$A = 2, 3, 4$ nuclear contact coefficients in the generalized contact formalism

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Outline

In collaboration with

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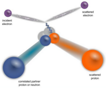
Outline

Short-range correlations in nuclei

Aim: understanding nuclear wave function behavior at high momenta and short distances go beyond nuclear shell model – see, for example, [Hen it et al., RMP, 2017] [CLAS and Hall A Collaborations, Jefferson Lab.]



Experimental evidences

- Inclusive (e, e') experiments [Egiyan *et al.*, 2003], [Fomin *et al.*, 2012]
- Exclusive reactions (NN knokout) [Hen *et al.*, 2014]
- (e, e') reactions at large Bjorken x_B values [Korover *et al.*, 2014]
- $\rightarrow n-p$ pairs dominates the high-momentum components of the nuclear wave function [Subedi, *et al.*, 2008],[Piasetzky *et al.*, 2006],[...]
- $\rightarrow$ all nuclei exhibit similar momentum distributions at high momenta

The "contact" hypothesis

- For $r_{12} \rightarrow 0$: $\Psi(1,\ldots,A) \approx \varphi(1,2)\tilde{\Psi}(3,\ldots,A)$
- Consequences: Two-body momentum distribution

$$n_2(k) \xrightarrow{k\rightarrow\infty} C_A \times |\tilde{\varphi}(k)|^2$$

- C_A "contact" coefficient [Tan, 2008], [Weiss *et al.*, 2015-2018], [Ciofi degli Atti, 2015], [Cruz-Torres *et al.*, 2021], ...
- This study: calculation of C_A for $^3\mathrm{He}$, $^3\mathrm{H}$, and $^4\mathrm{He}$ using various interactions
- Extension of [Cruz-Torres *et al.*, 2021]

Outline

Two-body Momentum/Density Distributions

Two-body momentum distributions – 2BMD

Pair $1, 2$: $\boldsymbol{k} = \frac{\boldsymbol{k}_1 - \boldsymbol{k}_2}{2}$ relative momentum; $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2$ total momentum

$$n^S_{N_1\,N_2}(k) = \int d\hat{\boldsymbol{k}} \int d\boldsymbol{K} d\boldsymbol{k}_3 \cdots d\boldsymbol{k}_A \, \Psi^\dagger(\boldsymbol{k}, \boldsymbol{K}, \boldsymbol{k}_3, \ldots, \boldsymbol{k}_A) P_{N_1\,N_2} P^S \Psi(\boldsymbol{k}, \boldsymbol{K}, \boldsymbol{k}_3, \ldots, \boldsymbol{k}_A)$$

Two-body density functions – 2BDF

Pair $1, 2$: $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ relative distance; $\boldsymbol{R} = \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}$ pair CM position

$$\rho^S_{N_1\,N_2}(r) = \int d\hat{\boldsymbol{r}} \int d\boldsymbol{R} d\boldsymbol{r}_3 \cdots d\boldsymbol{r}_A \, \Psi^\dagger(\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_A) P_{N_1\,N_2} P^S \Psi(\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_A)$$

$$N_1 N_2 = pp, pn, nn; \; P^{S=1} \equiv P^{^3S_1}; \; P^{S=0} \equiv P^{^1S_0}$$

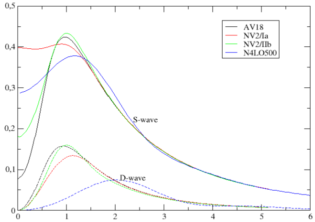
The generalized contact formalism

The "contact" hypothesis [Tan, 2008], [Ciofi degli Atti, 2015], [Weiss *et al.*, 2015] ...

$$\Psi \xrightarrow{r_{ij}\to 0} \sum_{LSJ,N_1N_2} \varphi^{LSJ}_{N_1N_2}(\mathbf{r}_{ij})\, A^{LSJ}_{ij}(\mathbf{R}_{ij}, \mathbf{r}_{k\neq ij}),$$

- $N_1N_2 \equiv pp, pn, nn$ (or better, $T, T_z = 00, 1+1, 10, 1-1$)
- LSJ angular-spin state of the pair
- $(-)^{L+S+T} = -1$ antisymmetry
- We study the $^3S_1$ and $^1S_0$ cases
- $^3S_1$ case ($T=0$): $\varphi \equiv \varphi_d$
- $^1S_0$ case ($T=1$): $\varphi \equiv \varphi_{^1S_0}$ at $E=0$ (virtual state)
- We can just distinguish the two cases specifying $S = 0, 1$

Deuteron u & w



$$n^S_{N_1N_2,\,A}(k) \xrightarrow{k\to\infty} \tilde{C}^S_{N_1N_2,\,A} \times |\tilde{\varphi}^S_{N_1N_2}(k)|^2$$

$$\rho^S_{N_1N_2,\,A}(r) \xrightarrow{r\to 0} C^S_{N_1N_2,\,A} \times |\varphi^S_{N_1N_2}(r)|^2$$

Normalization ($k_F = 1.3$ fm^{-1})

$$\int_{k_F}^{\infty} dk\, k^2 |\tilde{\varphi}^S_{N_1N_2}(k)|^2 = 1$$

Interactions

Name	DOF	O_χ	(R_S, R_L) or Λ	E range	L/NL	$B(^3\mathrm{He})$	$B(^4\mathrm{He})$
AV18/UIX	N	–	–	0–300 MeV	L	7.750	28.46
NV2+3/Ia*	π, N, Δ	N3LO	(0.8, 1.2) fm	0–125 MeV	L	7.730	28.24
NV2+3/Ib*	π, N, Δ	N3LO	(0.7, 1.0) fm	0–125 MeV	L	7.727	28.21
NV2+3/IIa*	π, N, Δ	N3LO	(0.8, 1.2) fm	0–200 MeV	L	7.728	28.08
NV2+3/IIb*	π, N, Δ	N3LO	(0.7, 1.0) fm	0–200 MeV	L	7.727	28.11
N2LO450	π, N	N2LO	450 MeV	0–300 MeV	NL	7.714	28.34
N2LO500	π, N	N2LO	500 MeV	0–300 MeV	NL	7.727	28.03
N2LO550	π, N	N2LO	550 MeV	0–300 MeV	NL	7.727	27.99
N3LO450	π, N	N3LO	450 MeV	0–300 MeV	NL	7.715	28.38
N3LO500	π, N	N3LO	500 MeV	0–300 MeV	NL	7.724	28.00
N3LO550	π, N	N3LO	550 MeV	0–300 MeV	NL	7.728	28.14
N4LO450	π, N	N4LO	450 MeV	0–300 MeV	NL	7.716	28.34
N4LO500	π, N	N4LO	500 MeV	0–300 MeV	NL	7.723	27.94
N4LO550	π, N	N4LO	550 MeV	0–300 MeV	NL	7.729	28.29

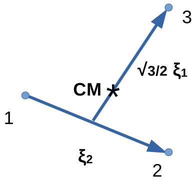
[Wiringa *et al.*, 1995], [Piarulli *et al.*, 2016], [Entem *et al.*, 2017]
All chiral interactions include local 3N potential at N2LO

The wave functions – $A = 3$ case

Definitions

- Jacobi vectors
  - $\boldsymbol{\xi}_1 = \sqrt{\frac{2}{3}}(\boldsymbol{r}_3 - \frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2})$
  - $\boldsymbol{\xi}_2 = \frac{1}{\sqrt{2}}(\boldsymbol{r}_2 - \boldsymbol{r}_1)$
- Hyperradius $\rho = \sqrt{\boldsymbol{\xi}_1^2 + \boldsymbol{\xi}_2^2}$
- Hyperangles $\Omega = \{\hat{\xi}_1, \hat{\xi}_2, \varphi_2\}$, $\cos\varphi_2 = \xi_2/\rho$

Hyperspherical Harmonic expansion of the wave function
[Kievsky *et al.*, 2008]
[Marcucci *et al.*, 2020]



Hamiltonian written in terms of the new coordinates

$$H = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{d^2\rho} + \frac{5}{\rho}\frac{\partial}{d\rho} - \frac{\Lambda^2(\Omega)}{\rho^2}\right] + V(\rho,\Omega)$$

Hyperspherical harmonics (HH) functions = eigenstates of $\Lambda^2(\Omega)$
Complete set in Ω space

The wave functions – $A = 3$ case

Expansion basis: HH $\times$ spin-isospin states

$K = \ell_1 + \ell_2 + 2n_2$ grand angular quantum number

$$\mathcal{H}_\mu(\Omega_{ijk}) = N_{[K]}\left\{\left[Y_{\ell_1}(\hat{\xi}_1)Y_{\ell_2}(\hat{\xi}_2)\right]_L\left[(s_is_j)_{S_a}s_k\right]_S\right\}_{j,j_z}\left[(t_it_j)_{T_a}t_k\right]_{T,T_z}$$

$$\times(\cos\varphi_2)^{\ell_2}(\sin\varphi_2)^{\ell_1}P_{n_2}^{\ell_1+\frac{1}{2},\ell_2+\frac{1}{2}}(\cos 2\varphi)$$

Expansion of the wave function

$$\Psi = \sum_\mu u_\mu(\rho)\sum_{perm}\mathcal{H}_\mu(\Omega_{ijk})$$

- $\mu \equiv \{\ell_1, \ell_2, L, S_a, S, T_a, T, n_2\}$
- Epansion of the hyperradial functions $u_\mu(\rho) = \sum_{l=1,N_L} a_{\mu,l} f_l(\rho)$
- $f_l(\rho) \sim L_{l-1}(\gamma\rho)\exp[-\gamma\rho/2]$
- We include states up to a given K_{max} and N_L

Convergence

^{3}He - N3LO500 case

K_{max}	N_L	$C_{np}^{s=1}$
80	16	4.118
80	20	4.113
80	24	4.112
80	28	4.112
100	28	4.112

Outline

Extraction of the contact coeffcients (CCs)

$$n^S_{N_1 N_2,\,A}(k) \xrightarrow{k\to\infty} \tilde{C}^S_{N_1 N_2,\,A} \times |\tilde{\varphi}^S_{N_1 N_2}(k)|^2$$
$$\rho^S_{N_1 N_2,\,A}(r) \xrightarrow{r\to 0} C^S_{N_1 N_2,\,A} \times |\varphi^S_{N_1 N_2}(r)|^2$$

Using the 2BDM

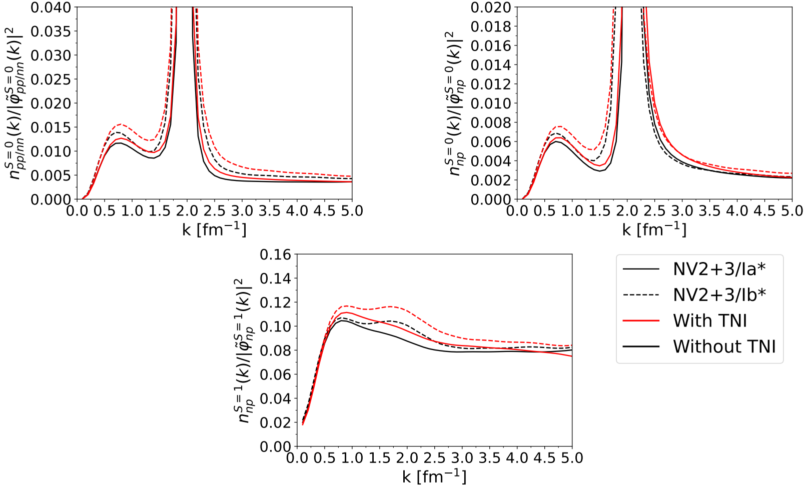
$$\tilde{C}^{S=0}_{pp/nn}(k) = \lim_{k\to\infty} \left\{ n^{S=0}_{pp/nn}(k) \;/\; |\tilde{\varphi}^{S=0}_{pp/nn}(k)|^2 \right\},$$
$$\tilde{C}^{S=1}_{pn}(k) = \lim_{k\to\infty} \left\{ n^{S=1}_{pn}(k) \;/\; |\tilde{\varphi}^{S=1}_{pn}(k)|^2 \right\},$$
$$\tilde{C}^{S=0}_{pn}(k) = \lim_{k\to\infty} \left\{ n^{S=0}_{pn}(k)/|\tilde{\varphi}^{S=0}_{pn}(k)|^2 \right\}.$$

Using the 2BDF

$$C^{S=0}_{pp/nn}(r) = \lim_{r\to 0} \left\{ \rho^{S=0}_{pp/nn}(r) \;/\; |\varphi^{S=0}_{pp/nn}(r)|^2 \right\},$$
$$C^{S=1}_{pn}(r) = \lim_{r\to 0} \left\{ \rho^{S=1}_{pn}(r) \;/\; |\varphi^{S=1}_{pn}(r)|^2 \right\},$$
$$C^{S=0}_{pn}(r) = \lim_{r\to 0} \left\{ \rho^{S=0}_{pn}(r)/|\varphi^{S=0}_{pn}(r)|^2 \right\}.$$

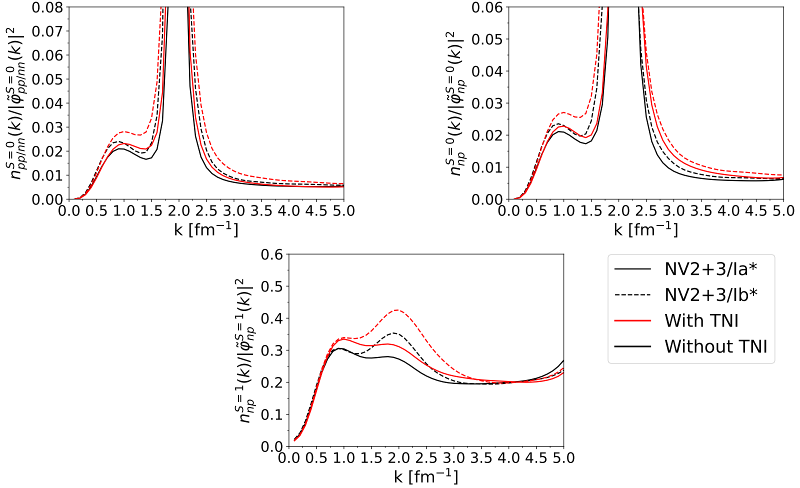
We disregard the difference between C_{pp} and C_{nn}

From the 2BMD - local potentials – ^{3}He



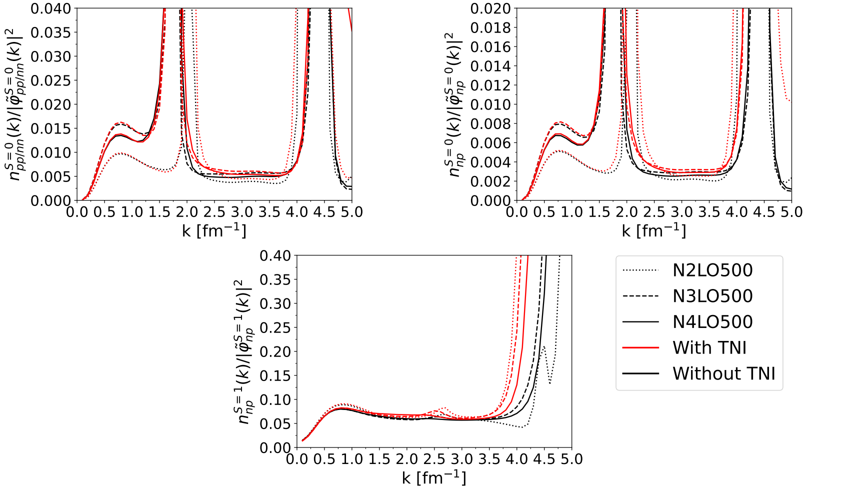
Extraction zone: $3.5 - 4.5$ fm^{-1}

From the 2BMD - local potentials – ^{4}He



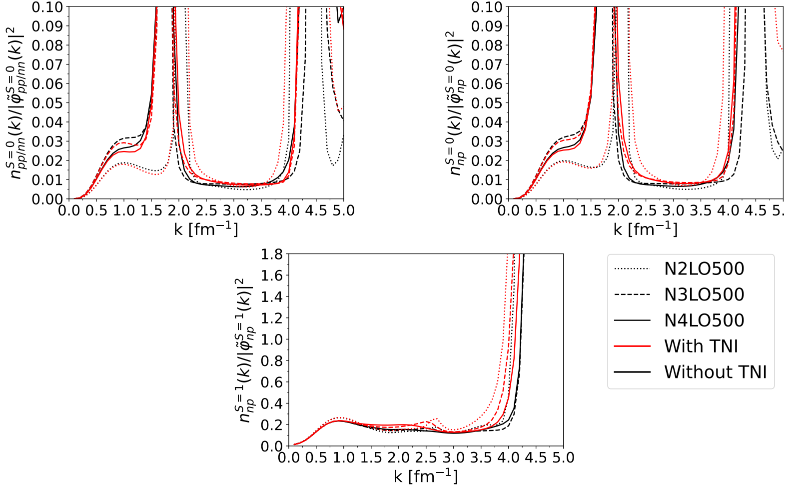
Extraction zone: 3.5 – 4.5 fm^{-1}

From the 2BMD - non-local potentials – ^{3}He



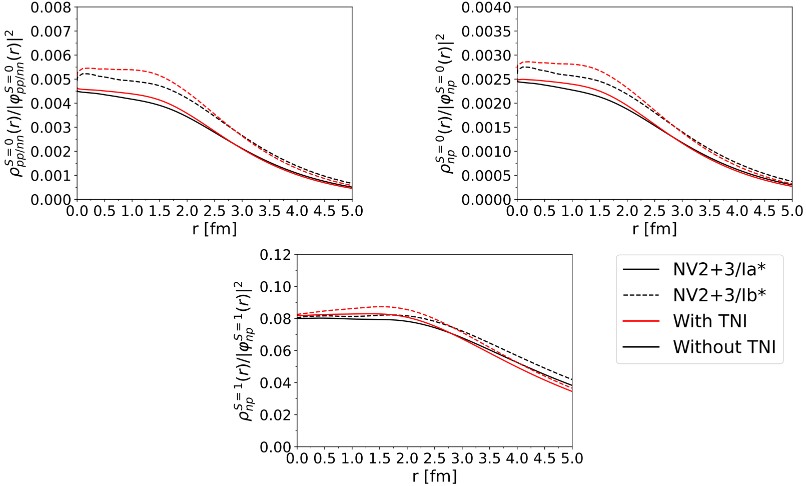
Extraction zone: $\approx 2.5 - 3.5$ fm^{-1}// divergences at high momenta are due to cutoff effects

From the 2BMD - non-local potentials – $^4\mathrm{He}$



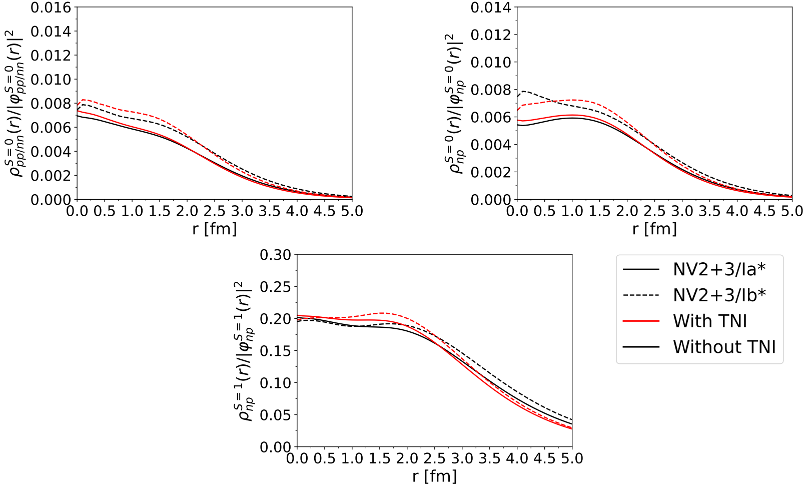
Extraction zone: $\approx 2.5 - 3.5$ fm^{-1}// divergences at high momenta are due to cutoff effects

From the 2BDF - local potentials – ^{3}He



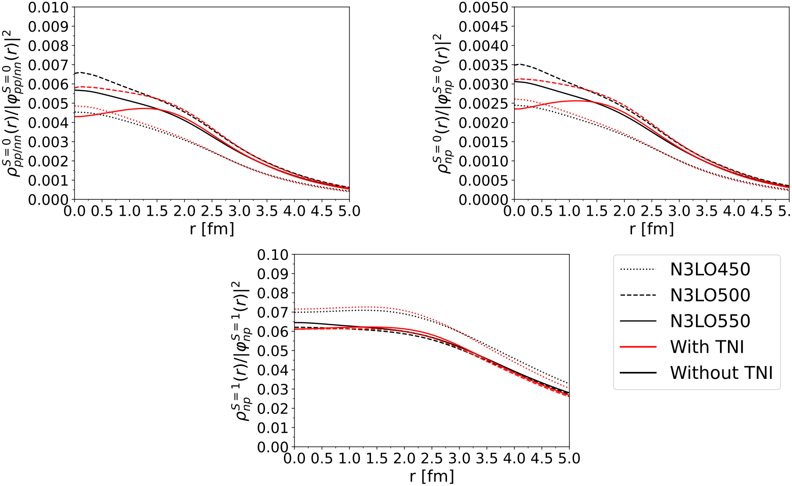
Extraction zone: 0 − 1 fm

From the 2BDF - local potentials – ^{4}He



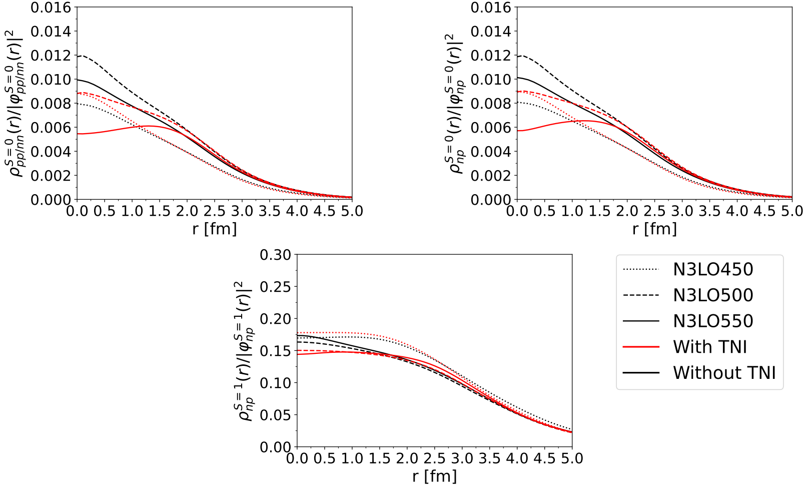
Extraction zone: $0 - 1$ fm

From the 2BDF - non-local potentials – $^3\mathrm{He}$



Extraction zone: 0 – 1 fm

From the 2BDF - non-local potentials – ^{4}He



Extraction zone: 0 – 1 fm

Numerical tests

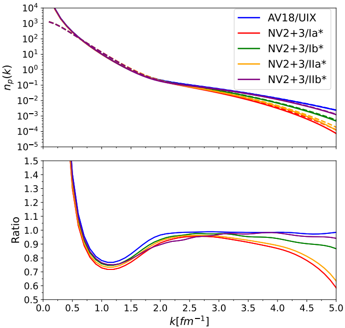
1) $C^S_{N_1N_2,A}$ should be the same extracted from either the 2BMD or the 2NDF (see later)

2) Relation between the 1-body MD and the 2BMD in the GCF [Weiss *et al.*. 2018]

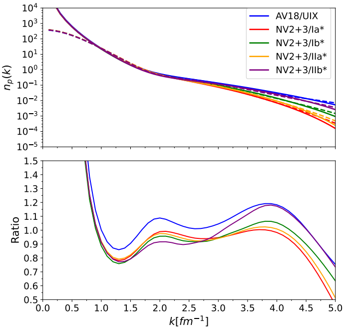
$$n^{\rm GCF}_{p,A}(\boldsymbol{k}) = 2\tilde{C}^{S=0}_{pp,A}|\tilde{\varphi}^{S=0}_{pp}(\boldsymbol{k})|^2 + \tilde{C}^{S=0}_{np,A}|\tilde{\varphi}^{S=0}_{np}(\boldsymbol{k})|^2 + \tilde{C}^{S=1}_{np,A}|\tilde{\varphi}^{S=1}_{np}(\boldsymbol{k})|^2$$

Local potentials

^{3}He (dashed lines: 1BMD)



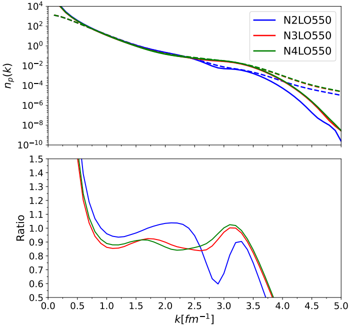
^{4}He (dashed lines: 1BMD)



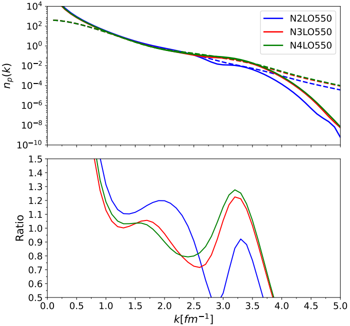
Numerical tests – non-local potentials

$$n^{\rm GCF}_{p,A}(\boldsymbol{k}) \;=\; 2\tilde{C}^{S=0}_{pp,A}|\tilde{\varphi}^{S=0}_{pp}(\boldsymbol{k})|^2 + \tilde{C}^{S=0}_{np,A}|\tilde{\varphi}^{S=0}_{np}(\boldsymbol{k})|^2 + \tilde{C}^{S=1}_{np,A}|\tilde{\varphi}^{S=1}_{np}(\boldsymbol{k})|^2$$

^{3}He (dashed lines: 1BMD)



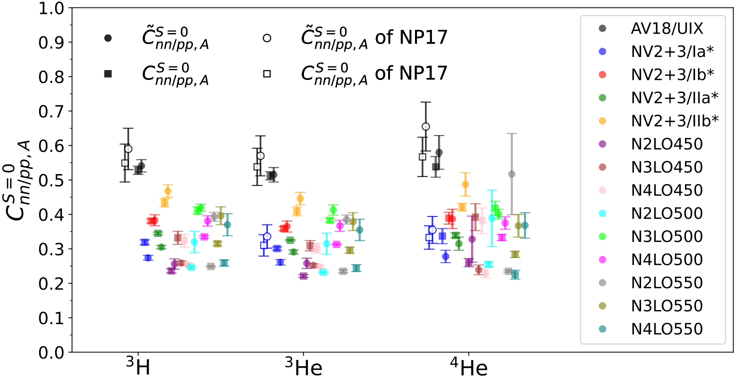
^{4}He (dashed lines: 1BMD)



Outline

Results for $C^{S=0}_{pp/nn,A}$

solid circles: from 2BMD; solid squares: from 2BDF; open symbols: [Cruz-Torres *et al.*, 2021]



Results for $C^{S=0}_{pp/nn,A}$

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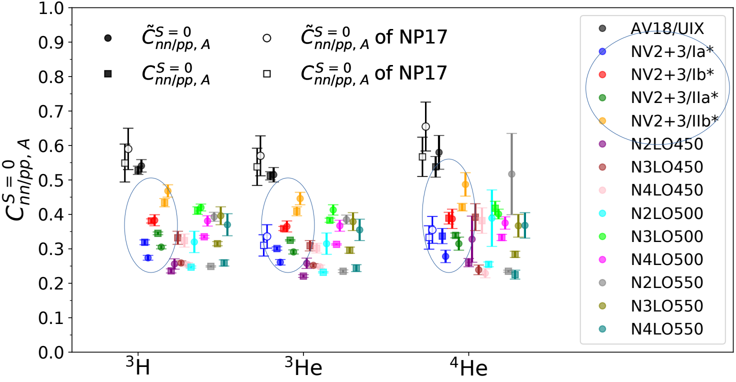

Results for $C^{S=0}_{pp/nn,A}$

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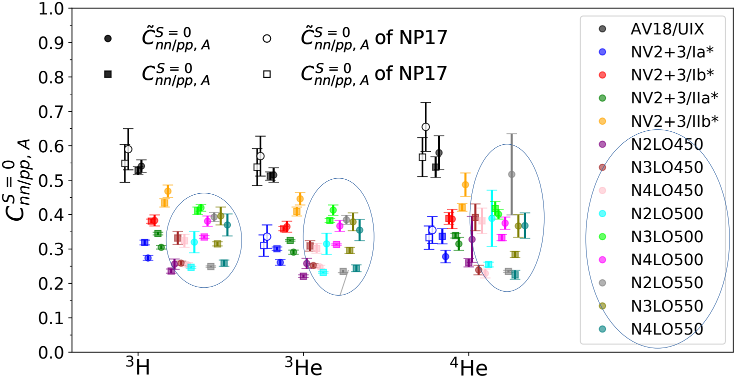
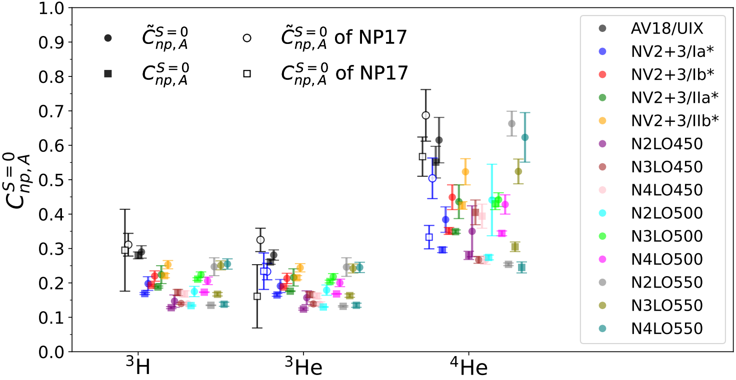


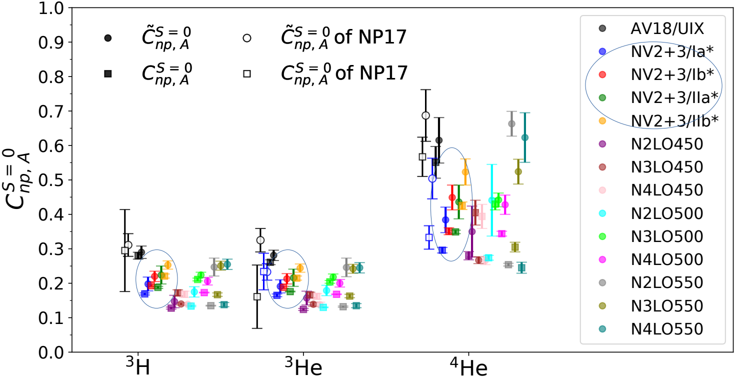
Results for $C^{S=0}_{pn,A}$

solid circles: from 2BMD; solid squares: from 2BDF; open symbols: [Cruz-Torres *et al.*, 2021]



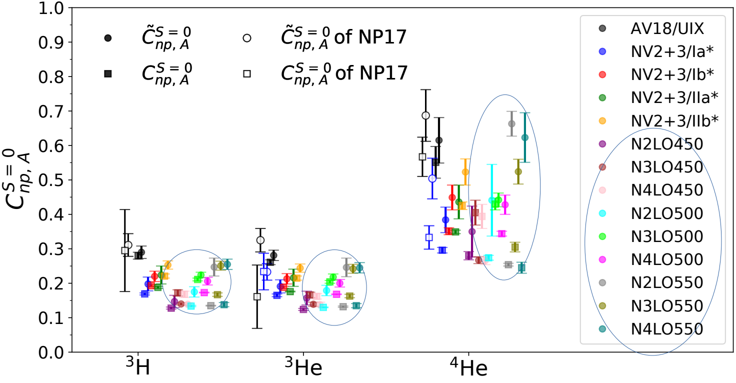
Results for $C^{S=0}_{pn,A}$

solid circles: from 2BMD; solid squares: from 2BDF; open symbols: [Cruz-Torres *et al.*, 2021]



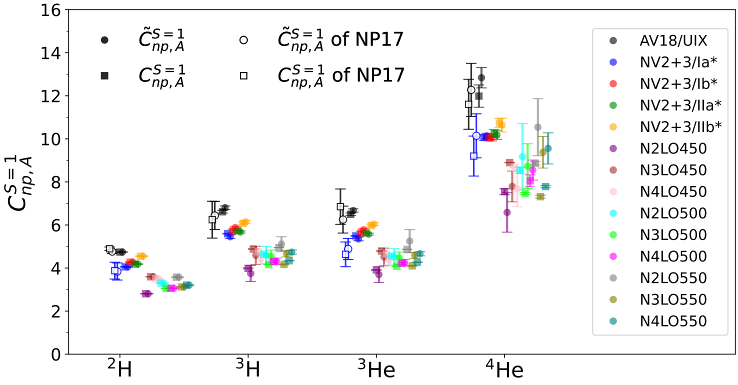
Results for $C_{pn,A}^{S=0}$

solid circles: from 2BMD; solid squares: from 2BDF; open symbols: [Cruz-Torres *et al.*, 2021]



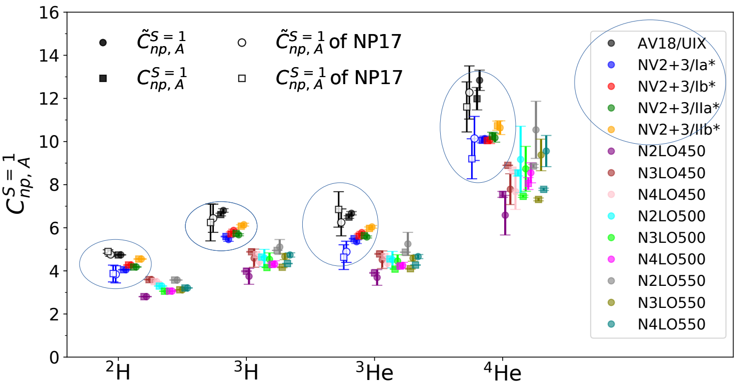
Results for $C_{pn,A}^{S=1}$

solid circles: from 2BMD; solid squares: from 2BDF; open symbols: [Cruz-Torres *et al.*, 2021]



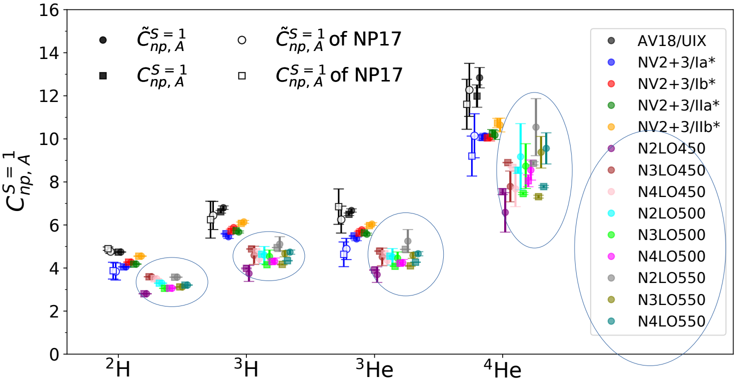
Results for $C^{S=1}_{pn,A}$

solid circles: from 2BMD; solid squares: from 2BDF; open symbols: [Cruz-Torres *et al.*, 2021]



Results for $C^{S=1}_{pn,A}$

solid circles: from 2BMD; solid squares: from 2BDF; open symbols: [Cruz-Torres *et al.*, 2021]



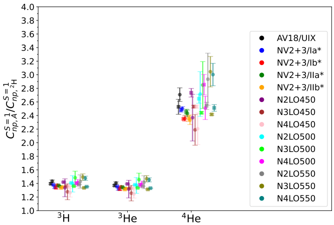
Ratio of contact coeffcients

Ratios of the extracted CCs with respect to a reference nucleus

$$\rho^S_{N_1N_2,\,A}(r)/\rho^S_{N_1N_2,\,A_0}(r) \xrightarrow{r\to 0} C^S_{N_1N_2,\,A}/C^S_{N_1N_2,\,A_0}$$

$$n^S_{N_1N_2,\,A}(k)/n^S_{N_1N_2,\,A_0}(k) \xrightarrow{k\to \infty} \tilde{C}^S_{N_1N_2,\,A}/\tilde{C}^S_{N_1N_2,\,A_0}$$

- For the $S=1$ CC: the deuteron
- For the $S=0$ CCs: ^{4}He
- Ratio $C^{S=1}_{pn,A}/C^{S=1}_{pn,d}$:



- Hypothesis: indipendent on the interaction [Cruz-Torres *et al.*, 2021], [Weiss *et al.*, 2015]
- $\rightarrow$ CCs of $A \gg 1$ nuclei:
  1. compute the CC for a soft potential, C^{soft}_A
  2. then, the CC of a realistic potential can be obtained using

$$C^{real}_A/C^{real}_{A_0} = C^{soft}_A/C^{soft}_{A_0}$$

  3. Knowing $C^{real}_{A_0}$ and $C^{soft}_{A_0}$, C^{real}_A can be calculated

Ratio of contact coeffcients & averages

Now, we want to compute these ratios, averaging over the interactions
$i = 1$ AV18/UIX $i = 2, \ldots, 5$ local potentials NV2+3 (4) $i = 6, \ldots, 14$ N2LO-N4LO potentials (9)

$$\langle C \rangle = \sum_i C_i P_i \qquad \delta^2 \langle C \rangle = \delta C_i P_i + \delta^2 C_{\rm syst} \qquad \delta^2 C_{\rm syst} = \left[\sum_i C_i^2 P_i\right] - \left[\sum_i C_i P_i\right]^2$$

Criterium 1

Average separately the local and non-local potentials

- $P_{1-5} = 1/10$
- $P_{5-14} = 1/18$

Criterium 2

Average separately the local and non-local potentials, excluding AV18/UIX

- $P_1 = 0$
- $P_{2-5} = 1/8$
- $P_{5-14} = 1/18$

Criterium 3

Treat all potentials equally

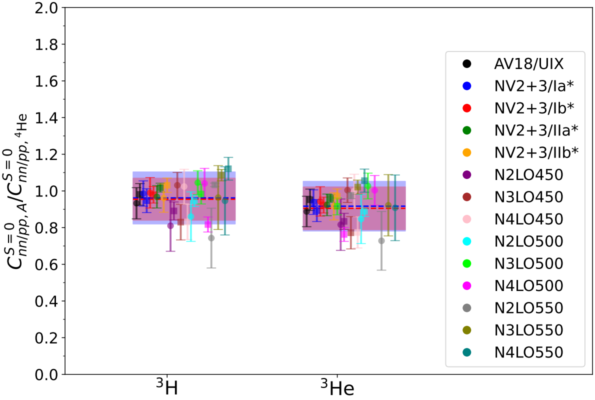
- $P_{1-14} = 1/14$

Criterium 4

Consider only N3LO potentials

- $P_1 = 0$
- $P_{2-5} = 1/8$
- $P_{6-8,12-14} = 0$
- $P_{9-11} = 1/6$

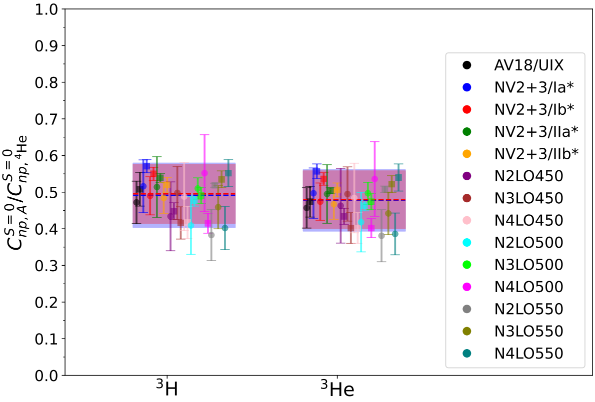
Ratio $C^{S=0}_{pp/nn,A}/C^{S=0}_{pp/nn,^4\mathrm{He}}$



Bands: average+error
violet: criterium 1
magenta: criterium 4

nn/pp $S=0$	Criterium 1	Criterium 2	Criterium 3	Criterium 4
^{3}H	0.978 ± 0.147	0.981 ± 0.147	0.978 ± 0.162	0.968 ± 0.119
^{3}He	0.932 ± 0.140	0.934 ± 0.139	0.933 ± 0.153	0.918 ± 0.123

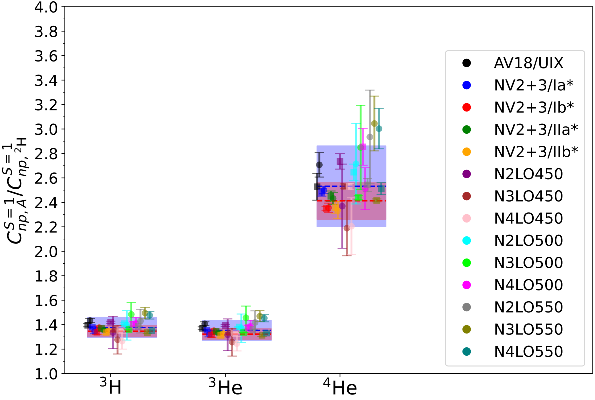
Ratio $C^{S=0}_{pn,A} / C^{S=0}_{pn,^4\mathrm{He}}$



Bands: average+error
violet: criterium 1
magenta: criterium 4

np $S = 0$	Criterium 1	Criterium 2	Criterium 3	Criterium 4
^{3}H	0.496 ± 0.083	0.499 ± 0.084	0.490 ± 0.086	0.501 ± 0.072
^{3}He	0.481 ± 0.079	0.485 ± 0.080	0.477 ± 0.082	0.485 ± 0.071

Ratio $C^{S=1}_{pn,A}/C^{S=1}_{pn,d}$



Bands: average+error
violet: criterium 1
magenta: criterium 4

$np\ S=1$	Criterium 1	Criterium 2	Criterium 3	Criterium 4
^{3}H	1.377 ± 0.084	1.371 ± 0.082	1.380 ± 0.092	1.347 ± 0.046
^{3}He	1.353 ± 0.082	1.348 ± 0.081	1.356 ± 0.090	1.323 ± 0.044
^{4}He	2.520 ± 0.332	2.499 ± 0.329	2.540 ± 0.360	2.400 ± 0.143

Outline

Conclusions and perspectives

Calculation of the CC for ^{2}H, ^{3}H, ^{3}He, and ^{4}He

- For local potentials, $\tilde{C}^S_{N_1N_2,A} = C^S_{N_1N_2,A}$ fairly well
- For non-local potentials, some discrepancies, most for $S = 0$
- N#LOXXX potentials: NN non-local, but TNI local, mismatch?
- Less problems with the ratios. Estimates:
  - $C^{S=1}_{pn,^3\mathrm{He}} / C^{S=1}_{pn,d} = 1.353 \pm 0.082$
  - $C^{S=1}_{pn,^4\mathrm{He}} / C^{S=1}_{pn,d} = 2.520 \pm 0.332$

Perspectives

- Extracted CCs to be used to analyze (e, e') experiments [CLAS and Hall A Collaborations, Jefferson Lab.], [. . .]
- Calculation of CCs for $A = 6$ (HH wave functions [Gnech *et al.*, 2020])
- Using other methods (Neural Networks, GFMC, AFMC, . . .)
- Use the local NN chiral potentials [Saha *et al.*, 2023] (consistency NN+3N int.)
- Long term: calculation of spectral functions

Thank you for your attention!