Topological phases and bosonization

(Or: symmetries and anomalies)

Topological phases

Landau approach to phases of matter: spontaneous symmetry breaking (SSB), local order parameter.

→ Phases classified by two groups (G,H): G symmetry of the system, H symmetry of ground state (GS)

'New' paradigm: topological phases

- Beyond Landau classification (distinguished from the trivial phase by properties other than ordinary symmetry breaking, where trivial phase is the one that has a product state representative that breaks no symmetries, e.g. atomic insulator)
- Quantum phases for gapped systems at T = 0 (i.e. T << Gap)
- Absence of local order parameter
- Different topological phases: systems (Hamiltonian or GS wave function) cannot be continuosly connected without closing the gap
- *Topological order (TO)*: topological GS degeneracy robust against local perturbations and/or fractionalized excitations
- *Boundary modes (bulk-boundary correspondence)*

Symmetry protected topological (SPT) phases

A d-dimensional G-SPT phase is a phase of matter which has *no topological order* (unique ground state) but still *different from the trivial phase ("vacuum") if the symmetry G is taken into account*. Characterized by:

- Gapped bulk
- No topological order
- Degrees of freedom on the boundary

They are classified. Free *fermionic* classification works fairly well. Non trivial *bosonic* SPTs are all interacting.

New tools to describe topological phases: TQFT (no local excitations in the bulk) and anomalies

Topological phases $\xrightarrow{\text{IR}}$ *TQFTs*

The general IR theory of a gapped system is a topological field theory

Symmetries still play a fundamental role

The concept of symmetry has greatly expanded in the last ten years: generalized symmetries

D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, Generalized global symmetries, Journal of High Energy Physics 2015 (2015)

TQFTs, anomalies and generalized symmetries

Effective field theory description: topological quantum field theory (bulk)

If the manifold has a boundary, the TQFT is usually not well-defined: there should be boundary degrees of freedom to compensate for this ‘anomaly’: anomaly inflow

Anomaly inflow ⟷ *Bulk-boundary correspondence*

Existence of boundary d.o.f. is an immediate consequence of anomaly

The converse is also true: given an anomalous d-dimensional theory, the anomaly can always be removed by a (d+1)-dimensional topological field theory

Classification of d-dim anomalies = Classification of (d+1)-dim SPTs

Topological GS degeneracy (TO) also follows from symmetry arguments: SSB of a generalized higher-form symmetry (which is a symmetry that acts on extended objects): ‘generalized Landau approach’

Examples: QHE and Superconductor

Bosonization

Quantum Hall Effect (QHE): paradigmatic example of topological phase (known since '80)

Bulk: for $k \neq 1$ topological order (FQHE)

Boundary: chiral compact boson = Weyl fermion for k=1

$$S = \int_Y \frac{k}{4\pi} ada + \frac{1}{4\pi} adA \quad \xleftrightarrow[\partial Y = X]{\text{Anomaly inflow}} \quad S = \frac{k}{2\pi}\int_X \partial_t\phi\partial_x\phi - H$$

Hydrodynamic description: *a* parametrizes particle conserved current $J =* \, da$

Superconductor: $\mathbb{Z}_2$ topologically ordered phase (known since 1911!)

Abelian Higgs model: Higgs condensate is Cooper pair field, charge 2 $\xrightarrow{\text{IR}}$ $S = \int_Y \frac{2}{2\pi} bda$ $\quad \mathbb{Z}_2$ gauge theory

SSB of electric one-form symmetry $\mathbb{Z}_2^{(1)}$ of electromagnetic field coupled to charged 2 matter
Magnetic symmetry of electromagnetic field confined: photon is massive

Example: Topological Insulator (TI)

3+1d topological insulator: T (time reversal)-SPT phase classified by a $\mathbb{Z}_2$ index

Bulk: electromagnetic θ-angle

$Z = e^{iS} \quad S = \frac{\theta}{8\pi^2}\int_Y F \wedge F$ $\quad \theta = 0$ *trivial insulator*, $\theta = \pi$ *topological insulator*

(Since $\theta \sim \theta + 2\pi, \theta = \pi$ is a T-invariant configuration)

Z is not T-invariant when $\partial Y \neq 0$

Boundary: massless Dirac fermion

$Z_\psi = |Z_\psi| e^{-\frac{i\pi}{2}\eta(X)} \quad \eta =$ APS eta invariant

Boundary modes massless because of T symmetry: breaking T gaps the system → trivial insulator if not time reversal

2+1d Dirac fermion has global T anomaly (Z is not real)

Anomaly inflow: APS index theorem

Witten (2016)

Partition function of the total system is well defined and T-invariant

Bosonization

Topological phases are also a nice framework to study bosonization (see QHE)

Gaiotto,Kapustin (2016) Bhardwaj, Gaiotto, Kapustin (2017), Thorngren (2018)

Bosonization: relation between a fermionic and a bosonic theory

$$Z_b \longleftrightarrow Z_f$$

Systematic approach: sum = gauging over spin structures (old result in 2d)

- *Summing over the spin structures is equivalent to gauging fermion parity* $\mathbb{Z}_2^f = (-1)^F$ ($F =$ # of fermions mod 2)
- The spin structure η can be thought of as a gauge field for $\mathbb{Z}_2^f$ with a twisted flux $d\eta = w_2(TX)$
- The bosonic theory has a dual $\mathbb{Z}_2^{(d-2)}$ symmetry: gauging it gives back the fermionic theory
- The dual $\mathbb{Z}_2^{(\mathrm{d}-2)}$ symmetry is anomalous, with a precise form for its 't Hooft anomaly

In 2+1d there are other methods to fermionize a bosonic theory, most notably the *flux attachment.* It amounts to couple the bosonic theory to a Chern-Simons gauge field that creates the fermionic statistics. Old technique in non-relativistic physics, can be applied also to critical massless QFTs, where it induces a duality web between critical QFTs

Our work

arXiv: 2406.01787

- Study of a proposed dynamics for surface degrees of freedom of TI. It is called ‘loop model’, a quadratic non-local theory that seems to be a 3dCFT with a critical line. It is a bosonic theory though: to correctly describe fermionic excitations, fermionization was needed. This shows how spin sectors and time reversal arise from the bosonic theory.

arXiv: 2503.02801

- We reviewed and compared two approaches of bosonization in 2+1d, sum over spin structures and flux attachment. By applying both to the loop model, explicit results are obtained, since the calculations can be done explicitly. Moreover, we combined the sum over spin structures with the dualities induced by flux attachment to get new bosonic dualities out of the known 2+1d duality web.

Thanks for the attention!