Developments in relativistic spin hydrodynamics

David Wagner

based mainly on

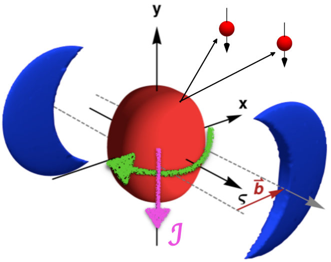
DW, Phys.Rev.D 111 (2025) 1, 016008
DW, MS, DHR, Phys.Rev.Res. 6 (2024) 4, 4
DW, N. Weickgenannt, DHR, Phys.Rev.D 106 (2022) 11, 116021

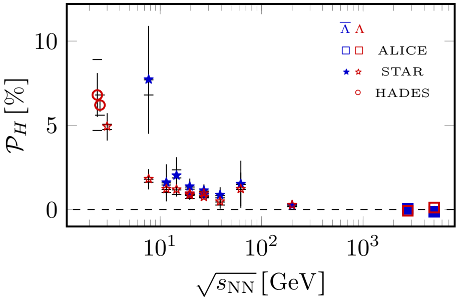
19.03.2025

Overview

Motivation & open questions

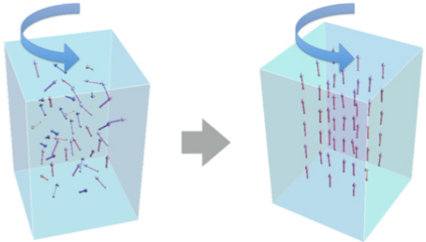
Global Λ-Polarization



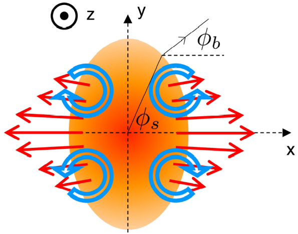


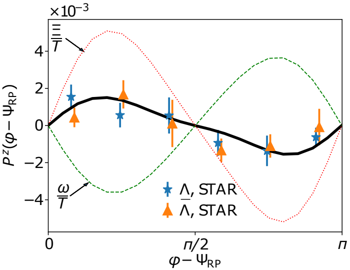
- "Global": Integrated polarization along the direction of orbital angular momentum
- Can be explained by assuming spins in equilibrium

→ "Polarization through rotation"
  - Analogous to Barnett effect



Local Λ-Polarization



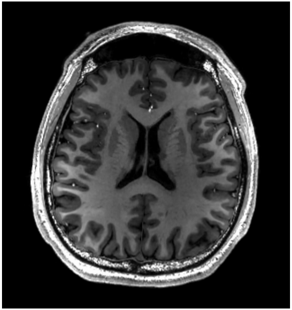


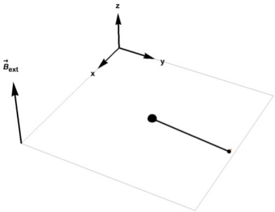
F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, A. Palermo, PRL 127, 272302 (2021)

- "Local": Angle-dependent polarization along beam-direction
- Can only be explained by incorporating shear effects
  - → Simple picture of equilibrated spins not complete
- Possible answer: develop a theory of **spin hydrodynamics** to describe polarization dynamics

Analogy: Magnetic resonance imaging (MRI)

- MRI: Large constant B-field in z-direction and short-lived alternating field in x, y-plane
- Identify materials by relaxation times T_1, T_2





https://en.wikipedia.org/wiki/Bloch_equations

Bloch equations

$$T_2 \dot{M}_{x,y} + M_{x,y} = \mu_2 \left(\mathbf{M} \times \mathbf{B}\right)_{x,y} \ ,$$
$$T_1 \dot{M}_z + M_z = \mu_1 \left(\mathbf{M} \times \mathbf{B}\right)_z + M_0 \ .$$

$\mu_1 := T_1 \frac{gq}{2m}$, $\mu_2 := T_2 \frac{gq}{2m}$

Deriving spin hydrodynamics

Spin Hydrodynamics: Basics

- Theory is based on conservation laws for energy-momentum tensor and total angular momentum tensor

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\lambda J^{\lambda\mu\nu} =: \hbar \partial_\lambda S^{\lambda\mu\nu} + T^{[\mu\nu]} = 0$$

$A^{[\mu}B^{\nu]} := A^\mu B^\nu - A^\nu B^\mu$

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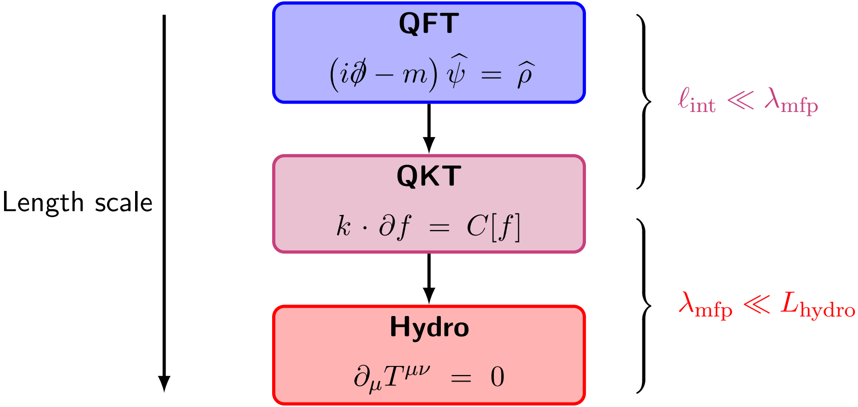
- **10** equations for **16**+**24** quantities
- Additional information about dissipative quantities has to be provided
- Here: Use *quantum kinetic theory* as the microscopic basis
- For small polarization: spin tensor can be described through **11** quantities:
  - the **spin potential** $\Omega_0^{\mu\nu} = u^{[\mu}\kappa_0^{\nu]} + \epsilon^{\mu\nu\alpha\beta}u_\alpha \omega_{0,\beta}$,
  - the **spin-shear-stress tensor** $\mathfrak{t}^{\mu\nu}$

$$A^{[\mu}B^{\nu]} := A^\mu B^\nu - A^\nu B^\mu$$
$$S^{\lambda\mu\nu} = A u^\lambda u^{[\mu}\kappa_0^{\nu]} + B u^\lambda \epsilon^{\mu\nu\alpha\beta}u_\alpha \omega_{0,\beta} + C u^{[\mu}\epsilon^{\nu]\lambda\alpha\beta}u_\alpha\omega_{0,\beta} + D\Delta^{\lambda[\mu}\kappa_0^{\nu]} + E\mathfrak{t}^{\lambda[\mu}u^{\nu]}$$

Spin hydrodynamics: Procedure



Dissipative spin hydrodynamics

DW, Phys.Rev.D 111 (2025) 1, 016008

$$\tau_\omega \dot{\omega}_0^{\langle\mu\rangle} + \omega_0^\mu = -\beta_0 \omega_K^\mu + \ell_{\omega\kappa} \epsilon^{\mu\nu\alpha\beta} u_\nu \nabla_\alpha \kappa_{0,\beta} + \cdots$$

$$\tau_\kappa \dot{\kappa}_0^{\langle\mu\rangle} + \kappa_0^\mu = -\beta_0 \dot{u}^\mu + \mathfrak{b} \nabla^\mu \alpha_0 + \frac{\tau_\kappa}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \nabla_\alpha \omega_{0,\beta} + \ell_{\kappa\mathfrak{t}} \Delta^\mu_\lambda \nabla_\nu \mathfrak{t}^{\nu\lambda} + \cdots$$

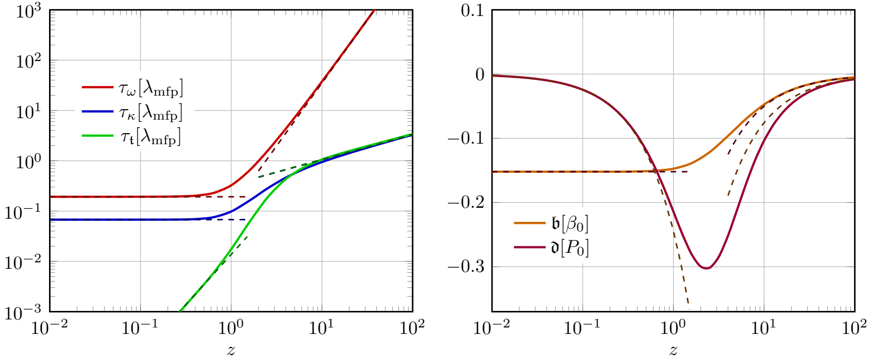
$$\tau_{\mathfrak{t}} \dot{\mathfrak{t}}^{\langle\mu\nu\rangle} + \mathfrak{t}^{\mu\nu} = \mathfrak{d} \beta_0 \sigma^{\mu\nu} + \ell_{\mathfrak{t}\kappa} \nabla^{\langle\mu} \kappa_0^{\nu\rangle} + \cdots$$

- System of *coupled relaxation-type equations*
  - Relaxation controlled by times τ_ω, τ_κ, $\tau_{\mathfrak{t}}$
  - At late times: Spin determined by fluid gradients: ω_K^μ, $\dot{u}^\mu$, $\sigma^{\mu\nu}$, $\nabla^\mu \alpha_0$

$\beta_0 := 1/T$, $\alpha_0 := \mu/T$

Relaxation times and first-order coefficients

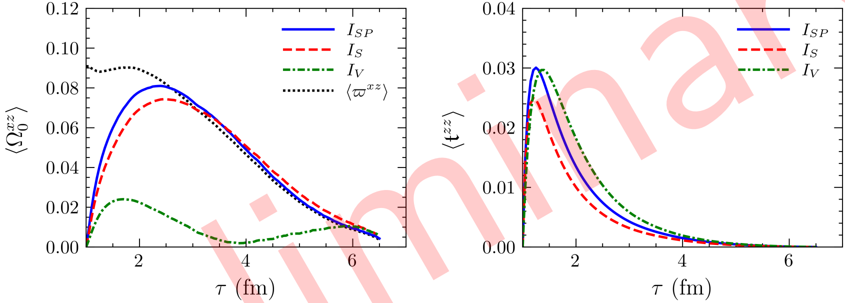
DW, Phys.Rev.D 111 (2025) 1, 016008



- τ_ω grows with z^2 compared to τ_κ and $\tau_{\mathfrak{t}}$
- $\tau_{\mathfrak{t}}$ vanishes for $z \to 0$

Numerical results (preliminary)

Sapna, S.K. Singh, DW et al., in progress



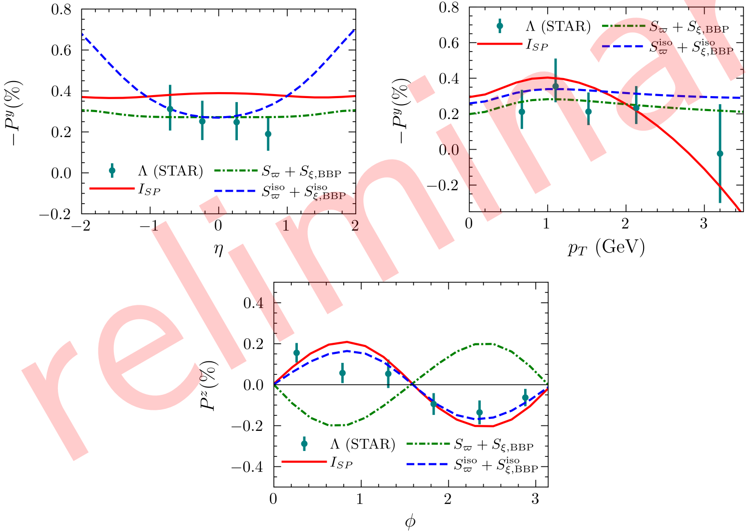
$$S : \mathcal{L}_{\text{int}} \sim G\left(\bar{\psi}\psi\right)^2 ,$$

$$V : \mathcal{L}_{\text{int}} \sim G\left(\bar{\psi}\gamma^\mu\psi\right)\left(\bar{\psi}\gamma_\mu\psi\right) ,$$

$$SP : \mathcal{L}_{\text{int}} \sim G\left[\left(\bar{\psi}\psi\right)^2 - \left(\bar{\psi}\gamma_5\psi\right)^2\right] .$$

3+1 simulation: Polarization

Sapna, S.K. Singh, DW et al., in progress

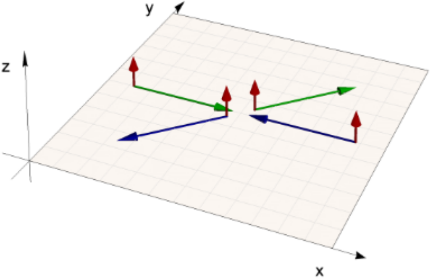


Summary and outlook

- Spin hydrodynamics derived from quantum kinetic theory shows promising results
- Future prospects:
  - Further study of spin hydro in various setups, in particular polarization dynamics
  - Expected to become even more relevant at lower collision energies

Appendix

Appendix: Angular momentum and collisions



W. Florkowski, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108, 103709 (2019)

- Assume that collisions take place in a point
  - → Total orbital angular momentum vanishes
  - → Spin is conserved on its own
  - → No exchange of spin and orbital angular momenta
- Collisions must be **nonlocal** for spin equilibration!
- Becomes manifest through a spacetime shift Δ^μ that is fixed by the microscopic interaction

Appendix: Equilibrium

- Local-equilibrium distribution function makes local collision term vanish (without spacetime shifts Δ^μ)
- Has to depend on the **collisional invariants**
  - $\rightarrow$ Charge, four-momentum and total angular momentum

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  - $\rightarrow$ Charge, four-momentum and total angular momentum

Local-equilibrium distribution function

$$f_{\text{eq}}(x, k, \mathfrak{s}) = \exp\left(\alpha_0 - \beta_0 E_{\mathbf{k}} + \frac{\hbar}{2}\Omega_{0,\mu\nu}\Sigma_{\mathfrak{s}}^{\mu\nu}\right)$$

- Necessary conditions on Lagrange multipliers for a vanishing **nonlocal** collision term: $\partial^\mu \alpha_0 = 0$, $\partial^{(\mu}(\beta_0 u^{\nu)}) = 0$, $\Omega_0^{\mu\nu} = -\frac{1}{2}\partial^{[\mu}(\beta_0 u^{\nu]})$
- Same conditions as for **global** equilibrium, where $k \cdot \partial f_{\text{eq}} = 0$

$\Sigma_{\mathfrak{s}}^{\mu\nu} := -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}k_\alpha \mathfrak{s}_\beta$, $E_{\mathbf{k}} := k \cdot u$

Appendix: Conserved currents in QKT

Conserved currents

$$\frac{1}{2}T^{(\mu\nu)} = \int \mathrm{d}\Gamma k^\mu k^\nu f \, ,$$

$$S^{\lambda\mu\nu} = \frac{1}{2m}\int \mathrm{d}\Gamma k^\lambda \epsilon^{\mu\nu\alpha\beta} k_\alpha \mathfrak{s}_\beta f \, .$$

$$T^{[\mu\nu]} = \frac{1}{2}\int [\mathrm{d}\Gamma]\widetilde{\mathcal{W}}\Delta^{[\mu} k^{\nu]} \left(f_1 f_2 - f f'\right)$$

Conservation laws

$$\int \mathrm{d}\Gamma k^\mu C[f] = 0$$

$$\frac{\hbar}{2m}\int \mathrm{d}\Gamma \epsilon^{\mu\nu\alpha\beta} k_\alpha \mathfrak{s}_\beta C[f] = \frac{\hbar}{m}\int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4} k^{[\mu}\mathcal{D}_{\mathcal{V}}^{\nu]}$$

$[\mathrm{d}\Gamma] \coloneqq \mathrm{d}\Gamma_1\, \mathrm{d}\Gamma_2\, \mathrm{d}\Gamma\, \mathrm{d}\Gamma'$

Appendix: Polarization in spin hydrodynamics

Local Polarization

$$S_0^\mu = \frac{2\sigma^2\hbar}{N(k)m}\int \mathrm{d}\Sigma_\lambda k^\lambda \left(u^\mu \omega_0^\nu k_\nu - E_{\mathbf{k}}\omega_0^\mu + \epsilon^{\mu\nu\alpha\beta}u_\nu k_\alpha \kappa_{0,\beta}\right) f_0 \widetilde{f}_0$$

$$\delta S^\mu = -\frac{2\sigma}{N(k)}\int \mathrm{d}\Sigma_\lambda k^\lambda K^{\mu\gamma}\Xi_{\gamma\alpha} f_0\widetilde{f}_0 \times \left(\mathfrak{x}_n \epsilon^{\alpha\beta\rho\sigma}u_\beta k_\rho n_\sigma + \mathfrak{x}_{\mathfrak{t}}\mathfrak{t}_\rho{}^{\langle\beta}\epsilon^{\gamma\rangle\alpha\sigma\rho}u_\sigma k_{\langle\beta}k_{\gamma\rangle}\right)$$

Global Polarization

$$\overline{S}_0^\mu = -\frac{2\sigma^2\hbar}{\overline{N}m}\int \mathrm{d}\Sigma_\lambda \left(J_{21}u^\mu\omega_0^\lambda + J_{20}\omega_0^\mu u^\lambda + J_{21}\epsilon^{\mu\nu\lambda\beta}u_\nu \kappa_{0,\beta}\right)$$

$$\delta\overline{S}^\mu = \frac{\sigma}{\overline{N}}\frac{1}{2}\int \mathrm{d}\Sigma_\lambda\, B_0 \epsilon^{\mu\lambda\alpha\beta}u_\alpha n_\beta$$

$\mathfrak{x}_n := \frac{1}{2}\sum_n \mathcal{H}_{\mathbf{k}n}^{(1,1)}\frac{\mathfrak{b}_n^{(1)}}{\varkappa}$, $\quad \mathfrak{x}_{\mathfrak{t}} := \frac{2}{3}\sum_n \mathcal{H}_{\mathbf{k}n}^{(1,2)}\frac{\mathfrak{d}_n}{\mathfrak{d}_0}$

Appendix: Nonlocal collisions

DW, NW, ES, 2306.05936 (2023)

Spacetime shifts

$$\Delta^\mu \quad := \quad -\frac{i\hbar}{8m}\frac{m^4}{\mathcal{W}} M^{\gamma_1\gamma_2\delta_1\delta_2} M^{\zeta_1\zeta_2\eta_1\eta_2} h_{1,\gamma_1\eta_1} h_{2,\gamma_2\eta_2} h'_{\zeta_2\delta_2} [h,\gamma^\mu]_{\zeta_1\delta_1}$$

- Depend on the transfer-matrix elements

$$\langle 11'|\,\widehat{t}\,|22'\rangle = \bar{u}_{1,\alpha}\bar{u}_{1',\beta}u_{2,\gamma}u_{2',\delta}M^{\alpha\beta\gamma\delta}$$

- Manifestly covariant
  - $\rightarrow$ no "no-jump" frame

$h := \frac{1}{4}(\mathbb{1} + \gamma_5 \not{s})(\not{k} + m)$

Appendix: Some interactions

Scalar interaction

$$M_{\alpha\alpha'\alpha_1\alpha_2} = \frac{2G}{\hbar}\left(\delta_{\alpha\alpha_1}\delta_{\alpha'\alpha_2} - \delta_{\alpha\alpha_2}\delta_{\alpha'\alpha_1}\right)$$

Thermal gluon exchange

$$M_{\alpha\alpha'\alpha_1\alpha_2} = \frac{2g}{\hbar}\left[\gamma^\mu_{\alpha\alpha_1}\frac{g_{\mu\nu}}{(k-k_1)^2 - m_{\text{th}}^2}\gamma^\nu_{\alpha'\alpha_2} - \gamma^\mu_{\alpha\alpha_2}\frac{g_{\mu\nu}}{(k-k_2)^2 - m_{\text{th}}^2}\gamma^\nu_{\alpha'\alpha_1}\right]$$

- $m_{\text{th}} = \sqrt{2N_C + N_f}\, gT/(3\sqrt{2})$

Appendix: Spin waves

- $\boldsymbol{\kappa}$ and $\boldsymbol{\omega}$ follow coupled relaxation equations
  $\rightarrow$ Disentangle longitudinal and transverse components

Longitudinal components: Decay

$$\tau_\kappa \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{\nabla} \cdot \boldsymbol{\kappa}) = -\boldsymbol{\nabla} \cdot \boldsymbol{\kappa} \,,$$

$$\tau_\omega \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{\nabla} \cdot \boldsymbol{\omega}) = -\boldsymbol{\nabla} \cdot \boldsymbol{\omega} \,,$$

Transverse components: Damped waves

$$\ddot{\boldsymbol{\kappa}} + a\dot{\boldsymbol{\kappa}} + b\boldsymbol{\kappa} - c_{\mathfrak{s}}^2 \Delta \boldsymbol{\kappa} = 0 \,,$$

$$\ddot{\boldsymbol{\omega}} + a\dot{\boldsymbol{\omega}} + b\boldsymbol{\omega} - c_{\mathfrak{s}}^2 \Delta \boldsymbol{\omega} = 0 \,,$$

V. E. Ambrus, R. Ryblewski, R. Singh, Phys. Rev. D 106, 014018(2022)
J. Hu, Z. Xu, Phys. Rev. D 107, 016010 (2023)
DW, M. Shokri, D. H. Rischke, Phys.Rev.Res. 6 (2024) 4, 4

$$a := \frac{\tau_\kappa + \tau_\omega}{\tau_\kappa \tau_\omega} \,, \qquad b := \frac{1}{\tau_\kappa \tau_\omega} \,, \qquad c_{\mathfrak{s}}^2 := \frac{\mu_\kappa \mu_\omega}{\tau_\kappa \tau_\omega} .$$

Appendix: Moment method

- Split distribution function $f = f_{\text{eq}} + \delta f$
- Perform moment expansion including spin degrees of freedom

Irreducible moments

$$\rho_r^{\mu_1 \cdots \mu_\ell}(x) := \int \mathrm{d}\Gamma E_{\mathbf{k}}^r k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \delta f(x, k, \mathfrak{s})$$

$$\tau_r^{\mu, \mu_1 \cdots \mu_\ell}(x) := \int \mathrm{d}\Gamma \mathfrak{s}^\mu E_{\mathbf{k}}^r k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} \delta f(x, k, \mathfrak{s})$$

- Equations of motion can be derived from Boltzmann equation
- Knowing the evolution of all moments is equivalent to solving the Boltzmann equation

$k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle} := \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} k^{\nu_1} \cdots k^{\nu_\ell}$

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Irreducible moments

Standard dissipation

$$\rho_r^{\mu_1 \cdots \mu_\ell}(x) := \int \mathrm{d}\Gamma E_{\mathbf{k}}^r k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \delta f(x, k, \mathfrak{s})$$

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Spin dissipation

- Equations of motion can be derived from Boltzmann equation
- Knowing the evolution of all moments is equivalent to solving the Boltzmann equation

$k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} := \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} k^{\nu_1} \cdots k^{\nu_\ell}$

Appendix: IReD

DW, A. Palermo, V. E. Ambruș, Phys. Rev. D **106**, 016013 (2022)
DW, Phys.Rev.D 111 (2025) 1, 016008)

- Basic idea: Power-counting scheme to second order in
  - Knudsen number $\mathrm{Kn} := \lambda_{\mathrm{mfp}}/L_{\mathrm{hydro}}$
  - inverse Reynolds numbers $\mathrm{Re}^{-1} \sim \delta f/f_{\mathrm{eq}}$
- Derive asymptotic (Navier-Stokes) relations to close the system

Asymptotic matching (example)

$$\rho_r^{\mu\nu} = \eta_r \sigma^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1}) = \frac{\eta_r}{\eta_0}\pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}\mathrm{Re}^{-1})$$

- The same procedure can be done for the moments $\tau_r^{\mu,\mu_1\cdots\mu_\ell}$
- Many moments can be related to ω_0^μ and κ_0^μ
  - No need to introduce more dynamical quantities
- Exception: tensor-valued moments $t_r^{\mu\nu} := \tau_{r,\alpha,\beta}{}^{\langle\mu}\epsilon^{\nu\rangle\alpha\beta\rho}u_\rho$
  - Additional dynamical quantity $\mathfrak{t}^{\mu\nu}$ is needed, $S^{\lambda\mu\nu} \sim \mathfrak{t}^{\lambda[\mu}u^{\nu]}$