle\tilde{K}|\tilde{K}\rangle}.$$

Calculation of the overlap from Modified Slavnov formula

Use free parameter of the twist decomposition

$$\bar{B}(u)/\mu = \tilde{B}(u)/\tilde{\mu},$$

$$\frac{\rho_1}{\kappa_-} = \frac{\tilde{\rho}_1}{\tilde{\kappa}_-}, \qquad \frac{\rho_2}{\kappa_-} = \frac{\tilde{\rho}_2}{\tilde{\kappa}_-},$$

$$\rho_1\rho_2 + \kappa_-\kappa_+ - (\kappa_1\rho_2 + \kappa_2\rho_1) = 0, \qquad \tilde{\rho}_1\tilde{\rho}_2 + \tilde{\kappa}_-\tilde{\kappa}_+ - (\tilde{\kappa}_1\tilde{\rho}_2 + \tilde{\kappa}_2\tilde{\rho}_1) = 0.$$

$$\left\langle \exp\left(Q^{(\ell)}(\bar\beta)\right)\right\rangle_K = \sum_{\tilde K}\left(\frac{\Lambda_K(0)}{\Lambda_{\tilde K}(0)}\right)^l \frac{\langle K|\tilde K\rangle\langle\tilde K|K\rangle}{\langle K|K\rangle\langle\tilde K|\tilde K\rangle}.$$

Calculation of the overlap from Modified Slavnov formula

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$$\bar B(u)/\mu = \tilde B(u)/\tilde\mu,$$

$$\frac{\rho_1}{\kappa_-} = \frac{\tilde\rho_1}{\tilde\kappa_-}, \qquad \frac{\rho_2}{\kappa_-} = \frac{\tilde\rho_2}{\tilde\kappa_-},$$

$$\rho_1\rho_2 + \kappa_-\kappa_+ - (\kappa_1\rho_2 + \kappa_2\rho_1) = 0, \qquad \tilde\rho_1\tilde\rho_2 + \tilde\kappa_-\tilde\kappa_+ - (\tilde\kappa_1\tilde\rho_2 + \tilde\kappa_2\tilde\rho_1) = 0.$$

$$\langle\tilde K, \bar v|\bar u, K\rangle = \langle 0|\tilde C(\bar v)\bar B(\bar u)|0\rangle = (\mu/\tilde\mu)^L\langle 0|\tilde C(\bar v)\tilde B(\bar u)|0\rangle = (\mu/\tilde\mu)^L S_{\tilde K}(\bar v|\bar u).$$

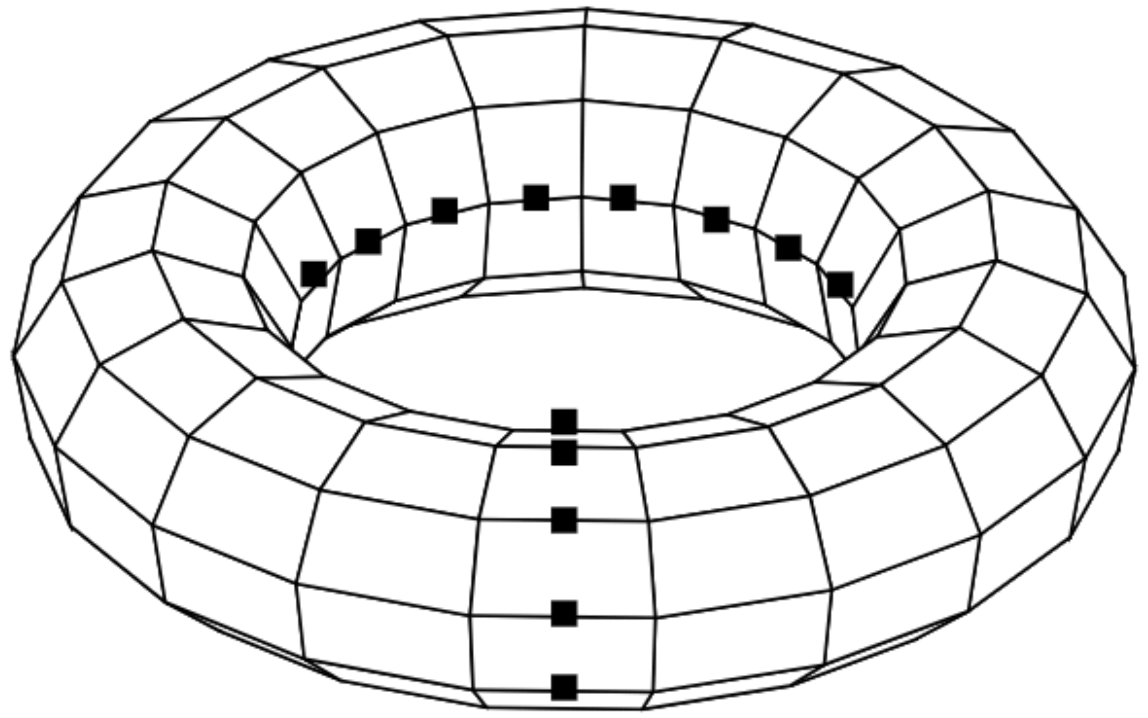
$$\langle K, \bar u|\bar v, \tilde K\rangle = \langle 0|\bar C(\bar u)\tilde B(\bar v)|0\rangle = (\tilde\mu/\mu)^L\langle 0|\bar C(\bar u)\bar B(\bar v)|0\rangle = (\tilde\mu/\mu)^L S_K(\bar u|\bar v).$$

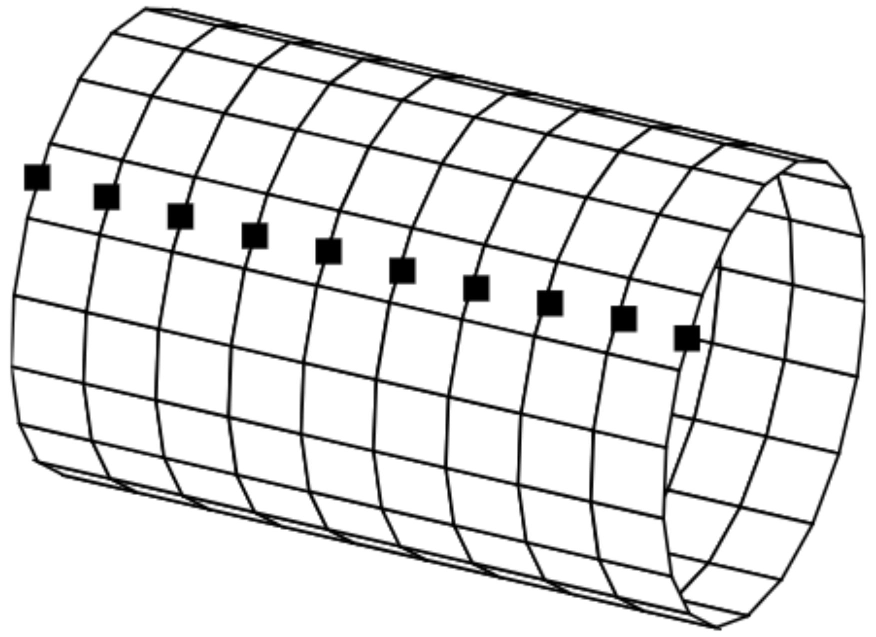
Close formulas fonctions of Bethe roots

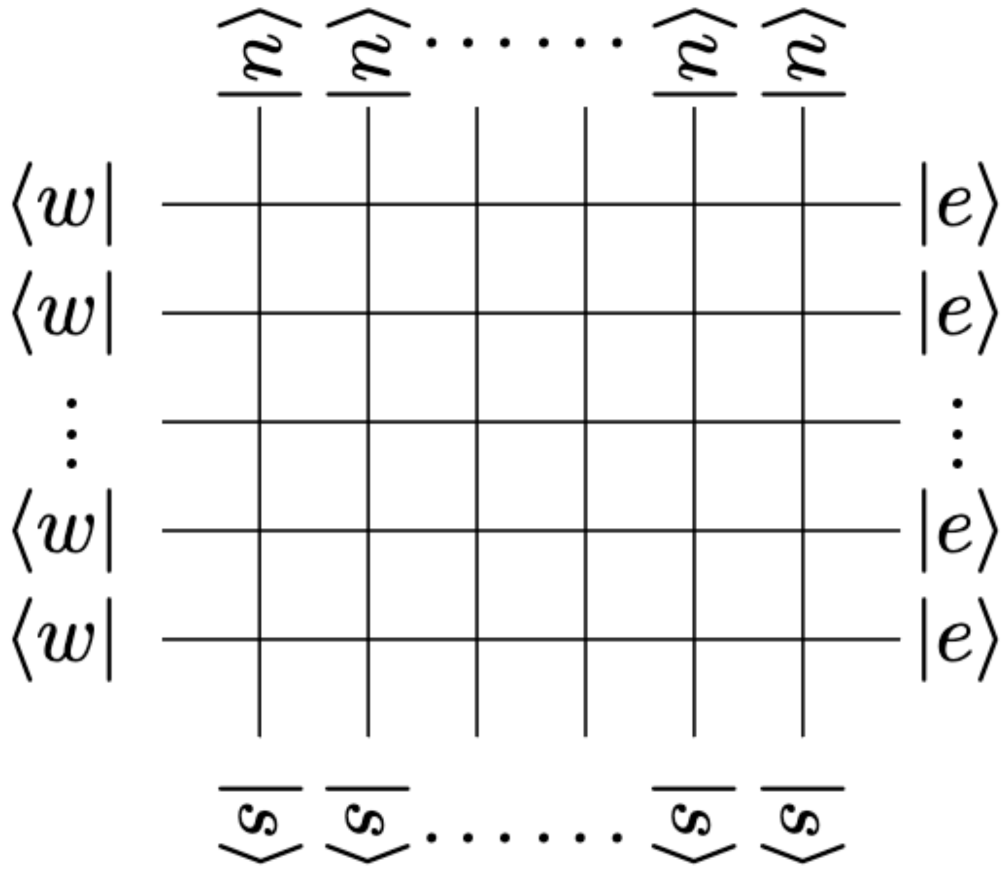
$$\Big\langle \exp\left(Q^{(\ell)}(\bar\beta)\right)\Big\rangle_K = \sum_{\bar v} \frac{1}{g(\bar u,\bar v)g(\bar v,\bar u)}\left(\frac{\Lambda_K(0|\bar u)}{\Lambda_{\tilde K}(0|\bar v)}\right)^{\ell} \frac{\det_L\left(c\frac{\partial \Lambda_K(v_k|\bar u)}{\partial u_j}\right)\det_L\left(c\frac{\partial \Lambda_{\tilde K}(u_k|\bar v)}{\partial v_j}\right)}{\det_L\left(c\frac{\partial \mathcal{Y}_K(u_k|\bar u)}{\partial u_j}\right)\det_L\left(c\frac{\partial \mathcal{Y}_{\tilde K}(v_k|\bar v)}{\partial v_j}\right)},$$

$$\Big\langle \exp\left(Q^{(\ell)}(\bar\beta)\right)\Big\rangle_K = \frac{1}{L!}\oint_{\bar v} \frac{d\bar z}{(2\pi)^L}\frac{1}{g(\bar u,\bar z)g(\bar z,\bar u)}\left(\frac{\Lambda_K(0|\bar u)}{\Lambda_{\tilde K}(0|\bar z)}\right)^{\ell} \frac{\det_L\left(c\frac{\partial \Lambda_K(z_k|\bar u)}{\partial u_j}\right)\det_L\left(c\frac{\partial \Lambda_{\tilde K}(u_k|\bar z)}{\partial z_j}\right)}{\det_L\left(c\frac{\partial \mathcal{Y}_K(u_k|\bar u)}{\partial u_j}\right)\prod_{j=1}^{L}\mathcal{Y}_{\tilde K}(z_j|\bar z)}.$$

6-vertex model



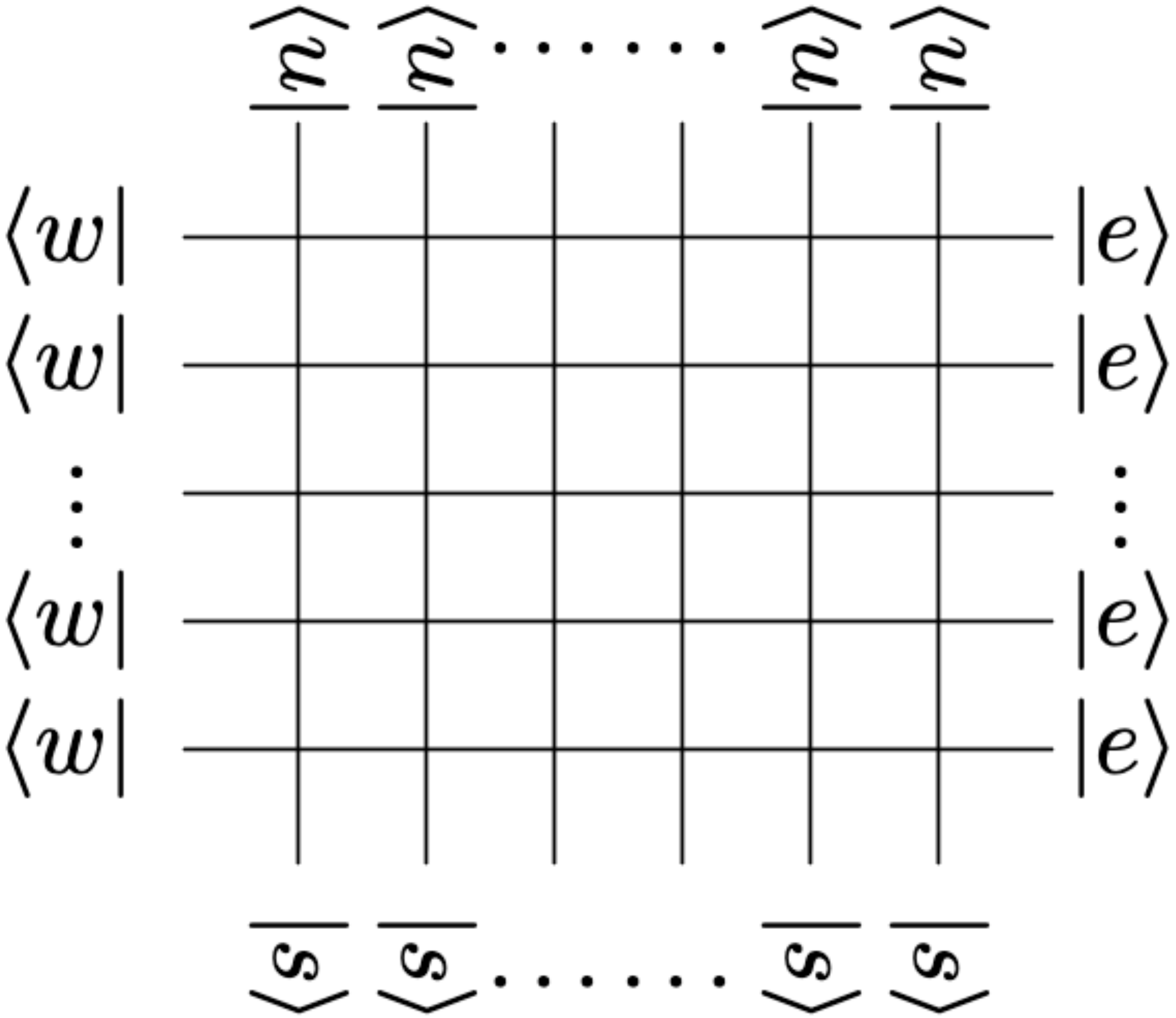




Definition 1.1. *The partition function of the inhomogeneous six vertex model on the $n \times m$ rectangular lattice with general boundary conditions is*

$$Z_{nm}(\bar{u}|\bar{v}|B|\hat{C}) = \operatorname*{tr}_{\bar{a},\bar{b}} \left(\prod_{i=1}^{n} B_{a_i} \prod_{j=1}^{m} \hat{C}_{b_j} \prod_{i=1}^{n} \prod_{j=1}^{m} R_{a_i b_j}(u_i - v_j) \right). \qquad (1.7)$$

Rectangular 6-vertex model with GB



(B, Pimenta, Slavnov 2024)

Definition 1.2. *We define the following twist*

$$B = |e\rangle \otimes \langle w| \quad \text{and} \quad \hat{C} = |s\rangle \otimes \langle n| \,.$$

We can rewrite the partition function with the following formula

$$Z_{nm}(\bar{u}|\bar{v}|B|\hat{C}) = \langle W| \otimes \langle N| \left(\prod_{i=1}^{n} \prod_{j=1}^{m} R_{a_i b_j}(u_i - v_j) \right) |E\rangle \otimes |S\rangle$$

with

$$\langle N| = \bigotimes_{j=1}^{m} \langle n|_{b_j} \,, \quad \langle W| = \bigotimes_{i=1}^{n} \langle w|_{a_i} \,, \quad |S\rangle = \bigotimes_{j=1}^{m} |s\rangle_{b_j} \quad \text{and} \quad |E\rangle = \bigotimes_{i=1}^{n} |e\rangle_{a_i} \,.$$

Proposition 1.1. *[BPS24] The partition function of the inhomogeneous rational six-vertex model with general boundary conditions is*

$$Z_{nm}(\bar{u}|\bar{v}|B|\hat{C}) = \frac{\mathrm{tr}(B)^n \mathrm{tr}(\hat{C})^m}{(1-\beta)^m} \frac{1}{g(\bar{u},\bar{v})} K^{(\beta)}_{nm}(\bar{u}|\bar{v}) \tag{1.12}$$

with

$$\beta = 1 - \frac{\mathrm{tr}(B)\mathrm{tr}(\hat{C})}{\mathrm{tr}(B\hat{C})}, \quad \mathrm{tr}(B) = \langle w|e\rangle\,, \quad \mathrm{tr}(\hat{C}) = \langle n|s\rangle \quad \text{and} \quad \mathrm{tr}(B\hat{C}) = \langle n|e\rangle\,\langle w|s\rangle\,. \tag{1.13}$$

(Gorsky, Zabrodin, Zotov, 2014)

The modified Izergin determinant can be expressed with two formulae [BSV18]

$$K^{(\beta)}_{nm}(\bar{u}|\bar{v}) = \det_{1\leqslant k,l\leqslant m}\left(-\beta\delta_{kl} + \frac{f(\bar{u},v_k)f(v_k,\bar{v}_k)}{h(v_k,v_l)}\right) \tag{1.14}$$

$$= (1-\beta)^{m-n} \det_{1\leqslant k,l\leqslant n}\left(f(u_k,\bar{v})\delta_{ij} - \beta\frac{f(u_k,\bar{u}_k)}{h(u_k,u_l)}\right). \tag{1.15}$$

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Homogeneous limit

(Cornillault, B to appear)

(Foda, Wheeler 2012)

$$x = u - v.$$

$$Z_{nm}(x|B|\hat{C}) = \frac{\mathrm{tr}(B)^n \mathrm{tr}(\hat{C})^m}{(1-\beta)^{\min(n,m)}} \times \det_{1\leqslant i,j\leqslant \min(n,m)}\left(\binom{d+i+j-2}{i-1}\left(\left(1+\frac{x}{c}\right)^{d+i+j-1} - \beta\left(\frac{x}{c}\right)^{d+i+j-1}\right)\right). \tag{2.7}$$

with $d = |n-m|$.

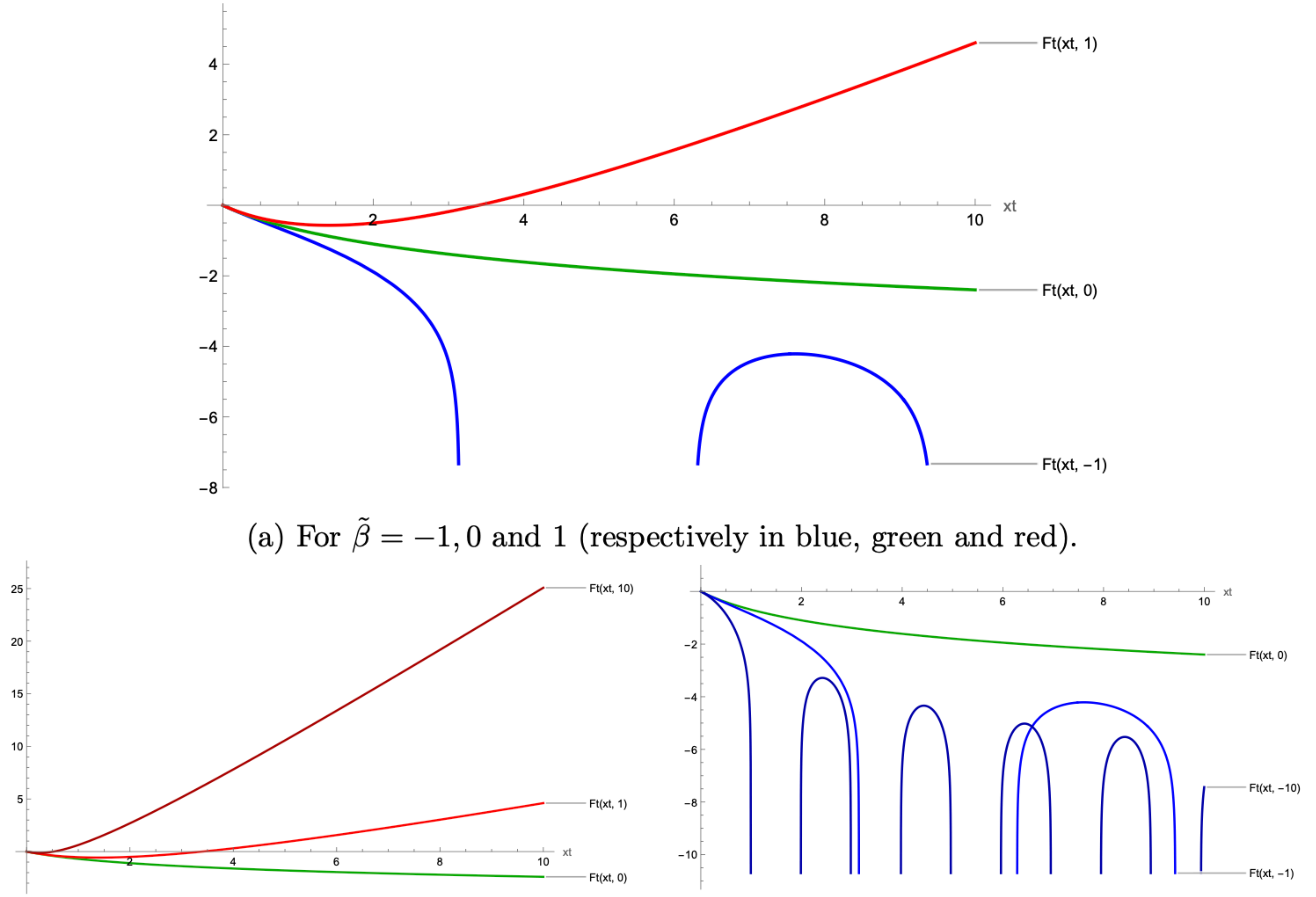
Thermodynamic limit

(Cornillault, B to appear) (Korepin Zinn-Justin 00)

$$\ln(Z_{nm}(x|B|\hat{C})) = -nm\frac{F(x)}{k_B T} + o(nm)$$

$$\tilde{\beta} = \begin{cases} 0 & \text{if } d > 1 \\ 3\beta & \text{if } d = 1 \\ 3\beta(\beta - 2) & \text{if } d = 0 \end{cases}$$

$$\tilde{F}(\tilde{x}) = \frac{F(c\tilde{x})}{k_B T} = \begin{cases} \ln\left(\frac{\sinh\left(\sqrt{\tilde{\beta}}\tilde{x}\right)}{\sqrt{\tilde{\beta}}\tilde{x}}\frac{1}{1+\tilde{x}}\right) & \text{if } \tilde{\beta} > 0 \\ \ln\left(\frac{1}{1+\tilde{x}}\right) & \text{if } \tilde{\beta} = 0 \\ \ln\left(\frac{\sin\left(\sqrt{-\tilde{\beta}}\tilde{x}\right)}{\sqrt{-\tilde{\beta}}\tilde{x}}\frac{1}{1+\tilde{x}}\right) & \text{if } \tilde{\beta} < 0 \end{cases}.$$



(a) For $\tilde{\beta} = -1, 0$ and 1 (respectively in blue, green and red).

(b) For $\tilde{\beta} = 0, 1$ and 10 (respectively in green, red and dark-red).

(c) For $\tilde{\beta} = 0, -1$ and -10 (respectively in green, blue and dark-blue).

Figure 4: Graphs of $\tilde{F}$ for different values of $\tilde{\beta}$. "xt" corresponds to $\tilde{x}$ and "Ft" to $\tilde{F}$

Some open problems

- XXZ and XYZ cases
- Thermodynamical limit
- Open boundaries cases
- Higher rank cases
- « Et au-delà ? »

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Grazie per l'attenzione !