UNIVERSITY OF BIRMINGHAM

LEVERHULME TRUST

Quantum and Classical Dynamics with Random Permutation Circuits

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Motivation: quantum many-body dynamics and thermalization

not simple in eigenbasis of H
short-range correlated

Quantum quench: $|\Psi_0\rangle \to |\Psi_t\rangle = \mathrm{e}^{-itH}|\Psi_0\rangle$

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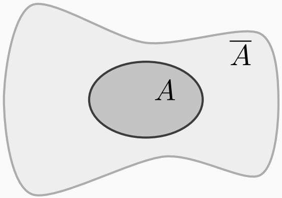

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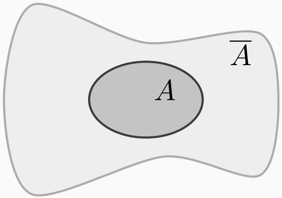

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Many subtleties: conservation laws, localisation, scars, ...

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H: local

$|\Psi_0\rangle$: **not** simple in eigenbasis of H, short-range correlated

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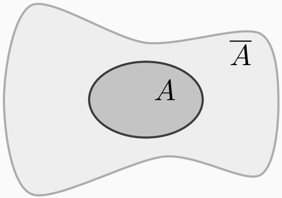

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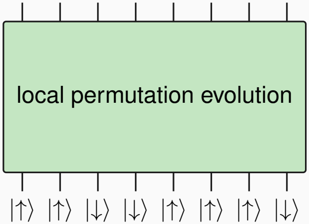
Q: What happens under classical deterministic dynamics? $\Leftarrow \exists$ a basis : $|\underline{s}\rangle \to |\underline{s}'\rangle$

Classical deterministic dynamics

$|\Psi_t\rangle =$



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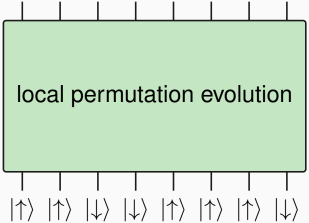

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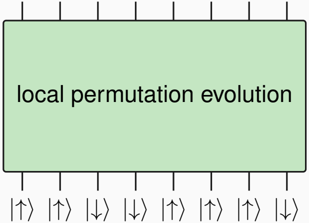

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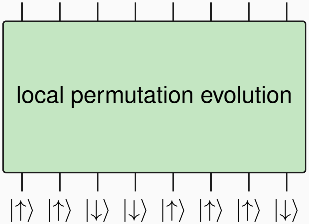

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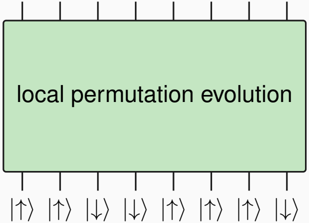
$|\Psi_t\rangle =$ 

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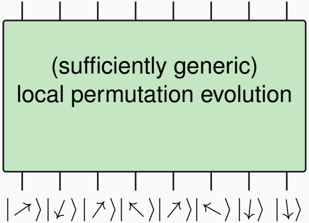
$|\Psi_t\rangle =$ 

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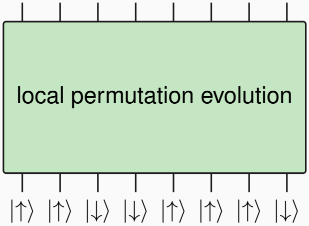

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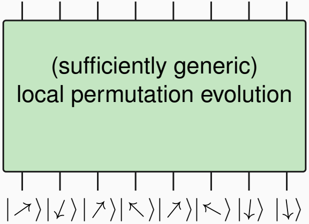

$\neq$ product state $\Longrightarrow$ $\langle\Psi_t|\mathcal{O}_A|\Psi_t\rangle \xrightarrow[?]{t\to\infty} \mathrm{tr}[\rho_{\mathrm{th}}\mathcal{O}_A]$

How “generic” can classical deterministic evolution be?

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chain of L qudits with locally applied randomly chosen permutations

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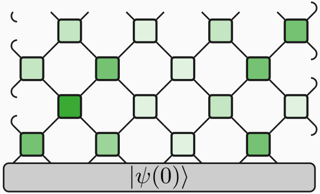

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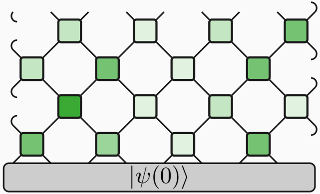

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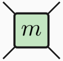
How do the averaged quantities (e.g. OTOCs and purity) behave?

Technical tool: Averages over the permutation group I

Basic building block:

$$\mathcal{Q}^{(m)} := \frac{1}{d^2!} \sum_{U \in S(d^2)} \underbrace{U \otimes U \otimes U \otimes \cdots \otimes U}_{m} =$$



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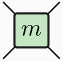
$$\mathcal{Q}^{(m)} := \frac{1}{d^2!} \sum_{U \in S(d^2)} \underbrace{U \otimes U \otimes U \otimes \cdots \otimes U}_{m} =$$ 

Detour: Uniform averages over $d \times d$ unitary matrices

$$:= \int_{U(d)} \mathrm{d}U \; \underbrace{U \otimes U^* \otimes U \otimes \cdots \otimes U^*}_{2n}$$

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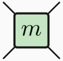
$$\mathcal{Q}^{(m)} := \frac{1}{d^2!} \sum_{U \in S(d^2)} \underbrace{U \otimes U \otimes U \otimes \cdots \otimes U}_{m} =$$ 

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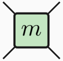
$$\mathcal{Q}^{(m)} := \frac{1}{d^2!} \sum_{U \in S(d^2)} \underbrace{U \otimes U \otimes U \otimes \cdots \otimes U}_{m} =$$ 

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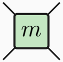
Basic building block: $\mathcal{Q}^{(m)} := \frac{1}{d^2!} \sum_{U \in S(d^2)} \underbrace{U \otimes U \otimes U \otimes \cdots \otimes U}_{m} =$ 

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- Symmetry under reshuffling of replicas: $|\sigma\rangle = \sum_{s_1,\ldots,s_n=0}^{d-1} \left| s_1 s_{\sigma(1)} s_2 s_{\sigma(2)} \ldots s_n s_{\sigma(n)} \right\rangle \quad \forall \sigma \in S(n)$

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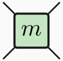

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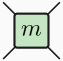

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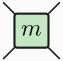

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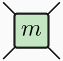

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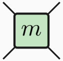

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Example: $2n = m = 4$

$B_4 = 15$ partition states (and $2! = 2$ permutation states)

$\pi_1 = \{\{0,1,2,3\}\}$ $\quad$ $\pi_2 = \{\{0,1,2\},\{3\}\}$

$\pi_3 = \{\{0,1,3\},\{2\}\}$ $\quad$ $\pi_4 = \{\{0,2,3\},\{1\}\}$

$\pi_5 = \{\{1,2,3\},\{0\}\}$ $\quad$ $\pi_6 = \{\{0,1\},\{2,3\}\}$

$\pi_7 = \{\{0,3\},\{1,2\}\}$ $\quad$ $\pi_8 = \{\{0,2\},\{1,3\}\}$

$\pi_9 = \{\{0,1\},\{2\},\{3\}\}$ $\quad$ $\pi_{10} = \{\{0,2\},\{1\},\{3\}\}$

$\pi_{11} = \{\{0,3\},\{1\},\{2\}\}$ $\quad$ $\pi_{12} = \{\{1,3\},\{0\},\{2\}\}$

$\pi_{13} = \{\{1,2\},\{0\},\{3\}\}$ $\quad$ $\pi_{14} = \{\{2,3\},\{0\},\{1\}\}$

$\pi_{15} = \{\{0\},\{1\},\{2\},\{3\}\}$

Example: $2n = m = 4$

$B_4 = 15$ partition states (and $2! = 2$ permutation states)

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$\pi_2 = \{\{0,1,2\},\{3\}\}$

$\pi_3 = \{\{0,1,3\},\{2\}\}$

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$$|\bigcirc\rangle = \frac{1}{d}\sum_{a,b=0}^{d-1} |aabb\rangle$$

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Averaged infinite-temperature two-point correlation function I

$$C_{\mu\nu}(x,t) = \frac{1}{\mathrm{tr}\,\mathbb{1}} \left\langle \mathrm{tr}[\mathcal{O}_\mu(x,t)\mathcal{O}_\nu(0,0)] \right\rangle_{\mathrm{Q}}$$

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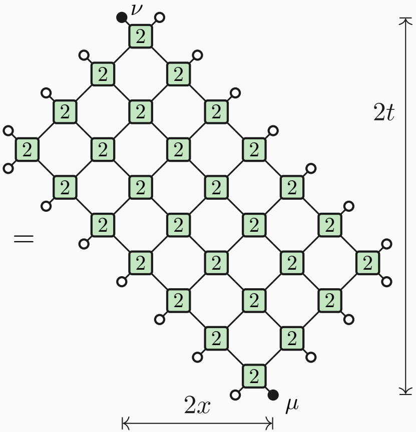
$=$



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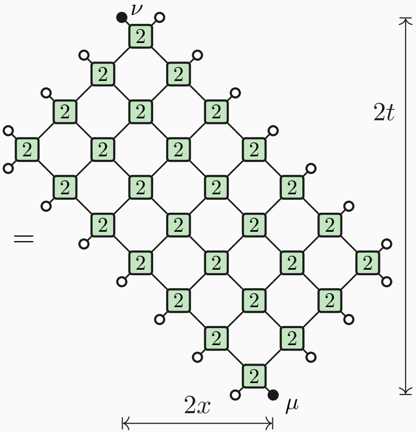

$= \mathcal{Q}^{(2)}$

$$|\bigcirc\rangle = \frac{1}{\sqrt{d}} \sum_{s=0}^{d-1} |ss\rangle \propto |\pi_1\rangle$$

$$|\bullet_\mu\rangle = \mathcal{O}_\mu \otimes \mathbb{1}\,|\bigcirc\rangle$$

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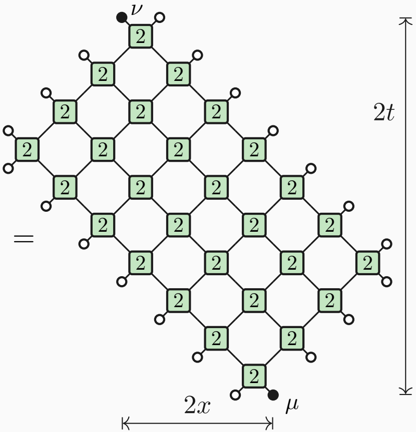
$=$ 

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$B_2 = 2 \implies C_{\mu\nu}(x,t)$ nontrivial

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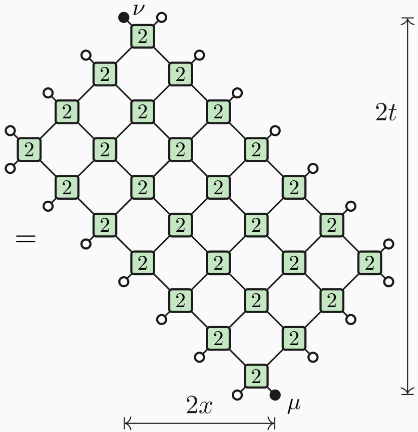
$=$ 

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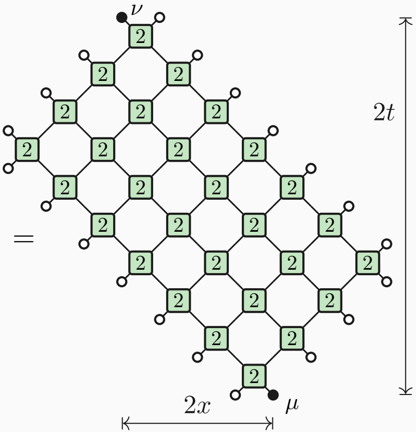

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First step: project $|\bullet_{\mu,\nu}\rangle$ to $\{\, |\bigcirc\rangle\,,\ |-\rangle \}$

Averaged infinite-temperature two-point correlation function II

$$C_{\mu\nu}(x,t) = \frac{o_\mu o_\nu}{(1-1/d)^2}\left[C_{--}(x,t) - \frac{1}{d}\right]$$

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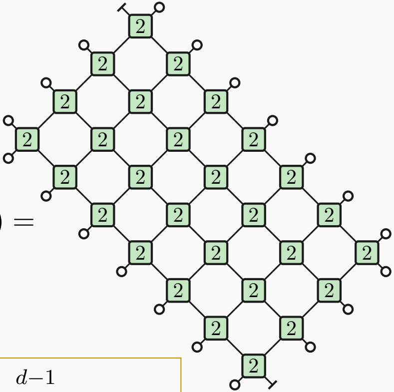

$$o_\mu = \frac{1}{d^{\frac{3}{2}}}\sum_{i,j=0}^{d-1} \langle i|\mathcal{O}_\mu|j\rangle$$

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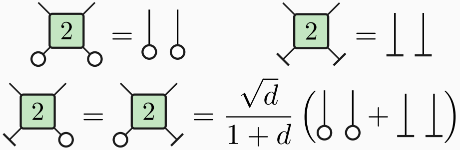

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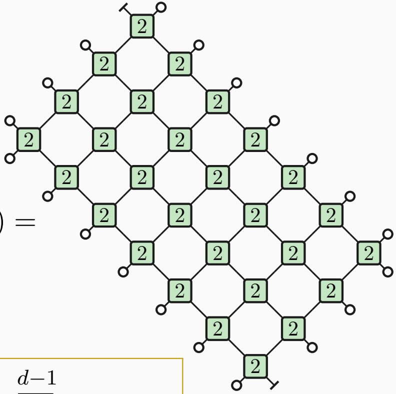

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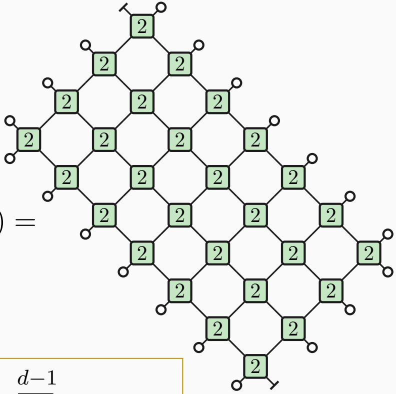

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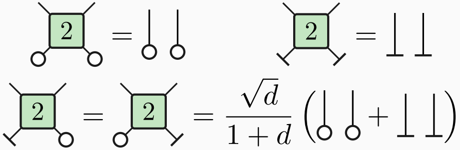

Analogous to $2n = 4$ for random unitaries:

$$\left.\frac{C_{\mu\nu}(x,t)}{o_\mu o_\nu}\right|_{d\to d^2} = \left\langle \left(\frac{\mathrm{tr}[O_\mu(x,t)O_\nu(0,0)]}{\mathrm{tr}\,\mathbb{1}}\right)^2 \right\rangle_{\mathrm{U}}$$

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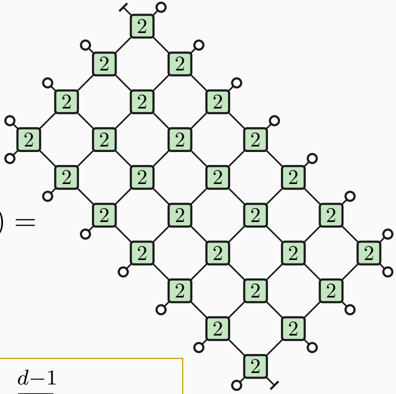
$C_{--}(x,t) =$ 

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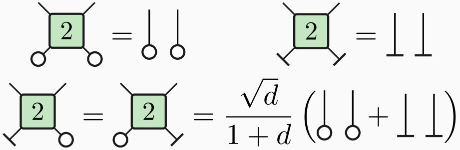

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Asymptotic scaling:

$$C_{\mu\nu}(x,t) \simeq \frac{o_\mu o_\nu c(d)}{t^2}\left(\frac{4/d}{(1+1/d)^2}\right)^{2t} \mathrm{e}^{-2x^2/t}$$

Out-of-time-ordered correlation functions I

$$O_{\mu\nu}(x,t) = \frac{1}{2} \frac{\left\langle \mathrm{tr}\left[\left| [\mathcal{O}_\mu(x,t), \mathcal{O}_\nu(0,0)] \right|^2 \right] \right\rangle_{\mathrm{Q}}}{\mathrm{tr}\, \mathbb{1}}$$

Out-of-time-ordered correlation functions I

$$O_{\mu\nu}(x,t) = \underbrace{\frac{\left\langle \mathrm{tr}\left[\mathcal{O}_\mu^2(x,t)\mathcal{O}_\nu^2(0,0)\right]\right\rangle_{\mathrm{Q}}}{\mathrm{tr}\,\mathbb{1}}}_{\tilde{O}^{(1)}_{\mu\nu}(x,t)\,=1+\text{subleading terms}} - \underbrace{\frac{\langle \mathrm{tr}[\mathcal{O}_\mu(x,t)\mathcal{O}_\nu(0,0)\mathcal{O}_\mu(x,t)\mathcal{O}_\nu(0,0)]\rangle_{\mathrm{Q}}}{\mathrm{tr}\,\mathbb{1}}}_{\tilde{O}^{(2)}_{\mu\nu}(x,t)}$$

$$\mathrm{tr}[\mathcal{O}_\mu] = 0$$
$$\mathrm{tr}\left[\mathcal{O}_\mu^2\right] = d$$

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$\tilde{O}^{(1)}_{\mu\nu}(x,t) = 1 +$ subleading terms

$\tilde{O}^{(2)}_{\mu\nu}(x,t)$ is a $m = 4$ quantity

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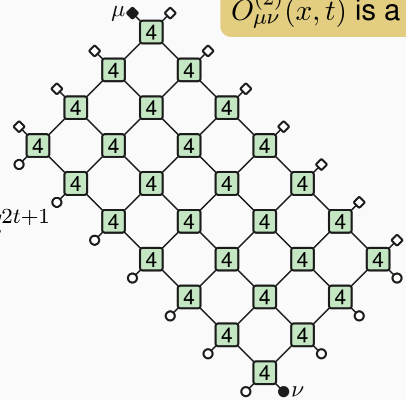

$$\tilde{O}^{(2)}_{\mu\nu}(x,t) = d^{2t+1}$$

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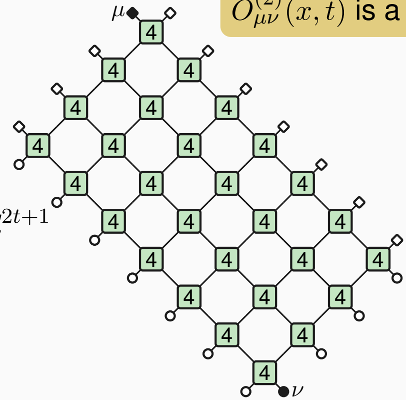
$B_4 = 15$ states!

$\tilde{O}^{(2)}_{\mu\nu}(x,t) = d^{2t+1}$



$$|\bigcirc\rangle = \frac{1}{d}\sum_{a,b=0}^{d-1}|aabb\rangle \qquad |\square\rangle = \frac{1}{d}\sum_{a,b=0}^{d-1}|abba\rangle$$

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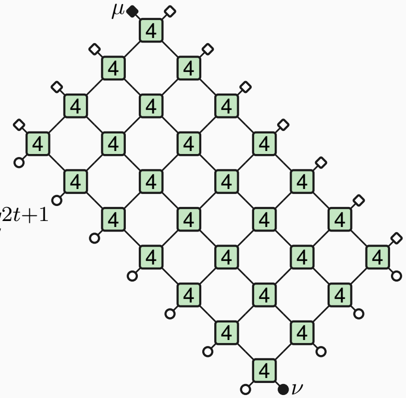
$$|\bullet_\nu\rangle = (\mathcal{O}_\nu\otimes\mathbb{1})^{\otimes 2}\,|\bigcirc\rangle \qquad |\blacksquare_\mu\rangle = (\mathcal{O}_\mu\otimes\mathbb{1})^{\otimes 2}\,|\square\rangle$$

Out-of-time-ordered correlation functions II

Problem: generic $|\blacksquare_\mu\rangle$, $|\CIRCLE_\nu\rangle$ are projected to all 15 states $\Rightarrow \tilde{O}^{(2)}(x,t)$ hard to evaluate

$$\tilde{O}^{(2)}_{\mu\nu}(x,t) = d^{2t+1}$$



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Additional structure for $\mathcal{O}_\nu \to \mathcal{O}_\mathrm{d}$ diagonal

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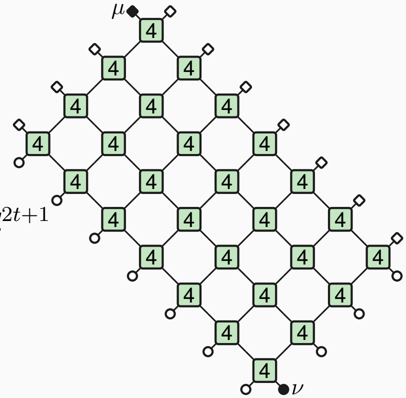

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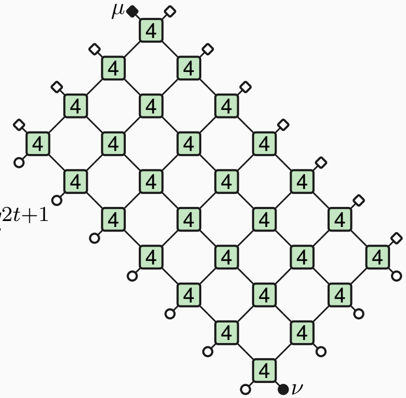

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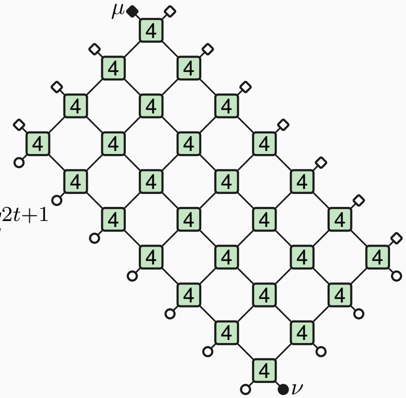

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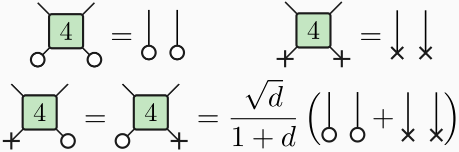
Action of $\mathcal{Q}^{(4)}$ on these states closes:



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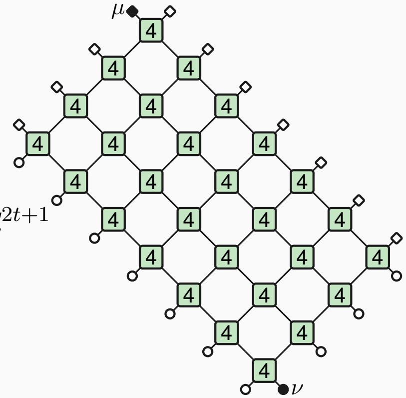

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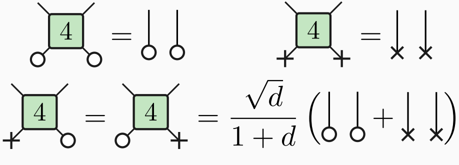

We can obtain the closed-form expression

Out-of-time-ordered correlation functions III

$$C_{\mu\nu}(x,t) = \frac{d^2 o_\mu o_\nu}{d-1} f(x,t) \qquad O_{\mu \mathrm{d}}(x,t) = \frac{1 - o_{\mu,2}}{d-1} \left[f(x,t)\right]_{d \mapsto \frac{1}{d}}$$

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$$o_\mu = \frac{1}{d^{\frac{3}{2}}}\sum_{i,j=0}^{d-1} \langle i|\mathcal{O}_\mu|j\rangle$$

$$o_{\mu,2} = \frac{1}{d}\sum_{i=0}^{d-1} \langle i|\mathcal{O}_\mu|i\rangle^2$$

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$$O^{(\mathrm{RU})}(x,t) = O_{\mu\mathrm{d}}(x,t)\big|_{o_{\mu,2}\mapsto 0, d\mapsto d^2}$$

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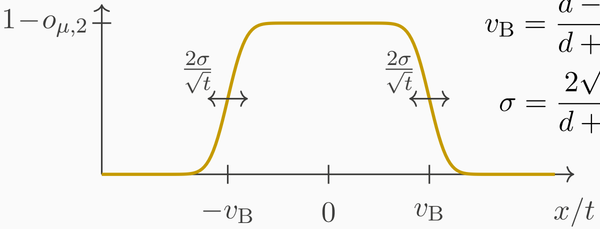
Asymptotic scaling:



$$v_\mathrm{B} = \frac{d-1}{d+1}$$

$$\sigma = \frac{2\sqrt{d}}{d+1}$$

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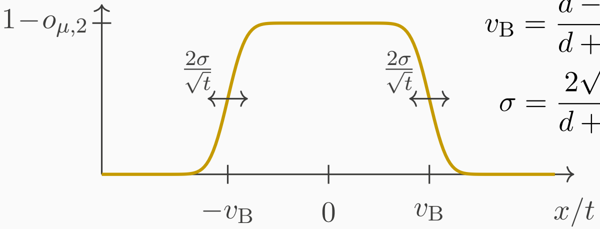
Asymptotic scaling:

Non-generic value inside the light-cone!



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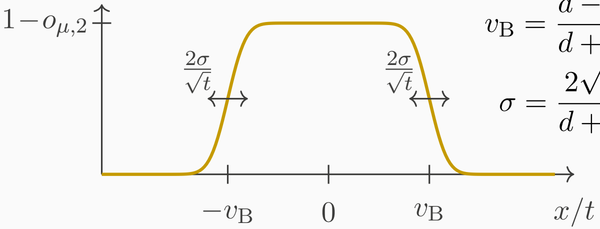
Asymptotic scaling:



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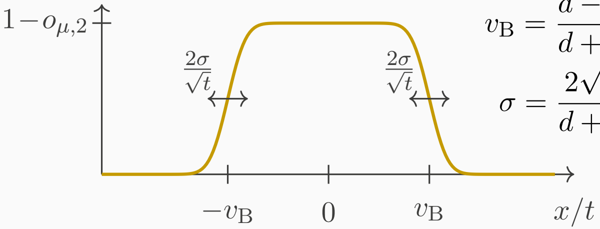

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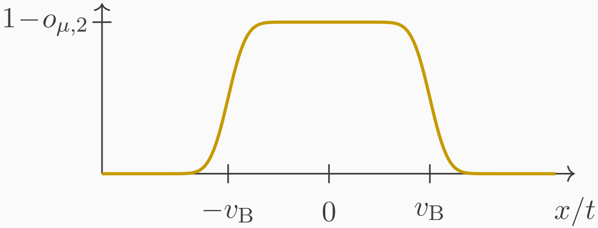

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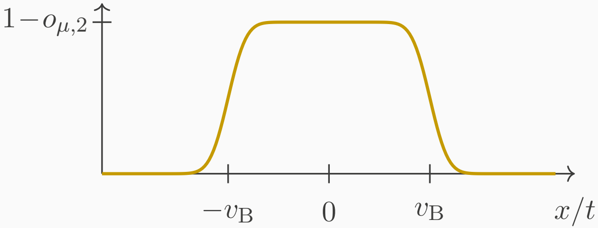

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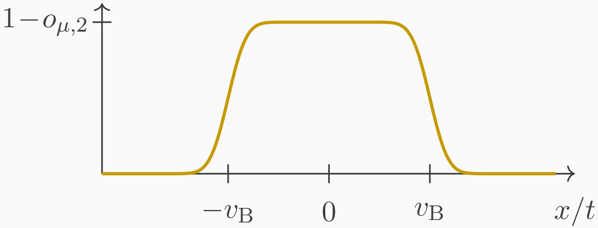

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Higher replica quantities: local operator entanglement

B. Bertini, KK, P. Kos, D. Malz, 2025, arXiv:2508.10890