3-α and 4-α particle systems and reactions in near-zero range Effective Field Theory

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October 22, 2025

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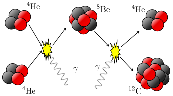
FONDAZIONE BRUNO KESSLER

1. Motivations
2. EFT inspired potential
3. Bound States with $A = 3, 4$ Bosons
4. Triple α Capture
5. $^{12}\mathrm{C}(\alpha, \gamma)^{16}\mathrm{O}$ Reaction
6. Conclusions

Motivations

- The nucleosynthesis of $^{12}\mathrm{C}$ and $^{16}\mathrm{O}$ in the universe stands as a fundamental issue within nuclear astrophysics

$$\alpha + \alpha + \alpha \Rightarrow {}^{12}\mathrm{C} + \gamma$$
$$^{12}\mathrm{C} + \alpha \Rightarrow {}^{16}\mathrm{O} + \gamma$$



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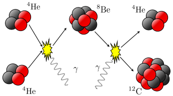

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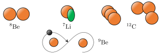
- Experimental evidence for the α cluster structure of some nuclei is well documented [M.Freer et al., Reviews of Modern Physics (2018)]



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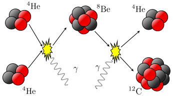

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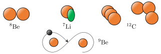

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- Effective field theories (EFTs) provides a controlled framework to exploit the separation of scales

Aim of the project

Our purpose is to describe some of these α-cluster nuclei and reactions in low energy range

- We use an EFT formulated with contact interactions among α particles

- Description of $^{12}\mathrm{C}$ and $^{16}\mathrm{O}$ excited and bound states within α cluster EFT approach
- $\alpha + \alpha + \alpha \rightarrow {}^{12}\mathrm{C} + \gamma$
- $^{12}\mathrm{C} + \alpha \rightarrow {}^{16}\mathrm{O} + \gamma$

The Cluster EFT potential

The theory is constructed by analyzing the independent monomials allowed by the low-energy spatial symmetries of the underlying fundamental theory. Developing this procedure, we constructed a $\alpha\alpha$ contact potential up to N2LO

$$
\begin{aligned}
V_{\text{eff}}(\mathbf{r}) = & \frac{4\alpha}{r}\,\text{erf}\left(\frac{r}{r_\alpha\sqrt{2}}\right) \\
& + a_0^2 C_1 \delta_{a_0}^{(3)}(\mathbf{r}) + a^4 C_2 \nabla^2 \delta_a^{(3)}(\mathbf{r}) \\
& + a^6 C_3 \nabla^4 \delta_a^{(3)}(\mathbf{r}) - C_5 \nabla^2 \delta_a^{(3)}(\mathbf{r}) \left(\frac{1}{2}\overleftrightarrow{\nabla}\right)^2 \\
& + C_4 \left(\frac{l(l+1)}{a^4} + \frac{2}{a^2}\left(\frac{1}{2}\overleftrightarrow{\nabla}\right)^2\right)\delta_a^{(3)}(\mathbf{r})
\end{aligned}
$$

where $\delta_a^{(3)}(\mathbf{r}) = e^{-(\mathbf{r}/2a)^2}$, l indicates the angular quantum number, $r_\alpha = 1.44$ fm is the α particle radius and $a_0 = \frac{\hbar c}{\Lambda_0}$,$a = \frac{\hbar c}{\Lambda}$.

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Fitting LECs

- We performed a fit on the S-wave and D-wave $\alpha\alpha$ scattering data of [AFZAL,et al. Rev. Mod. Phys. 41, 247-273 (1969)] up to 5 MeV
- Two-body scattering state is calculated by Kohn's variational principle
- We used the routine of minimization MIGRAD (Minuit routine of the Cern library)

Fit strategy

- First, for fixed cutoff of higher order terms, we fit the LO LEC from the position of $^8\mathrm{Be}$ resonance
- In the next step we fit the other LECs and the LO cutoff from the width of $^8\mathrm{Be}$ resonance and from the phase shift data

FIT Results

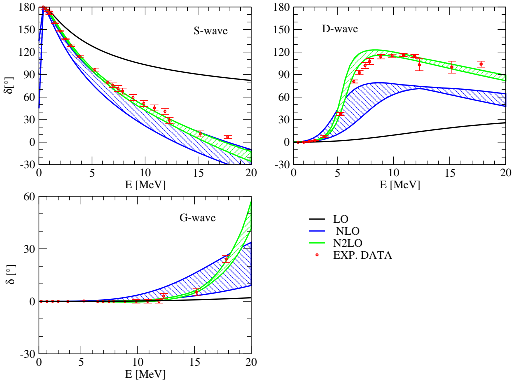
Using a total of 28 experimental data we obtain a $\chi^2/d.o.f \sim 1$ for each cutoff value analyzed

LECs	$\Lambda = 120$	$\Lambda = 130$	$\Lambda = 140$	$\Lambda = 150$
C_1	61171.417	59459.301	60631.730	64520.692
C_2	-3.318	-13.884	-29.350	-101.4158
C_3	-3.089	-4.218	-5.834	-9.619
C_4	8.453	9.933	10.361	43.485
Λ_0	388.789	355.706	350.332	264.395
χ^2	29.275	19.587	20.838	22.180

- Non-natural values of Λ_0 and C_1 can be explained as an attempt of the theory to describe the resonance correctly
- Increasing the values of Λ, Λ_0 has a decreasing trend
- The LECs as Λ increases go to non-natural scales

Phase shift

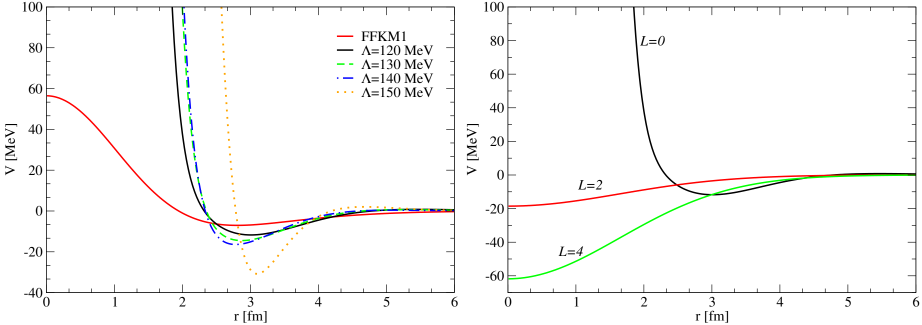
- Very good agreement with the data at low energies
- We also reproduce energies up to ~ 15 MeV and the trend of the G-wave ($l = 4$)



Order-by-order study of the phase shifts compared with experimental data

- The bands reflect the variation of the theoretical predictions when the cutoff Λ is varied in the range 120–150 MeV

The resulting LO term of the effective potential is extremely repulsive at short distances (repulsion reaches 60 000 MeV) and rapidly decreases to zero at a distance of $\sim$ 2 fm



$\Rightarrow$ Due to the strong repulsion of the LO term of the potential, the convergence is achieved with high values of the variational parameters

Bound States with $A = 3, 4$ Bosons (L. Marcucci's talk)

Wave Function Decomposition

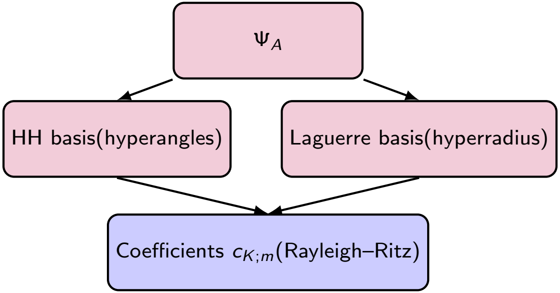
$$\Psi_A = \sum_{K,m} c_{K;m} \, |K, m\rangle$$

with basis states

$$\langle \rho, \Omega | K, m \rangle = \underbrace{f_m(\rho)}_{\text{Radial (Laguerre)}} \underbrace{\sum_p \mathcal{Y}^K_{[\alpha]}(\Omega^{(p)})}_{\text{Angular (HH)}} .$$

- **Angular part (HH)**
  - Depends on hyperangles φ_2 ($A = 3$), φ_2, φ_3 ($A = 4$)
  - Channels labeled by angular momenta $[\alpha]$
  - Symmetry constraints $\Rightarrow$ allowed l_i, parity
- **Radial part (Laguerre)**

  $f_m(\rho) \sim L_m^{(D-1)}(\gamma\rho) \, e^{-\gamma\rho/2}$
  - γ optimized variationally
- **Coefficients** $c_{K;m}$: from Rayleigh–Ritz$\Rightarrow$ generalized eigenvalue problem



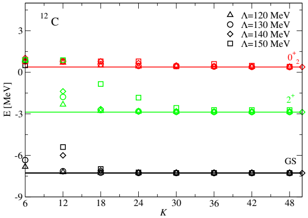
^{12}C excited and bound states

We include a three-body force of the form,

$$V_3(\rho) = (V_{03}\hat{P}_{L=0} + V_{23}\hat{P}_{L=2})e^{-(\rho^2/2a_3^2)}$$

where $\hat{P}_{L=0,2}$ are the $L = 0, 2$ waves projectors, tuning the V_{03}, V_{23} and a_3 on the binding energy of the ground state, of the 2^+ excited state and of the Hoyle state, respectively

	$\Lambda = 120$	$\Lambda = 130$	$\Lambda = 140$	$\Lambda = 150$
V_{03} [MeV]	-19.44	-18.75	-18.58	-18.74
V_{23} [MeV]	-12.50	-11.68	-11.04	-10.69
a_3 [fm]	3.283088	3.222431	3.058219	3.135534

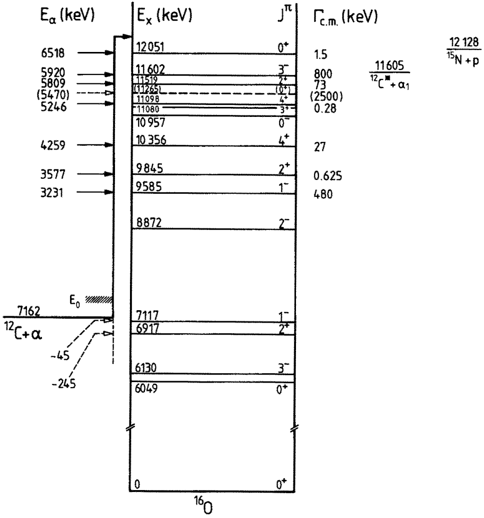


M	$E_{G.S.}$ [MeV]	E_{2^+} [MeV]	E_{Hoyle} [MeV]
20	-7.1724	-2.7764	0.4068
30	-7.2664	-2.8664	0.3861
40	-7.2755	-2.8750	0.3843
50	-7.2764	-2.8762	0.3841

Stability of ^{12}C state energies as a function of the number of Laguerre polynomials M

^{16}O excited and bound states

The ^{16}O nucleus described as a 4-α system has an experimental binding energy of -14.437 MeV



We include a four-body force of the form,

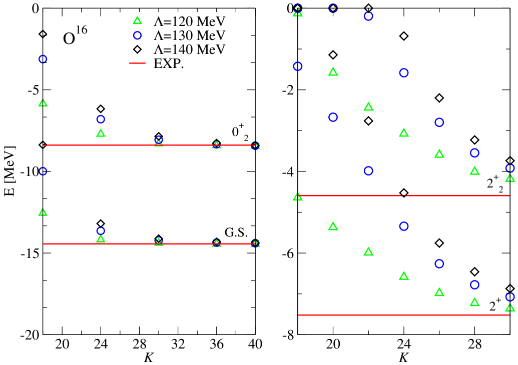
$$V_4(\rho) = (V_{04}\hat{P}_{L=0} + V_{24}\hat{P}_{L=2})e^{-(\rho^2/2a_4^2)}$$

$\hat{P}_{L=0}, \hat{P}_{L=2} = L = 0, 2$ waves projectors, tuning V_{04} and a_4 on the binding energy of the ground state and of the first excited state.

	$\Lambda = 120$	$\Lambda = 130$	$\Lambda = 140$
V_{04} [MeV]	234.41	207.85	185.10
V_{24} [MeV]	236.80	193.42	160.00
a_4 [fm]	2.542874	2.542874	2.542874

- The HH basis needed for convergence is quite large, since the LO two-body potential is strongly repulsive at short distances
- Stability checked by extrapolating results with increasing K

Convergence and Stability Checks



M	GS	0_2^+
26	-14.298	-8.253
28	-14.315	-8.266
30	-14.322	-8.272

M	2^+	2_2^+
26	-7.227	-4.003
28	-7.227	-4.003
30	-7.228	-4.009

^{16}O ground and excited state energy as a function of M for $\Lambda = 140$ MeV.

J^π	EXP.
0^+	-14.44
0_2^+	-8.39
2^+	-7.52
2_2^+	-4.592

	2^+	2_2^+
$\Lambda = 120$	-7.557	-4.447
$\Lambda = 130$	-7.515	-4.479
$\Lambda = 140$	-7.491	-4.500

Extrapolated ^{16}O 2^+ excited state energies in MeV for $\bar{K} = \infty$ and for different Λ values in MeV.

Triple α Capture

We want to study the process

$$\gamma + {}^{12}\mathrm{C}(L=2) \rightarrow 3\alpha(L=0)$$

We adopt the adiabatic approximation

- Solve the 3-α system by expanding in adiabatic channels depending on the hyperradius ρ
- Off-diagonal couplings suppressed (or damped) beyond a certain ρ_0
- Advantages: simplifies boundary conditions; captures long-range Coulomb effects in an approximate way

Other works / comparisons

- Katsuma (2024): Faddeev-HHR + R-matrix expansion; finds the derived triple-α rates are in accord with standard evaluations for $0.08 \leq T \leq 3$, but suppressed by 10^{-4} at $T = 0.05$ [Katsuma, arXiv:2411.03600 (2025)]
- Nguyen, Nunes et al. (2011–2013): full three-body model with hyperspherical harmonics + R-matrix propagation; they compute triple- rates, finding agreement with NACRE at higher T, but strong enhancement at low $T \lesssim 0.07$ [Nguyen et al., Phys. Rev. Lett. 106:042502 (2011); Phys. Rev. C 87:054605 (2013)]

Adiabatic approximation

$3\alpha(L=0)$ wave function calculated using the adiabatic method

$\Psi^{LM}_{3\alpha} = \sum_{\nu=1}^{N_A} \frac{u_\nu(\rho)}{\rho^{5/2}} \Phi^{LM}_\nu(\rho,\Omega)$ $\Phi^{LM}_\nu(\rho,\Omega)$ =adiabatic functions calculated using an HH basis up to $K = K_M$

$$-\frac{\hbar^2}{m}\frac{d^2 u_\nu}{d\rho^2} + U_\nu(\rho) u_\nu + \sum_{\nu'} \left[B_{\nu\nu'}(\rho)\, u'_{\nu'} + C_{\nu\nu'}(\rho)\, u_{\nu'} \right] = E\, u_\nu$$

For $\nu \neq \nu'$ $B_{\nu\nu'}, C_{\nu\nu'}$ *multiplied by* $\exp[-(\rho/\rho_0)^4]$, *with* $\rho_0 = 200$ *fm*

Internal solution ($0 < \rho < 200$ fm)

- Boundary: $u_\nu(0) = 0$
- Full coupled equations
- Numerical integration (Numerov)

Matching solution ($\rho \sim 200$ fm)

- Couplings suppressed ($B, C \to 0$)
- Two analytic basis functions:

$$F^R_\nu \sim \sin(z - \eta_\nu \ln 2z),$$
$$G^R_\nu \sim \cos(z - \eta_\nu \ln 2z)$$

- Construct regular/irregular solutions

Asymptotic solution ($\rho > 400$ fm)

- Effective potential: $U_\nu + C_{\nu\nu} \sim \frac{A_\nu}{\rho} + \frac{B_\nu}{\rho^{3/2}} + \frac{C_\nu}{\rho^2}$ note $B_{\nu\nu} = 0$
- Boundary condition: $u^{(\nu_0)}_\nu = \delta_{\nu_0\nu} F_\nu + R_{\nu_0\nu} G_\nu$
- Physical states: $w^{\nu_0}_\nu \sim \delta_{\nu_0\nu} F_\nu + T_{\nu_0\nu}(G_\nu + iF_\nu)$

Quantities of interest

Disintegration cross-section

$$d\sigma_\gamma \propto |\langle \psi_{^{12}\mathrm{C}(L=2)}||BE_2|\Psi_{3\alpha(L=0)}\rangle|^2$$

$\gamma +^{12}\mathrm{C}(L=2) \rightarrow 3\alpha(L=0)$ process, only E2 contribution $\Rightarrow BE_2 = \sum_{i=1}^N r_i^2 Y_{2m}(\hat{r}_i)$

rate $3\alpha(L=0) \rightarrow \gamma +^{12}\mathrm{C}(L=2)$

$$R(E) = 3!\,\mathcal{N}_A^2 G_N \frac{8\pi}{(\mu_2\mu_3)^{3/2}} \frac{E_\gamma^2}{E^2}\sigma_{\text{dis}}$$

- $G_N = 10$ statistical factor
- $\mu_2 = m/2$ and $\mu_3 = 2m/3$ reduced masses
- $E_\gamma = E + \Delta B,\ \Delta B \approx 2.85$ MeV
- $\mathcal{N}_A$ Avogadro number

Energy averaged rate

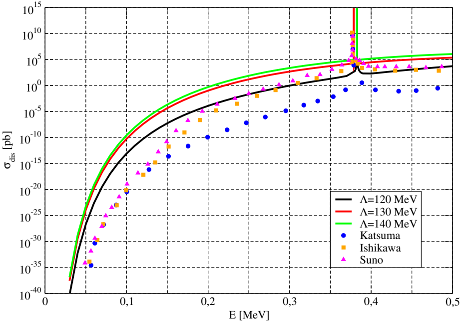
$$\overline{R}(T) = \frac{1}{2}\frac{1}{(k_B T)^3}\int_0^\infty dE\, E^2 R(E)\, e^{-E/k_B T}$$

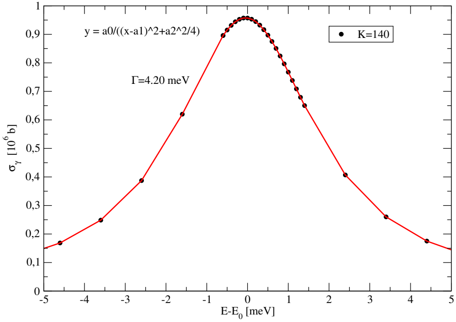
- k_B Boltzmann constant

Cross-section

PRELIMINARY

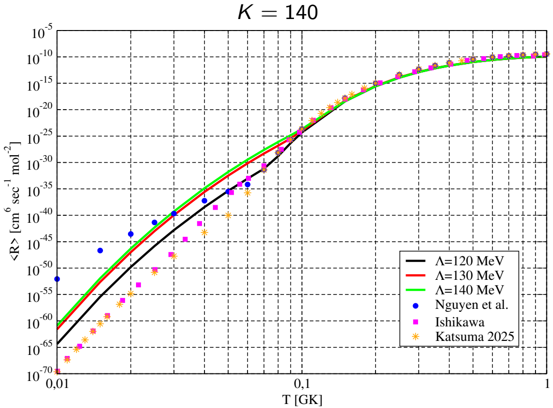
$K = 140$





Average rate

PRELIMINARY



Wave Function and Transition Matrix Element

We study the radiative capture

$$^{12}\mathrm{C} + \alpha \;\rightarrow\; ^{16}\mathrm{O} + \gamma,$$

essential to determine the stellar C/O ratio in the universe

Transition Matrix Element

$$\langle \Psi_f | H_{\mathrm{em}} | \Psi_i \rangle = -\frac{eZ}{\sqrt{2k}} \sum_L \sqrt{4\pi}\, i^L\, \hat{L}\, e^{i\delta_L}\, d^L_{0,-\lambda}(-\theta)\, E_L.$$

Dominant multipole: $L = 2 \Rightarrow E_2$, k = photon momentum, λ = photon helicity, δ_L = scattering phase shift

Scattering Wave Function and Kohn Method (L. Marcucci's talk)

The total wave function is written as

$$\Psi = \sum_L [u_L(\rho)\, \mathcal{Y}_L(\Omega)] + F_L(k\rho) + R_L\, G_L(k\rho)$$

$u_L(\rho)$ = internal wave function in the nuclear region (expanded as a bound-state–like basis, vanishing for large ρ),
F_L, G_L =regular and irregular Coulomb functions (external region)
The Kohn variational principle fixes $R_L = \tan \delta_L$

Cross-section and Astrophysical S Factor

Cross-section

$$\sigma(E) = \frac{8\pi}{5}\frac{k^5}{v}\frac{1}{1+\frac{k}{k_0}}\frac{q_t^2}{30}\left|\tilde{E}_2\right|^2 \quad \Rightarrow \quad \sigma \propto \frac{k^5}{v}\left|\tilde{E}_2\right|^2.$$

k = photon momentum, v = relative velocity,
$k_0 = \sqrt{2\mu E_0}/\hbar$, $\tilde{E}_2$ = reduced $E2$ matrix element

Astrophysical S Factor

$$S(E) = \sigma(E)\, E\, e^{2\pi\eta}, \qquad \eta = \frac{Z_1 Z_2 e^2}{\hbar v}$$

η = Sommerfeld parameter (Coulomb barrier).
$S(E)$ isolates the nuclear contribution from the Coulomb suppression

Some Previous Theoretical Works

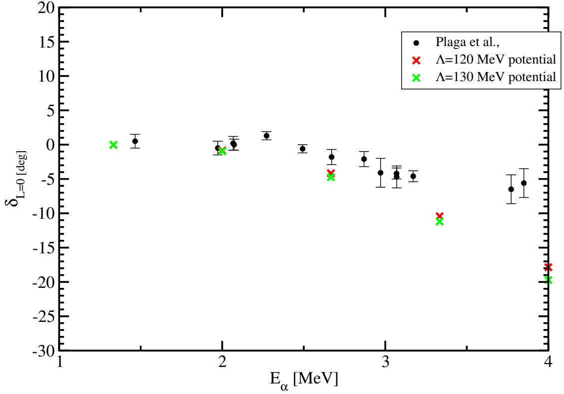
Dufour & Descouvemont, *Phys. Rev. C* **78**, 015808 (2008): microscopic cluster (GCM) calculation combined with an R-matrix treatment of the E_2 capture

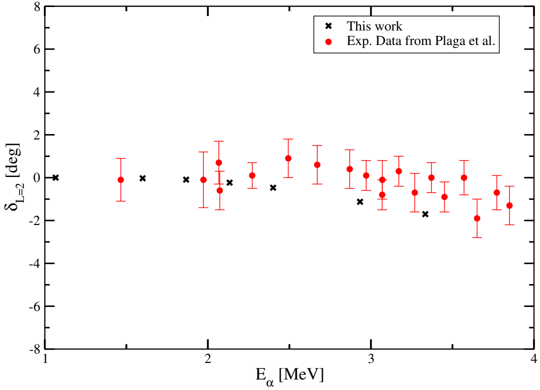
Katsuma, *Astrophys. J.* **745**, 192 (2012): direct-capture potential model with $\alpha + {}^{12}$C potentials fitted to elastic data

deBoer *et al.*, *Rev. Mod. Phys.* **89**, 035007 (2017): R-matrix analysis approach

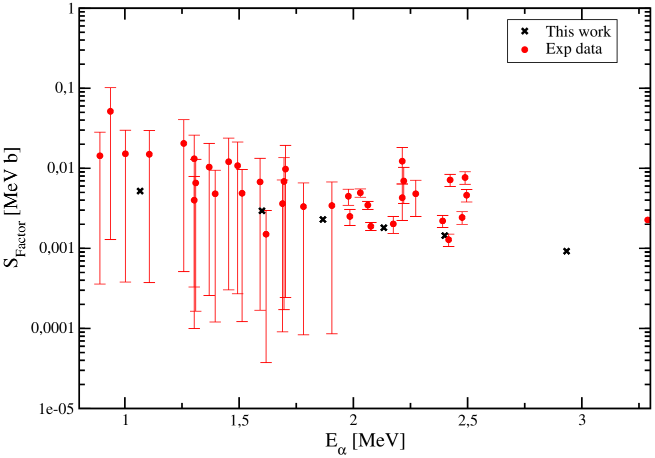
Phase-shift

PRELIMINARY





Low Energy *S*-Factor PRELIMINARY



- Only a few experimental data are available at low energies, and they are affected by large uncertainties [Fey, Schürmann *et al.*, Assunção *et al.*, Ouellet *et al.*, Kunz *et al.*, Redder *et al.*, Roters *et al.*, Makii *et al.*, and Plaga *et al.*].
- Despite the scarcity and uncertainty of the experimental data, our theoretical predictions fall within the region indicated by the available measurements

Conclusions

- We studied bound and excited states of 3α and 4α systems using short-range EFT-inspired potentials
  - LECs of the α–α potential were fitted to phase shifts, resonance position, and width
  - The 3α LECs were tuned to reproduce the ground and first excited states of ^{12}C
  - The 4α force ensures correct energies for ^{16}O
- Preliminary adiabatic-approximation results for 3α are consistent with trends in the literature
- Preliminary $^{12}\mathbf{C} + \alpha$ results at low energy agree with available experimental data
- Ongoing developments aim at improving accuracy and enabling a quantitative comparison with experiment