Chiral Nuclear Forces at High Order

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ERATO Exploratory Research for Advanced Technology

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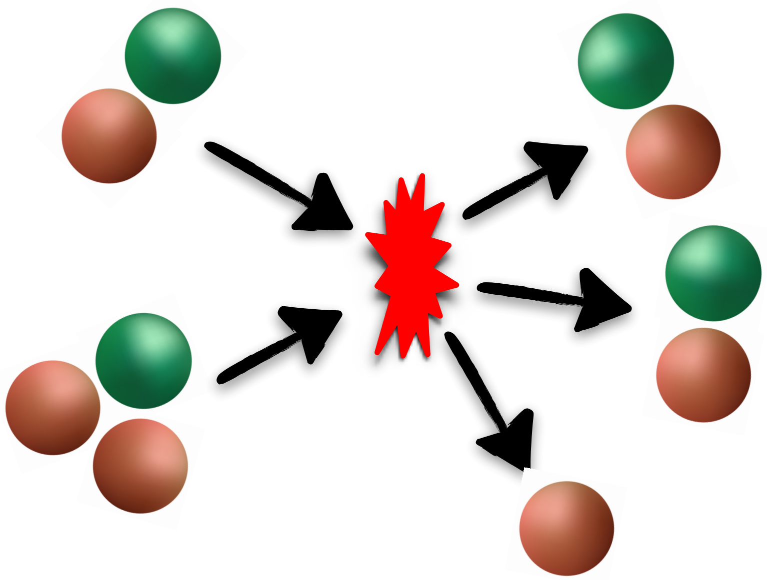
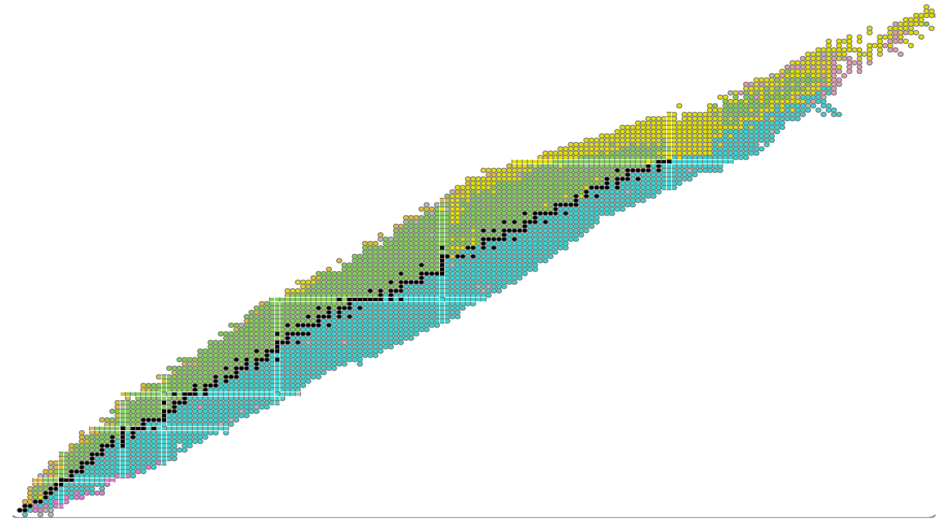
In collaboration with Evgeny Epelbaum

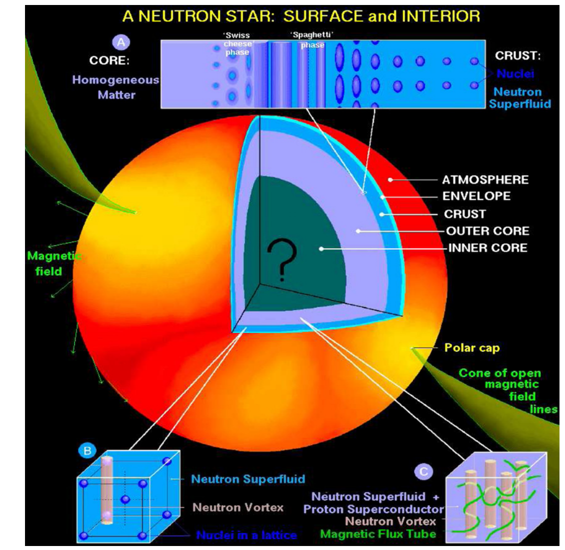
Outline

- Nuclear forces up to N^3LO
- 3PE contributions to NN within unitary transformation method
- Gradient-flow regularizaton in chiral EFT
- Status report on construction of 3NF at N^3LO





Livechart, IAEA: https://www-nds.iaea.org

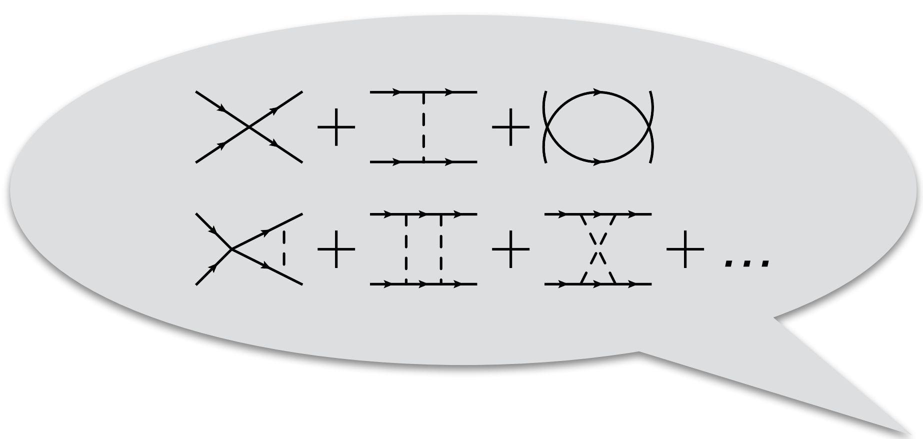


Lattimer: NAR54 (2010) 101

QM A-body problem

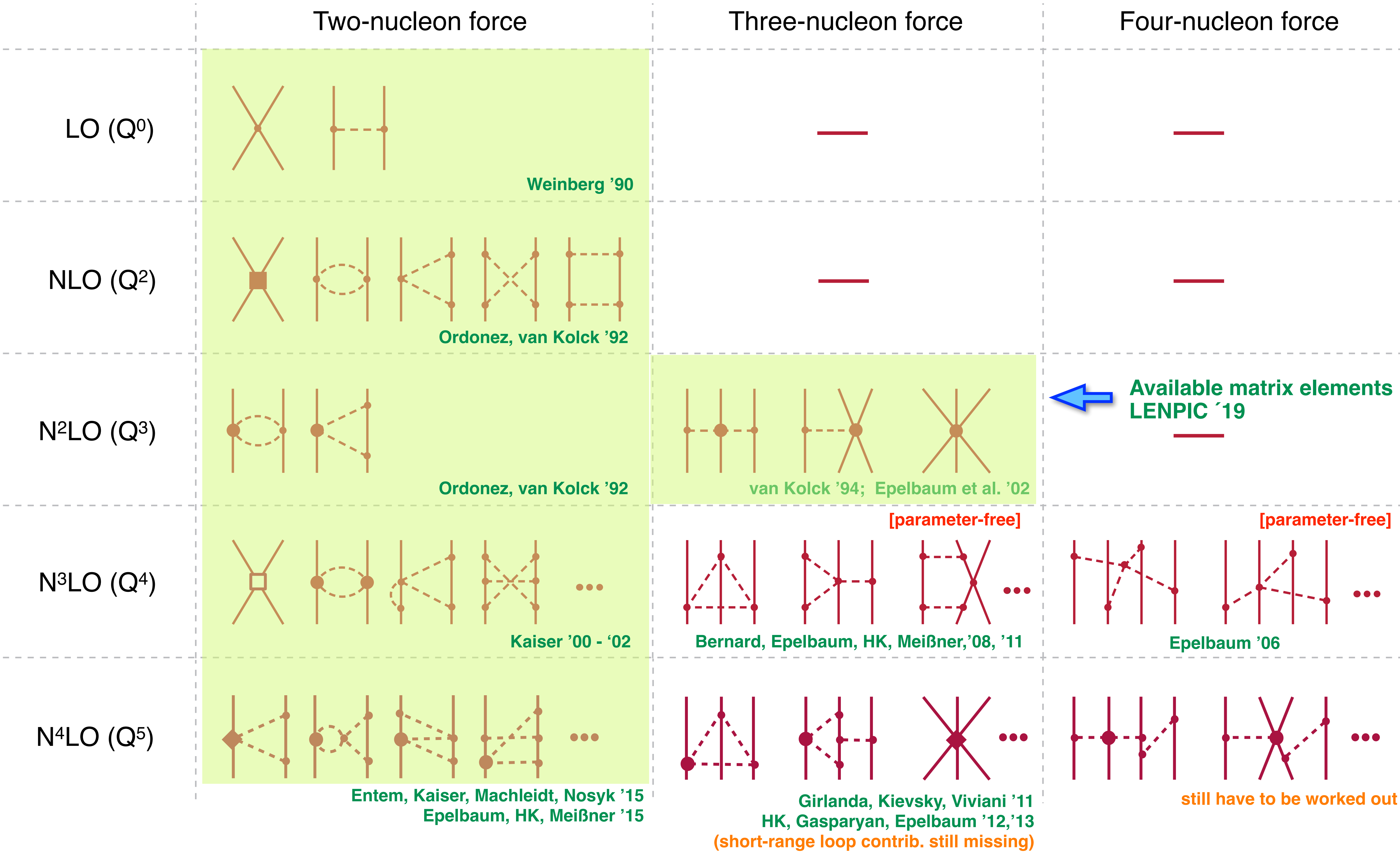
$$\left[\left(\sum_{i=1}^{A}\frac{-\vec{\nabla}_i^2}{2m_N}+\mathcal{O}(m_N^{-3})\right)+\underbrace{V_{2N}+V_{3N}+V_{4N}+\ldots}_{\textit{derived within ChPT}}\right]|\Psi\rangle = E|\Psi\rangle$$

Weinberg '91



Chiral EFT is a systematic tool for derivation of nuclear forces below pion-production threshold

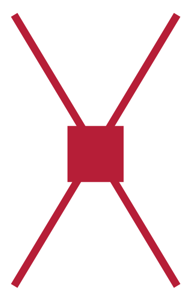
Chiral Expansion of the Nuclear Forces



Adjustable Parameters in NN

Reinert, HK, Epelbaum PRL126 (2021) 092501

Couplings of short-range interactions are fixed from NN - data



LO [Q^0]: 2 operators (S-waves)
NLO [Q^2]: + 7 operators (S-, P-waves and ε_1)
N^2LO [Q^3]: no new terms
N^3LO [Q^4]: + 12 operators (S-, P-, D-waves and ε_1, ε_2)
N^4LO [Q^5]: + 5 IB operators
N^4LO^+ [Q^6]: + 4 operators (F-waves)

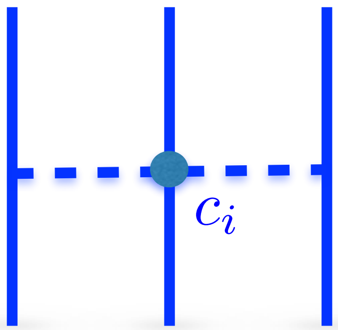
# of adjustable LECs = 25 IC + 5 IB + 3 πN constants = 33 parameters

Summary on NN

- Employed a Bayesian approach to account for statistical and systematic uncertainties
- Extracted πN couplings from NN data within chiral EFT
- Achieved a statistically perfect description of NN data
  χ^2/dat = 1.005 for ~5000 data in the energy range E_{lab} = 0 - 280 MeV

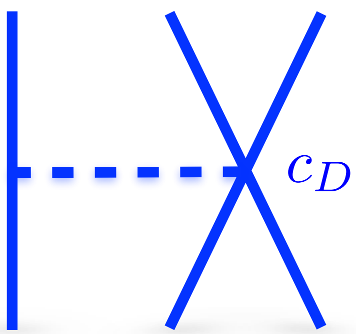
Three-Nucleon Force at N²LO

Epelbaum et al. EPJA56 (2020) 92; Maris et al. PRC103 (2021) 054001



c_i´s are extracted from solutions of Roy-Steiner equation in pion-nucleon scattering: **Hoferichter et al. PRL115 (2015) 192301**

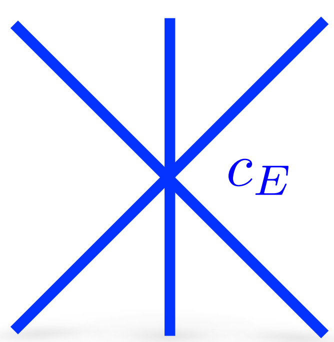
$$c_1 = -0.74\,\mathrm{GeV}^{-1} \quad c_3 = -3.61\,\mathrm{GeV}^{-1} \quad c_4 = 2.44\,\mathrm{GeV}^{-1}$$

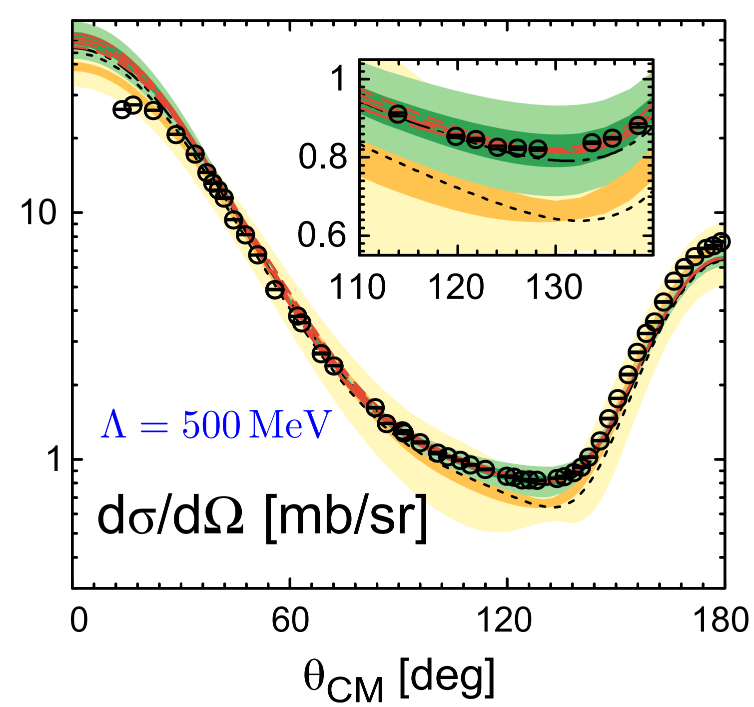


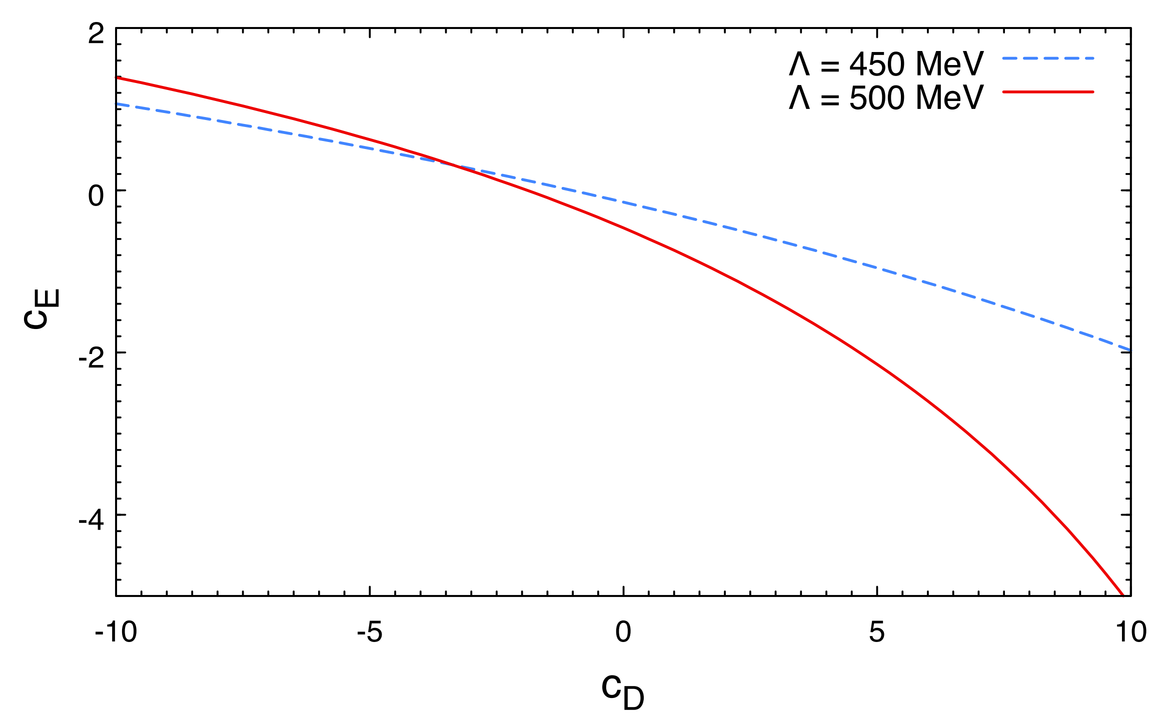
Requirement to reproduce ³H correlates c_D & c_E

c_D is fitted to the minimum of Nd-scattering cross section at $E_{\mathrm{lab}}^{N} = 70\,\mathrm{MeV}$

Sekiguchi et al. PRC65 (2002) 034003

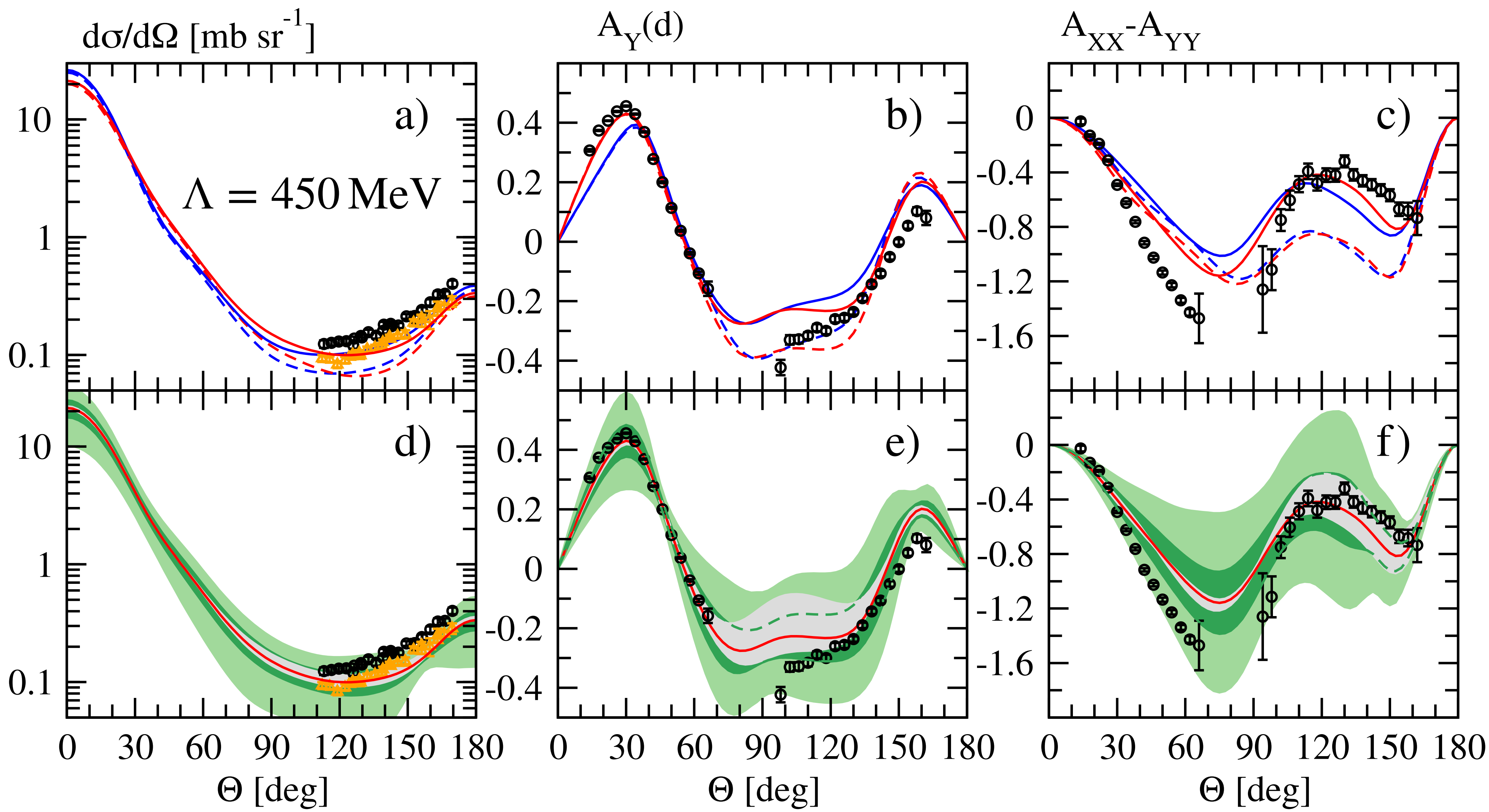






Nd Scattering at Low Energy

Differential cross section and selected analyzing powers of elastic Nd scattering at E_N = 200 MeV



NN (N^2LO) NN (N^4LO^+) NN (N^2LO) +3NF NN (N^4LO+) +3NF

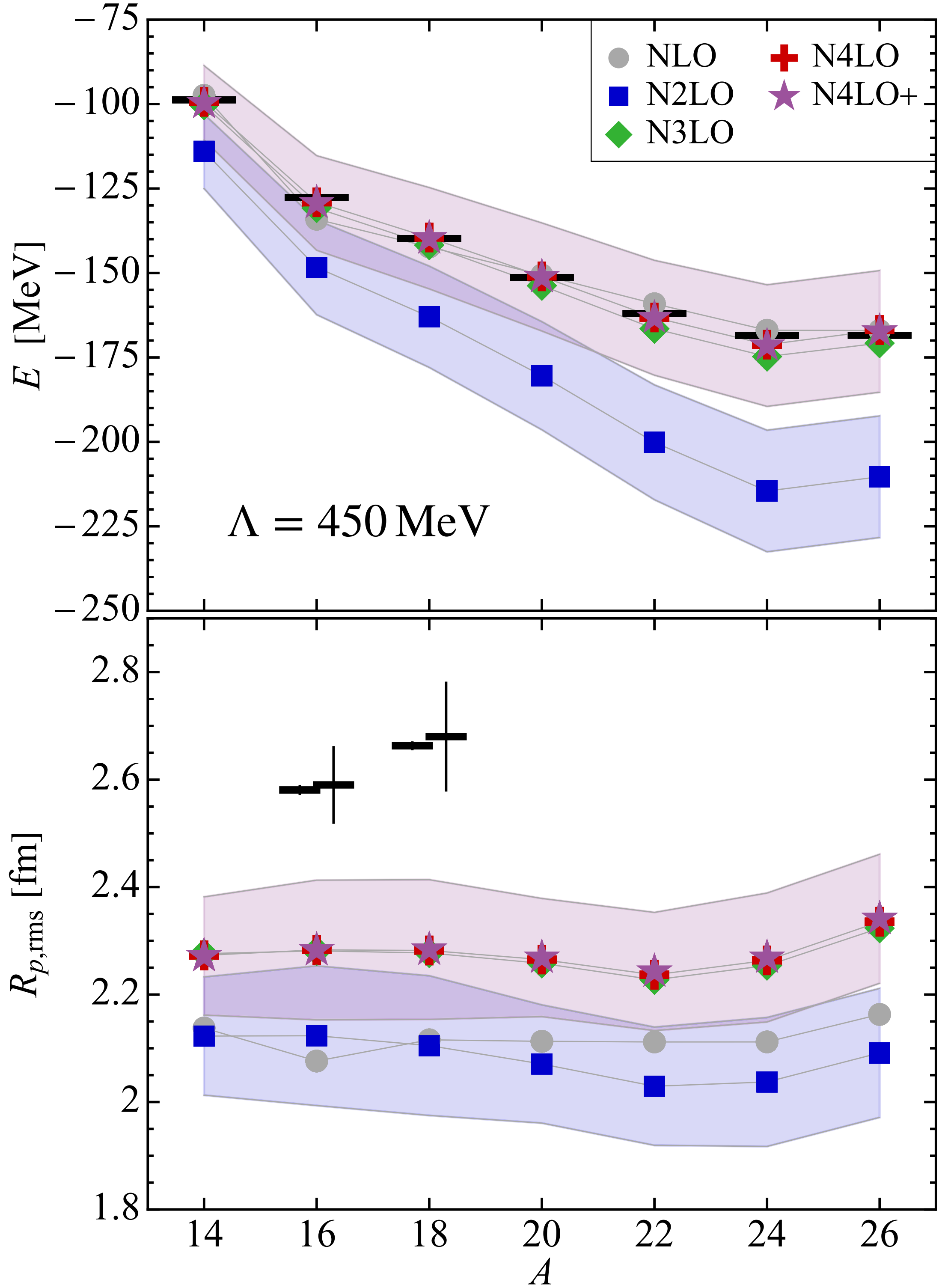
Light (dark) green band correspond to truncation errors at 95% (68%) DoB

- Similar to phenomenological forces $d\sigma/d\Omega$ is slightly underestimated

Even Oxigen Isotopes

Ground state energies and point-proton radii for even oxygen isotopes within IM-NCSM

Maris et al. PRC 106 (2022) 064002



SRG flow parameter $\alpha = 0.08\ \mathrm{fm}^4$

The error bands show chiral truncation uncertainties with 95% DoB for N^2LO and N^4LO+

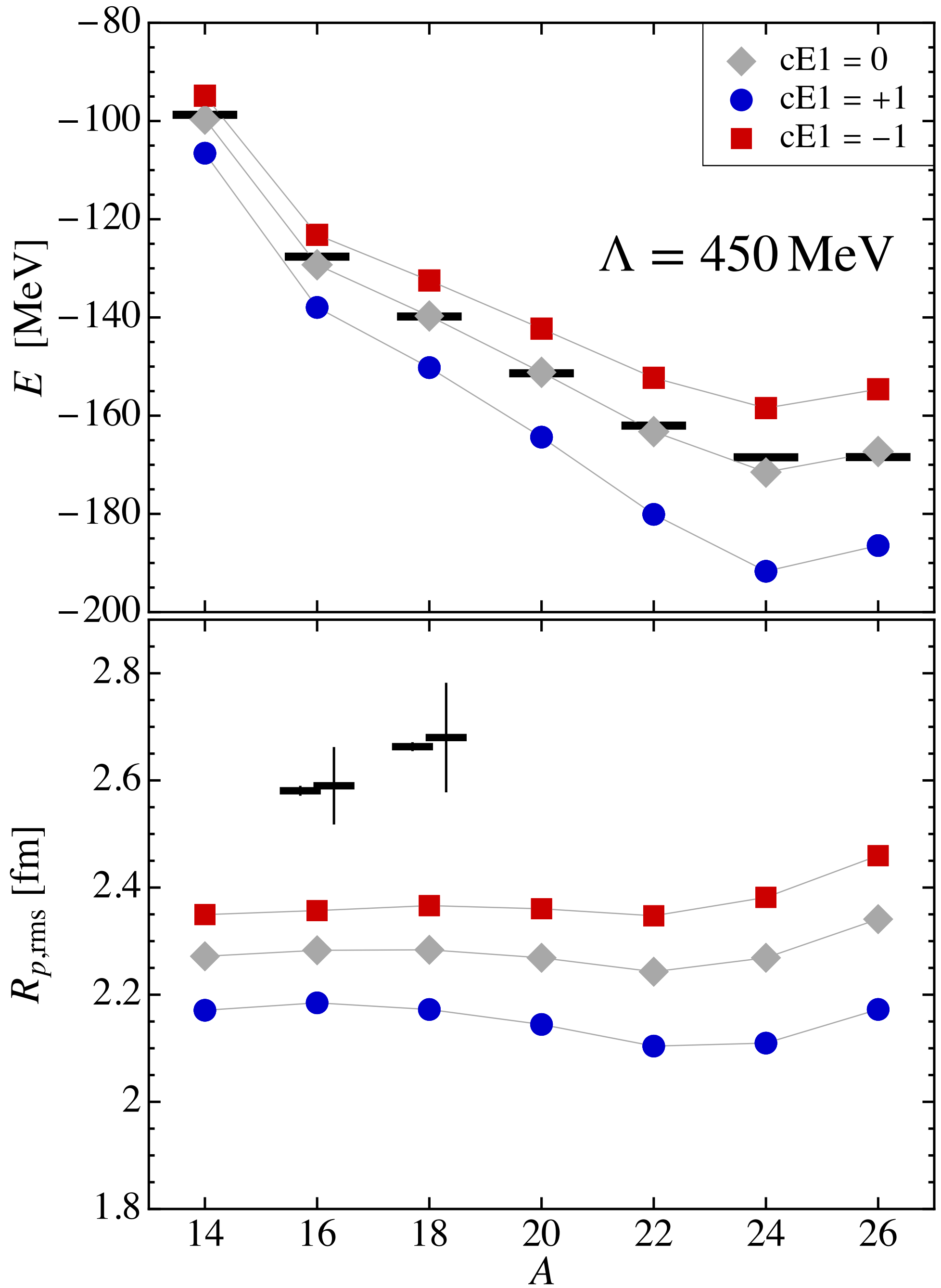
N^2LO leads to significant overbinding but higher orders stabilize ground state energy predictions

Radii are underestimated even at highest order

Systematic Deviation for Radii

Ground state energies and point proton radii for even oxygen isotopes within IM-NCSM

Maris et al. PRC 106 (2022) 064002



To explore sensitivity on higher order 3NF's we introduce spin-isospin independent 3NF:

$$V_{3\mathrm{N}} = -E_1 q_1^2, \quad E_1 = \frac{c_{E_1}}{F_\pi^4 \Lambda_\chi^3}$$

Girlanda et al. PRC84 (2011) 014001; PRC102 (2020) 019903
Epelbaum, HK, Reinert, Front. In Phys. 8 (2020) 98

Higher order terms have the potential to significantly affect energies and radii

Systematic underprediction of radii

Remains to be seen whether consistent inclusion of higher order 3NF's and charge density lead to satisfactory description of energies and radii

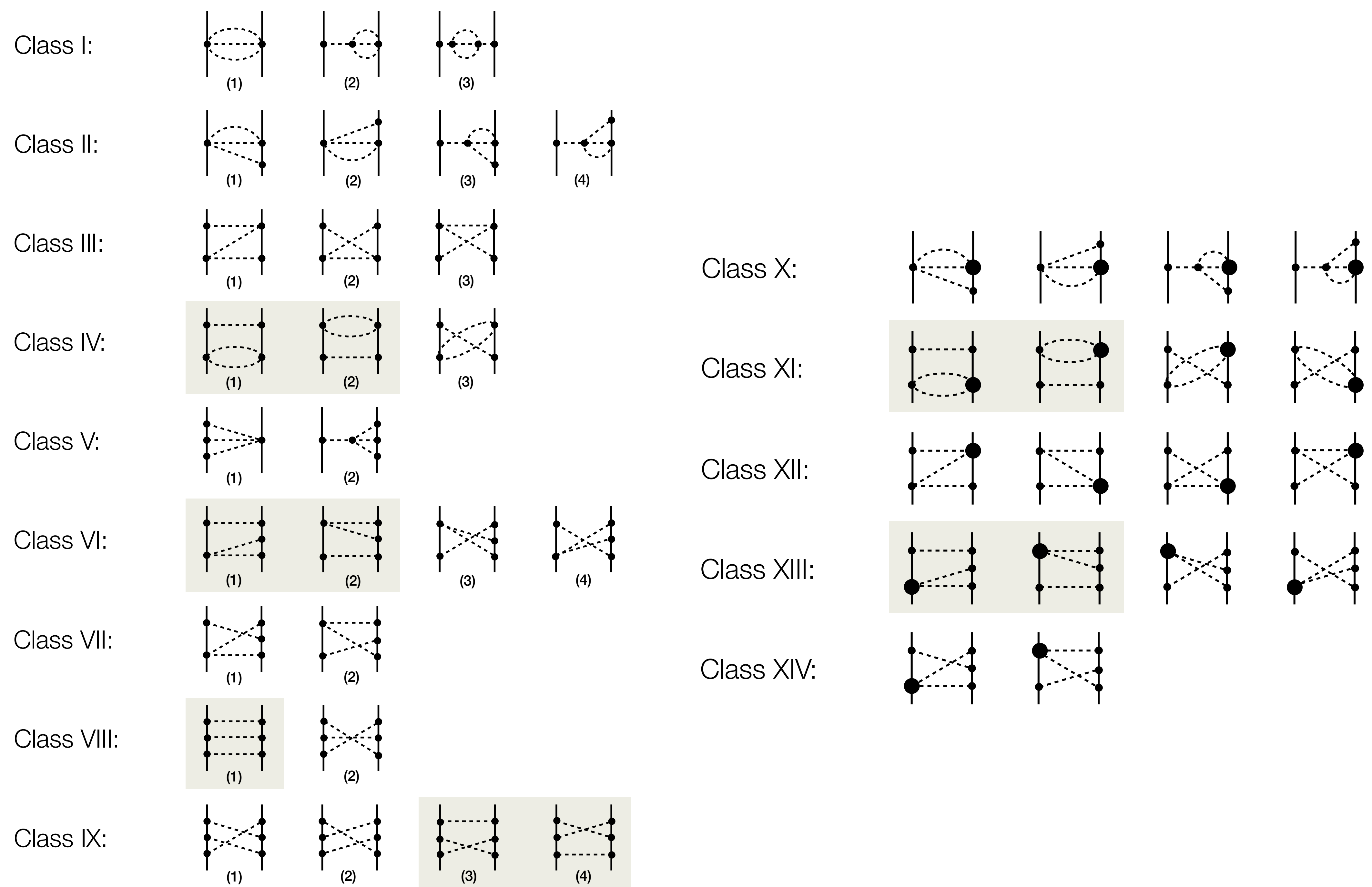
→ work in progress

Possible Improvements in NN Sector

1/m correction to 2PE is scheme dependent → Scheme-dependence of 3PE

3PE calculated by Kaiser `00 - `02 can not be used in unitary transformation approach

N^3LO	N^4LO

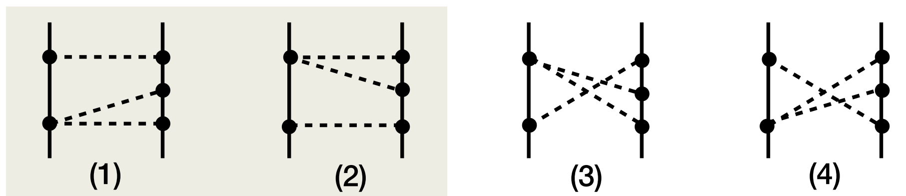


3PE contributions to NN at N³LO

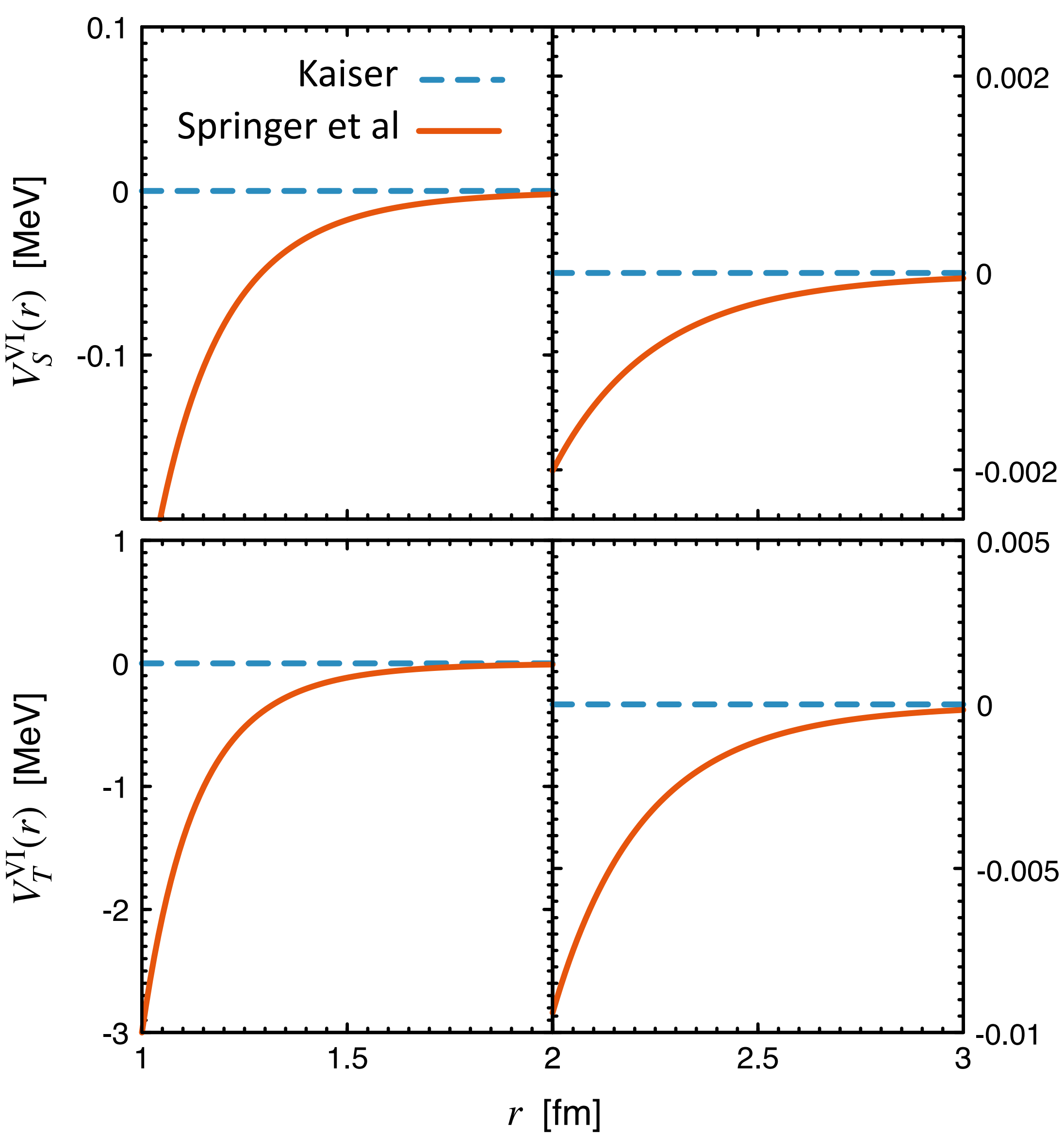
3PE within Unitary Transformation Method (UTM): Springer, HK, Epelbaum arXiv:2505.02034

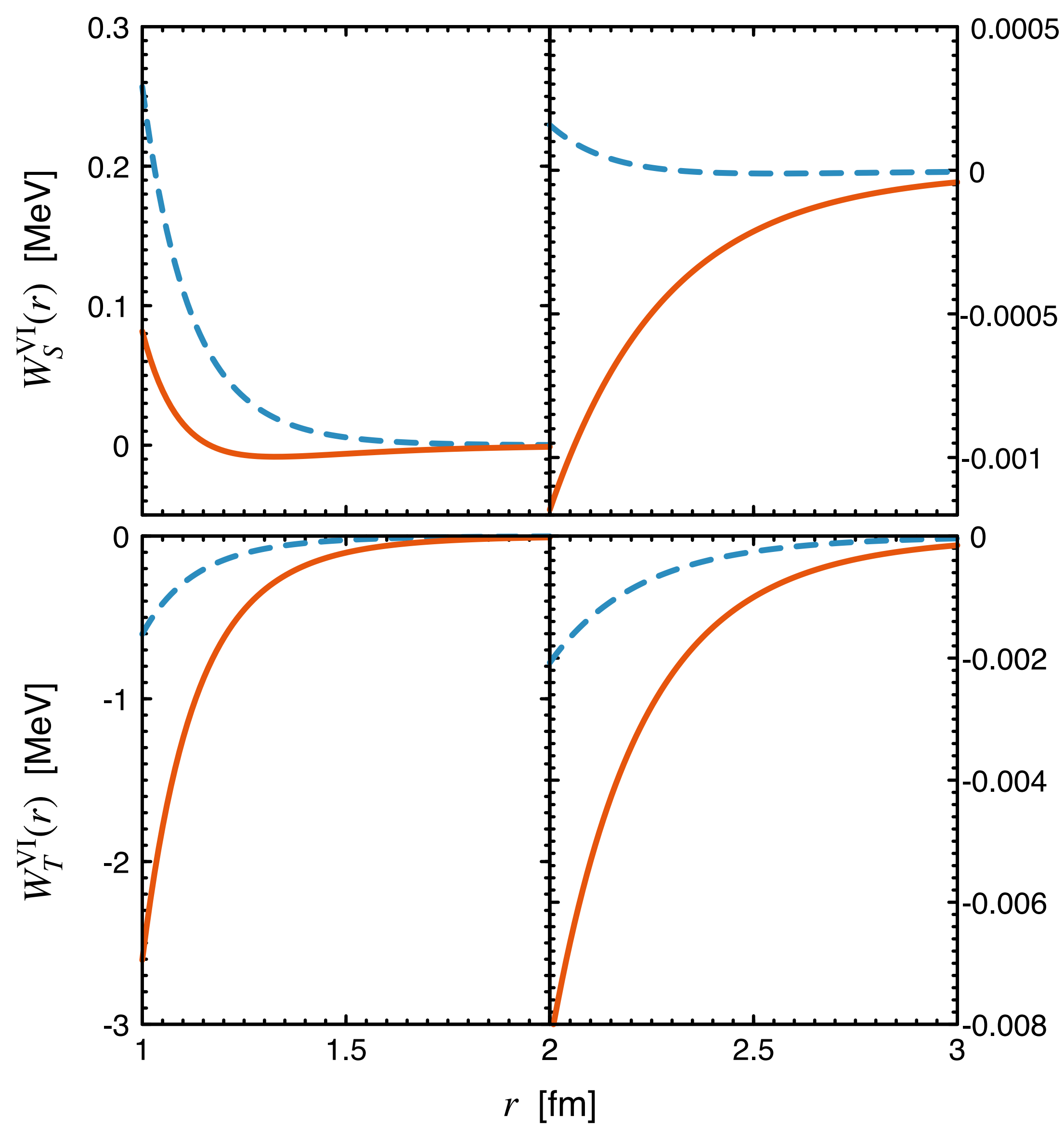
$$V_{3\pi}(\vec{r}) = V_C(r) + \tau_1 \cdot \tau_2 W_C(r) + \vec{\sigma}_1 \cdot \vec{\sigma}_2 \left[V_S(r) + \tau_1 \cdot \tau_2 W_S(r)\right] + S_{12}(\hat{r})\left[V_T(r) + \tau_1 \cdot \tau_2 W_T(r)\right]$$

Class VI:



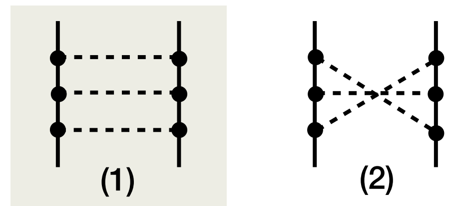
Scheme-dependent deviations between Kaiser and Springer et al.



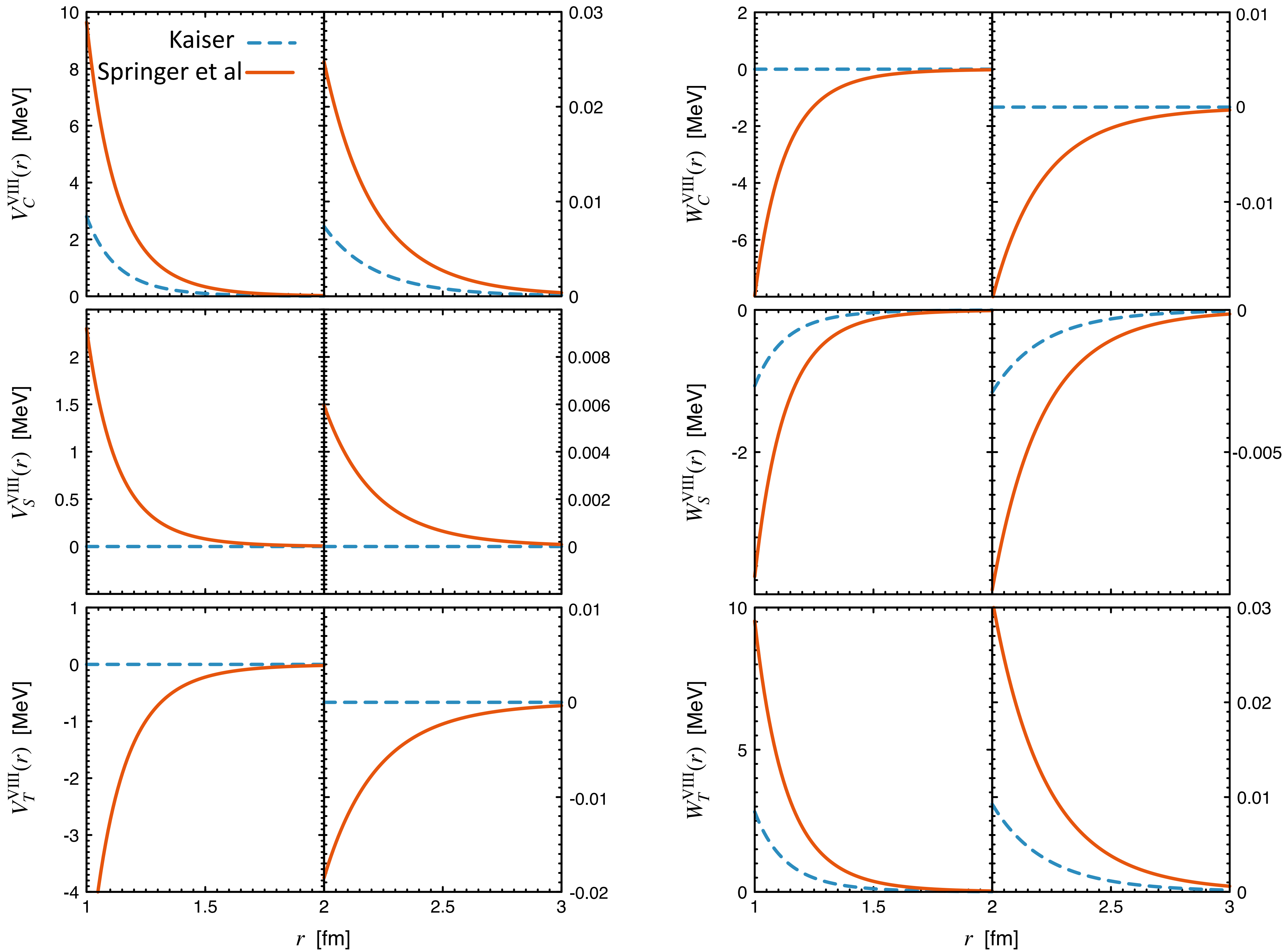


Deviations Between Two Schemes

Class VIII:

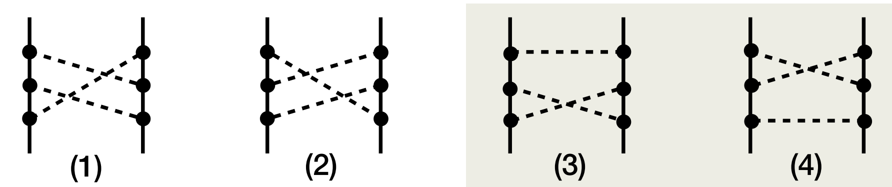


Scheme-dependent deviations between Kaiser and Springer et al.

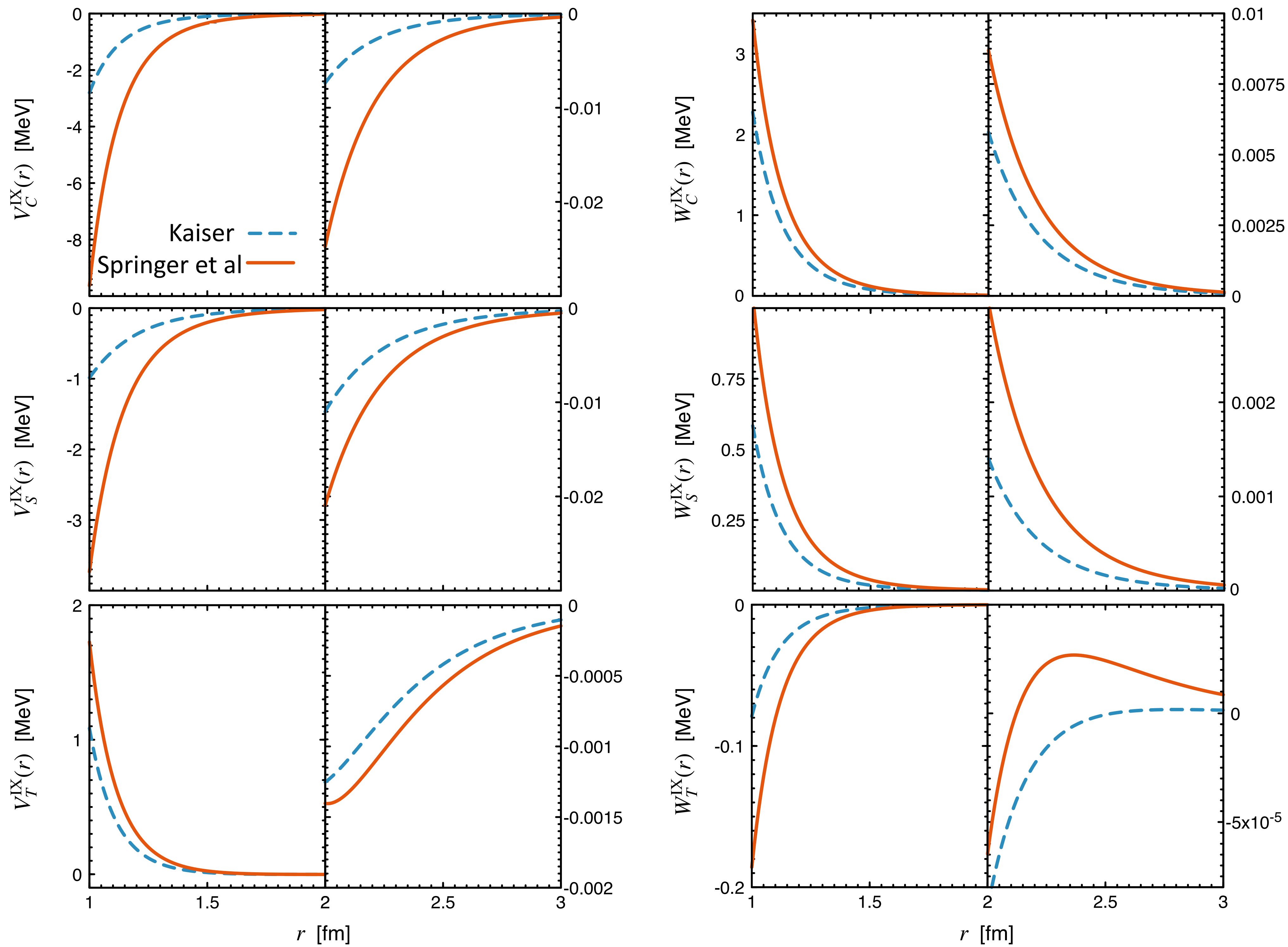


Deviations Between Two Schemes

Class IX:



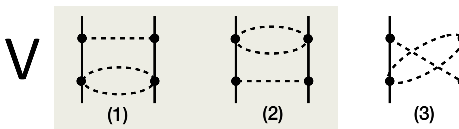
Scheme-dependent deviations between Kaiser and Springer et al.



Two Schemes Results: Summarized

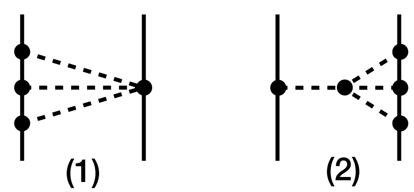
- For the Classes VI, VIII and IX we get for most of the potentials stronger 3PE contributions

- Despite reducible-like diagrams we do not see any deviation for the Class IV



- We reproduced all results of Kaiser with one exception:

Class V:



Different sign in **Kaiser PRC 62 (2000) 024001, Eq. (8)**

$$\mathrm{Im}\, W_T^{\mathrm{V}}(i\mu) = \frac{1}{\mu^2}\mathrm{Im}\, W_S^{\mathrm{V}}(i\mu) - \frac{g_{\mathrm{A}}^4\left(\mu^2 - M_\pi^2\right)^{-1}}{\mu^2\left(8\pi F_\pi^2\right)^3}\iint\limits_{z^2<1} d\omega_1 d\omega_2 \left[\left(6\mu^2 + 2M_\pi^2\right)(\omega_1+\omega_2)\right.$$
$$\left. -\mu\left(4\mu^2+3M_\pi^2\right)\right]\left[\left(\left(\mu^2+M_\pi^2\right)\left(2\omega_1 - \frac{\mu}{2}\right) - 2\mu\omega_1\omega_2\right)\frac{\arccos(-z)}{l_1 l_2\sqrt{1-z^2}} + \mu + 2z\omega_1\frac{l_2}{l_1}\right]$$

- At N[4]LO we don't see any deviation for all classes of diagrams

Remains to be seen if we observe an evidence of 3PE from NN scattering data.

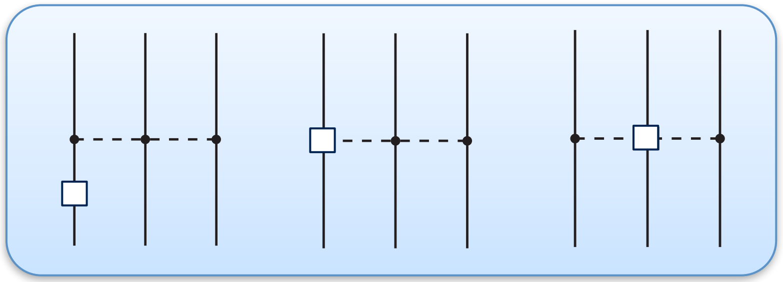
Work in progress

Chiral 3NF at N^3LO

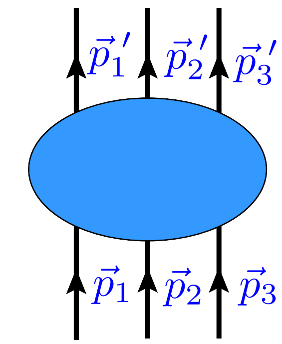
Call for Consistent Regularization

Violation of chiral symmetry due to different regularizations: Dim. reg. vs cutoff reg.

Epelbaum, HK, Reinert, Front. in Phys. 8 (2020) 98



← 1/m - corrections to TPE 3NF $\sim g_A^2$



$$V_{2\pi,1/m}^{g_A^2} = i\frac{g_A^2}{32mF_\pi^4}\frac{\vec{\sigma}_1\cdot\vec{q}_1\,\vec{\sigma}_3\cdot\vec{q}_3}{(q_1^2+M_\pi^2)(q_3^2+M_\pi^2)}\tau_1\cdot(\tau_2\times\tau_3)(2\vec{k}_1\cdot\vec{q}_3+4\vec{k}_3\cdot\vec{q}_3+i\,[\vec{q}_1\times\vec{q}_3]\cdot\vec{\sigma}_2)$$

$$\vec{q}_i = \vec{p}_i{}' - \vec{p}_i$$

$$\vec{k}_i = \frac{1}{2}(\vec{p}_i{}' + \vec{p}_i)$$

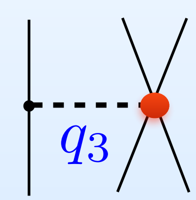
Naive local cut-off regularization of the current and potential

$$V_{2\pi,1/m}^{g_A^2,\Lambda} = V_{2\pi,1/m}^{g_A^2}\exp\left(-\frac{q_1^2+M_\pi^2}{\Lambda^2}\right)\exp\left(-\frac{q_3^2+M_\pi^2}{\Lambda^2}\right) \quad \& \quad V_{1\pi}^{Q^0,\Lambda} = -\frac{g_A^2}{4F_\pi^2}\tau_1\cdot\tau_2\frac{\vec{q}_1\cdot\vec{\sigma}_1\vec{q}_1\cdot\vec{\sigma}_2}{q_1^2+M_\pi^2}\exp\left(-\frac{q_1^2+M_\pi^2}{\Lambda^2}\right)$$

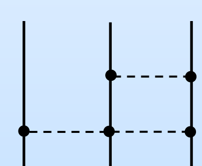
First iteration with OPE NN potential

$$V_{2\pi,1/m}^{g_A^2,\Lambda}\frac{1}{E-H_0+i\epsilon}V_{1\pi}^{Q^0,\Lambda} + V_{1\pi}^{Q^0,\Lambda}\frac{1}{E-H_0+i\epsilon}V_{2\pi,1/m}^{g_A^2,\Lambda} = \Lambda\frac{g_A^4}{128\sqrt{2}\pi^{3/2}F_\pi^6}(\tau_2\cdot\tau_3-\tau_1\cdot\tau_3)\frac{\vec{q}_2\cdot\vec{\sigma}_2\vec{q}_3\cdot\vec{\sigma}_3}{q_3^2+M_\pi^2}+\ldots$$

No such D-like term in chiral Lagrangian



The problematic divergence is canceled by the one $V_{2\pi-1\pi}$ if calculated via cutoff regularization

In dim. reg. $V_{2\pi-1\pi}$ =  + … is finite

Symmetry Preserving Regulator

HK, Epelbaum, PRC 110 (2024) 4, 044004

Gradient-Flow Equation (GFE)

Balitsky, Yung, PL168B (1986) 113; Irwin, Manton, PLB 385 (1996) 187

Yang-Mills gradient flow in QCD: **Lüscher, JHEP 04 (2013) 123**

$\partial_\tau B_\mu = D_\nu G_{\nu\mu}$ with $B_\mu|_{\tau=0} = A_\mu$ & $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$

B_μ is a regularized gluon field

- Apply this idea to ChPT: **HK, Epelbaum, PRC 110 (2024) 4, 044004**

(Proposed in various talks by D. Kaplan for nuclear forces)

Introduce a smoothed pion field W with $W|_{\tau=0} = U$ satisfying GFE

$\partial_\tau W = i\, w\, \mathrm{EOM}(\tau)\, w$ with $w = \sqrt{W}$ and $\mathrm{EOM}(\tau) = [D_\mu, w_\mu] + \frac{i}{2}\chi_- - \frac{i}{4}\mathrm{Tr}(\chi_-)$

$$w_\mu = i(w^\dagger(\partial_\mu - i\, r_\mu)w - w(\partial_\mu - i\, l_\mu)w^\dagger), \quad \chi_- = w^\dagger \chi w^\dagger - w\chi^\dagger w, \quad \chi = 2B(s + ip)$$

Note: The shape of regularization is dictated by the choice of the right-hand side of GFE

- Our choice is motivated by a Gaussian regularization of one-pion-exchange in NN

Gradient-Flow Equation

Analytic solution is possible of $1/F$ - expanded gradient flow equation:

$$W = 1 + i\boldsymbol{\tau}\cdot\boldsymbol{\phi}(1-\alpha\boldsymbol{\phi}^2) - \frac{\boldsymbol{\phi}^2}{2}\left[1 + \left(\frac{1}{4} - 2\alpha\right)\boldsymbol{\phi}^2\right] + \mathcal{O}(\phi^5), \quad \phi_b = \sum_{n=0}^{\infty}\frac{1}{F^n}\phi_b^{(n)}$$

In the absence of external sources we have

$$\left[\partial_\tau - (\partial_\mu^x\partial_\mu^x - M^2)\right]\phi_b^{(1)}(x,\tau) = 0, \quad \phi_b^{(1)}(x,0) = \pi_b(x)$$

$$\left[\partial_\tau - (\partial_\mu^x\partial_\mu^x - M^2)\right]\phi_b^{(3)}(x,\tau) = (1-2\alpha)\partial_\mu\boldsymbol{\phi}^{(1)}\cdot\partial_\mu\boldsymbol{\phi}^{(1)}\phi_b^{(1)} - 4\alpha\partial_\mu\boldsymbol{\phi}^{(1)}\cdot\boldsymbol{\phi}^{(1)}\partial_\mu\phi_b^{(1)}$$
$$+\frac{M^2}{2}(1-4\alpha)\boldsymbol{\phi}^{(1)}\cdot\boldsymbol{\phi}^{(1)}\phi_b^{(1)}, \quad \phi_b^{(3)}(x,0) = 0$$

Iterative solution in momentum space: $\tilde{\phi}^{(n)}(q,\tau) = \int d^4x\, e^{iq\cdot x}\phi_b^{(n)}(x,\tau)$

$$\tilde{\phi}_b^{(1)}(q) = e^{-\tau(q^2+M^2)}\tilde{\pi}_b(q)$$

$$\tilde{\phi}_b^{(3)}(q) = \int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}(2\pi)^4\delta(q-q_1-q_2-q_3)\int_0^\tau ds\, e^{-(\tau-s)(q^2+M^2)}e^{-s\sum_{j=1}^3(q_j^2+M^2)}$$
$$\times\left[4\alpha\, q_1\cdot q_3 - (1-2\alpha)q_1\cdot q_2 + \frac{M^2}{2}(1-4\alpha)\right]\tilde{\pi}(q_1)\cdot\tilde{\pi}(q_2)\tilde{\pi}_b(q_3)$$

Integration over momenta of pion fields with Gaussian prefactor introduces smearing

Properties under Chiral Transformation

Replace all pion fields in pion-nucleon Lagrangians $\mathscr{L}^{(1)}_{\pi N}, \ldots, \mathscr{L}^{(4)}_{\pi N}$: $U \to W$

$$\mathscr{L}^{(1)}_{\pi N} = N^\dagger \Big(D^0 + g\, u \cdot S \Big) N \to N^\dagger \Big(D^0_w + g\, w \cdot S \Big) N$$

Chiral transformation: by induction, one can show

$$U \to RUL^\dagger \longrightarrow W \to RWL^\dagger$$

- Regularized pion fields transform under τ - independent transformations

- Nucleon fields transform in τ - dependent way

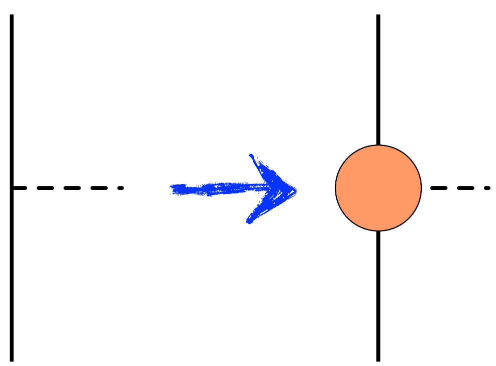
$$N \to KN, \quad K = \sqrt{LU^\dagger R^\dagger} R \sqrt{U} \longrightarrow N \to K_\tau N, \quad K_\tau = \sqrt{LW^\dagger R^\dagger} R \sqrt{W}$$

Regularization for Nuclear Forces

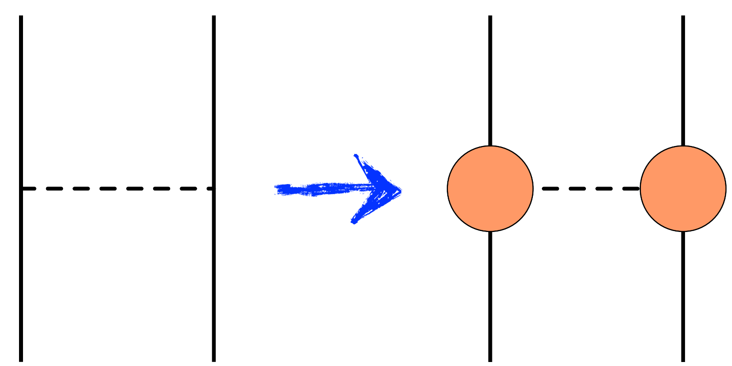
To regularize long-range part of the nuclear forces and currents

- Leave pionic Lagrangians $\mathscr{L}_{\pi}^{(2)}$ & $\mathscr{L}_{\pi}^{(4)}$ unregularized (essential)
- Replace all pion fields in pion-nucleon Lagrangians $\mathscr{L}_{\pi N}^{(1)}, \ldots, \mathscr{L}_{\pi N}^{(4)}$: $U \to W$

$$\mathscr{L}_{\pi N}^{(1)} = N^{\dagger}\Big(D^0 + g\, u \cdot S\Big)N \to N^{\dagger}\Big(D_w^0 + g\, w \cdot S\Big)N$$



$\sim e^{-\tau(q^2+M^2)}$

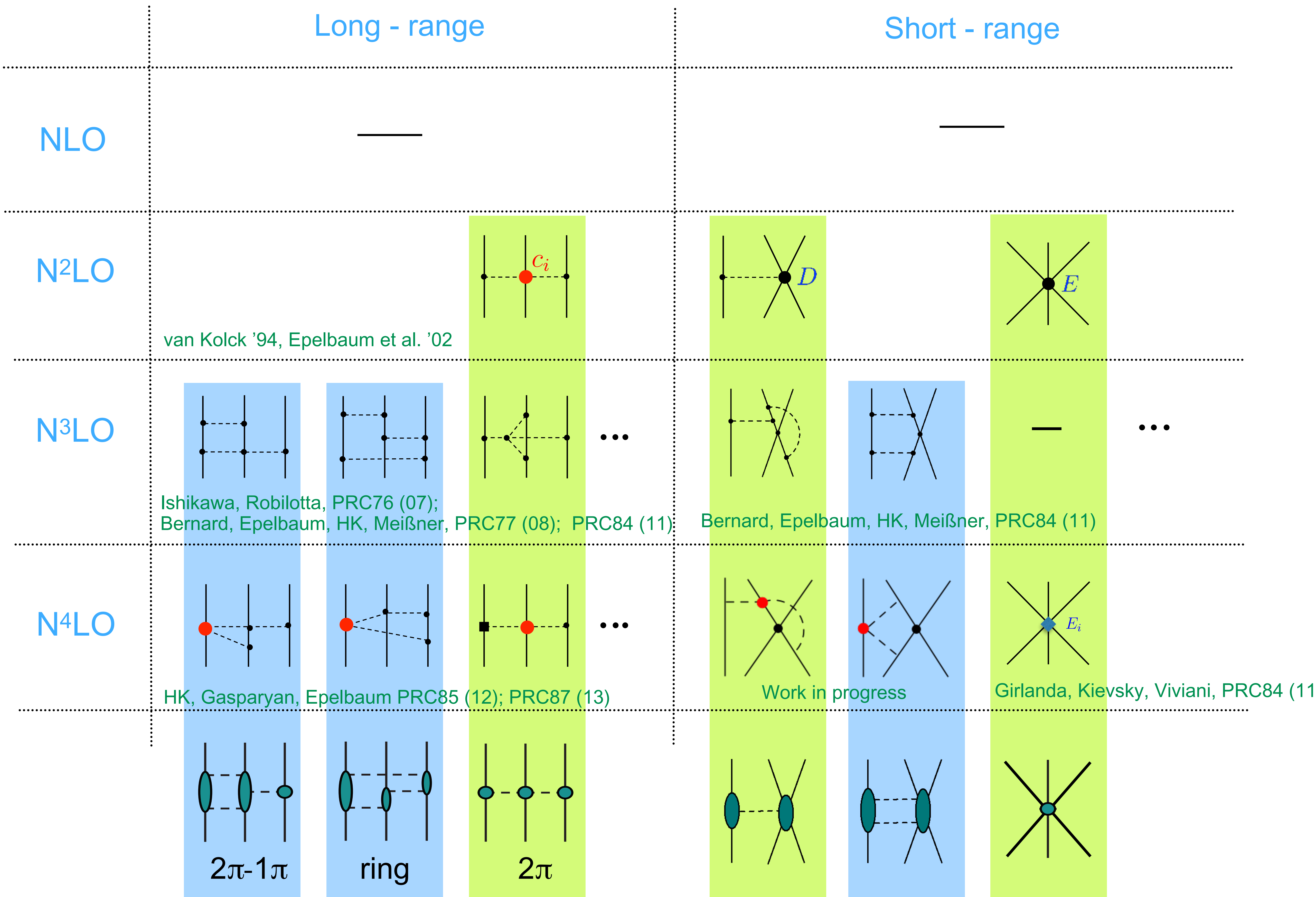


$\sim e^{-2\tau(q^2+M^2)}\dfrac{1}{q^2+M^2}$

For $\tau = \dfrac{1}{2\Lambda^2}$ this regulator reproduces SMS regularization of OPE

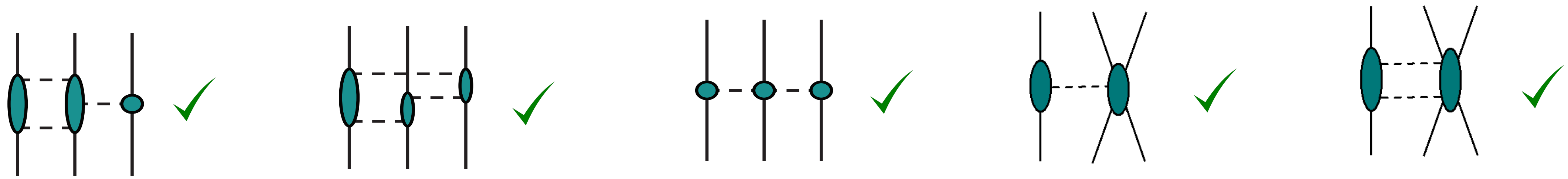
Status Report on 3NF

3NF up to N^{4}LO



Status Report on 3N at N³LO

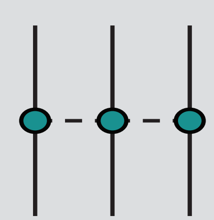
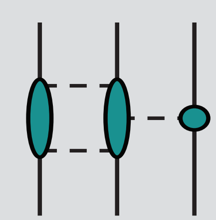
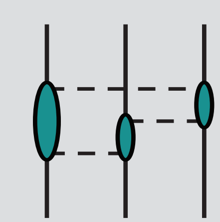
- We calculated all long- and short-range contributions to 3NF & 4NF at N³LO



3NF's are given in terms of integrals over Schwinger parameters

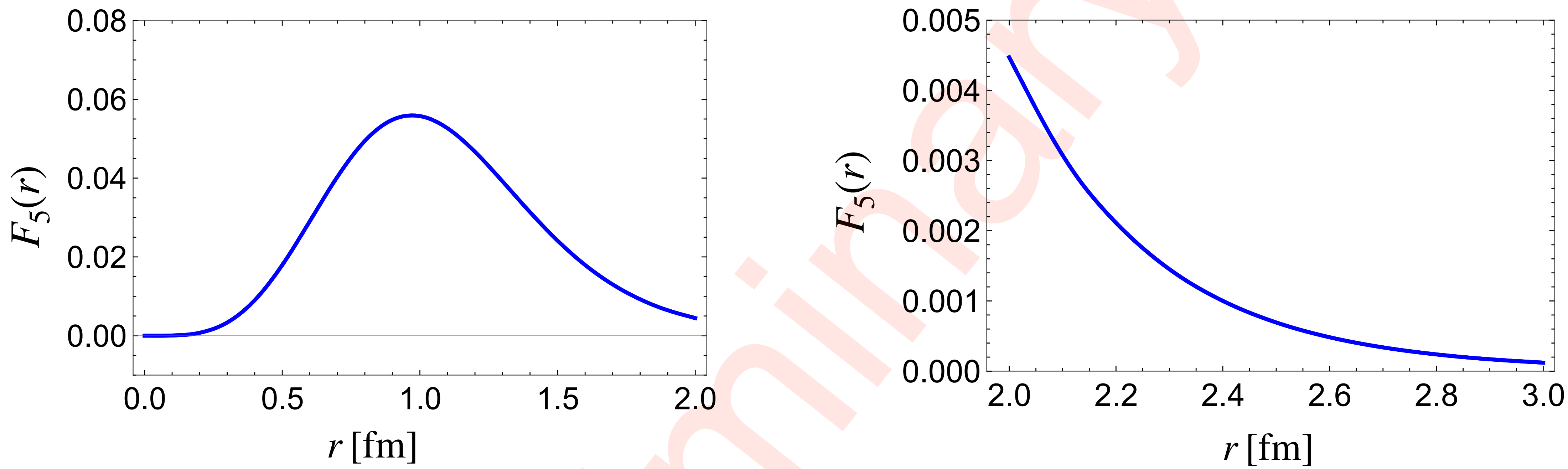
$$V_{3N}^{2\pi-1\pi} = \tau_1 \cdot \tau_2 \times \tau_3 \vec{q}_1 \cdot \vec{\sigma}_1 \times \vec{\sigma}_2 \vec{q}_3 \cdot \vec{\sigma}_3 \frac{e^{-\frac{q_3^2+M_\pi^2}{\Lambda^2}}}{q_3^2+M_\pi^2}\left(-\frac{g_A^4}{F_\pi^6}\frac{q_1}{2048\pi}\int_0^\infty d\lambda \, \mathrm{erfi}\left(\frac{q_1\lambda}{2\Lambda\sqrt{2+\lambda}}\right)\frac{\exp\left(-\frac{q_1^2+4M_\pi^2}{4\Lambda^2}(2+\lambda)\right)}{2+\lambda} + \dots\right) + \dots$$

Dimension of integrals over Schwinger parameters depends on topology

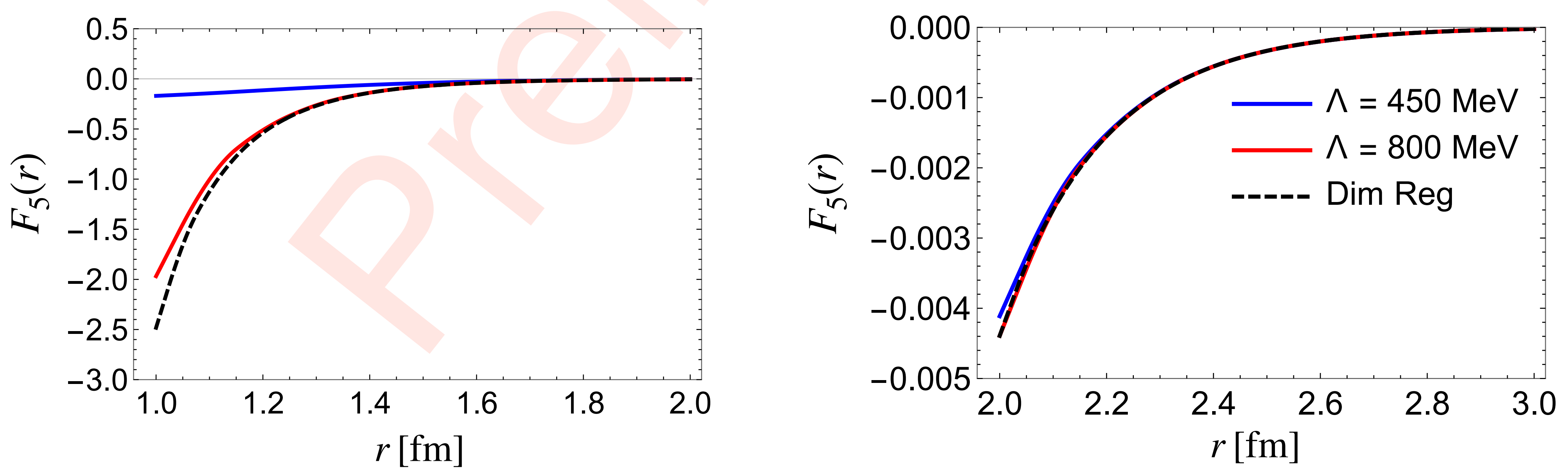
Space			
Momentum	2	1	3
Coordinate	4	1	0

Selected Profile Functions

$V_{3N}^{\text{ring}} = F_1(r_{12}, r_{23}, r_{13}) + \ldots + \tau_2 \cdot \tau_3 \vec{\sigma}_1 \cdot \vec{\sigma}_2 F_5(r_{12}, r_{23}, r_{13}) + \ldots$ $\quad F_5(r) = F_5(r, r, r)$ [MeV]

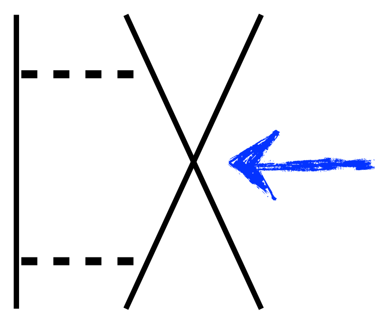


At $\Lambda \to \infty$ regularized 3NF reproduce dim. reg. results from **Bernard et al. PRC77 (08)**



Short Range 3NF at N^{3}LO

Complication in calculation of short-range 3NF due to non-local regulator of LO NN



Non-local regulator of short-range NN at LO introduces additional momentum in loop functions

Structure functions of short-range 3NF can become complex

Time-reversal transformation (T): $\vec{\sigma}_j \to -\vec{\sigma}_j,\ \tau_j^y \to -\tau_j^y,\ \vec{q}_j \to \vec{q}_j,\ \vec{k}_j \to -\vec{k}_j$

Hermitian conjugation (h.c.): $\vec{\sigma}_j \to \vec{\sigma}_j,\ \boldsymbol{\tau}_j \to \boldsymbol{\tau}_j,\ \vec{q}_j \to -\vec{q}_j,\ \vec{k}_j \to \vec{k}_j$

$$\exp\Big(-\frac{(2(\vec{k}_2-\vec{k}_3)+\vec{q}_2)^2}{8\Lambda^2}\Big)+\exp\Big(-\frac{(2(\vec{k}_2-\vec{k}_3)-\vec{q}_2)^2}{8\Lambda^2}\Big)$$ Invariant under T and h.c.

$$i\left[\exp\Big(-\frac{(2(\vec{k}_2-\vec{k}_3)+\vec{q}_2)^2}{8\Lambda^2}\Big)-\exp\Big(-\frac{(2(\vec{k}_2-\vec{k}_3)-\vec{q}_2)^2}{8\Lambda^2}\Big)\right]$$ Invariant under T and h.c.

- Combination of these functions are allowed to appear in structure functions

→ Structure functions might be complex: not related to unitarity cut (phase)

Short Range 3NF at N^3LO

Complex structure functions of short-range part of 3NF require complex PWD

Solution 1: Is there a nucleon-field transformation which would make 3NF's real?

Idea: Constrain field transformations needed to make interactions instant

Every ϵ_{ijk} in field transformations should be accompanied with an „i"

→ Indeed, we achieved with these transformations an instant 3NF and get real structure functions for short-range 3NF

Solution 2: Change the regulator of short-range NN interaction at LO to local one

→ Short-range 3NF's at N^3LO becomes local and automatically real

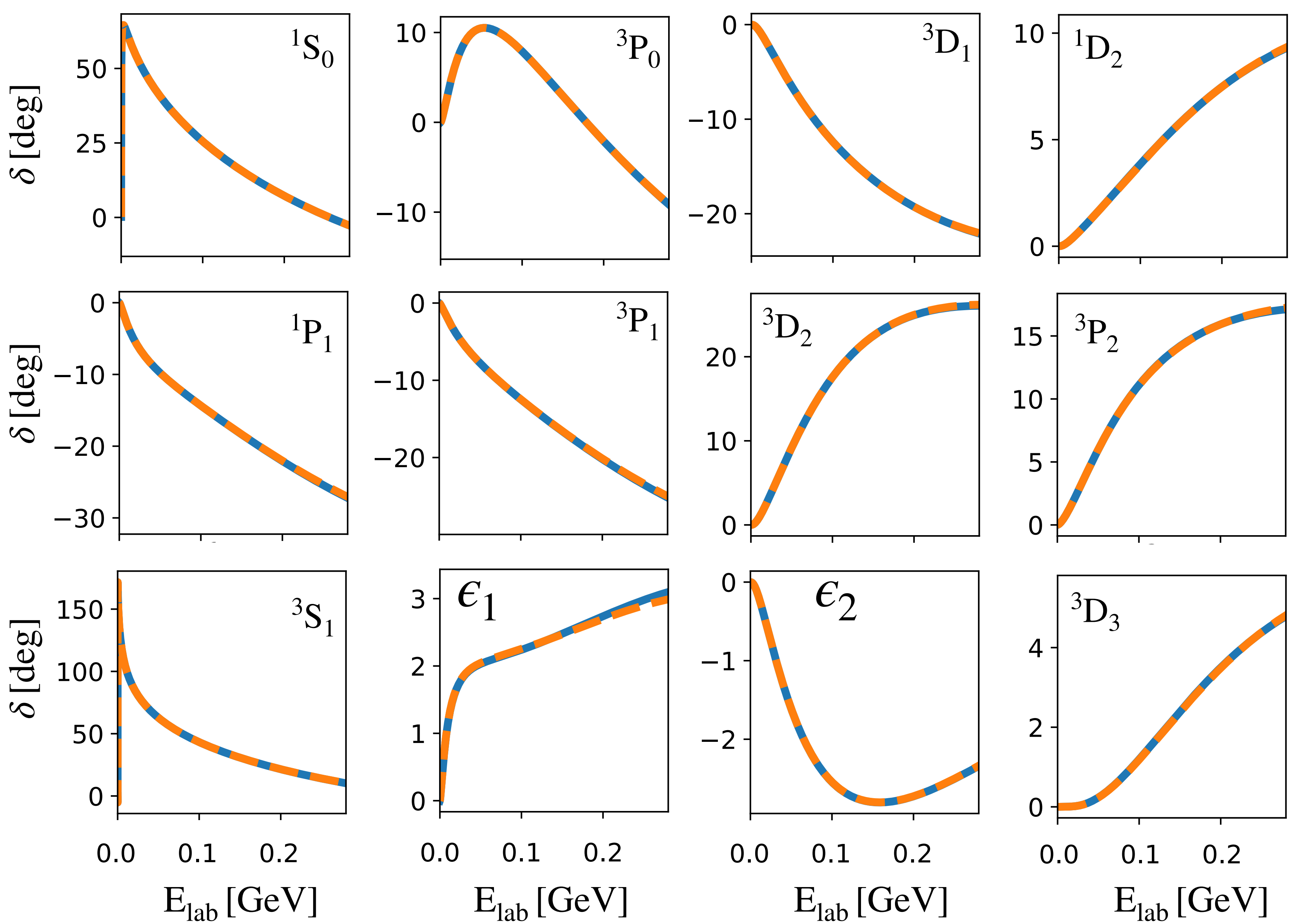
→ Expressions for local short-range 3NF's at N^3LO are simpler

→ PWD of local 3NF's is less expensive

But: we need to generate a new NN force

NN phase shifts and mixing angles

Heihoff et al. : forthcoming



$\Lambda = 450\,\mathrm{MeV}$

- Quality of nuclear force does not change when we change the regulator of the LO short-range NN interaction
- Local regularization of the LO short-range NN leads to simpler 3NF at N^3LO

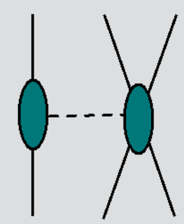
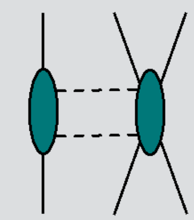
Local short-range regulator at LO: $\exp(-q^2/\Lambda^2)$, $\chi^2 = 1.0069$

Non-local short-range regulator at LO: $\exp\big(-(p'^2 + p^2)/\Lambda^2\big), \chi^2 = 1.0062$

Short Range 3NF at N^3LO

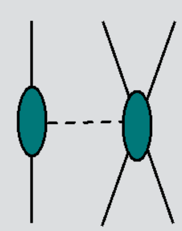
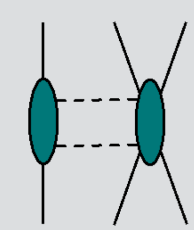
We followed both paths and provide two versions of 3NF

Version 1: Non-local short-range 3NF which can be used with SMS potential

Space		
Momentum	1	1

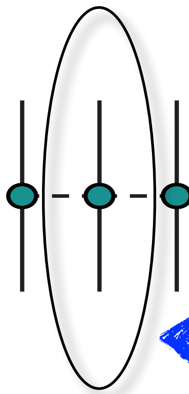
2.4 MB

Version 2: Local short-range 3NF to be used with the new NN potential

Space		
Momentum	1	1
Coordinate	0	0

0.4 MB

Pion-Nucleon Scattering up to Q^3

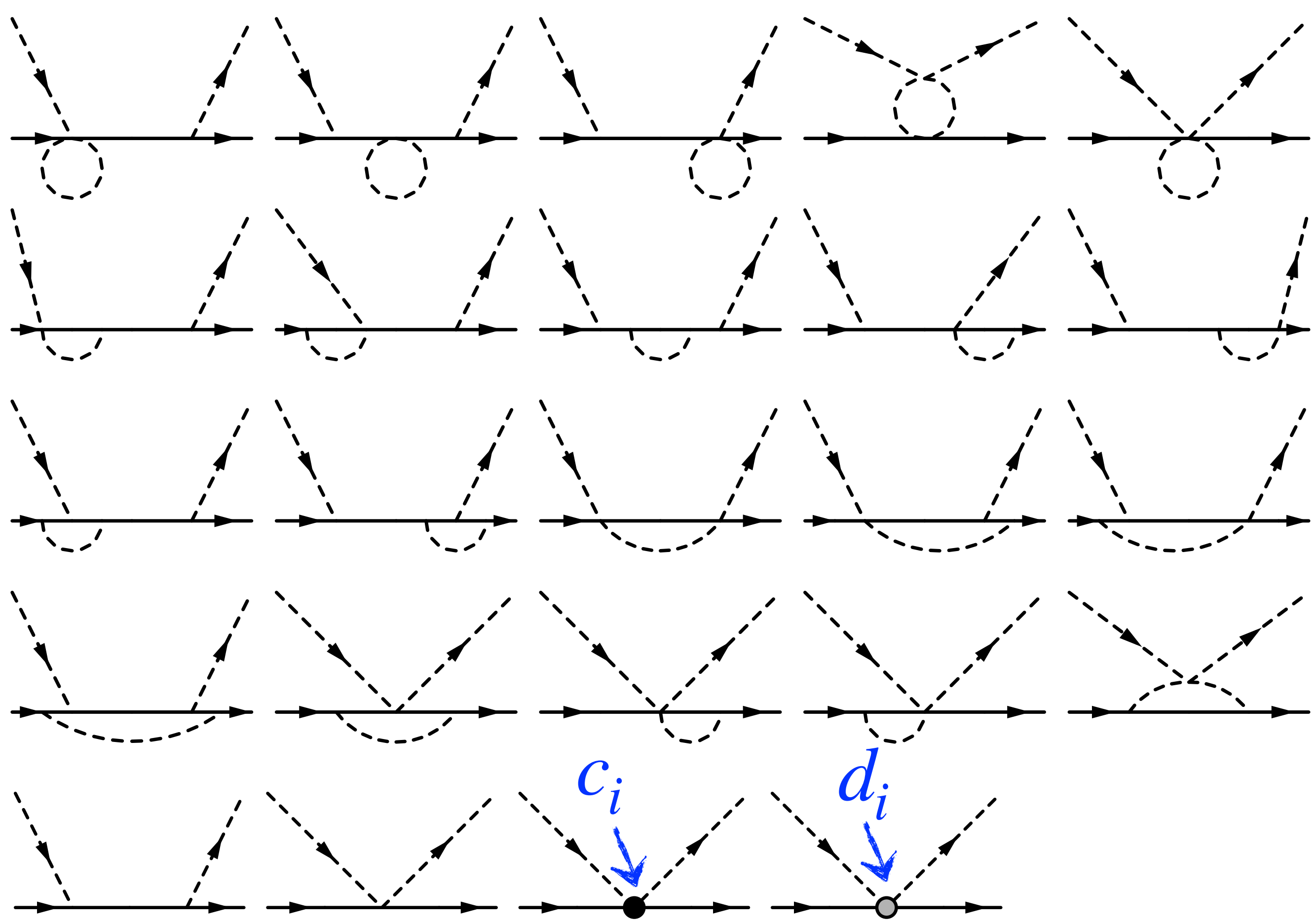


- TPE topology includes pion-nucleon amplitude as a subprocess

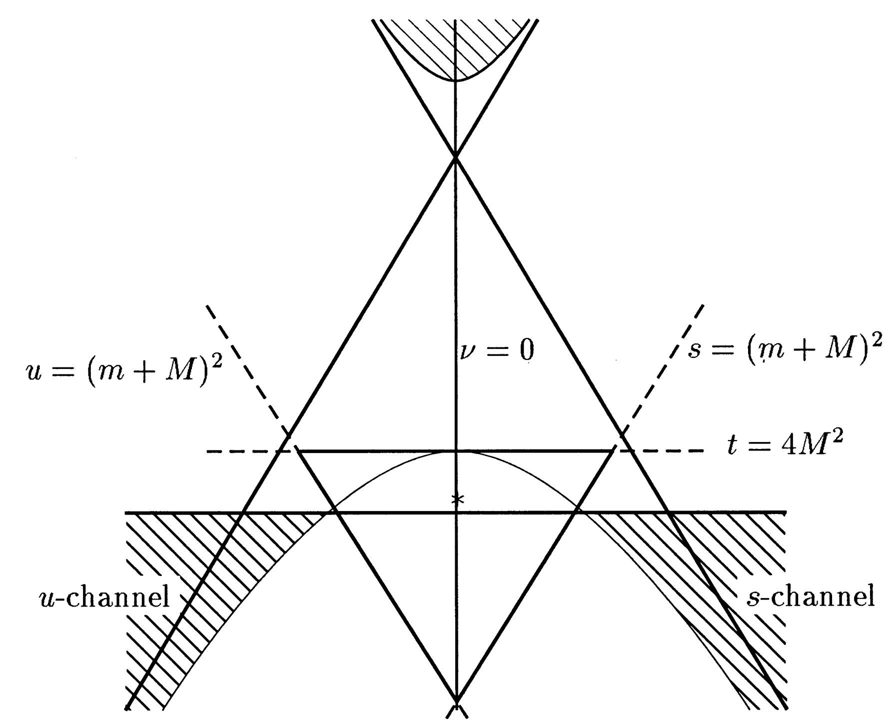
Pion-nucleon amplitude with gradient-flow regulator depends on c_i's

Calculation of pion-nucleon scattering with gradient-flow regulator required

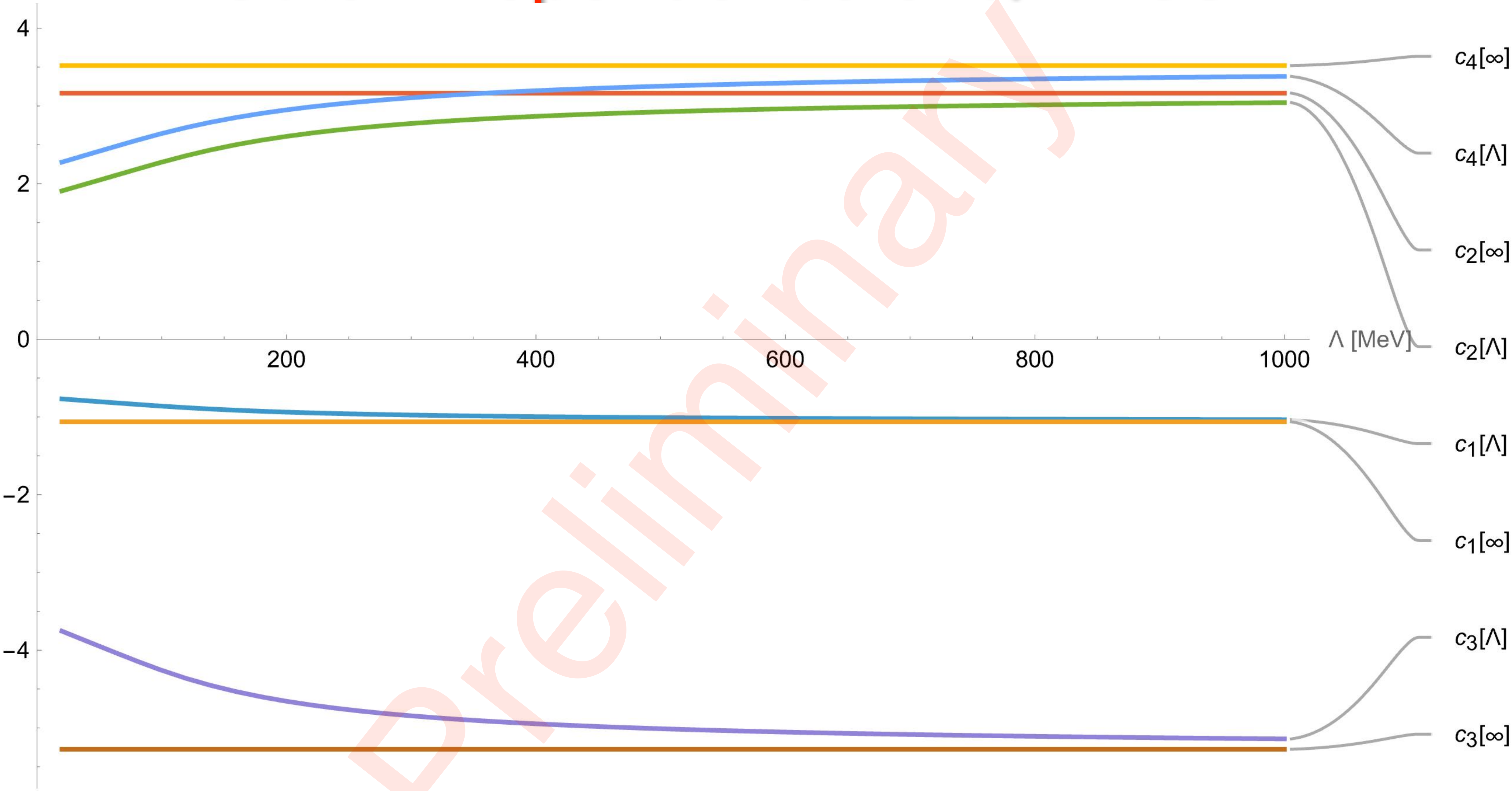
→ Patrick Walkowiak's master thesis



Fit LECs to pion-nucleon sub-threshold coefficients which are determined from Roy-Steiner equation



Cutoff Dependence of c_i LECs



- Saturation towards dim-reg results ($\Lambda \to \infty$) is fast
- For $\Lambda \sim 500\,\mathrm{MeV}$ the absolute value of c_i is smaller compared to c_i in dim-reg.

Summary

- 3PE contribution to NN has been calculated within unitary transformation approach
- Calculation of gradient-flow regularized 3NF at N^3LO is finished
  - Two versions for short-range 3NF at N^3LO
    - With non-local regulator in LO NN (SMS potential)
    - With local regulator in LO NN (new NN required)

Outlook

- Partial wave decomposition (PWD): K. Hebeler, A. Nogga & K. Topolnicki

  PWD is computationally more expensive, due to higher dimension of integrals over Schwinger parameters

Path-Integral Framework for Derivation of Nuclear Forces

HK, Epelbaum, PRC110 (2024) 4, 044003

Illustration fo Yukawa Model

We start with generating functional:

$$Z[\eta^\dagger, \eta] = \int [DN^\dagger][DN][D\pi] \exp\left(i \int d^4x \left(\mathscr{L} + \eta^\dagger(x) N(x) + N^\dagger(x) \eta(x) \right) \right)$$

Yukawa toy-model:

$$\mathscr{L} = N^\dagger \left(i \frac{\partial}{\partial x_0} + \frac{\vec{\nabla}^2}{2m} + \frac{g}{2F} \vec{\sigma} \cdot \vec{\nabla} \boldsymbol{\pi} \cdot \boldsymbol{\tau} \right) N + \frac{1}{2} \left(\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} - M^2 \boldsymbol{\pi}^2 \right)$$

- Perform a Gaussian path-integral over the pion fields

$$Z[\eta^\dagger, \eta] = \int [DN^\dagger][DN] \exp\left(i S_N + i \int d^4x \left(\eta^\dagger(x) N(x) + N^\dagger(x) \eta(x) \right) \right)$$

$$S_N = \int d^4x \, N^\dagger(x) \left(i \frac{\partial}{\partial x_0} + \frac{\vec{\nabla}^2}{2m} \right) N(x) \; - \; V_{NN}$$

← Non-instant one-pion-exchange interaction

$$V_{NN} = -\frac{g^2}{8F^2} \int d^4x \, d^4y \, \vec{\nabla}_x \cdot \left[N^\dagger(x) \vec{\sigma} \boldsymbol{\tau} \right] N(x) \, \Delta_F(x-y) \, \vec{\nabla}_y \cdot \left[N^\dagger(y) \vec{\sigma} \boldsymbol{\tau} \right] N(y)$$

with non-instant pion propagator: $\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{-i q \cdot x}}{q^2 - M^2 + i\epsilon}$

Instant Interactions from Path-Integral

To transform V_{NN} into an instant form we rewrite a pion propagator

$$\frac{1}{q_0^2-\omega_q^2} = -\frac{1}{\omega_q^2} + \frac{1}{q_0^2-\omega_q^2} + \frac{1}{\omega_q^2} = -\frac{1}{\omega_q^2} + q_0^2 \frac{1}{\omega_q^2}\frac{1}{q_0^2-\omega_q^2}, \quad \omega_q = \sqrt{\vec{q}^2+M^2}$$

In coordinate space this corresponds to $\Delta_F(x) = \Delta_S(x) - \frac{\partial^2}{\partial x_0^2}\Delta_{FS}(x)$ with

$$\Delta_S(x) = -\int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq\cdot x}}{\omega_q^2} = -\delta(x_0)\int \frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{q}\cdot\vec{x}}}{\omega_q^2}, \quad \Delta_{FS}(x) = \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq\cdot x}}{\omega_q^2(q_0^2-\omega_q^2)}$$

- The decomposition $\Delta_F(x) = \Delta_S(x) - \frac{\partial^2}{\partial x_0^2}\Delta_{FS}(x)$ can be generalized

  $G(x) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot x}\tilde{G}(q_0^2, q^2)$ and $\tilde{G}(q_0^2, q^2)$ is differentiable at $q_0 = 0$

  Defining $G_S(x) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot x}\tilde{G}(0,q^2)$ and $G_{FS}(x) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot x}\frac{\tilde{G}(q_0^2, q^2) - \tilde{G}(0,q^2)}{q_0^2}$

$$\rightarrow \quad G(x) = G_S(x) - \frac{\partial^2}{\partial x_0^2}G_{FS}(x)$$

Instant Interactions from Path-Integral

Perform an instant decomposition of the pion propagator $\Delta_F(x) = \Delta_S(x) - \frac{\partial^2}{\partial x_0^2}\Delta_{FS}(x)$

$$V_{NN} = -\frac{g^2}{8F^2}\int d^4x\, d^4y \vec{\nabla}_x \cdot \left[N^\dagger(x)\vec{\sigma}\boldsymbol{\tau}\right]N(x)\,\Delta_F(x-y)\,\vec{\nabla}_y \cdot \left[N^\dagger(y)\vec{\sigma}\boldsymbol{\tau}\right]N(y)$$

$$\rightarrow \quad V_{NN} = V_{OPE} + V_{FS}$$

$$V_{OPE} = -\frac{g^2}{8F^2}\int d^4x\, d^4y \vec{\nabla}_x \cdot \left[N^\dagger(x)\vec{\sigma}\boldsymbol{\tau}\right]N(x)\,\Delta_S(x-y)\,\vec{\nabla}_y \cdot \left[N^\dagger(y)\vec{\sigma}\boldsymbol{\tau}\right]N(y)$$ is instant

$$V_{FS} = \frac{g^2}{8F^2}\int d^4x\, d^4y \vec{\nabla}_x \cdot \left[N^\dagger(x)\vec{\sigma}\boldsymbol{\tau}\right]N(x)\,\frac{\partial^2}{\partial x_0^2}\Delta_{FS}(x-y)\,\vec{\nabla}_y \cdot \left[N^\dagger(y)\vec{\sigma}\boldsymbol{\tau}\right]N(y)$$ is non-instant

V_{FS} is time-derivative dependent and thus can be eliminated by a non-polynomial field redefinition

$$N(x) \rightarrow N'(x) = N(x) + i\frac{g^2}{8F^2}\int d^4y\,[\vec{\sigma}\boldsymbol{\tau}N(x)]\cdot[\,\vec{\nabla}_x\frac{\partial}{\partial x_0}\Delta_{FS}(x-y)]\,\vec{\nabla}_y \cdot [N^\dagger(y)\vec{\sigma}\boldsymbol{\tau}N(y)]$$

$$N^\dagger(x) \rightarrow N'^\dagger(x) = N^\dagger(x) - i\frac{g^2}{8F^2}\int d^4y \vec{\nabla}_y \cdot [N^\dagger(y)\vec{\sigma}\boldsymbol{\tau}N(y)][\,\vec{\nabla}_y\frac{\partial}{\partial y_0}\Delta_{FS}(y-x)]\cdot[N^\dagger(x)\vec{\sigma}\boldsymbol{\tau}]$$

Instant Interactions from Path-Integral

Non-local field transformations remove time-derivative dependent two-nucleon interactions but generate time-derivative dependent three-nucleon interactions.

These contributions can be eliminated by similar field transformations

$$Z[\eta^\dagger,\eta] = \int [DN'^\dagger][DN'] \det\left(\frac{\delta(N'^\dagger,N')}{\delta(N^\dagger,N)}\right)\exp\left(i\,S_{N(N'^\dagger,N')} + i\int d^4x\left(\eta^\dagger(x)N(N'^\dagger,N')(x) + N(N'^\dagger,N')^\dagger(x)\eta(x)\right)\right)$$

$$\simeq \int [DN'^\dagger][DN'] \det\left(\frac{\delta(N'^\dagger,N')}{\delta(N^\dagger,N)}\right)\exp\left(i\,S_{N(N'^\dagger,N')} + i\int d^4x\left(\eta^\dagger(x)N'(x) + N'^\dagger(x)\eta(x)\right)\right)$$

Equivalence theorem: nucleon pole-structure is unaffected by the field-transf.

$$S_{N(N'^\dagger,N')} = \int d^4x\, N'^\dagger(x)\left(i\frac{\partial}{\partial x_0} + \frac{\vec{\nabla}^2}{2m}\right)N'(x) \;-\; V_{OPE} + \mathcal{O}(g^4)$$

$$V_{OPE} = -\frac{g^2}{8F^2}\int d^4x\, d^4y\, \vec{\nabla}_x \cdot \left[N'^\dagger(x)\vec{\sigma}\boldsymbol{\tau}\right]N'(x)\,\Delta_S(x-y)\,\vec{\nabla}_y \cdot \left[N'^\dagger(y)\vec{\sigma}\boldsymbol{\tau}\right]N'(y)$$

Instant one-pion-exchange interaction

One-Loop Corrections to Interaction

One loop corrections to NN & NNN interaction come from functional determinant

$$\det\left(\frac{\delta(N'^\dagger, N')}{\delta(N^\dagger, N)}\right) = \exp\left(\mathrm{Tr}\log\frac{\delta(N'^\dagger, N')}{\delta(N^\dagger, N)}\right)$$

Due to non-local structure of field transformations $\det\left(\frac{\delta(N'^\dagger, N')}{\delta(N^\dagger, N)}\right) \neq 1$

$$S_{N(N'^\dagger,N')} = \int d^4x\, N'^\dagger(x)\left(i\frac{\partial}{\partial x_0} + \frac{\vec{\nabla}^2}{2m} + \frac{3g^2M^3}{32\pi F^2}\right)N'(x) - V_{OPE} + \mathcal{O}(g^4)$$

Nucleon mass-shift **Langacker, Pagels, PRD 10 (1974) 2904**
is reproduced from functional determinant

Note: The Z-factor of the nucleon is equal to one. This is due to the replacement

$\eta^\dagger N + N^\dagger \eta \to \eta^\dagger N' + N'^\dagger \eta$ in the generating functional $Z[\eta^\dagger, \eta]$

The original Z-factor of the nucleon is reproduced if we remove this replacement

$$Z = 1 - \frac{9M^2g^2}{2F^2}\left(\bar{\lambda} + \frac{1}{16\pi^2}\left(\log\frac{M}{\mu} + \frac{1}{3} - \frac{\pi}{2}\frac{M}{\mu})\right)\right)$$

Path-integral Approach

We start with generating functional: **HK, Epelbaum, PRC110 (2024) 4, 044003**

$$Z[\eta^\dagger, \eta] = \int [DN^\dagger][DN][D\pi] \exp\left(i \int d^4x \left(\mathscr{L}_\pi + \mathscr{L}_{\pi N} + \mathscr{L}_{NN} + \mathscr{L}_{NNN} + \eta^\dagger(x) N(x) + N^\dagger(x)\eta(x) \right) \right)$$

- Integrate over pion fields via loop-expansion of the action

  → expansion of the action around the classical pion solution

- Perform instant decomposition of the remaining interactions between nucleons

- Perform nucleon-field redefinitions to eliminate non-instant part of the interaction

- Calculate functional determinant to get one-loop corrections to few-nucleon forces

Checks in dimensional regularization

Unitary transformation (Okubo) & path-integral approaches lead to the same chiral EFT nuclear forces up to N^4LO